Two Parameter Smooth Martingales on the Wiener Space

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Abstract We prove that two parameter smooth continuous martingales have $\omega$-modification and establish a Doob's inequality in terms of $(p, r)$-capacity for two parameter smooth martingales.

Keywords Two parameter smooth martingales, $\omega$-modifications, Doob's inequality, $(p, r)$-capacity

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1 Introduction

Quasi-sure analysis of Wiener functionals has been extensively developed since the pioneering work of P. Malliavin [7] (see [6, 8, 11, 12, 13, 15], etc). Recently, P. Malliavin and D. Nualart [8], J. Ren [11, 12] and Z. Liang [6], have studied the quasi-sure properties of one parameter smooth martingales and two smooth parameter martingales. In the present paper, we will follow their work, proving that two parameter smooth martingales have $\omega$-modification, and establish a Doob's inequality in terms of $(p, r)$-capacity for two parameter smooth martingales.

2 Preliminaries

We first recall and fix some notions and notations. Throughout this paper we shall work on the probability space $(X, H, \mu)$, where $X$ is the space of continuous maps from $\Pi \equiv [0, 1]^2$ to $\mathbb{R}$ and vanishing on the axes; $H$ is the Cameron-Martin subspace, i.e. $H = \{ f \in X, \frac{\partial^2 f}{\partial s \partial t} \text{ exists a.e. and } f \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial s \partial t} \, ds \, dt < +\infty \}$; $\mu$ is the two parameter standard Wiener measure defined on the Borel $\sigma$-field $B(X)$ (see [9, 16]). The parameter space for our two parameter stochastic processes is $\Pi$ in which a partial ordering is introduced by defining $z_1 \leq z_2$ for $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ in $\Pi$ when $s_1 \leq s_2$ and $t_1 \leq t_2$; $z_1 \wedge z_2$ when $s_1 \leq s_2$ and $t_1 \geq t_2$. Let $\{W_z, z \in \Pi\}$ be the coordinate Brownian sheet on $X$, $\mathcal{F}_z$ be the sub $\sigma$-field of $B(X)$ generated by the Brownian sheet up to $z$, i.e., $\mathcal{F}_z \equiv \sigma\{W_{z'}, z' \leq z\}$, $\mathcal{F}_z^1 \equiv \bigvee_{0 \leq t \leq 1} \mathcal{F}_{(s,t)} \equiv \mathcal{F}_{(s,1)}^1$, $\mathcal{F}_z^2 \equiv \bigvee_{0 \leq s \leq 1} \mathcal{F}_{(1,t)} \equiv \mathcal{F}_{(1,t)}$ for...
z = (s, t) ∈ Π. Then \( \{ F_z, z ∈ Π \} \) satisfies the usual conditions (F1) - (F4) as introduced in [1] and [2]. If \( f \) is a map from \( Π \) to \( R \), the increment of \( f \) on the rectangle \((z_1, z_2) \equiv \{ ξ | z_1 < ξ ≤ z_2 \}, z_1 = (s_1, t_1), z_2 = (s_2, t_2) = f((z_1, z_2)) \equiv f(s_1, t_1) - f(s_1, t_2) - f(s_2, t_1) + f(s_2, t_2) \) We denote the rectangle \((0, 2]\) by \( R_2 \). The basic definitions for two parameter martingales, \( i \)-martingales (\( i = 1, 2 \)) and surface integral (i.e., the stochastic integral of the first type and of the second type) are contained in [1] and [14].

Denote by \( W^p_{2r} \) the Sobolev subspace of order \( 2r \) and of power \( p \) over \((X, H, \mu)\) equipped with the norm: \( \| f \|_{p, 2r} := \| (I - L)^r f \|_p \), where \( L \) is the Ornstein-Uhlenbeck operator on \( X \) (see [3, 7, 13]). By Meyer's inequalities, there exists a constant \( A_{p, 2r} > 0 \) such that

\[
A_{p, 2r}^{-1} \| f \|_{p, 2r} \leq \| f \|_{p, 2r} \leq A_{p, 2r} \| f \|_{p, 2r}
\]

for all \( f \in W^p_{2r} \), where

\[
\| f \|_{p, 2r} := \| \nabla^{2r} f \|_p + \| f \|_p,
\]

\( \nabla \) being the gradient operator (see [3, 7, 13]).

Given an open set \( O \) in \( X \), its \((p, r)\)-capacity is defined as (see [7])

\[
C_{p, r}(O) = \inf \{ \| u \|_{p, 2r}; u ≥ 1, u - a.e. on O \},
\]

and for any subset \( A \) of \( X \), the capacity is defined to be

\[
C_{p, r}(A) = \inf \{ C_{p, r}(O); O is open and O ⊆ A \}.
\]

If \( C_{p, r}(A) = 0 \) for all \( p ≥ 2 \) and for all \( r ∈ N \), then \( A \) is called a slim set (see [1]). If some property holds except on a slim set, then we say that it holds quasi-surely (abbr. q.S.). It is well known that for any element \( f \) in \( W_∞(R) = \bigcap W^p_{2r} \), we can find a function \( f^* \) such that

1. \( f^* = f - a.e.; \)
2. for each pair \((p, r)\), \( ε > 0 \), there exists an open set \( O \) with \( C_{p, r}(O) < ε \) such that \( f^* \) is continuous on \( X \setminus O \). \( f^* \) is referred to as a redefinition of \( f \). Obviously, any two redefinitions of a function coincide except on a slim set. Any function with property (2) above is called \( oo \)-quasi continuous.

In addition to the J.Ren's Kolmogorovs criterion, the important tool we will use is the concept of \( oo \)-modification of a random field.

**Definition 2.1** (See [3, 7, 10]) Let \( \{ X(t), t ∈ D \} \) be a random field, where \( D \) is a domain in \( R^d \). A random field \( \{ \tilde{X}(t), t ∈ D \} \) is called an \( oo \)-modification of \( \{ X(t), t ∈ D \} \) if

i. \( \tilde{X}(t) = X(t), a.e. for each t ∈ D; \)
ii. \( \tilde{X}(, ) \) are continuous in \( D \) q.S.;
iii. \( \tilde{X}(t, ) \) is \( oo \)-quasi continuous for each \( t ∈ D \).

**Theorem 2.1** (See[3, 10, 11]) Suppose that for any pair \((p, r)\) we can find an even \( \beta(p, r) \) and two positive constants \( c = c(p, r), α = α(p, r) \) such that

i. \( X(t) ∈ W_∞^p \) for each \( t ∈ D; \)
ii. \( (X(t) - X(s))^β ∈ W_∞^p \) for each \((t, s) ∈ D × D; \)
iii. \( \| (X(t) - X(s))^β \|_{p, 2r} ≤ C\| t - s \|^{α + d} \) for each \((t, s) ∈ D × D \)

where \( \| t - s \| = \sum_{j=1}^d |t_j - s_j| \).

Then \( \{ X(t), t ∈ D \} \) has an \( oo \)-modification.
Theorem 2.2 (See [3, 10, 11]) If two processes $X_1(t, \omega), X_2(t, \omega)$ satisfy the condition of Theorem 2.1 and if $X_1(t, \omega) \leq X_2(t, \omega)$ a.e for every $t$, then $\tilde{X}_1(t, \omega) \leq \tilde{X}_2(t, \omega)$ a.e.

In addition, we will use the following Fa'ade Bruno’s inequalities (see [10]) and Tchebyshev-type inequality on capacity.

Theorem 2.3 For any fixed $p \geq 2$, $r \in N$ and $n \geq 2$, there exists a constant $c = c(n, p, r)$ such that

$$\|g^n\|_{p, 2r} \leq c\|g\|_{4, 2p, 2r}^{2p} \max_{0 \leq a \leq 2r} [E|g|^{(n-a)2rp}]^{\frac{1}{2p}}.$$ 

Theorem 2.4 Suppose that $u \in W^p_{2r}$, $u^*$ is its refinement. Then

$$C_{p, r}(\|u^*\| > \varepsilon) \leq \frac{1}{\varepsilon}\|u\|_{p, 2r}$$

for any $\varepsilon > 0$.

Now we shall define two parameter smooth martingales. Let $L^2_W$ be that of $\mathcal{F}_t$-predictable processes $\phi = \{\phi(z), z \in \Pi\}$ for which $E\{\int_{R_t} \phi(\eta)^2 d\eta\} < +\infty$ for all $z \in \Pi$, and $L^2_{WW}$ be the class of all processes $\psi = \{\psi(\xi, \eta), \xi, \eta \in \Pi\}$ satisfying: (i) $\psi$ is predictable, (ii) $\psi(\xi, \eta) = 0$ unless $\xi \land \eta$, (iii) $E\{\int_{R_t} \psi(\xi, \eta)^2 d\xi d\eta\} < +\infty$ for all $z \in \Pi$. By [1] and [14], for any square integrable martingale $N$ which vanishes on the axes, we have

$$N_z = \phi \cdot W_z + \psi \cdot WW_z$$

where $\phi \in L^2_W$ and $\psi \in L^2_{WW}$.

Following P. Malliavin and D. Nualart [8], we say that $N$ is smooth if the following conditions are fulfilled:

(c.1) $\phi(z) \in W^\infty(R)$ for all almost $z \in \Pi$ and $\phi \in L^2_W$, $\int_0^1 \int_0^1 \|\phi(z)\|_{p, 2r} d\xi d\eta < +\infty$ for all $p, r$.

(c.2) $\psi(\xi, \eta) \in W^\infty(R)$ for all almost $\xi, \eta \in \Pi$, $\psi \in L^2_{WW}$ and for all $p, r$, $\int_{\Pi \times \Pi} \|\psi(\xi, \eta)\|^2_{p, 2r} d\xi d\eta < +\infty$.

3 Main Results

Theorem 3.1 Let $N$ be a two parameter smooth martingale represented as (2). Then

$$\{N(z), z \in \Pi\}$$

has an $oo$-modification.

For simplicity, throughout this paper, all the constants depending only on $N$, $p$, but not on $n$ and the parameter $z$, will be denoted by $c$. To prove Theorem 3.1 we need the following proposition.

Proposition 3.1 For any square integrable martingale $N$ represented as (2), we have

$$L(N_z) = \int_{R_t} (L\phi - \frac{1}{2}\phi)(\eta) dW_\eta + \int_{R_t \times R_t} (L\psi - \psi)(\xi, \eta) dW_\xi dW_\eta$$

for any $z \in \Pi$.

Proof By the definition of $L$ (see [3]) and Stroock's commutation formula, we have

$$L \left( \int_{R_t} \phi(\eta) dW_\eta \right) = \int_{R_t} \left( L\phi - \frac{1}{2}\phi \right)(\eta) dW_\eta.$$
On the other hand, by the stochastic Fubini's Theorem 2.6 of [1], we get
\[
\int \int_{R_\times R_\times R} \psi(\xi, \eta)dW_\xi dW_\eta = \int_{R_\times R} \left( \int_{R_\times R} \psi(\xi, \eta)dW_\eta \right) dW_\xi.
\]
Combining this formula with (4), we obtain
\[
L \left( \int \int_{R_\times R_\times R} \psi(\xi, \eta)dW_\xi dW_\eta \right) = \int \int_{R_\times R_\times R} (L\psi - \psi)(\xi, \eta)dW_\xi dW_\eta.
\]
Hence this proof is finished.

**Proof of Theorem 3.1** By [1], for any \( z \in \Pi \), we have
\[
\left( \int \int_{R_\times R_\times R} \psi(\xi, \eta)dW_\xi dW_\eta \right)(z) = \int \int_{R_\times R_\times R} \psi(\xi, \eta)^2 d\xi d\eta,
\]
\[
\left( \int \int_{R_\times R} \psi(\eta)dW_\eta \right)(z) = \int_{R_\times R} \phi(\eta)^2 d\eta.
\]
Hence by Proposition 3.1,
\[
E[(I - L)N_z]^p \leq cE \left| \int_{R_\times R} \left( \frac{3}{2} \phi - L\phi \right)(\eta)dW_\eta \right|^p + cE \left| \int \int_{R_\times R_\times R} (2\psi - L\psi)(\xi, \eta)dW_\xi dW_\eta \right|^p
\]
(by Burkholder's inequality for two parameter continuous martingales, see [2])
\[
\leq cE \left( \int_{R_\times R} \left( \frac{3}{2} \phi - L\phi \right)^2(\xi)d\xi \right)^{\frac{p}{2}} + cE \left( \int \int_{R_\times R_\times R} (2\psi - L\psi)^2 d\xi d\eta \right)^{\frac{p}{2}}
\]
(by Hölder's inequality)
\[
\leq c \int_0^1 \int_0^1 \|\phi\|^p_{p,2} d\xi + c \int_\Pi \Pi \|\psi\|^p_{p,2} d\xi d\eta.
\]
Therefore we get
\[
\sup_z \|N_z\|^p_{p,2} \leq c \int_0^1 \int_0^1 \|\phi\|^p_{p,2} d\xi + c \int_\Pi \Pi \|\psi\|^p_{p,2} d\xi d\eta < +\infty
\]
(by (c.1) and (c.2)).

And in the same way (by Proposition 3.1),
\[
\sup_z E[(I - L)^r N_z]^p \leq c \int_0^1 \int_0^1 \|\phi(\eta)\|^p_{p,2} d\eta + c \int_\Pi \Pi \|\psi(\xi, \eta)\|^p_{p,2} d\xi d\eta < +\infty
\]
for all \( p, r \).

Let \( \tilde{M}_z \equiv \phi \cdot W_z \) and \( \tilde{M}_z \equiv \psi \cdot WW_z \), for any \( z \leq z' \). We have
\[
N_{z'} - N_z = \tilde{M}(D_1) + \tilde{M}(D_2) + \tilde{M}(D_1) + \tilde{M}(D_2)
\]
where \( D_1 = (0, s'] \times (t, t'], D_2 = (s, s'] \times (0, t) \) for \( z = (s, t) \) and \( z' = (s', t') \in \Pi \).
By the definition of the stochastic integral of second type, we have
\[
\overline{M}(D_1) = \int \int_{D_1 \times D_2} \psi(\xi, \eta) dW_\xi dW_\eta,
\]
\[
\overline{M}(D_2) = \int \int_{R_{\gamma_1} \times D_2} \psi(\xi, \eta) dW_\xi dW_\eta.
\]

Using again Burkholder's inequality and Hölder's inequality, we get
\[
\|\overline{M}(D_1)\|_{p,2}^p \leq c\mathbb{E} \left( \int_{D_1} \left| \frac{3}{2} \phi - L\phi \right|^2 d\xi \right)^{\frac{p}{2}}
\leq cm(D_1)^{\frac{p}{2} - 1} \int_0^1 \int_0^1 \mathbb{E} \left| \frac{3}{2} \phi - L\phi \right|^p d\xi
\leq \left( c \int_0^1 \int_0^1 \|\phi(\xi)\|_{p,2}^p d\xi \right) |t - t'|^{\frac{p}{2} - 1}
\leq c|t - t'|^{\frac{p}{2} - 1},
\]
(by (c.1))

\[
\|\overline{M}(D_1)\|_{p,2}^p \leq c\mathbb{E} \left( \int \int_{D_1 \times D_2} |2\psi - \psi|^2 d\xi d\eta \right)^{\frac{p}{2}}
\leq cm(D_1 \times D_2)^{\frac{p}{2} - 1} \int \int_{D_1 \times D_2} \mathbb{E}|2\psi - \psi|^p d\xi d\eta
\leq c|t - t'|^{\frac{p}{2} - 1}.
\]

Similarly,
\[
\|\overline{M}(D_2)\|_{p,2}^p \leq c|s - s'|^{\frac{p}{2} - 1},
\]
\[
\|\overline{M}(D_2)\|_{p,2r}^p \leq c|s - s'|^{\frac{p}{2} - 1}.
\]

A combination of (6), (8), (9), (10) and (11) implies
\[
\|N_{s'} - N_s\|_{p,2}^p \leq c|s - s'|^{\frac{p}{2} - 1} + c|t - t'|^{\frac{p}{2} - 1}.
\]

And in general we can prove by Proposition 3.1 that
\[
\|N_{s'} - N_s\|_{p,2r}^p \leq c|s - s'|^{\frac{p}{2} - 1} + c|t - t'|^{\frac{p}{2} - 1}
\]
for all \(p, r\). For the case of \(z \wedge z'\), the method of proof of (12) is completely similar to that of the case \(z \leq z'\), we don't give the details.

Hence by (5), (10) and Theorem 2.3 we can choose \(n\) sufficiently large so that
\[
\|(N_{s't'} - N_{st})^n\|_{p,2r} \leq c'_n(|s' - s|^{2 + \alpha} + |t' - t|^{2 + \alpha})
\]
with some positive constants \(c'_n\) and \(\alpha\). Therefore the conditions of Theorem 2.1 are fulfilled, and then \(\{N_z, z \in \Pi\}\) has \(\infty\)-modification, and the proof of Theorem 3.1 is finished.
We denote \( \infty \)-modification of \( N \) by \( \tilde{N} \). Now we present Doob's maximal inequality in terms of \((p, r)\)-capacity for two parameter smooth martingales.

**Theorem 3.2** If \( \{N_{z}, z \in \Pi\} \) is a two parameter smooth martingale represented as \((2)\), then

\[
C_{p, r} \left( \sup_{z \in \Pi} |\tilde{N}(z, \omega)| \geq \varepsilon \right) \leq \frac{1}{\varepsilon} \left( \frac{p}{p-1} \right)^{2} \|N_{(1, 1)}(\omega)\|_{p, 2r} \tag{13}
\]

for any \( \varepsilon > 0, p > 1 \) and \( r > 0 \).

Before proving this theorem, we will use the following result.

**Proposition 3.2** Let \( f \in \cap L_{p} \) and \( x_{t} = P_{t}f \), where \( \{P_{t}, t \geq 0\} \) is the Ornstein-Uhlenbeck \( p \)-semigroup generated by \( L \). Then \( \{x_{t}, t \geq 0\} \) satisfies the conditions of Theorem 2.2 and for any \( p \geq 1 \) and integer \( r \) we have

\[
((I - L)^{-r} f)^{*} = \int_{0}^{\infty} e^{-t} t^{r-1} \tilde{x}_{t} dt \tag{14}
\]

The proof of Proposition 3.2 is contained in [11].

**Proof of Theorem 3.2** We divide the proof into two steps.

**Step 1** We first consider the case: \( r \) is an integer. By the lower continuity of capacity (see [5]) and the quasi-sure continuity of the trajectories it is sufficient to prove (13) for \( z \in \{(s_{i}, t_{j})\}_{i,j=1}^{\infty} \), where \( (s_{i}, t_{j}) \in \Pi \).

\[
C_{p, r} \left( \sup_{z \in \Pi} |\tilde{N}(z, \omega)| \geq \varepsilon \right)
= C_{p, r} \left( \sup_{i,j} |\tilde{N}_{s_{i}, t_{j}}(\omega)| \geq \varepsilon \right)
\]

(by Definition 2.1 and the definition of \( f^{*} \))

\[
= C_{p, r} \left( \sup_{i,j} |(N_{s_{i}, t_{j}}(\omega))^{*}| \geq \varepsilon \right)
= C_{p, r} \left( \sup_{i,j} |((I - L)^{-r}(I - L)^{r}N_{s_{i}, t_{j}}(\omega))^{*}| \geq \varepsilon \right)
\]

(by \((14)\))

\[
= C_{p, r} \left( \sup_{i,j} \left| \int_{0}^{\infty} e^{-t} t^{r-1}(p_{t}(I - L)^{r}N_{s_{i}, t_{j}}(\omega))^{*} dt \right| \geq \varepsilon \right)
\leq C_{p, r} \left( \int_{0}^{\infty} \sup_{i,j} e^{-t} t^{r-1}(p_{t}(I - L)^{r}N_{s_{i}, t_{j}}(\omega))^{*} dt \geq \varepsilon \right)
\]

(by the Markov property of \( p_{t} \) and Theorem 2.2)
\[
\leq C_{p,r} \left( \int_0^\infty e^{-t} t^{r-1} \left( p_t \sup_{i,j} |(I-L)^r N_{s,t_j}(\omega)| \right)^\ast \, dt \geq \varepsilon \right)
\]
(by (14))

\[
= C_{p,r} \left( \left( (I-L)^{-r} \sup_{i,j} |(I-L)^r N_{s,t_j}(\omega)| \right)^\ast \geq \varepsilon \right)
\]
(by Theorem 2.4)

\[
\leq \frac{1}{\varepsilon} \left\| \sup_{i,j} |(I-L)^r N_{s,t_j}| \right\|_p
\]
(Noting that by using Proposition 3.1, \((I-L)^r N\) is a two parameter martingale and by using Doob's inequality for two parameter martingales, see [4])

\[
\leq \frac{1}{\varepsilon} \left( \frac{p}{p-1} \right)^2 \|N(1,1)(\omega)\|_{p,2r}.
\]

**Step 2** For the case where \(r\) is not an integer. Let \(C_{t,n}\) denote the \(n\)th Wiener's homogeneous chaos until \(t\). Then \(L_2(F_{s}^1)\) and \(L_2(F_{s}^2)\) for \(z = (s,t)\) admit decompositions into direct sums,

\[
L_2(F_{s}^1) = C_{s,0} \oplus C_{s,1} \oplus C_{s,2} \oplus \cdots \oplus C_{s,n} \oplus \cdots
\]

and

\[
L_2(F_{s}^2) = C_{t,0} \oplus C_{t,1} \oplus C_{t,2} \oplus \cdots \oplus C_{t,n} \oplus \cdots
\]

respectively.

For any fixed \(s_0\) and \(t_0\), write the decompositions of \(N_{s't_0} - N_{s_0t_0}\) and \(N_{s_0t'} - N_{s_0t}\) as

\[
N_{s't_0} - N_{s_0t_0} = G_1^1(s,s',t_0) + G_2^1(s,s',t_0) + \cdots + G_n^1(s,s',t_0) + \cdots
\]

and

\[
N_{s_0t'} - N_{s_0t} = G_1^2(s_0,t,t') + G_2^2(s_0,t,t') + \cdots + G_n^2(s_0,t,t') + \cdots
\]

respectively, for any \(s < s'\) and \(t < t'\).

By [1], that \(N\) is a two parameter martingale is equivalent to say that \(N\) is both a 1-martingale and a 2-martingale. On the other hand, that \(N\) is a 1-martingale is equivalent to say that \(G_1^1(s,s',t_0)\) is orthogonal to \(C_{s,n}\) for every \(n\), and that \(N\) is a 2-martingale is equivalent to say that \(G_2^2(s_0,t,t')\) is orthogonal to \(C_{t,n}\) for every \(n\). This implies \(\left(1 + \frac{n}{2}\right)^r G_n^1(s,s',t_0)\) and \(\left(1 + \frac{n}{2}\right)^r G_n^2(s_0,t,t')\) are orthogonal to \(C_{s,n}\) and \(C_{t,n}\) respectively, showing that \((I-L)^r N\) is both a 1-martingale and a 2-martingale. Hence \((I-L)^r N\) is a two parameter martingale. By the Markov property and semigroup's property of \((I-L)^r N\), similarly to the proof of Step 1, we have

\[
C_{p,r} \left( \sup_{x \in \Omega} |\tilde{N}_x| \geq \varepsilon \right) \leq C_{p,r} \left( \left( (I-L)^{-r} \left( \sup_{i,j} |(I-L)^r N_{s,t_j}(\omega)| \right) \right)^\ast \geq \varepsilon \right)
\]

\[
\leq \frac{1}{\varepsilon} \left\| \sup_{i,j} |(I-L)^r N_{s,t_j}| \right\|_p
\]

\[
\leq \frac{1}{p} \left( \frac{p}{p-1} \right)^2 \frac{\|N(1,1)(\omega)\|_{p,2r}}{\varepsilon}.
\]
Therefore the proof of Theorem 3.2 is completed.

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