Optimal control of the insurance company with proportional reinsurance policy under solvency constraints

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A B S T R A C T

This paper considers the optimal control problem of the insurance company with proportional reinsurance policy under solvency constraints. The management of the company controls the reinsurance rate and dividends payout processes to maximize the expected present value of the dividend until the time of bankruptcy. This is a mixed singular-regular control problem. However, the optimal dividend payout barrier may be too low to be acceptable. The company may be prohibited to pay dividend according to external reasons because this low dividend payout barrier will result in bankruptcy soon. Therefore, some constraints on the insurance company’s dividend policy will be imposed. One reasonable and normal constraint is that if \( b \) is the minimum dividend barrier, then the bankrupt probability should not be larger than some predetermined \( \varepsilon \) within the time horizon \( T \). This paper is to work out the optimal control policy of the insurance company under the solvency constraints.

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1. Introduction

In this paper, we consider an insurance company in which the dividend payout and the risk exposure are controlled by the management. We assume the company can reduce its risk exposure by proportional insurance policy for simplicity. We associate the value of the company with the expected present value of the dividends payout until the time of bankruptcy. However, the optimal dividend payout barrier may be unacceptably low. The company may be prohibited to pay dividend at such a low barrier according to external reasons. To solve the problem, some constraints will be imposed on the insurance company’s solvency constraints.


Unfortunately, there are very few results concerning on optimal control of the insurance company with solvency constraints. According to Miller Modigliani, the management of the company chooses the maximum of shareholders’ return as their goals. However, if the optimal dividend payout barrier is too low to be acceptable and it will result in bankruptcy shortly, then the company may be prohibited to pay dividend at such a low barrier according to external reasons. To solve the problem, some constraints will be imposed on the insurance company’s...
dividend policy. One reasonable and normal constraint is that if $b$ is the minimum dividend barrier, then $b$ is such that bankruptcy probability is not larger than some predetermined $\varepsilon$ within the time horizon $T$. The optimal control problem on minimum the bankrupt probability is discussed in Cadenillas et al. (2006) and Paulsen (2003) solves the optimal dividend payout for diffusions with solvency constraints. By these ideas we further discuss the optimal control of the insurance company with proportional reinsurance policy under solvency constraints.

The paper is organized as follows: In Section 2, we establish the control model of the insurance company with proportional reinsurance policy under solvency constraints. In Section 3, we give a rigorous probability proof on the bankruptcy probability is decreasing with respect to the dividend barrier $b$, which is the right economic point but we cannot find rigorous mathematical proof in existing literatures. The fact is a basis of solving the control problem we consider. The solutions of the HJB equation and existence of the dividend barrier $b$ of the insurance company with proportional reinsurance policy under solvency constraints is given in Section 4. The optimal return function and the optimal control policy are established in Section 5. The conclusion is given in Section 6.

2. Control model of the insurance company with solvency constraints

We consider an insurance company with proportional reinsurance policy. In this case, the company’s management can accommodate the profit and the risk by choosing dividend payout and the reinsurance rate. In this paper we will consider the linear Brownian motion model. In this model, if there are no dividends payout to control the risk, then the liquid reserves of the company evolve according to the following stochastic differential equation,

$$dR_t = \mu a(t) dt + \sigma a(t) dW_t,$$

where $W_t$ is a standard Brownian Motion, $1-a(t) \in [0, 1]$ is the proportional reinsurance rate.

To give a mathematical foundation of the optimization problem, we fixed a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\{W_t, t \geq 0\}$ is a standard Brownian Motion on this probability space. $\mathcal{F}_t$ represents the information available at time $t$ and any decision made up to time $t$ is based on this information $\mathcal{F}_t$. In our model, we denote $L_t$ as the cumulative amount of dividends paid from time 0 to time $t$. We assume that the process $\{L_t, t \geq 0\}$ is $(\mathcal{F}_t, t \geq 0)$-adapted, increasing and right-continuous with left limits.

A control policy $\pi$ is described by stochastic process $\{a_t, \mathcal{L}_t \} \in \mathcal{P}$, where $\mathcal{P}$ denotes the whole set of the admissible policies according to the solvency constraints. Given a control policy $\pi$, we assume that the liquid reserves of the insurance company are modeled by the following equation,

$$dR_t^\pi = \mu a_t(t) dt + \sigma a_t(t) dW_t - dL^\pi_t, \quad R_0^\pi = x,$$

where the $R_0^\pi = x$ means that the initial liquid reserve is $x$ under control policy $\pi$.

In this case, we assume the company needs to keep its reserves above 0. The company is considered bankrupt as soon as the reserves fall below 0. We define the time of bankruptcy as $\tau = \inf\{t \geq 0 : R_t^\pi < 0\}$, $\tau$ is an $\mathcal{F}_t$-stopping time.

So the management of the insurance company should maximize the expected present value of the dividends payout by control policy $\pi$. Hauggaard and Taksar (1999) and He and Liang (2008) and other authors proved that there exist a dividend barrier $b_0$, control policy $\pi_{b_0}$ and the time $\tau^{b_0}$ of bankruptcy maximizing the expected present value of the dividends payout before bankruptcy,

$$J(x, \pi) = E \left( \int_0^T e^{-\delta s} dL^\pi_s \right),$$

$$J(x, \pi_{b_0}) = \sup_{\pi \in \mathcal{P}} J(x, \pi), \quad \pi \in \mathcal{P},$$

where $\delta$ denotes the discount rate. If the optimal dividend barrier is unacceptably low, then one reasonable and normal constraint imposed on the insurance company’s dividend policy is following: the dividend barrier $b_0$, control policy $\pi_{b_0}$ should solve the following problem with solvency constraints for given $\varepsilon > 0$.

$$J(x, \pi_{b_0}) = \sup_{\pi \in \mathcal{P}} J(x, \pi),$$

$$P\{\tau^{b_0} < T\} \leq \varepsilon,$$

where $\tau^{b_0}$ is the time $\tau^{b_0}$ of bankruptcy when the dividend barrier is $b_0$, the initial asset $x = b_0$ and the control policy is $\pi$. we call $b_0$ on (2.2) and (2.3) the dividend barrier without solvency constraints, and $b$ on (2.4) and (2.5) the dividend barrier with solvency constraints. If the $b$ exists, then we define the optimal return function of the insurance company with proportional reinsurance policy under solvency constraints (2.5) as follows:

$$V(x) := \sup_{\pi} \{V(x, b)\}, \quad \pi \in \mathcal{P},$$

where the sup is taken over $b$ and $b$ is the dividend barrier with solvency constraints, $V(x, b) := \inf_{\pi} J(x, \pi_{b_0})$. The main goal of the present paper is to solve the optimal control problem (2.4)–(2.6). For the goal we only need to firstly show that the bankruptcy probability $\psi^b(T, x) = P(\tau^{b_0} < T)$ is decreasing with respect to the dividend barrier $b$, then prove the existence of such the dividend barrier $b$ with solvency constraints, finally get $V(x)$ and the optimal strategy $\pi^* := \pi_{\pi_{b_0}}$, see Section 5) associated with $V(x)$ under the solvency constraints.

3. Properties of the bankrupt probability $\psi^b(T, b)$

Given a finite time horizon $T$ and a ruin tolerance $\varepsilon > 0$, we define

$$\psi^b(T, b) := \psi^b(T, x)|_{a=b} = P(\tau^{b_0} < T) \leq \varepsilon$$

for $b$, where $\psi^b(T, x) = P(\tau^{b_0} < T)$ is the bankruptcy probability of the initial asset $x$ and the dividend barrier $b$ is employed before time $T$. In this section we will prove the bankruptcy probability $\psi^b(T, b)$ is decreasing with respect to the dividend barrier $b$ and the existence of $b$ satisfying (3.1) by probability approach. We define $\inf \psi^b = \infty$ through this paper for convenience. The first result of this section is following.

**Theorem 3.1.** Let $\psi^b(T, x)$ be defined by (3.1). Then the function $\psi^b(T, b)$ is decreasing in $b$ for fixed $T$.

**Proof.** Assume $b_1 > b_2$ and $\{R^b_t\}_{t \geq 0}$ is the surplus process of the initial asset $x$ when a dividend barrier $b$ is employed. Define stopping times

$$\tau^{(1)} = \inf\{t : t > 0, R^{b_1}_{t+1} = R^{b_2}_{t+2}\},$$

$$\tau_1 = \inf\{t : t \geq \tau^{(1)}, R^{b_1}_{t+1} = b_2\}.$$ 

It is easy to see that

$$R^{b_1}_{\tau_1} = R^{b_2}_{\tau_2}, \quad \text{on } [0, \tau^{(1)}] \quad \text{and} \quad R^{b_1}_{\tau_1} = R^{b_2}_{\tau_2}.$$  

Using the strong Markov property of $\{R^b_t\}_{t \geq 0}$ at $\tau_1$ and $R^{b_1}_{\tau_1} = R^{b_2}_{\tau_1} = b_2$, we know that the surpluses $R^{b_1}_{\tau_1}$ and $R^{b_2}_{\tau_2}$ follow the
same stochastic differential equations with the same initial value $b_2$ on $[\tau^{(1)}, \tau_1]$. Therefore,

$$R^{b_1, b_1}_{\tau_1} = R^{b_2, b_2}_{\tau_2} \text{ on } [\tau^{(1)}, \tau_1].$$

Define

$$\bar{\tau} = \inf\{t : t > \tau, R^{b_1, b_1}_{\tau} = R^{b_2, b_2}_{\tau}\}.$$ 

Because the dividend barrier $b_2$ is employed at $\tau_1$, we have

$$R^{b_1, b_1}_{\tau_1} \geq R^{b_2, b_2}_{\tau_2} \text{ on } [\tau_1, \bar{\tau}) \text{ and } R^{b_1, b_1}_{\tau} = R^{b_2, b_2}_{\tau}.$$ 

So

$$R^{b_1, b_1}_{\bar{\tau}} \geq R^{b_2, b_2}_{\bar{\tau}} \text{ on } [\tau^{(1)}, \bar{\tau}).$$

Repeating the same procedure above, we can get stopping times $\bar{\tau}^{(k)}, k = 1, 2, \ldots, N$ such that

$$R^{b_1, b_1}_{\bar{\tau}^{(k)}} \geq R^{b_2, b_2}_{\bar{\tau}^{(k)}} \text{ on } [\tau^{(k)}, \bar{\tau}^{(k+1)}] \text{ and } \tau^{(N)}, \infty],$$

where

$$N = \inf\{k : k \geq 0 \text{ such that } R^{b_1, b_1}_{\bar{\tau}^{(k)}} = R^{b_2, b_2}_{\bar{\tau}^{(k)}},$$

or $R^{b_1, b_1}_{\tau} > R^{b_2, b_2}_{\tau}, \forall \tau \in [\tau^{(k)}, \infty]\}.$

Noting that

$$R^{b_1, b_1}_{\tau} = \sum_{k=0}^{N-1} I_{[\tau^{(k)}, \tau^{(k+1)}]}(t)R^{b_1, b_1}_{\tau} + I_{[\tau^{(N)}, \infty]}(t)R^{b_1, b_1}_{\bar{\tau}},$$

$$R^{b_2, b_2}_{\tau} = \sum_{k=0}^{N-1} I_{[\tau^{(k)}, \tau^{(k+1)}]}(t)R^{b_2, b_2}_{\tau} + I_{[\tau^{(N)}, \infty]}(t)R^{b_2, b_2}_{\bar{\tau}},$$

where $\tau^{(0)} = 0$, it is easy to see from (3.2) and (3.3) that

$$R^{b_1, b_1}_{\tau} \geq R^{b_2, b_2}_{\tau}, \forall \tau \geq 0.$$ 

Thus, by the definition of $\tau^{b}_x$, it easily follows that

$$\tau^{b}_1 \geq \tau^{b}_2,$$

which implies that $P(\tau^{b}_1 < T) \leq P(\tau^{b}_2 < T)$ for fixed $T$. So we complete the proof.

We now prove the second result of this section concerning on existence of the dividend barrier $b$ with solvency constraints (2.5).

**Theorem 3.2.** There exists at least $b > 0$ satisfying (2.5).

**Proof.** Define $\psi_b(T, x) = P(\tau^b_x \leq T)$ by (3.1). Using the same way as used in Theorem 3.1, we have the following

$$\psi_b(T, b) \leq \psi_b(T, b/2),$$

that is,

$$P(\tau^b_x \leq T) \leq P(\tau^b_{x/2} \leq T).$$

Assume $\{\tilde{R}_t\}_{t \geq 0}$ satisfies the following stochastic differential equation

$$d\tilde{R}_t = \mu(\tilde{R}_t)dt + \sigma(\tilde{R}_t)dW_t, \quad \tilde{R}_0 = b/2.$$ 

By comparison theorem on SDE (see Ikeda and Watanabe (1981)), we have

$$\{\tau^b_x \leq T\} \subseteq \{\exists t \leq T \text{ such that } \tilde{R}_t = 0 \text{ or } \tilde{R}_t = b\}.$$ 

Thus,

$$P(\tau^b_{x/2} \leq T) \leq P(\exists t \leq T \text{ such that } \tilde{R}_t = 0 \text{ or } \tilde{R}_t = b)$$

$$\leq P(\sup_{0 \leq t \leq T} \tilde{R}_t \geq b) + P(\inf_{0 \leq t \leq T} \tilde{R}_t \leq 0)$$

$$\leq 2P(\sup_{0 \leq t \leq T} \tilde{R}_t \geq b)$$

$$= 2E^Q[\int_0^T \mu(\tilde{R}_t)dt \geq b)]$$

$$\leq 2E^Q[\int_0^T \mu^2(\tilde{R}_t)dt]^{1/2}Q(\sup_{0 \leq t \leq T} \tilde{R}_t \geq b)^{1/2},$$

(3.6)

where

$$M_x = \exp \left\{ \int_0^t \frac{\mu}{\sigma} dW_s + \frac{1}{2} \int_0^t \frac{\mu^2}{\sigma^2} ds \right\}, \quad t \leq T,$$

d$P = M_t dQ$.

and $E^Q$ denotes expectation with respect to the measure $Q$.

Since $\mu$ and $\sigma$ are constants, $\{M_t\}$ is a martingale w.r.t. $F_t$ and $E^Q[\int_0^T 1_{\{\tilde{R}_t \geq b\}} d\tilde{W}_t] = C(T) < +\infty$. Moreover, by the Girsanov theorem, $Q$ is a probability measure on $F_T$, the process $\tilde{W}_t := \int_0^t \frac{\mu}{\sigma} ds + \tilde{W}_t$, $t \leq T$ is a Brownian motion w.r.t. $Q$, and

d$\tilde{R}_t = \sigma(\tilde{R}_t)d\tilde{W}_t$.

Therefore, by Chebyshev inequality and B-D-G inequalities (see Ikeda and Watanabe (1981)), there exist some constant $C$ such that

$$Q \left\{ \sup_{0 \leq t \leq T} \tilde{R}_t \geq b \right\} \leq Q \left\{ \sup_{0 \leq t \leq T} \int_0^t \sigma(\tilde{R}_t)d\tilde{W}_t \right\} \leq \frac{4E^Q \left\{ \sup_{0 \leq t \leq T} \int_0^t \sigma(\tilde{R}_t)d\tilde{W}_t \right\}^2}{b^2}$$

$$\leq \frac{4CE^Q \left\{ \int_0^T \sigma^2(\tilde{R}_t)^2 ds \right\}^2}{b^2}$$

$$\leq \frac{4CT^2}{b^2}.$$ 

(3.7)

The inequalities (3.6) and (3.7) imply that $\psi_b(T, b) \rightarrow 0$, as $b \rightarrow +\infty$. So for any given $\epsilon > 0$, there exists $b > 0$ such that $\psi_b(T, b) \leq \epsilon$. Thus, we complete the proof.

4. The solutions of the HJB equation

**Lemma 4.1.** Let $f(x) \in C^2$ satisfy the following HJB equation and boundary conditions,

$$\max_{x \in [0, 1]} \left[ \frac{1}{2} \sigma^2 a^2 f''(x) + \mu x a f'(x) - cf(x) \right] = 0, \quad 0 \leq x \leq b_0,$$

$$f'(x) = 1, \quad f''(x) = 0, \quad f(0) = 0.$$ 

Then the following conditions are valid.

$$\max_{x \geq 0} Lf(x) \leq 0, \quad f'(x) \geq 1, \quad f(0) = 0,$$

where $b_0$ is a variable to be specified later and $L = \frac{1}{2} \sigma^2 a^2 \frac{d^2}{dx^2} + \mu a \frac{d}{dx} - c.$
Lemma 4.2. Let $b > b_0$ be a predetermined variable, $g \in C^4(R_+)$, $g \in C^4(R_+ \setminus \{b\})$ satisfy the following HJB equation and boundary conditions,

$$
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 g''(x) + \mu a g'(x) \right] = 0, \quad \text{for } 0 \leq x \leq b,
$$

(4.2)

$g'(x) = 1$, for $x \geq b$.

$g''(x) = 0$, for $x > b$.

$g(0) = 0$.

Then the following conditions are valid.

$$
\max \mathcal{L} g(x) \leq 0, \quad \text{for } x \geq 0,
$$

(4.3)

$g'(x) \geq 1$, for $x \geq b$.

$g(0) = 0$.

where $b_0$ and $\mathcal{L}$ are same as in Lemma 4.1.

Proof. Using the same approach in Højgaard and Taksar (1999), if the max of (4.2) is attained in the interior of the control region, then, by differentiating w.r.t. $a$, we can find the maximizing function

$$
a(x) = -\frac{\mu g'(x)}{\sigma^2 g''(x)}. \quad (4.4)
$$

If $a(x)$ belongs to the interval $[0, 1]$, putting the expression (4.4) into (4.2), we guess that the solution of (4.2) is

$$
g_1(x) = C_1x^{\gamma} + C_2 \quad (4.5)
$$

and

$$
a(x) = -\frac{\mu x}{\sigma^2(1 - \gamma)}.
$$

where $\gamma = \frac{c}{c+\frac{\sigma^2}{\mu}}$.

The validity of the solution requires $a(x) \in [0, 1]$, which means that $x < x_0 = \left(\frac{1 - \gamma}{\mu}\right)^{\frac{\gamma}{2}}$. By (4.5) and $g(0) = 0$,

$$
g_1(x) = C_1x^{\gamma}, \quad \text{for } 0 \leq x \leq x_0.
$$

On the other hand, if $x_0 \leq x < b$, we have $a(x) = 1$ and (4.2) becomes

$$
\frac{1}{2} \sigma^2 g''(x) + \mu g'(x) - cg(x) = 0,
$$

the solution of (4.6) is

$$
g_2(x) = C_3 \exp^{d_+x} + C_4 \exp^{d_-x} \quad \text{for } x_0 \leq x < b,
$$

where $d_+ = \frac{\mu - \sqrt{\mu^2 + 2\sigma^2}}{\sigma}$, $d_- = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2}}{\sigma}$.

Since $g \in C^4$, the following conditions are valid.

$$
g_1(x_0) = g_2(x_0),
$$

$$
g_1'(x_0) = g_2'(x_0).
$$

Then, we get

$$
C_3 = C_1 \frac{d_+ x_0^{\gamma} - d_- x_0^{\gamma - 1}}{(d_+ - d_-) x_0^{\gamma - 1}} = C_1 A,
$$

(4.7)

$$
C_4 = C_1 \frac{d_+ x_0^{\gamma - 1} - d_- x_0^{\gamma}}{(d_+ - d_-) x_0^{\gamma - 1}} = C_1 B.
$$

For $x \geq b$, the solution has the following form,

$$
g_3(x) = x - b + g_2(b).
$$

Using $g_2'(b) = 1$,

$$
C_1 = \frac{1}{d_- \exp^{d_-b} + d_+ \exp^{d_+ b}} \left( d_+ \exp^{d_-b} - d_- \exp^{d_+ b} \right) = \frac{1}{Ad_- \exp^{d_-b} + Bd_+ \exp^{d_+ b}}.
$$

We claim that

$$
g_2''(b) \geq 0. \quad (4.9)
$$

In order to prove this, we use $f(x)$ in Lemma 4.1 and notice that $A$ in (4.7) and $B$ in (4.8) have the same expression both in $f(x)$ and $g(x)$. First, we want to prove that $A < 0$ and $B > 0$. Since $f''(b_0) = 1$ and $f''(b_0) = 0$,

$$
C' \left( Ad_- \exp^{d_-b} + Bd_+ \exp^{d_+ b} \right) = 1, \quad (4.10)
$$

$$
C' \left( Ad_+ \exp^{d_-b} + Bd_- \exp^{d_+ b} \right) = 0. \quad (4.11)
$$

From (4.11), we know that $AB < 0$, together with (4.10), $A < 0$ and $B > 0$ are valid for $d_- < 0$, $d_+ > 0$ and $C_1'' > 0$ in $f(x)$. Moreover,

$$
l(b) = g_2''(b) = \frac{A d_- \exp^{d_-b} + B d_+ \exp^{d_+ b}}{Ad_- \exp^{d_-b} + Bd_+ \exp^{d_+ b}} = \frac{\partial l(b)}{\partial d} \left( A d_- \exp^{d_-b} + B d_+ \exp^{d_+ b} \right) > 0
$$

holds for $A < 0$, $B > 0$, $d_- < 0$ and $d_+ > 0$.

Since $f''(b_0) = 0$, we have $g_2''(b_0) = 0$, so for $b > b_0$,

$$
g_2''(b) > l(b_0) = g_2''(b_0) = 0.
$$

The problem remained is to prove that the solution $g$ satisfies (4.3). It suffices to prove the following condition:

$$
\max_{a \in [0,1]} \left[ \frac{1}{2 le H \sigma^2} g''(x) + \mu a g'(x) \right] \leq 0, \quad \text{for } x \geq x_0.
$$

We can use the same approach in Højgaard and Taksar (1999) to prove the above claims sentence by sentence (see Højgaard and Taksar [1999] for more details), here the condition for $x \geq b$ should be proved due to the lack of second order continuity of $g(x)$ at $x = b$.

For $x \geq b$, using (4.6) and (4.9) we have

$$
\max \mathcal{L} g_3(x) = \mu a - cg_3(x) \leq \mu - c(x - b + g_2(b)) \leq \mu - c g_2(b) \leq 0.
$$

Thus, we complete the proof. \qed

Lemma 4.3. For $b \geq b_0$, $\frac{d}{db} g(b, x) \leq 0$ is valid.

Proof. For $b \geq b_0$, $A < 0$ and $B > 0$ we have

$$
\frac{\partial}{\partial b} g_1(b, x) = -\gamma' \left( A d_- \exp^{d_-b} + B d_+ \exp^{d_+ b} \right) \leq 0, \quad \text{for } 0 \leq x \leq x_0,
$$

$$
\frac{\partial}{\partial b} g_2(b, x) = -\gamma' \left( A d_- \exp^{d_-b} + B d_+ \exp^{d_+ b} \right) \leq 0,
$$

for $x_0 \leq x \leq b$,

$$
\frac{\partial}{\partial b} g_3(b, x) = -\gamma' \left( A d_- \exp^{d_-b} + B d_+ \exp^{d_+ b} \right) \leq 0,
$$

for $x \geq b$.

where $\gamma' = \frac{c}{c+\frac{\sigma^2}{\mu}}$.

The proof thus has been done. \qed
5. The optimal return function \( V(x, b^*) \) and the optimal control policy \( \pi^* \)

In this section, based on Theorems 3.1 and 3.2, we assume that \( b \) is the minimum dividend barrier satisfying (2.5). Let

\[
a(x) = \begin{cases} 
\frac{\mu x}{\sigma^2(1 - \gamma)}, & x < x_0, \\
1, & x \geq x_0
\end{cases}
\]

where \( \gamma = \frac{c + \frac{\mu x}{\sigma^2}}{\sigma^2} \) and \( x_0 = \frac{1 - \gamma}{\mu} \).

The main results of this paper is the following.

**Theorem 5.1.** Let \( b_0 \) be the optimal dividend payout barrier without solvency constraints. Then, if \( b_0 \geq b \), the optimal return function \( V(x) = V(x, b^*) \) with \( b^* = b_0 \) is \( f \) in Lemma 4.1, i.e., for any admissible control policy \( \pi \), \( V(x, b^*) \geq f(x, \pi) \). Furthermore, the optimal control policy \( \pi^* (= \pi_{b_0}) \) is as follows:

\[
\begin{align*}
R_s^\pi &= x + \int_0^t \mu a(R_s^\pi) \, ds + \int_0^t \sigma a(R_s^\pi) \, dW_s - L_t^s, \\
R_s^\pi &\leq b_0, \\
\int_0^\infty \mathbb{1}_{\{R_t^\pi < b_0\}}(t) \, dt &\leq 0.
\end{align*}
\]

If \( b_0 < b \), the optimal return function \( V(x) = V(x, b^*) \) with \( b^* = b \) in Lemma 4.2, i.e., for any admissible control policy \( \pi \), \( V(x, b^*) \geq f(x, \pi) \). Furthermore, the optimal control policy \( \pi^* (= \pi_b) \) is as follows:

\[
\begin{align*}
R_s^\pi &= x + \int_0^t \mu a(R_s^\pi) \, ds + \int_0^t \sigma a(R_s^\pi) \, dW_s - L_t^s, \\
R_s^\pi &\leq b, \\
\int_0^\infty \mathbb{1}_{\{R_t^\pi < b\}}(t) \, dt &\leq 0.
\end{align*}
\]

**Proof.** If \( b_0 \geq b \), the complete proof is given in Højgaard and Taksar (1999) (see Højgaard and Taksar (1999) for details), and, by Theorem 3.1, if we let \( b_0 = b \), then \( b^* \) also satisfies (2.5).

If \( b_0 < b \), the management of the company is prohibited from paying dividend at \( b_0 \) according to the solvency constraints. Paying dividend at a barrier larger than \( b \) is the admissible control policy. We first prove that for any dividend barrier \( b_1 \geq b \), \( g \) in Lemma 4.2 is the optimal return function and the optimal control policy is in (5.1) replacing \( b_1 \) by \( b_1 \), then choose \( b^* \) from the dividend barriers \( \{b_1 : b_1 \geq b\} \) is the best and paying dividend is the optimal control policy.

Fixed a policy \( \pi \), let \( A = \{s : L_s^\pi \neq L_s^\pi\} \), \( \hat{t} = \sum_{A \subseteq \Lambda} (L_s^\pi - L_{s-}^\pi) \) be the discontinuous part of \( L_s^\pi \) and \( L_s^\pi = L_s^\pi - \hat{t} \) be the continuous part of \( L_s^\pi \). Let \( \tau \) be the first time that the corresponding reserves \( \hat{R}_t \) defined by (2.1) hit \(( -\infty, \varepsilon) \). Then, by generalized Itô formula,

\[
ce^{-c(t+\tau^t)}g(R_{t+\tau^t}) = g(x) + \int_0^{t+\tau^t} e^{-c\tau}L^\pi g(R_{s}) \, ds \\
+ \int_0^{t+\tau^t} \sigma a(R_s^\pi) g(R_s^\pi) \, dW_s - \int_0^{t+\tau^t} e^{-c\tau}g(R_s^\pi) \, dL_s^\pi \\
+ \sum_{s \in A \subseteq \Lambda} e^{-c\tau}[g(R_s^\pi) - g(R_{s-}^\pi) - g'(R_{s-}^\pi)(R_s^\pi - R_{s-}^\pi)] \\
= g(x) + \int_0^{t+\tau^t} e^{-c\tau}L^\pi g(R_{s}) \, ds \\
+ \int_0^{t+\tau^t} \sigma a(R_s^\pi) g(R_s^\pi) \, dW_s - \int_0^{t+\tau^t} e^{-c\tau}g(R_s^\pi) \, dL_s^\pi \\
+ \sum_{s \in A \subseteq \Lambda} e^{-c\tau}[g(R_s^\pi) - g(R_{s-}^\pi)]
\]

where

\[
L = \frac{1}{2} \sigma^2 \frac{d^2}{dx^2} + \alpha \mu \frac{d}{dx} - c.
\]

By the same way as used in Lemma 4.2, we can prove that the second term on the right hand side is non-positive. Since \( g'(R_s^\pi) \leq g'(\varepsilon) \), the third term is a square integrable martingale, therefore, by taking expectations at both sides of (5.2) and letting \( \varepsilon \to 0 \),

\[
E \left\{ e^{-c(t+\tau^t)} g(R_{t+\tau^t}) \right\} \leq g(x) - E \left\{ \int_0^{t+\tau^t} e^{-c\tau}g'(R_s^\pi) \, dL_s^\pi \right\} \\
+ \mathbb{E} \left\{ \sum_{s \in A \subseteq \Lambda} e^{-c\tau}[g(R_s^\pi) - g(R_{s-}^\pi)] \right\}.
\]

(5.3)

Since \( g'(R_s^\pi) \geq 1 \) for \( x \geq b \),

\[
g(R_{t+\tau^t}) - g(R_{s-}^\pi) \leq -(L_s^\pi - L_{s-}^\pi),
\]

which, together with (5.3), implies that

\[
E \left\{ e^{-c(t+\tau^t)} g(R_{t+\tau^t}) \right\} + E \left\{ \int_0^{t+\tau^t} e^{-c\tau}dL_s^\pi \right\} \leq g(x).
\]

(5.5)

**Remark.** The dividend barrier is \( b \), so the management can only pay out dividend at \( x \geq b \). The condition \( g'(R_s^\pi) \geq 1 \) for \( x \geq b \) can guarantee that (5.4) and (5.5) are valid.

By the definition of \( \tau \) and \( g(0) = 0 \), it is easy to prove that

\[
\lim_{t \to \infty} e^{-c(t+\tau^t)} g(R_{t+\tau^t}) = e^{-ct} g(0) I_{(t < \infty)} \\
+ \lim_{t \to \infty} \inf e^{-ct} g(R_t) I_{(t = \infty)} \geq 0.
\]

(5.6)

So, we deduce from (5.3) and (5.6) that

\[
J(x, \pi) = E \left\{ \int_0^{\infty} e^{-ct} dL_t^{\pi} \right\} \leq g(x).
\]

If we choose the control policy as in (5.1), the inequalities above become equalities. Furthermore,

\[
\lim_{t \to \infty} e^{-c(t+\tau^t)} g(R_{t+\tau^t}) = e^{-ct} g(0) I_{(t < \infty)} \\
+ \lim_{t \to \infty} \inf e^{-ct} g(R_t) I_{(t = \infty)} = 0
\]

due to \( g'(x) = 1 \) for \( x \geq b \). Then,

\[
J(x, \pi^*) = E \left\{ \int_0^{\infty} e^{-ct} dL_t^{\pi^*} \right\} = g(x).
\]

Therefore, noting that \( \frac{\partial}{\partial x} g(b, x) \leq 0 \) is valid for \( b \geq b_0 \) in Lemma 4.3, paying dividend at the minimum dividend barrier \( b \) is the optimal choice among all the admissible policies \( \{b_1 : b_1 \geq b\} \), the \( b \) satisfies (2.6). Moreover, the \( b \) satisfies (2.5). So if we let \( b^* = b \), then \( \pi^* = \pi_b \) and \( V(x, b^*) \) solve the problem (2.4)-(2.6). The proof thus has been done. \( \square \)

6. Conclusion

In this paper, we consider the optimal control problem of the insurance company with solvency constraints. The management of the company controls the reinsurance rate, dividends payout process to maximize the expected present value of the dividend until the time of bankruptcy. This is a mixed singular-regular control problem. However, the optimal dividend payout barrier may be unacceptable low. The company may be prohibited to pay dividend at such a low barrier according to external reasons. In
fact, the authorities bring out some constraints on the insurance company’s dividend policy. One normal constraint is that if $b$ is the minimum dividend barrier, the probability that the reserve will be negative within a time horizon $T$ is not larger than some predetermined $\varepsilon$. We come to the conclusion that if the predetermined dividend barrier is larger than the traditional optimal dividend barrier, paying everything beyond the barrier is the optimal policy. Also, we extend the HJB methods to find an approach to determine the minimum dividend barrier according to the solvency constraints.

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