Exit problems for nonlinear stochastic evolution equations on Hilbert spaces

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Abstract. This paper extends exit theorems of Da Prato and Zabczyk to nonconstant diffusion coefficients. It uses extensively general, exponential estimates due to Peszat.

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In the present paper we study the exit problems for the following nonlinear stochastic evolution equations with nonconstant diffusion coefficients,

$$\begin{aligned}
&dX^\varepsilon = (AX^\varepsilon + F(t, X^\varepsilon))dt + \sqrt{\sigma} B(X^\varepsilon) dW, \\
&X^\varepsilon(0) - x \in \mathbf{H}.
\end{aligned}$$

(0.1)

In eq. (0.1), $A$ is the generator of a $C_0$-semigroup $S(\cdot)$ on a separable space $\mathbf{H}$, that is, $S(\cdot)$ is a strongly continuous semigroup in $L(\mathbf{H}, \mathbf{H})$, $F$ and $B$ act from $\mathbf{H}$ into $\mathbf{H}$ and from $\mathbf{H}$ into $L(\mathbf{H}, \mathbf{H})$, respectively. And $W$ stands for a cylindrical Wiener process on $\mathbf{H}$. By the solution $X^\varepsilon$ we understand the so-called mild solution, that is, the solution of the integral equation

$$X^\varepsilon(t) = S(t)x + \int_0^t S(t - s) F(s, X^\varepsilon(s)) ds + \sqrt{\sigma} \int_0^t S(t - s) B(X^\varepsilon(s)) dW(s).$$

(0.2)

We assume that $F(\cdot, 0) = 0$. Then 0 is an equilibrium state for the deterministic equation

$$z' = Az + F(\cdot, z), \quad z(0) = x.$$

(0.3)

Let $z(\cdot, x)$ denote the solution of eq. (0.3) and assume that there exists an open bounded neighborhood $D \subset \mathbf{H}$ of 0 which is uniformly attracted to 0 by eq. (0.3):

$$\forall \varepsilon > 0, \quad \exists T > 0, \quad \|z(t, x)\| \leq \varepsilon, \quad \forall t \geq T, \quad x \in D.$$  

(0.4)

Note that $z(0, x) = X^0(\cdot, x)$. So the assumption (0.4) implies that $X^0(t, x) \rightarrow 0 \text{ as } t \rightarrow +\infty$. However, for $\forall \varepsilon > 0$ the behavior of $X^\varepsilon(\cdot, x)$ will be completely different. Under the influence of some random perturbations the solution $X^\varepsilon(\cdot, x)$, starting from $D$, will eventually reach the boundary $\partial D$.

To explain this denote $\tau^{x, \varepsilon}$ the exit time of process $X^\varepsilon(\cdot, x)$ from $D$:

$$\tau^{x, \varepsilon} \equiv \inf \{t \geq 0; X^\varepsilon(t, x) \in \partial D\}.$$

(0.5)

It is intuitively (under the assumption of (0.4)) clear that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}(\tau^{x, \varepsilon}) = +\infty.$$

(0.6)
Observing the above phenomena, one asks immediately the following questions:

(1) how to calculate the rate of divergence in eq. (0.6)?

(2) \( \forall x \in D, \epsilon > 0, \text{ the fact } E(\tau^{x,\epsilon}) < +\infty \) holds?

The problems have been considered already by many authors. Freidlin, Wentzell and Zabczyk gave a complete solution to such questions for ordinary stochastic equations in refs. [1—5]. Da Prato and Zabczyk solved the problems\(^6\) only for infinite dimensional stochastic evolution equations with Gaussian perturbations (additive perturbations),

\[ dX^\epsilon = (AX^\epsilon + F(X^\epsilon))dt + \sqrt{\epsilon}BdW, \]

the diffusion coefficient \( B \) being a constant, by applying a large deviations result for a family of Gaussian measure on a Banach space. The general case has not been studied.

In our investigations we are motivated by the exponential tail estimates for infinite dimensional stochastic evolutions with non-additive random perturbations (non-Gaussian perturbations) obtained by Chow, Kyprianou\(^9\) and Peszat\(^10,11\). Using these estimates and the strong Markov property as main tools, we get an almost complete answer to the problems for eq. (0.1) in the general case. An alternate approach would be to use some general technique of refs. [7, 8].

1 Notations and formulation of the main results

Let \((H, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space. By \( | \cdot | \) we denote the norm on \( H \). The operator norm on \( L(H) (\equiv L(H, H)) \) and the Hilbert-Schmidt norm on \( L_2(H, H) \), a separable Hilbert space of all Hilbert-Schmidt operators from \( H \) into itself, are denoted by \( \| \cdot \| \) and \( \| \cdot \|_2 \), respectively. Denote by \((A, D(A))\) the generator of a \( C_0 \)-semigroup \( S(t) \) on \( H \). From now on we make the assumptions on \( S(t)\):

\[ k_T \equiv \left( \int_0^T t^{1-a-1/p_0} \| S(t) \|_{L_2(H)}^2 dt \right)^{1/p_0} < +\infty, \]

\[ \Theta_T \equiv \left( \int_0^T \| S(t) \|_{L_2(H)}^2 dt \right)^{1/2} < +\infty. \]

Let \((\Omega, \mathcal{F}, P)\) be probability space with a right-continuous increasing family \( \{\mathcal{F}\}_{t \geq 0} \) of sub-\( \sigma \)-field of \( \mathcal{F} \) each containing \( P \)-null sets. Let \{\( e_k \)\} be an orthogonal basis in \( H \), and let \{\( W_k \)\} be a sequence of independent, real valued \( \{\mathcal{F}\}_{t \geq 0} \)-Wiener processes. By a cylindrical Wiener process on \( H \) we mean the series

\[ W(\cdot) = \sum_{k=1}^{+\infty} W_k(\cdot)e_k. \tag{1.1} \]

This series does not converge in \( H \) but in an arbitrary Hilbert space \( U_1 \) containing \( H \) with a Hilbert-Schmidt embedding. It is well known\(^8,12,13,15\) that the formula (1.1) defines a \( Q_1 \)-Wiener process on \( U_1 \) with \( \text{Tr} Q_1 < +\infty \), that is, the trace of \( Q_1 \) is finite, and for \( a, b \in H \), the process

\[ \langle a, W(t) \rangle = \sum_{k=1}^{+\infty} \langle a, e_k \rangle W_k(t) \]
is a real valued Wiener process and
\[ E[a, W(t)|0, W(s)] = (t \wedge s)(a, b). \]

In this paper we consider the stochastic equations (0.1). The stochastic integral in eq. (0.1) is the stochastic Ito integral with respect to cylindrical Wiener process \( W(\cdot) \) in the sense of Da Prato and Zabczyk. For details see refs. [8, 12—15]1. The solution of (0.1), if it exists and is unique, is denoted by \( X^x(., x) \).

Let us also consider the associated control system
\[ f' = (Af + F(t, f)) + B(f)u, \quad f(0) = x, \quad (1.2) \]
and denote by \( f_{x,u} \) the solution of eq. (1.2). We define
\[ K_T^x(r) \equiv \left\{ f \in C([0, T]; \mathcal{H}) : f = f_{x,u} : \frac{1}{2} \int_0^T |u(s)|^2 ds \leq r \right\}, \]
and
\[ \varepsilon \equiv \inf \left\{ \frac{1}{2} \int_0^T |u(s)|^2 ds : f_{x,u}(T) \in (\hat{D})^c, T > 0 \right\}. \quad (1.3) \]
Interpreting the integral \( \int_0^T |u(s)|^2 ds \) as energy dissipated by the control \( u \), we can say that \( \varepsilon \) is the minimal energy required by the control system (1.2) to transfer the equilibrium state \( 0 \) outside \( \hat{D} \). We will call \( \varepsilon \) the upper exit rate. For any \( r > 0 \), let
\[ e_r \equiv \inf \left\{ \frac{1}{2} \int_0^T |u(s)|^2 ds : f_{x,u}(T) \in (D)^c, T > 0, \|x\| \leq r \right\}. \quad (1.4) \]
We call the number
\[ \underline{\varepsilon} \equiv \lim_{r \downarrow 0} e_r \quad (1.5) \]
the lower exit rate. Note that always \( \underline{\varepsilon} \leq \varepsilon \). Set \( D_0 = \{ x \in D : \varepsilon^x(t) \in D, \forall t \geq 0 \} \).

Now, we formulate a list of assumptions on \( F, A \) and \( B \).

(C.1) There exists a function \( \Psi \in L^2_{\text{loc}}([0, \infty); \mathbb{R}) \) such that, for \( \forall h, g \in \mathcal{H} \) and \( t \geq 0 \),
\[ |F(t, h) - F(t, g)| \leq |\Psi(t)||h - g|, \quad \text{and} \quad F(\cdot, 0) = 0. \]

(C.2) For each \( T > 0 \),
\[ \beta_T \equiv \int_0^T t^{-2\alpha} \sup_{h \in \mathcal{H}} \|S(t)B(h)\|^2 dt < +\infty. \]

(C.3) There exist functions \( \Phi \in L^2_{\text{loc}}([0, \infty); \mathbb{R}) \), where \( \Phi \) satisfies
\[ \int_0^T t^{-2\alpha} \Phi^2(s) ds < +\infty \quad \text{for any} \quad T > 0, \]
such that, for all \( h, g \in \mathcal{H} \) and \( t > 0 \),
\[ \|S(t)(B(h) - B(g))\| \leq |\Phi(t)||h - g|. \]

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\text{(C.4)} For each $T > 0$,\[
\lim_{n \to \infty} \int_0^T t^{-2\alpha} \sup_{h \in \mathcal{H}} \|S(t) B(h)(H_n - I)\|^2 dt = 0.
\]

\text{(C.5)} The mapping $F: [0, T] \times \mathcal{H} \to \mathcal{H}$, $(t, x) \to F(t, x)$ is measurable from $([0, T] \times \mathcal{H}$, \(\mathcal{B}(0, T) \times \mathcal{B}(\mathcal{H})\)) into $\mathcal{H}$, $\mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(E)$ denotes the Borel $\sigma$-field of $E$.

\text{(C.6)} The mapping $B: \mathcal{H} \to L(H)$, $x \to B(x)$ is measurable from $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ into $(L(H), \mathcal{B}(L(H)))$.

The theorem below, proved by Peszat$^{10,11}$, ensures the existence and uniqueness of the solutions of eqs. (0.1) and (1.2), and the compactness of the sets $K_T(r)$.

\textbf{Theorem 1.1.} Assume (C.1)—(C.6). Then

\begin{enumerate}[(i)]
  \item \forall x \in \mathcal{H}$ and \forall \varepsilon > 0$ there exists the unique continuous solution of eq. (0.1),
  \item \forall x \in \mathcal{H}$, \forall T > 0$ and $u$ with $\int_0^T |u(s)|^2 ds < +\infty$, there exists the unique solution of eq. (1.2),
  \item \forall x \in \mathcal{H}$, \forall T > 0$ and $r < +\infty$, $K_T^r(r)$ is compact in $C([0, T]; \mathcal{H})$.
\end{enumerate}

We can now state our main results.

\textbf{Theorem 1.2.} Assume (C.1)—(C.6). If $\forall \varepsilon > 0$, $\forall C > 0$ and $\forall x \in D$, the stochastic process $W_{\varepsilon}(\cdot) \equiv \int_0^T S(t-s) B(x(s, x)) dW(t)$ satisfies $0 < P(|W_\varepsilon(T_0)| \leq C) < 1$ for some $T_0 > 0$, then $E(\tau^{\varepsilon, r}) < +\infty$.

\textbf{Theorem 1.3.} Assume (C.1)—(C.6) and (0.4). Then we have

\begin{align}
\limsup_{\varepsilon \downarrow 0} \varepsilon \log E(\tau^{\varepsilon, r}) & \leq \delta, \quad x \in D, \\
\liminf_{\varepsilon \downarrow 0} \varepsilon \log E(\tau^{\varepsilon, r}) & \geq \delta, \quad x \in D_0.
\end{align}

2 Large deviation principle for eq. (0.1)

In this subsection we review a large deviation result for eq. (0.1). The theorem was proved by Peszat, Chow and Kwapień$^{9,10,11,16}$ in the general case. This result will be proved extremely useful in proof of Theorem 1.3.

\textbf{Theorem 2.1.} Assume (C.1)—(C.4). Then

\begin{enumerate}[(i)]
  \item $\forall \delta > 0$, $\forall T > 0$ and $\forall \gamma > 0$, $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0]$, $\forall u \in L^2(0, T; \mathcal{H})$ satisfying $\int_0^T |u(s)|^2 ds \leq r^2$ and $\forall x \in \mathcal{H}$ one has
    \[ P\left\{ \sup_{0 \leq t \leq T} |X^\varepsilon(t, x) - f^{\varepsilon,n}(t)| \leq \delta \right\} \geq \exp \left\{ -\frac{1}{\varepsilon} \left( \frac{1}{2} \int_0^T |u(s)|^2 ds + \gamma \right) \right\}, \]
  \item $\forall \delta > 0$, $\forall T > 0$, $\forall \gamma > 0$, $\forall R > 0$, $\forall r > 0$ and $\forall x_0 \in \mathcal{H}$, $\exists \varepsilon_0 > 0$ such that $\forall x$ with $|x - x_0| \leq R$ and $\forall \varepsilon \in (0, \varepsilon_0]$ one has
    \[ P\left\{ \text{dist}_{C([0, T]; \mathcal{H})}(X^\varepsilon(\cdot, x), K_T^r(r)) \leq \delta \right\} \geq 1 - \exp \left\{ -\frac{1}{\varepsilon} \left( r - \gamma \right) \right\}. \]
\end{enumerate}
3 Regularity, Markov and strong Markov properties

In this subsection we are concerned with the following equation under hypotheses (C.1)—(C.6),

$$X(t) = S(t - s)\xi + \int_s^t S(t - r)F(r, X(r))dr + \int_s^t S(t - r)B(X(r))dW(r), \quad t \in [s, T]. \tag{3.1}$$

Let us remark from the definition of stochastic integration that all the results on stochastic integral and stochastic equations obtained for the time interval [0, T] can be generalized in a natural way to intervals [s, T], s \in [0, T], with the \(\sigma\)-field \(\mathcal{F}_s\) playing the role of the \(\sigma\)-field \(\mathcal{F}_0\), and \(\{W(t) - W(s)\}_{t \geq s}\) the role of the Wiener process. In particular for any \(s \in [0, T]\) and for an arbitrary \(H\)-valued, \(\mathcal{F}_s\)-measurable random variable \(\xi\), there exists a unique solution \(X(t), t \in [s, T]\), of eq. (3.1). This solution will be denoted as \(X(\cdot, s; \xi)\). If \(x\) is an element of \(H\) and \(\xi = x\), \(P\)-a.s., the solution of eq. (3.1) is denoted as \(X(\cdot, s; x)\). To shorten notation, let \(C(T)\) and \(C_n(T)\) denote constants, depending only on \(T\), that may be different from one formula to another.

Now, we first give the regularity of solution on initial data in the sense specified in the following theorem.

**Theorem 3.1.** Assume (C.1)—(C.6). Then, for arbitrary \(\xi, \eta \in L^2(\Omega, \mathcal{F}_s, P)\) and \(0 \leq s \leq s' \leq t \leq t' \leq T\), there exists a constant \(C(T) > 0\) such that the following estimates hold:

$$\mathbb{E}\left(\sup_{r \in [s, T]} |X(r, s; \xi)|^2\right) \leq C(T)(1 + \mathbb{E}|\xi|^2), \tag{3.2}$$

$$\mathbb{E}\left(\sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^2\right) \leq C(T)\mathbb{E}(|\xi - \eta|^2), \tag{3.3}$$

$$\mathbb{E}\left(\left|X(t', s; \xi) - X(t, s; \xi)\right|^2\right) \leq C(T)\mathbb{E}\left(\left|S(t' - t)X(t, s; \xi) - X(t, s; \xi)\right|^2\right) + |t' - t| + \int_t^{t'} r^{-2\alpha} \sup_{h \in H} \|S(r)B(h)\|^2 dr, \tag{3.4}$$

$$\mathbb{E}\left(\left|X(t, s'; \xi) - X(t, s; \xi)\right|^2\right) \leq C(T)\mathbb{E}\left(\left|S(s' - s)\xi - \xi\right|^2\right) + |s' - s| + \int_s^{s'} r^{-2\alpha} \sup_{h \in H} \|S(r)B(h)\|^2 dr. \tag{3.5}$$

In addition, if \(p > 2\), \(\xi, \eta \in L^p(\Omega, \mathcal{F}_s, P)\) and \(\Psi, \Phi \in L^p_{\text{loc}}(0, +\infty; \mathbb{R})\), then

$$\mathbb{E}\left(\sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^2\right) \leq C(T)(\mathbb{E}(|\xi - \eta|^p))^\frac{2}{p}. \tag{3.6}$$
Proof. We firstly prove (3.2). Set \( M_T \equiv \sup_{t \in [0, T]} \|S(t)\|. \) By eq. (3.1) we have
\[
\mathbb{E}(|X(t, s; \xi)|^2) \leq 3M_T^2 \mathbb{E}|\xi|^2 + 3\mathbb{E}\left( \int_s^t |S(t - r)F(r, X(r, s; \xi))|dr \right)^2 \\
+ 3\mathbb{E}\left( \int_s^t |S(t - r)B(X(r, s; \xi))dW(r)|^2 \right)
\equiv I_1 + I_2 + I_3,
\]
(3.7)
Using (C.1), (C.2) and H"{o}lder’s inequality, we obtain
\[
I_2(t) \leq C(T)\Theta_T \mathbb{E}\left( \int_s^t |\Psi(r)|^2 \left( \sup_{0 \leq r' \leq r} |X(r', s; \xi)|^2 \right)dr \right).
\]
By (C.2), (C.3), (C.5), (C.6) and properties of stochastic integral (see §4.4 of ref. [8]), we get
\[
I_3(t) = \mathbb{E}\left( \int_s^t \|S(t - r)B(X(r, s; \xi))\|^2 dr \right)
\leq C(T)\mathbb{E}\left( \int_s^t \|S(t - r)B(\xi)\|^2 dr \right) \\
+ C(T)\mathbb{E}\left( \int_s^t |\Phi(t - r)|^2 |X(r, s; \xi) - \xi|^2 dr \right)
\leq C(T)T^{2\beta} + C_1(T)\mathbb{E}|\xi|^2 \\
+ C_2(T)\mathbb{E}\left( \int_s^t |\Phi(r)|^2 \sup_{0 \leq r' \leq r} |X(r', s; \xi)|^2 dr \right)
\leq C(T)(1 + \mathbb{E}|\xi|^2) + C(T)\mathbb{E}\left( \int_s^t |\Phi(r)|^2 \sup_{0 \leq r' \leq r} |X(r', s; \xi)|^2 dr \right),
\]
Combining the above estimates for \( I_1, I_2(t) \) and \( I_3(t) \) yields
\[
\mathbb{E}(|X(t, s; \xi)|^2) \leq C(T)(1 + \mathbb{E}|\xi|^2) + C(T)\mathbb{E}\left( \int_s^t \left[ |\Psi^2(r) + \Phi^2(r)| \sup_{0 \leq r' \leq r} |X(r', s; \xi) - X(r', s; \eta)|^2 \right] dr \right),
\]
which, by using Gronwall’s lemma, implies the inequality (3.2).

By proceeding as the proof of (3.2) we see that
\[
\mathbb{E}\left( |X(t, s; \xi) - X(t, s; \eta)|^2 \right)
\leq C(T)\mathbb{E}|\xi - \eta|^2 + C(T)\int_s^t \left[ |\Psi^2(r) + \Phi^2(r)| \sup_{0 \leq r' \leq r} |X(r', s; \xi) - X(r', s; \eta)|^2 \right] dr,
\]
(3.8)
which, again by using Gronwall’s lemma, completes the proof of (3.3).

To prove (3.4), by eq. (3.1), we have the following identity,
\[
X(t', s; \xi) - X(t, s; \xi) = X(t', t; X(t, s; \xi)) - X(t, s; \xi)
= (S(t' - t) - 1)X(t, s; \xi) + \int_t^{t'} S(t - r)F(r, X(r, s; \xi))dr \\
+ \int_s^{t'} S(t - r)B(X(r, s; \xi))dW(r),
\]
it, by (3.2), follows
\[
\mathbb{E} \left( |X(t', s; \xi) - X(t, s; \xi)|^2 \right) \\
\leq 3 \mathbb{E} \left( |(S(t' - t) - 1) X(t, s; \xi)|^2 \right) \\
+ 3C(T) \int_t^{t'} \|S(t - r)||^2 dr \int_t^{t'} \psi^2(r) \mathbb{E}|X(r, s; \xi)|^2 dr \\
+ 3C(T) \int_t^{t'} \mathbb{E}\left\|S(t - r) B(X(r, s; \xi))\right\|^2_2 dr \\
\leq C(T) \mathbb{E} \left( |(S(t' - t) - 1) X(t, s; \xi)|^2 \right) \\
+ C(T) |t' - t| + C(T) T^{2\alpha} \int_t^{t'} r^{-2\alpha} \sup_{h \in \mathcal{H}} \|S(r) B(h)\|^2_2 dr,
\]
which implies (3.4).

To estimate (3.5), we note that
\[
X(t, s'; \xi) - X(t, s; \xi) = X(t, s'; \xi) - X(t, s'; X(s'; s; \xi)),
\]
and so, by (3.3),
\[
\mathbb{E} \left( |X(t, s'; \xi) - X(t, s; \xi)|^2 \right) \leq C(T) \mathbb{E} \left( \xi - X(s', s; \xi)^2 \right). \tag{3.9}
\]
But
\[
X(s', s; \xi) - \xi = S(s' - s)\xi - \xi + \int_s^{s'} S(s' - r)F(r, X(r, s; \xi)) dr \\
+ \int_s^{s'} S(s' - r)B(X(r, s; \xi))dW(r). \tag{3.10}
\]
So, in the same way as for (3.4), by (3.9) and (3.10) we complete the proof of (3.5).

Finally we show (3.6). By using Hölder’s inequality, for $p > 2$, we have
\[
\mathbb{E} \left( \sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^p \right) \leq \left( \mathbb{E} \sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^2 \right)^{\frac{p}{2}}. \tag{3.11}
\]
On the other hand, by using Doob’s inequality for $\mathcal{H}$-valued martingale (see ref. [8], p. 182 for more details) we have
\[
\mathbb{E} \left( \sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^p \right) \\
\leq C(T) \mathbb{E} |\xi|^p + C(T) \int_s^T \mathbb{E} \left( |X(r, s; \xi) - X(r, s; \eta)|^p \right) dr,
\]
therefore, the last inequality and Gronwall’s lemma imply
\[
\mathbb{E} \left( \sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^p \right) \leq C(T) \mathbb{E} |\xi - \eta|^p. \tag{3.12}
\]
Substituting (3.12) into (3.11), we get
\[
\mathbb{E} \left( \sup_{r \in [s, T]} |X(r, s; \xi) - X(r, s; \eta)|^2 \right) \leq C(T) \left( \mathbb{E} |\xi - \eta|^p \right)^{\frac{2}{p}},
\]
This implies (3.6). So the proof of the theorem is complete.

Secondly, we study the Markov and strong Markov properties of solutions of eq. (3.1). It is not difficult to see that the rather routine proof of existence [8, 17] and the conditions (C.5) and (C.6), that the process \( X(t, s; x) \) is measurable with respect to \( \sigma \left\{ W(u) - W(s); u \geq s \right\} \) and therefore is independent of \( \mathcal{F}_s \). We define for \( \varphi \in B_b(H) \), the Banach space of all real bounded Borel functions, endowed with the sup norm, and \( 0 \leq s \leq t \leq T \) and \( x \in H \),

\[
P_{s,t} \varphi(x) = \mathbb{E}_\varphi(X(t, s; x)).
\]  
(3.13)

From Theorem 3.1 we know that the family \( \{ P_{s,t} \} \) has Feller property, that is, for arbitrary \( \varphi \in C_b(H) \), the \( P_{s,t} \varphi(x) \) is a continuous function with respect to \( (s, t, x) \). The function

\[
P(s, x; t, \Gamma) = P_{s,t} \chi_{\Gamma}(x) = P(X(t, s; x) \in \Gamma)
\]

is called the transition function of solution of eq. (3.1), where \( \chi_{\Gamma}(x) \) is the indicator function of \( \Gamma \). We will show the following theorem, which states that the processes \( \left\{ X(t, u, \xi), t \in [u, T] \right\} \) are Markov with transition operator \( P_{s,t} \), \( 0 \leq s \leq t \leq T \).

**Theorem 3.2.** Assume (C.1)–(C.6), and let \( X(t, s; \xi) \) be the solution of eq. (3.1). Then, for arbitrary \( \varphi \in B_b(H) \) and \( 0 \leq u \leq s \leq t \leq T \),

\[
\mathbb{E}_s \left( \varphi(X(t, u; \xi)) \middle| \mathcal{F}_s \right) = P_{s,t}(\varphi)(X(s, u; \xi)), \quad \text{P-a.s.}
\]  
(3.14)

**Proof.** By the uniqueness of eq. (3.1) we obtain

\[
X(t, u; \xi) = X(t, s; X(s, u; \xi)), \quad \text{P-a.s.}
\]

Denote \( X(s, u; \xi) \) by \( \eta \). Then eq. (3.14) can be written as

\[
\mathbb{E}_s \left( \varphi(X(t, s; \eta)) \middle| \mathcal{F}_s \right) = P_{s,t} \varphi(\eta), \quad \text{P-a.s.}
\]  
(3.15)

On the other hand, if eq. (3.15) holds for all \( \varphi \in C_b(H) \), then, by the classical result from measure theory on separable space (see ref. [8], p. 16, Proposition 1.2 for details), it holds for all \( \varphi \in B_b(H) \). Therefore, it suffices to show eq. (3.15) holds for \( \varphi \in C_b(H) \) and \( \sigma(X(s, u; \xi)) \)-measurable random variable \( \eta \).

To this end note that for \( \eta = \sum_{h=1}^{N} x_h \chi_{\Gamma_h} \left( X(s, u; \xi) \right) \), P-a.s., where \( \{ \Gamma_1, \Gamma_2, \ldots, \Gamma_N \} \) is a partition of \( H \) and \( \{ x_h \}_{h=1}^{N} \) are some elements in \( H \), we have

\[
X(t, s; \eta) = \sum_{h=1}^{N} X(t, s; x_h) \chi_{\Gamma_h} \left( X(s, u; \xi) \right), \quad \text{P-a.s.}
\]  
(3.16)

Since \( X(t, s; x_h) \) is \( \sigma \left\{ W(u) - W(s); u \geq s \right\} \)-measurable, it is independent of \( \mathcal{F}_s \). Taking into account \( \chi_{\Gamma_h} \left( X(s, u; \xi) \right) \) is \( \mathcal{F}_s \)-measurable, we deduce from eq. (3.16) that

\[
\mathbb{E}_s \left( \varphi(X(t, s; \eta)) \middle| \mathcal{F}_s \right) = \sum_{h=1}^{N} \mathbb{E}_s \left( X(t, s; x_h) \chi_{\Gamma_h} \left( X(s, u; \xi) \right) \middle| \mathcal{F}_s \right) = \sum_{h=1}^{N} \mathbb{E}_s \left( X(t, s; x_h) \chi_{\Gamma_h} \left( X(s, u; \xi) \right) \right)
\]
\[\begin{align*}
E\left[\sum_{k=1}^{N} \mathbb{P}_{s,t} \varphi(x_k) \chi_{X(s,u;\xi)}\right]
= \sum_{k=1}^{N} \mathbb{P}_{s,t} \varphi(x_k) \chi_{X(s,u;\xi)}
= \mathbb{P}_{s,t} \varphi(\eta), \quad \text{P-a.s.}
\end{align*}\]

(3.17)

Now, for \(\eta\) satisfying \(E|\eta|^2 < +\infty\) we can find a sequence of simple random variables \(\{\eta_n\}_{n=1}^{\infty}\) for which eq. (3.15) holds and \(E|\eta - \eta_n|^2 \to 0\), as \(n \to \infty\). And from Theorem 3.1, there exists a subsequence \(\{\eta_{n_k}\}\) such that \(\eta_{n_k} \to \eta\) and \(X(t,s;\eta_{n_k}) \to X(t,s;\eta)\), P-a.s., as \(k \to +\infty\). We let eq. (3.15) hold for \(\eta_{n_k}\), then, again by using Theorem 3.1 and taking the limit in eq. (3.15), we see that eq. (3.15) holds if \(E|\eta|^2 < +\infty\).

The final step is to consider arbitrary \(\sigma(X(s,u;\xi))\)-measurable random variable \(\eta\). In this case the solution \(X(s,u;\eta)\) is considered as a limit of solution \(X(s,u;\eta_n)\) corresponding to \(\eta_n = \eta X([0,t] \leq n), \ n = 1, 2, \cdots\). Therefore, taking the limit in eq. (3.15) is possible. And so, the proof is complete.

If we define \(P^{s,x}\) as follows,
\[P^{s,x}(A) = P\left(X(s + \tau, s; x) \in A\right), \quad \exists \mathcal{A} \in \mathcal{B}(C([0, +\infty); \mathcal{H})), \tag{3.18}\]
then, as a direct consequence of Theorem 3.2, we have the following corollary.

**Corollary 3.1.** Assume (C.1)—(C.6), and let \(X(\cdot, s; \xi)\) be the solution of eq. (3.1). Then, for arbitrary \(\mathcal{A}, \mathcal{B} \in \mathcal{B}(C([0, +\infty); \mathcal{H}))\) and Borel function \(\varphi: C([0, +\infty); \mathcal{H}) \to [0, +\infty)\), as in Theorem 3.2,
\[E\left(\varphi(X(\cdot, s; u; \xi))\chi_{\mathcal{A}}\right) = E\left(E^{\tau,X(s,u;\xi)}(\varphi)\chi_{\mathcal{A}}\right), \quad s \geq u \geq 0.\]

Finally, we will end this subsection by showing the solutions of eq. (3.1) are also strong Markov processes in the sense that for an arbitrary stopping time \(\tau \geq u\) and Borel function \(\varphi\), as in Corollary 3.1,
\[E\left(\varphi(X(\tau + \tau, u; \xi))\chi_{\mathcal{F}_\tau}\right) = E^{\tau,X(\tau,u;\xi)}(\varphi), \quad \text{P-a.s. on } \tau < +\infty, \tag{3.19}\]
where the \(\sigma\)-field \(\mathcal{F}_\tau = \left\{\mathcal{A} \in \mathcal{F} | \mathcal{A} \cap \{\tau \leq t\} \subseteq \mathcal{F}_t, t \geq 0\right\}\), and we recall that if \(\tau \leq \sigma\) are stopping times then \(\mathcal{F}_\tau \subset \mathcal{F}_\sigma\) (see ref. [18] for more details).

**Theorem 3.3.** Assume (C.1)—(C.6). Then the solutions of eq. (3.1) are strong Markov processes.

**Proof.** By eq. (3.19), it is enough to show that for arbitrary \(A \in \mathcal{F}_\tau\),
\[E\left(\varphi\left(X(\tau + \tau, u; \xi)\right)\chi_{A \cap \{\tau < +\infty\}}\right) = E\left(E^{\tau,X(\tau,u;\xi)}(\varphi)\chi_{A \cap \{\tau < +\infty\}}\right). \tag{3.20}\]
For arbitrary stopping time \(\tau\) we define a sequence of stopping times as follows:
\[\tau_n = \sum_{k=1}^{n} \frac{k}{2n} X_k \mathbb{1}_{[\tau < \tau_n]} + (\tau_n) \mathbb{1}_{[\tau = \tau_n]}.\]

Then \(\tau_n \downarrow \tau\) as \(n \uparrow +\infty\). Since \(\tau_n \geq \tau, \mathcal{F}_{\tau_n} \supseteq \mathcal{F}_\tau\) and \(A \in \mathcal{F}_{\tau_n}\), for \(n = 1, 2, \cdots\). Therefore, by Corollary 3.1, we have
\[E\left(\varphi\left(X(\tau_n + \tau_n, u; \xi)\right)\chi_{A \cap \{\tau < +\infty\}}\right) = E\left(E^{\tau_n,X(\tau_n,u;\xi)}(\varphi)\chi_{A \cap \{\tau < +\infty\}}\right). \tag{3.21}\]
We first show that one can pass in eq. (3.21) to the limit for arbitrary \( \varphi : C([0, +\infty); H) \to \mathbb{R} \) of the form \( \varphi(f) = \varphi(f(t)), \ f \in C([0, +\infty); H), \ \varphi \in C_b(H), \ t \in [a, T]. \) In this special case eq. (3.21) becomes

\[
\mathbb{E}\left( \left[ X(\tau + t, u, \xi) \right]_{\mathcal{A}}(\tau < +\infty) \right) = \mathbb{E}\left[ \left[ \left( \tau_{\tau_{n} + t} \right) \varphi(X(\tau_{n}, u, \xi))\right]_{\mathcal{A}}(\tau < +\infty) \right].
\]

(3.22)

Since the family \( \mathbb{P}_{\tau_{n}} \) has Feller property and \( X(\cdot, u; \xi) \) has continuous trajectories, by taking the limit in eq. (3.22), we obtain

\[
\mathbb{E}\left[ \left[ \varphi(X(\tau + t, u, \xi))\right]_{\mathcal{A}}(\tau < +\infty) \right] = \mathbb{E}\left[ \left[ \left( \tau_{\tau_{n} + t} \right) \varphi(X(\tau_{n}, u, \xi))\right]_{\mathcal{A}}(\tau < +\infty) \right]
\]

(3.23)

for \( \varphi \in C_b(H) \) and therefore for arbitrary Borel function \( \varphi. \)

Secondly, by induction, eq. (3.23) holds for the indicator functions of all the cylindrical sets \( \{f \in C([0, +\infty); H) : f(t_{j}) \in I_{j}, j = 1, \cdots, n\}, \) \( I_{j} \in \mathcal{B}(H), \) \( j = 1, \cdots, n. \) And so, eq. (3.23) holds for all monotone limits of the indicator functions and thus for all Borel function \( \varphi \in \mathcal{B}(C([0, +\infty); H)). \) Therefore, the proof is complete.

4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following lemma.

**Lemma 4.1.** Let \( Y \) be a nonnegative random variable and \( p \in (0,1) \). If \( Y \) satisfies

\[
P(Y \geq k) \leq p^{k} \text{ for } k = 0, 1, \cdots, \text{ then } \mathbb{E}(Y) \leq \frac{1}{1 - p}.
\]

**Proof.** We have

\[
\mathbb{E}(Y) - \int_{0}^{\infty} P(Y \geq t)dt = -\sum_{k=0}^{\infty} \int_{k}^{k+1} P(Y \geq t)dt \\
\leq \sum_{k=0}^{\infty} P(Y \geq k) \leq \sum_{k=0}^{\infty} p^{k} = \frac{1}{1 - p}.
\]

**Proof of Theorem 1.2.** Since \( D \) is an open bounded neighborhood of \( 0 \) and \( F \) is Lipschitz continuous, we can assume that \( D = \{x \in H : ||x|| \leq R\} \) for some \( R > 0 \) and \( F \) is bounded. We claim that there exists \( \varphi \in H^{\ast} = H, \ |\varphi| = 1, \) such that \( \forall \ C > 0, \)

\[
P(\{\varphi(W_{D}(T_{0})) > C\}) > 0.
\]

(4.1)

For this we assume contrary that \( \forall \ varphi \in H^{\ast} = H, \ \exists C_{0} > 0, \ such that \)

\[
|\varphi(W_{D}(T_{0}))| = |\langle \varphi, W_{D}(T_{0}) \rangle| \leq C_{0}, \ \text{ P-a.s.}
\]

(4.2)

By Riesz's representation theorem, there exists a unique linear functional \( f \in H^{\ast} \) such that

\[
f(x) = \langle W_{D}(T_{0}), x, \ \ x \in H,
\]

which, by the inequality (4.2), implies

\[
|W_{D}(T_{0})|^{2} \leq C_{0}, \ \text{ P-a.s.}
\]

This contradicts \( P(|W_{D}(T_{0})| > C) > 0 \) for any \( C > 0. \) Define \( q_{C}(x) \) by

\[
q_{C}(x) = P(\tau_{x, x} > T_{0}) \ \text{ for } x \in H.
\]

Then we have

\[
q_{C}(x) \leq P(|X_{T_{0}}(T_{0}, x)| \leq R) \leq P(|\varphi(X_{T_{0}, x})| \leq R).
\]

(4.3)
Since $X^\varepsilon(\cdot, x)$ is a solution of eq. (0.2) and $F$ is bounded, there exists $R_0 > 0$ such that
\[ |\varepsilon X^\varepsilon(T_0, x)| \geq \sqrt{\varepsilon}|W_F(T_0)| - R_0, \quad \text{P-a.s.} \quad (4.4) \]
Consequently, a combination of (4.1), (4.3) and (4.4) yields
\[ q_\varepsilon(x) \leq \mathbb{P}\left( |W_F(T_0)| \leq \frac{R + R_0}{\sqrt{\varepsilon}} \right) \equiv \rho_\varepsilon < 1, \quad \forall x \in D. \quad (4.5) \]
We note that for arbitrary $k = 0, 1, \ldots,$
\[ \mathbb{P}(\tau^{x, \varepsilon} > (1 + k)T_0) = \mathbb{P}(A_k \cap B_k), \]
where
\[ A_k = \left\{ |X^\varepsilon(t, x)| < R, \quad \forall t \in [0, kT_0] \right\} \in \mathcal{F}_{kT_0}, \]
\[ B_k = \left\{ |X^\varepsilon(kT_0 + s, x)| < R, \forall s \in [0, T_0] \right\}. \]
It follows
\[ \mathbb{P}(\tau^{x, \varepsilon} > (1 + k)T_0) = \mathbb{E}\left[ \mathbb{P}(A_k \cap B_k \mid \mathcal{F}_{kT_0}) \right] = \mathbb{E}\left[ \chi_{A_k} \mathbb{P}(B_k \mid \mathcal{F}_{kT_0}) \right]. \quad (4.6) \]
On the other hand, we deduce the Markov property that
\[ \mathbb{P}(B_k \mid \mathcal{F}_{kT_0}) = \mathbb{P}(X^\varepsilon(kT_0 + s, x) \in \Gamma \mid \mathcal{F}_{kT_0}) \]
\[ = \mathbb{P}(X^\varepsilon(kT_0, x) \in \Gamma) \leq q_\varepsilon(\mathbb{E}(X^\varepsilon(kT_0, x))) \leq \rho_\varepsilon, \quad \text{P-a.s.} \]
where $\Gamma = \left\{ f \in C([0, +\infty); H) : |f(t)| \leq R, \forall t \in [0, T_0] \right\}.$ Therefore we obtain
\[ \mathbb{P}(\tau^{x, \varepsilon} > (1 + k)T_0) \leq \rho_\varepsilon \mathbb{P}(A_k) \leq \rho_\varepsilon \mathbb{P}(\tau^{x, \varepsilon} > kT_0), \]
which, by induction, produces
\[ \mathbb{P}(\tau^{x, \varepsilon} > kT_0) \leq \rho_\varepsilon^k, \quad k = 0, 1, \ldots. \]
Then $\mathbb{E}(\tau^{x, \varepsilon}) \leq \frac{T_0}{1 - \rho_\varepsilon} < +\infty$ due to the last inequality and Lemma 4.1. This completes the proof.

5 Proof of Theorem 1.3

We first prove inequality (1.6). For arbitrary fixed control $u_0$ such that $f^{0, u_0}(T) \in (\mathbb{D})^c$, by (1.3), it is enough to show that for all $x \in D,$
\[ \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}(\tau^{x, \varepsilon}) \leq \frac{1}{2} \int_0^T |u_0(s)|^2 ds \equiv \lambda. \quad (5.1) \]
By Theorem 1.1 the solution $f^{x, u_0}(t)$ of eq. (1.2) is continuous with respect to $(t, x),$ then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that
\[ |x| < \delta_1 \Rightarrow \text{dist}_H(f^{x, u_0}(T), \partial D) \geq \delta_2, \quad (5.2) \]
which, by the definition of $\tau^{x, \varepsilon}$, implies
\[ \mathbb{P}(\tau^{x, \varepsilon} > T) \leq \mathbb{P}(X^\varepsilon(t, x) \in D, \forall t \in [0, T]) \leq \mathbb{P}\left( \sup_{t \in [0, T]} |X^\varepsilon(t, x) - f^{x, u_0}(t)| \geq \delta_2 \right). \]
or, equivalently
\[
P(\tau^{x,\varepsilon} < T) \leq P\left( \sup_{t \in [0,T]} |X^\varepsilon(t,x) - f^x,\mu_0(t)| < \delta_2 \right).
\] (5.3)

This by (i) of Theorem 2.1 implies that for any \( \gamma > 0 \), \( \exists \varepsilon_0 > 0 \) such that \( \forall \varepsilon \in (0,\varepsilon_0] \), and \( x \in H \) with \( |x| \leq \delta_1 \),
\[
q_\varepsilon(x) = P(\tau^{x,\varepsilon} < T) \geq e^{-\tilde{\Phi}(\lambda + \gamma)}.
\] (5.4)

On the other hand, by assumption (0.4) and (3.3) of Theorem 3.1 there exist \( T_1 > 0 \) and \( p_1 \) with \( 0 < p_1 < 1 \) such that for all \( x \in D \) and all sufficiently small \( \varepsilon > 0 \),
\[
P\left( |X^\varepsilon(T_1,x)| \leq \delta_1 \right) \geq p_1.
\] (5.5)

Hence by the Markov property, (5.4) and (5.5) we obtain
\[
P(\tau^{x,\varepsilon} < T + T_1)
\geq P\left[ |X^\varepsilon(T_1)| \leq \delta_1, \text{ and } X^\varepsilon(T_1 + s) \in \partial D \text{ for some } s \in [0,T] \right]
= E\left[ P\left( X^\varepsilon(T_1 + s) \in \partial D \text{ for some } s \in [0,T] | \mathcal{F}_{T_1} \right) | \chi_{\{ |X^\varepsilon(T_1,x)| \leq \delta_1 \}} \right]
= E\left[ P\left( \tau^{X^\varepsilon(T_1,x),\varepsilon} < T \right) | \chi_{\{ |X^\varepsilon(T_1,x)| \leq \delta_1 \}} \right]
\geq p_1 e^{-\tilde{\Phi}(\lambda + \gamma)}, \quad x \in D,
\]
and so
\[
P(\tau^{x,\varepsilon} \geq T + T_1) \leq p \equiv 1 - p_1 e^{-\tilde{\Phi}(\lambda + \gamma)}, \quad x \in D.
\] (5.6)

Then by induction and the Markov property we get
\[
P\left( \tau^{x,\varepsilon} \geq k(T + T_1) \right) \leq p^k, \quad \text{for } k = 0, 1, \cdots,
\]
which, by Lemma 4.1, yields
\[
E(\tau^{x,\varepsilon}) \leq \frac{T + T_1}{1 - p} = (T + T_1) \frac{1}{p_1} e^{\tilde{\Phi}(\lambda + \gamma)},
\]
therefore
\[
\limsup_{\varepsilon \to 0} \varepsilon \log E(\tau^{x,\varepsilon}) \leq \lambda + \gamma.
\]

Letting \( \gamma \to 0 \) in the last inequality, we complete the proof of (1.6).

Now we turn to proof of inequality (1.7). For any fixed \( \gamma > 0 \), by (1.5), we can choose \( r > 0 \) such that \( e_r > e - \gamma \) and \( S = \{ x | |x| \leq r \} \subseteq D \). Then there exists \( M_1 > 0 \) such that
\[
E(\tau^{x,\varepsilon}) \geq M_1 e^{\tilde{\Phi}(e_r - \gamma)}, \quad \forall x \in S.
\] (5.7)

By (0.4), \( \forall x \in D_0 \) and \( \forall r > 0, \exists T > 0 \) such that
\[
z^\varepsilon(t) \in D \quad \text{and} \quad |z^\varepsilon(t)| \leq r, \quad \forall t \geq T.
\] (5.8)

On the other hand, similar to that of Proposition 6 of ref. [6], we have \( \forall \delta > 0, \forall T > 0, \forall R > 0 \) and \( x \) with \( |x| \leq R \),
\[
\lim_{\varepsilon \to 0} P\left( \sup_{s \leq T} |X^\varepsilon(s,x) - z^\varepsilon(s)| \leq \delta \right) = 1.
\] (5.9)
Then, by (5.8) and (5.9), we get \( \gamma > 0, \exists \rho \in (0, 1), \exists \varepsilon_0 > 0 \) and \( T_1 > 0 \) such that, \( \forall \varepsilon \in (0, \varepsilon_0] \),

\[
P\left( \left| X^\varepsilon(T_1, x) \right| \leq \gamma \right) - p > 0, \ \forall x \in D_0.
\]

(5.10)

If (5.7) holds for \( x \in S \), then we deduce from (5.10) and the Markov property that for any \( x \in D_0 \),

\[
E(\tau^{x, \varepsilon}) \geq E\left( \tau^{x, \varepsilon} | X^\varepsilon(T_1, x) | \leq \gamma \right)
\]

\[
= E\left[ E\left( \tau^{x, \varepsilon} | X^\varepsilon(T_1, x) | \leq \gamma \right) \right]
\]

\[
= E\left[ E\left( \tau^{x, \varepsilon}(T_1, x) | X^\varepsilon(T_1, x) | \leq \gamma \right) \right]
\]

\[
\geq pM\varepsilon^{1/(\varepsilon - \gamma)} \geq pM\varepsilon^{1/(\varepsilon - \gamma)}.
\]

which implies

\[
\liminf_{\varepsilon \downarrow 0} \varepsilon \log E(\tau^{x, \varepsilon}) \geq c,
\]

Therefore it is enough to prove (5.7). For this we choose \( r > r_0 > 0 \) and define stopping times

\[
r_0 = \sigma_{r_0}^{x, \varepsilon} < \sigma_{r_0}^{x, \varepsilon} < \cdots
\]

as follows:

\[
\sigma^{-1}_{k+1} = \inf \left\{ t > \sigma_k^{x, \varepsilon} : \left| X^\varepsilon(t, x) \right| = r, \ \exists s \in [\sigma_k^{x, \varepsilon}, t] \text{ such that } \left| X^\varepsilon(s, x) \right| = r_0 \right\}
\]

for \( k = 0, 1, \cdots, \), and \( x \in S \). Then there exists \( \varepsilon_0 > 0 \) such that \( \forall \varepsilon \in (0, \varepsilon_0] \),

\[
p_0(x) \equiv P(\sigma^{x, \varepsilon}_k < \tau^{x, \varepsilon}) \geq (1 - e^{-\frac{1}{k}(\varepsilon - \gamma)})^k,
\]

(5.11)

for \( k = 0, 1, \cdots, \), and \( x \in S \). We first estimate \( p_0(x) \).

\[
p_0(x) = P(\sigma^{x, \varepsilon}_T < \tau^{x, \varepsilon})
\]

\[
= P\left( \left| X^\varepsilon(s, x) \right| = r_0, \text{ for some } s < \tau^{x, \varepsilon} \right),
\]

\[
q_0(x) \equiv 1 - p_0(x) = P\left( \left| X^\varepsilon(s, x) \right| > r_0, \ \forall s < \tau^{x, \varepsilon} \right),
\]

therefore, for arbitrary \( T > 0 \),

\[
q_0(x) \leq P(\tau^{x, \varepsilon} \leq T) + P\left( \left| X^\varepsilon(s, x) \right| > r_0, \ \forall s \leq T \text{ and } T < \tau^{x, \varepsilon} \right)
\]

\[
\leq P(\tau^{x, \varepsilon} \leq T) + P\left( X^\varepsilon(s, x) \in K, \ \forall s \in [0, T] \right)
\]

\[
= I_1 + I_2,
\]

where \( K \equiv \bar{D} \cap \{ x : |x| > r_0 \} \).

In order to estimate \( I_2 \), we need the following.

**Proposition 5.1.** Assume \( (C.1) \rightarrow (C.4) \) and (0.4). Then \( \forall r > 0, \exists \delta(r) > 0, \exists T_1 > 0, \exists K_\delta \supseteq K \) and \( C(\subseteq C(T_1, \delta)) > 0 \) such that if \( T > T_1 \) and \( f^{x, u}(s) \in K_\delta, \forall s \in [0, T] \), then

\[
\frac{1}{2} \int_0^T |u(s)|^2 ds \geq C(t - T_1), \ \forall t \in [T_1, T].
\]

**Proof.** Let \( K_\delta = \left\{ x \in H | \text{dist}_H(x, K) \leq \delta \right\} \) and \( \delta < r/3 \). Since \( x^{\varepsilon}(t) \) is continuous in \((t, x)\), therefore, by (0.4), there exist \( T_1 > 0 \) such that \( \forall t \geq T_1 \) and \( x \in S, |x^{\varepsilon}(t)| < r/3 \). Let \( x \in S \) and \( f^{x, u} \in K_\delta \), then, by (C.1)–(C.4), \( \forall t \in [0, T_1] \),

\[
| x^{\varepsilon}(t) - f^{x, u}(t) |^2 \leq M \int_0^{T_1} | \psi^2(t) dt + T_1^2 \beta | \int_0^{T_1} | u(t) |^2 dt.
\]

Consequently,

\[
\frac{r^2}{9} \leq \sup_{t \in [0, T_1]} | x^{\varepsilon}(t) - f^{x, u}(t) |^2 \leq e^{MT_1} \int_0^{T_1} | \psi^2(t) dt + T_1^2 \beta \int_0^{T_1} | u(t) |^2 dt.
\]
which implies \[
\frac{1}{2} \int_0^{T_1} |u(t)|^2 dt \geq \frac{1}{2} r^2 \frac{T_1^{-2\alpha} \beta_1^{-1} e^{-\lambda T_1}}{\beta_1} \int_0^{T_1} \psi(t) dt \equiv C > 0.
\]
By a simple induction, for arbitrary \( j = 1, 2, \cdots \) such that \( jT_1 \leq T \) we have \[
\frac{1}{2} \int_0^{jT_1} |u(t)|^2 dt \geq jC,
\]
thus the proposition is proved.

We choose \( T = jC > 2L \). By Proposition 5.1, if \( f^{\epsilon,x} \in K^T_\epsilon(2L) \), then \( f^{\epsilon,x} \in (K_\delta)^\epsilon \). Therefore, by (ii) of Theorem 2.1, \( \exists \varepsilon_1 \) such that \( \forall x \in S \) and \( \varepsilon \in (0, \varepsilon_1] \),
\[
I_2 \leq P\left( d \text{ist}_{.[0,T_1;H]}(X^\epsilon(\cdot,x), K^T_\epsilon(2L)) \geq \frac{r_0}{2} \right) \leq e^{-\frac{T_1}{\lambda}(\epsilon_1 - \gamma)}.
\]  
(5.12)
Since \( K^T_\epsilon(\epsilon / 2) \) is compact in \( C([0,T];H) \) due to Theorem 1.1, by the definition of \( \epsilon_1 \), there exists \( \rho > 0 \) such that
\[
\left\{ \tau^{x,\epsilon} \leq T \right\} \subseteq \left\{ \text{dist}_{C([0,T_1];H)}(X^\epsilon(\cdot,x), K^T_\epsilon(\epsilon / 2)) > \rho \right\}.
\]
Then, by (ii) of Theorem 2.1, \( \exists \varepsilon_2 \) such that \( \forall x \in S \) and \( \varepsilon \in (0, \varepsilon_2] \),
\[
I_1 \leq P\left( \text{dist}_{C([0,T_1];H)}(X^\epsilon(\cdot,x), K^T_\epsilon(\epsilon / 2)) > \rho \right) \leq e^{-\frac{T_1}{\lambda}(\epsilon_1 - \gamma)}.
\]  
(5.13)
A combination of (5.12) and (5.13) completes the estimate \( p^\epsilon_1(x) \) with \( \varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2 \) and \( x \in S \).

By strong Markov property and the definition of \( \sigma^x_\epsilon \), we have
\[
p^\epsilon_{k+1}(x) = P(\sigma^x_\epsilon < \tau^{x,\epsilon}) = E\left[ X^\epsilon(\sigma^x_\epsilon, x) | \sigma^x_\epsilon < \tau^{x,\epsilon} \right] \]
\[
= E\left[ X^\epsilon(\sigma^x_\epsilon, x) | \sigma^x_\epsilon < \tau^{x,\epsilon} \right] \]
\[
= E\left[ X^\epsilon(\sigma^x_\epsilon, x) | \sigma^x_\epsilon < \tau^{x,\epsilon} \right] \]
\[
\geq \left( 1 - e^{-\frac{T_1}{\lambda}(\epsilon_1 - \gamma)} \right) p^\epsilon_1(x), \forall x \in S \text{ and } \forall \varepsilon \in (0, \varepsilon_0].
\]
By induction, we complete the proof of (5.11). Define \( A^\epsilon_k = \Omega, A^\epsilon_k = \{ \sigma^x_\epsilon < \tau^{x,\epsilon} \}, k = 1, 2, \cdots, x \in S \) and \( \varepsilon \in (0, \varepsilon_0] \). Then
\[
P(A^\epsilon_k) \geq \left( 1 - e^{-\frac{T_1}{\lambda}(\epsilon_1 - \gamma)} \right)^k.
\]
Let \( \xi^x_\epsilon \equiv \inf \left\{ s \geq 0, |X^\epsilon(s, \sigma^x_\epsilon) - X^\epsilon(\sigma^x_\epsilon, x)| \geq \frac{r - r_0}{2} \right\}, \) if \( \sigma^x_\epsilon < \infty \),
\[
\xi^x_\epsilon = +\infty, \text{ otherwise}.
\]
It is clear that
\[
\tau^{x,\epsilon} \geq \xi^x_\epsilon + \xi^x_\epsilon + \cdots + \xi^x_\epsilon \text{ on } A^\epsilon_k \setminus A^\epsilon_{k+1}, k = 0, 1, \cdots.
\]
Consequently,
\[
E(\tau^{x,\epsilon}) \geq \sum_{k=0}^{\infty} E \left( \xi^x_\epsilon + \xi^x_\epsilon + \cdots + \xi^x_\epsilon \right) \chi_{\{ A^\epsilon_0 \setminus A^\epsilon_1 \}}
\]
\[
= E(\xi^x_\epsilon) + \sum_{k=0}^{\infty} E(\xi^x_\epsilon).
\]  
(5.14)
By (0.4) and (0.6), without loss of generality, we can assume \( \exists M_1 > 0 \) such that \( \forall \varepsilon \in (0, \varepsilon_0] \) and \( x \in S \), \( E(\xi^\varepsilon_k x_{A_k^\varepsilon}) \geq M_1 \). Then, again by strong Markov property,

\[
E(\xi^\varepsilon_k x_{A_k^\varepsilon}) = E \left[ \chi_{A_k^\varepsilon} \cdot E(\xi^\varepsilon_{k-1} | \mathcal{F}_{\sigma_{\varepsilon_{k-1}}}^\varepsilon) \right] \\
= E \left[ A_k^\varepsilon \cdot E(\xi^\varepsilon_{k-1} | \sigma_{\varepsilon_{k-1}}, x) \right] \\
\geq M_1 P(A_k^\varepsilon). \tag{5.15}
\]

Putting (5.15) into (5.14) we get

\[
E(\tau^\varepsilon) \geq M_1 \sum_{k=0}^{\infty} P(A_k^\varepsilon) \geq M_1 \sum_{k=0}^{\infty} (1 - e^{-\varepsilon(X_{\varepsilon} - \gamma)})^k = M_1 e^{-\varepsilon(X_{\varepsilon} - \gamma)}, \forall x \in S.
\]

Thus we complete the proof of Theorem 1.3.

6 Examples

In order to see the generality of our results, we present two examples being in our setting but in Da Prato and Zabczyk's [0.9] in this subsection.

**Example 6.1.** Let \( H = L^2(0, \pi) \) and \( A \) be the Laplace operator with the Dirichlet boundary conditions:

\[
D(A) = H^2(0, \pi) \cap H_0^1(0, \pi), \\
AX(\xi) = \frac{\partial^2 X}{\partial \xi^2}, \quad X \in D(A), \; \xi \in [0, \pi].
\]

\( S(\cdot) \) stands for the semigroup generated by \( A \). We assume that the nonlinearities \( F \) and \( B \) are given by

\[
\begin{cases}
F(t, X(\xi)) = f(t, X(\xi)), & X \in L^2(0, \pi), \; \xi \in [0, \pi], \\
B(X[Y])(\xi) = g(X(\xi))Y(\xi), & X, Y \in L^2(0, \pi), \; \xi \in [0, \pi],
\end{cases}
\]

where \( f \) and \( g \) are real functions defined on \( [0, +\infty) \times \mathbb{R} \) and \( \mathbb{R} \), respectively. If \( f \) and \( g \) satisfy the conditions:

(i) \( \exists \; \Psi \in L^2_{loc}([0, +\infty), \mathbb{R}) \) such that \( \forall u, v \in \mathbb{R} \) and \( \forall t \geq 0 \),

\[
|f(t, 0)| = 0, \quad |f(t, u) - f(t, v)| \leq |\Psi(t)||u - v|,
\]

(ii) \( \exists \) a constant \( L \) such that \( u, v \in \mathbb{R} \),

\[
|g(u) - g(v)| \leq L|u - v| \quad \text{and} \quad |g(u)| \leq L,
\]

then by an easy computation we get

\[
\|S(t)[B(x) - B(y)]\|_2 \leq L \frac{t}{\sqrt{\pi}} \|S(t)\|_2 \|x - y\|, \; \forall x, y \in H \; \text{and} \; t \geq 0.
\]

And eq. (0.1) can be written equivalently as the following (stochastic heat equations):

\[
\begin{cases}
\frac{\partial X^\varepsilon}{\partial t}(t, \xi) = \frac{\partial^2 X^\varepsilon}{\partial \xi^2}(t, \xi) + f(t, X^\varepsilon(\xi)) + \varepsilon g(X^\varepsilon(\xi)) \tilde{W}(t, \xi), \\
X^\varepsilon(0, \cdot) = x \in H = L^2(0, \pi), \\
X^\varepsilon(t, 0) = X^\varepsilon(t, \pi), \; \forall t \geq 0.
\end{cases}
\]

An easy computation shows that all assumptions of Theorem 1.3 are fulfilled.
Example 6.2. Let \( g(x) = 2 + \sin x \) in the above Example 6.1, \( W(\cdot) \) is a \( Q \)-Wiener process in \( H \) such that

\[
Q e_k(\xi) = \lambda_k e_k(\xi), \ k = 1, 2, \ldots; \ \lambda_k > 0,
\]

where \( \{e_k(\xi)\} = \{\sqrt{\frac{2}{\pi}} \sin(k\xi)\} \) is an orthogonal basis in \( H \), \( \{\lambda_k\} \) satisfy \( \sum_{k=1}^{\infty} \frac{\lambda_k}{k^2} < +\infty \). Then

\[
A e_k = -k^2 e_k, \ k = 1, \ldots. \quad \text{Let } Y_t = \int_0^t S(t-s)dW(s), \ \text{P-a.s., and } \bar{Y}_t = \int_0^t S(t-s)B(X^\alpha(s,x))dW(s), \ \text{P-a.s.}
\]

Since \( W \) can be represented as \( W(t,\xi) = \sum_{k=1}^{\infty} \sqrt{\lambda_k e_k(\xi)} \beta_k(t) \), \( \{\beta_k(t)\} \) are independent Wiener processes,

\[
Y_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k e_k(\xi)} \int_0^t e^{-k^2(t-s)}d\beta_k(s), \ \text{P-a.s.,}
\]

\[
\bar{Y}_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k e_k(\xi)} \int_0^t [2 + \sin(X^\alpha(s,x))] e^{-k^2(t-s)}d\beta_k(s), \ \text{P-a.s.}
\]

Moreover,

\[
|\bar{Y}_t|^2 = \sum_{k=1}^{\infty} \lambda_k \left( \int_0^t [2 + \sin(X^\alpha(s,x))] e^{-k^2(t-s)}d\beta_k(s) \right)^2, \ \text{P-a.s.,}
\]

\[
|Y_t|^2 = \sum_{k=1}^{\infty} \lambda_k \left( \int_0^t e^{-k^2(t-s)}d\beta_k(s) \right)^2, \ \text{P-a.s.}
\]

Note that these series are convergent in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) due to \( \sum_{k=1}^{\infty} \frac{\lambda_k}{k^2} < +\infty \). Therefore

\[
|\bar{Y}_t| \leq |\bar{Y}_t| \leq 3|Y_t|, \ \forall t \geq 0, \ \text{P-a.s.}
\]

On the other hand, since \( Y \) is Gaussian process, we have for any \( C > 0 \),

\[
\mathbb{P}(|\bar{Y}_t| > C) \geq \mathbb{P}(|Y_t| > C) > 0.
\]

Hence all assumptions of Theorem 1.2 are fulfilled.

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