TWO-PERSON KNAPSACK GAME

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Abstract. In this paper, we study a two-person knapsack game. Two investors, each with an individual budget, bid on a common pool of potential projects. To undertake a project, investors have their own cost estimation to be charged against their budgets. Associated with each project, there is a potential market profit that can be taken by the only bidder or shared proportionally by both bidders. The objective function of each investor is assumed to be a linear combination of the two bidders’ profits. Both investors act in a selfish manner with best response to optimize their own objective functions by choosing portfolios under the budget restriction. We show that pure Nash equilibrium exists under certain conditions. In this case, no investor can improve the objective by changing individual bid unilaterally. A pseudo polynomial-time algorithm is presented for generating a pure Nash equilibrium. We also investigate the price of anarchy (the ratio of the worst Nash equilibrium to the social optimum) associated with a simplified two-person knapsack game.

1. Introduction. In this paper, we propose a two-person knapsack game model. Suppose that there exist two noncooperative investors who bid on a common pool of potential business projects. Each investor has an individual cost base for each project to be charged against an individual budget. Should an investor be the sole bidder of a particular project, an expected market profit will be rewarded to this investor. If both investors bid on the same project, then the expected market profit will be shared in proportion to each investor’s market power. Assume that each investor optimizes an objective function related to both investors’ profits and the investor acts in a selfish manner with best response to the opponent’s bidding. We are interested in knowing if there exists a state in which no investor can change individual bid unilaterally to improve his objective. We are also interested in finding such a state when it exists.

This study may enhance our understanding of some common business practices such as two fast food companies may consider opening new stores in ten possible shopping malls, two petroleum companies may bid on adding new gas stations near
fifteen populous subdivisions, or two apparel companies may invest on building outlet stores in five towns. In each of such cases, a company has its own cost base to add a new facility subject to the company’s budget constraint. The profit will come from the market of potential customers in the area. Should both companies invest on the same location, market share is proportional to the market power of each company (say, name brand effect). The objective of each company may differ a lot - one may consider maximizing its own profit while others may consider suppressing opponent’s profit or a combination of both. It depends on the developing stage and corporate mind setting of a particular company. A newcomer or a conservative company tends to survive by maximizing its own profit, an aggressive company cares more about the gap of profits between two investors, and a well-established dominating company may care about minimizing the opponent’s profit. As long as each investor acts in a selfish manner in response to the opponent’s decision to optimize a given objective function, we are interested in knowing how should an investment decision be made by each investor.

The structure of receiving a reward from an investment cost subject to a budget constraint links us to the classic knapsack problem (KP for short). In fact, investment is a primary application of KP [6, 15] where one investor with a budget decide how to choose from some profitable projects such that the overall return on investment is as large as possible. In the real world, there are always more than one investor, and their interaction leads to a complex situation beyond the scope of KP. The noncooperative nature of the investors updating their portfolios (selection of projects) to improve his own individual objective values puts the problem into a noncooperative two-person game. Whether this decision making process terminates with a stable state in which no investor can improve individual objective by changing bids unilaterally leads us to the realm of Nash equilibrium [12] in the game theory. Hence we name our model a two-person knapsack game.

In special case that the selection of a particular project results in the same contribution (depending only on the number of investors who actually select this project) to the objective function of both investors, the two-person knapsack game problem can be reduced to a congestion game [16], for which Monderer and Shapley [11] proved that there exists at least one pure Nash equilibrium solution using the so-called “potential functions.” But a general two-person knapsack game may not be treated as a congestion game because each project could contribute differently toward individual objective functions. In the case where each investor selects only one project and the individual objective value decreases as the project is selected by more investors, Milchtaich [10] showed that a pure Nash equilibrium exists. However, the two-person knapsack game allows each investor to select multiple projects and individual objective functions may not decrease in terms of the number of investors who select the project.

One of our major goals is to investigate the existence conditions for pure Nash equilibrium solutions of a two-person knapsack game. We are also interested in designing an algorithm to find such a Nash equilibrium, when it exists. The two-person knapsack game is an NP-hard problem since each investor actually faces a classical knapsack problem, which is NP-hard [4], when the opponent’s decision is fixed. Moreover, verifying if a given state of the two-person knapsack game is a Nash equilibrium is not easier than solving a knapsack problem. Most known algorithms for solving the knapsack problem are based on the branch-and-bound scheme [5, 13], dynamic programming approach [1, 9], and a mixture of both [8]. In particular, a
dynamic programming-based pseudo polynomial-time algorithm can solve knapsack problems in $O(nc)$ running time, where $c$ is the capacity of the knapsack and $n$ is the number of objects for selection. This result motivates us to develop a pseudo polynomial-time algorithm for finding a Nash equilibrium solution of the two-person knapsack game.

Even though a pure Nash equilibrium may exist in a two-person knapsack game under certain conditions, the noncooperative nature of the game and selfish manner of each investor may prevail at the sacrifice of social welfare. In order to study the impact of investment decisions to the whole society led by different objectives, we follow the pioneering work of Koutsoupias and Papadimitriou [7] to quantify the Nash equilibrium solutions in a simplified knapsack game using the concept of “price of anarchy”, defined as the ratio of the worst Nash equilibrium to the social optimum. This concepts has been adopted in many other fields such as traffic routing [17, 18], load balancing [7, 3, 2] and facility location problem [19].

If it is not specially mentioned, in this paper, a Nash equilibrium solution means a pure Nash equilibrium. The rest of the paper is organized as follows. In Section 2, we present the model and notations of the two-person knapsack game. The existence of a Nash equilibrium is studied in Section 3. A pseudo polynomial-time algorithm for finding a Nash equilibrium is presented in Section 4. The concept of the price of anarchy is employed and analyzed for a simplified knapsack game in Section 5. Summary of results and concluding remarks are included in Section 6.

2. Model and notations. Our knapsack game model can be stated as follows. Suppose that there exist two players (investors), say player $i$ for $i = 1, 2$. Each of them has a budget limit, say $c_i$ for player $i$, and is interested in investing on $n$ potential projects, say project $k$ for $k = 1, 2, \ldots, n$. To include project $k$ in player $i$’s portfolio, there associates a cost $w_{ik}$ for the player. If player $i$ is the sole investor of project $k$, a profit of $p_k$ will be rewarded to the player, no matter $i = 1$ or $2$. If both players select project $k$, then the profit becomes $(1 - \alpha_i)p_k$ for player $i$, where $0 \leq \alpha_i \leq 1$. Notice that $\alpha_1 + \alpha_2$ is not required to be 1 in our model, because the presence of two similar facilities in a location may either stimulate new demands for a bigger market or reduce the total needs for the market. Without loss of generality, we may assume that $c_i, p_k$ and $w_{ik}$ are nonnegative integers for $i = 1, 2$ and $k = 1, 2, \ldots, n$.

A state of the two-person knapsack game can be described as an ordered pair $(S_1, S_2)$, where $S_i$ is the set of projects selected by player $i$ for $i = 1, 2$. Let $w_1(S_1)$ be the total cost for player $i$ to undertake all projects in the set $S_i$. Then we say a state $(S_1, S_2)$ is feasible to the knapsack game if and only if $w_1(S_1) \leq c_1$ and $w_2(S_2) \leq c_2$. For a given feasible state $(S_1, S_2)$, we let $P_i(S_1, S_2)$ represent the total obtainable profit for player $i$, $i = 1, 2$, at this state. Then it is not difficult to verify that

\begin{align}
P_1(S_1, S_2) &= p(S_1) - \alpha_1 p(S_1 \cap S_2), \\
P_2(S_1, S_2) &= p(S_2) - \alpha_2 p(S_1 \cap S_2),
\end{align}

(1)

where $p(S)$ represents the sum of profits $p_k$ with project $k$ being included in the set $S$. In this paper, we allow that the objective function of each player to be a linear combination of the profits of both players. To be more specific, the objective function value of player $i$ at state $(S_1, S_2)$ is given by

\begin{equation}
O_i(S_1, S_2) = \beta_{ii} P_i(S_1, S_2) - \beta_{ij} P_j(S_1, S_2),
\end{equation}

(2)
where \( i, j = 1, 2 \) and \( j \neq i \), \( \beta_{ii} \geq 0 \) and \( \beta_{ij} \) is a real number.

The nonnegativity of \( \beta_{ii} \) indicates that both players value their own profits positively. When \( \beta_{ij} = 0 \), player \( i \) minds only his own profit. When \( \beta_{ij} > 0 \), player \( i \) is concerned about the weighted gap between his and opponent’s obtainable profits. When \( \beta_{ij} < 0 \), player \( i \) is interested in the weighted total obtainable profits for both players. In the latter case, we assume that \( \beta_{ii} > -\beta_{ij} \) to indicate that player \( i \) minds his profit more than the opponent’s. Substituting (1) into the objectives functions, we have

\[
O_1(S_1, S_2) = \beta_{11}p(S_1) - \beta_{12}p(S_2) - \Delta_1 p(S_1 \cap S_2)
O_2(S_1, S_2) = \beta_{22}p(S_2) - \beta_{21}p(S_1) - \Delta_2 p(S_1 \cap S_2),
\]

where \( \Delta_i = \alpha_i \beta_{ii} - \alpha_j \beta_{ij}, \) for \( i = 1, 2 \) and \( j \neq i \).

A state of the knapsack game is called a Nash equilibrium (or a Nash equilibrium solution) if it is a feasible state at which no player can improve his individual objective value by changing his portfolio unilaterally. A formal definition is given below.

**Definition 2.1.** A feasible state \((S_1, S_2)\) is a Nash equilibrium of the two-person knapsack game if and only if \(O_1(S_1, S_2) \geq O_1(S_1', S_2)\) and \(O_2(S_1, S_2) \geq O_2(S_1', S_2')\) for any feasible states \((S_1', S_2')\) and \((S_1, S_2')\).

3. **Existence of Nash equilibrium.** In this section, we present an existence condition of Nash equilibrium solutions in a two-person knapsack game. Although our knapsack game is not a potential game as defined by Monderer and Shapley [11], the idea of using “potential functions” to prove the existence of Nash equilibrium can still be used here.

**Definition 3.1.** For a two-person knapsack game, the potential function of the game is a real-valued function defined over the set of feasible states such that its value increases strictly when any player shifts to a new state for an improved objective value.

To show our major theorem which completely characterizes the existence of Nash equilibrium solutions using the product value of \(\Delta_1\) and \(\Delta_2\), some technical lemmas are presented first.

**Lemma 3.2.** If \(\Delta_1 = \Delta_2 = 0\) for the two-person knapsack game, then \(\Phi_1(S_1, S_2) \equiv p(S_1) + p(S_2)\) is a potential function.

*Proof.* In this case, equation (3) becomes

\[
O_1(S_1, S_2) = \beta_{11}p(S_1) - \beta_{12}p(S_2),
O_2(S_1, S_2) = \beta_{22}p(S_2) - \beta_{21}p(S_1).
\]

If there is any player, say player 1, who can improve his objective by shifting to a new feasible state, say \(S_1'\), then \(O_1(S_1, S_2) < O_1(S_1', S_2)\). Consequently, \(\beta_{11}p(S_1) < \beta_{11}p(S_1')\). Since \(\beta_{11} \geq 0\) is assumed in the model, we know \(p(S_1) < p(S_1')\), which further implies that \(\Phi_1(S_1, S_2) < \Phi_1(S_1', S_2)\). The conclusion follows from Definition 3.1. \(\square\)

**Lemma 3.3.** If \(\Delta_1 = 0\) and \(\Delta_2 \neq 0\) for the two-person knapsack game, then \(\Phi_2(S_1, S_2) \equiv M p(S_1) + \beta_{22} p(S_2) - \Delta_2 p(S_1 \cap S_2)\) is a potential function, where \(M = \max\{|\Delta_1|, |\Delta_2|\} \sum_{k=1}^{n} p_k + 1\). If \(\Delta_1 \neq 0\) and \(\Delta_2 = 0\), then \(\Phi_2(S_1, S_2) = M p(S_2) + \beta_{11} p(S_1) - \Delta_1 p(S_1 \cap S_2)\) is a potential function.
Proof. We only need to show that $\Phi_2$ is a potential function. When $\Delta_1 = 0$ and $\Delta_2 \neq 0$, we have

$$O_1(S_1, S_2) = \beta_{11}p(S_1) - \beta_{12}p(S_2),$$
$$O_2(S_1, S_2) = \beta_{22}p(S_2) - \beta_{21}p(S_1) - \Delta_2p(S_1 \cap S_2).$$

If player 1 can change to a new state $S'_1$ for an improved objective value, then $O_1(S_1, S_2) < O_1(S'_1, S_2)$. Consequently, we have $p(S_1) < p(S'_1)$. As we assumed in the model, $p_k$ is a positive integer for $k = 1, 2, ..., n$. Hence $p(S'_1) \geq p(S_1) + 1$. Noticing that

$$\Phi_2(S_1, S_2) = Mp(S_1) + \beta_{22}p(S_2) - \Delta_2p(S_1 \cap S_2),$$
$$\Phi_2(S'_1, S_2) = Mp(S'_1) + \beta_{22}p(S_2) - \Delta_2p(S'_1 \cap S_2),$$

and $M = \max\{|\Delta_1|, |\Delta_2|\} \sum_{k=1}^np_k + 1 > 0$, we have

$$\Phi_2(S'_1, S_2) - \Phi_2(S_1, S_2) = M(p(S'_1) - p(S_1)) - \Delta_2(p(S'_1 \cap S_2) - p(S_1 \cap S_2)) \geq M - |\Delta_2|\sum_{k=1}^np_k > 0.$$

Now, if player 2 gets an improved objective value by changing from $S_2$ to $S'_2$, then $O_2(S_1, S_2) < O_2(S'_1, S_2)$. This further implies that $\beta_{22}p(S_2) - \Delta_2p(S_1 \cap S_2) < \beta_{22}p(S'_2) - \Delta_2p(S'_1 \cap S'_2)$. Therefore, we have $\Phi_2(S_1, S_2) < \Phi_2(S_1, S'_2)$. Hence $\Phi_2(S_1, S_2)$ is a potential function.

A similar proof follows for $\Phi_2(S_1, S_2)$. 

Lemma 3.4. If $\Delta_1 > 0$ and $\Delta_2 > 0$ for the two-person knapsack game, then $\Phi_3(S_1, S_2) \equiv \Delta_2\beta_{11}p(S_1) + \Delta_1\beta_{22}p(S_2) - \Delta_1\Delta_2p(S_1 \cap S_2)$ is a potential function.

Proof. If player 1 changes from $S_1$ to a new state $S'_1$ with an improved objective value, then $O_1(S_1, S_2) < O_1(S'_1, S_2)$. Equation (3) implies that

$$\beta_{11}p(S_1) - \Delta_1p(S_1 \cap S_2) < \beta_{11}p(S'_1) - \Delta_1p(S'_1 \cap S_2).$$

Since $\Delta_2 > 0$, we have

$$\Phi_3(S'_1, S_2) - \Phi_3(S_1, S_2) = \Delta_2[(\beta_{11}p(S'_1) - \Delta_1p(S'_1 \cap S_2)) - (\beta_{11}p(S_1) - \Delta_1p(S_1 \cap S_2))] > 0.$$

If player 2 improves his objective value by changing from state $S_2$ to $S'_2$, the symmetric structure of the problem setting takes care of the rest of the proof. 

Lemma 3.5. If $\Delta_1 < 0$ and $\Delta_2 < 0$ for the two-person knapsack game, then $\Phi_4(S_1, S_2) \equiv -\Phi_3(S_1, S_2)$ is a potential function.

Proof. With the same argument as in Lemma 3.4, by noticing the fact that $\Delta_1 < 0$ and $\Delta_2 < 0$, we can easily construct a proof.

Combining the previous lemmas, we show that there always exists a potential function $\Phi$ for the two-person knapsack game when $\Delta_1\Delta_2 \geq 0$. Let us start from any current feasible state of the game, say $(S^*_1, S^*_2)$, if it is not a Nash equilibrium, then at least one player can move to a new feasible state, say $(S^{k+1}_1, S^{k+1}_2)$, with an improved objective value

$$\Phi(S^{k+1}_1, S^{k+1}_2) > \Phi(S^*_1, S^*_2).$$

If $(S^{k+1}_1, S^{k+1}_2)$ is a Nash equilibrium, then we are done. Otherwise we can repeat this procedure until a Nash equilibrium is obtained. Since the value of the potential function increases strictly, no feasible state will be visited more than once in this
process. As the number of feasible states is finite for a two-person knapsack game, we can eventually terminate this process with a Nash equilibrium. This leads us to the following existence theorem.

**Theorem 3.6.** If $\Delta_1 \Delta_2 \geq 0$ for the two-person knapsack game, then there exists at least one Nash equilibrium of the game.

Note that the potential function at a Nash equilibrium of the knapsack game may or may not achieve the maximum value. But a feasible state at which the potential function achieves its maximum value must be a Nash equilibrium. Otherwise, the process in the above argument will continue to reach a new feasible state with a potential value strictly bigger than the maximum potential. Consequently, we have the following corollary:

**Corollary 1.** If $\Delta_1 \Delta_2 \geq 0$ for the two-person knapsack game, then a feasible state with the maximum potential value is a Nash equilibrium of the game.

We emphasize that Corollary 1 presents a sufficient condition for identifying a Nash equilibrium, but it is not a necessary condition. Consider a simple example with two players, three projects, and the following parameters:

$$
p_1 = 2, p_2 = 3, p_3 = 4, w_{11} = w_{12} = w_{13} = c_1, w_{21} = w_{22} = c_2, w_{23} = c_2 + 1,
\alpha_1 = \alpha_2 = 1/2, \beta_{11} = \beta_{22} = 1, \beta_{12} = \beta_{21} = 0.
$$

Then $\Delta_1 = \Delta_2 = 1/2$ and the potential function $\Phi_3(S_1, S_2) = \frac{1}{2}p(S_1) + \frac{1}{2}p(S_2) - \frac{1}{4}p(S_1 \cap S_2)$. It can be verified that both the state $\{(3), \{1\}\}$ and state $\{(2), \{3\}\}$ are Nash equilibria of the game. However, $\Phi_3(\{3\}, \{1\}) = 3$ and $\Phi_3(\{2\}, \{3\}) = 3.5$.

Now we turn our attention to the case of $\Delta_1 \Delta_2 < 0$, i.e., “$\Delta_1 > 0$ and $\Delta_2 < 0$" or “$\Delta_1 < 0$ and $\Delta_2 > 0$".

**Theorem 3.7.** If $\Delta_1 \Delta_2 < 0$ for the two-person knapsack game, then there exists at least one instance without Nash equilibrium.

*Proof.* Without loss of generality, we may assume that $\Delta_1 > 0$ and $\Delta_2 < 0$. Consider a simple instance with two projects. Suppose that $p_1 = p_2 = 1$ and $w_{11} = w_{12} = c_1, w_{21} = w_{22} = c_2$, then each player can select at most one project. With the assumption of $\alpha_2 \geq 0, \beta_{11} \geq 0$ and $\beta_{11} \geq -\beta_{12}$, the condition of $\Delta_1 > 0$ implies that $\beta_{11} > 0$. We show that there exists no Nash equilibrium for the game under this setting.

(i) Obviously, the state $(\emptyset, \emptyset)$ is not a Nash equilibrium.

(ii) Any state in the form of $(\emptyset, S_2)$ is not a Nash equilibrium, no matter $S_2 = \{1\}$ or $\{2\}$. Since the objective value of player 1 in this situation is $-\beta_{12}p(S_2)$ at $(\emptyset, S_2)$, player 1 can select the project which is not in $S_2$ to strictly increase the objective value to $\beta_{11} - \beta_{12}p(S_2)$. Similarly, we can show that $(S_1, \emptyset)$ is not a Nash equilibrium.

(iii) Any state in the form of $(S_1, S_2)$ with non-empty sets $S_2 \neq S_1$ is not a Nash equilibrium. Notice that the objective value of player 2 in this situation is $\beta_{22}p(S_2) - \beta_{21}$ at $(S_1, S_2)$. Since $\beta_{22} \geq 0, p(S_2) \leq 1$ and $\Delta_2 < 0$, player 2 can replace his selection by $S_2' = S_1$ to strictly increase the objective value to $\beta_{22} - \beta_{21} - \Delta_2$ at the state $(S_1, S_2)$.

(iv) Any state in the form of $(S_1, S_2)$ with non-empty sets $S_1 = S_2$ is not a Nash equilibrium. At this state, the objective value of player 1 is $\beta_{11} - \beta_{12} - \Delta_1$. Since $\Delta_1 > 0$, player 1 can change to select the project that is not currently in $S_1$ to strictly increase the objective value to $\beta_{11} - \beta_{12}$.

□
4. Finding Nash equilibrium. From the existence theorem, we know that Nash equilibrium solutions of a two-person knapsack game indeed exist as long as $\Delta_1 \Delta_2 \geq 0$. The question is how to find one. In this section we present a dynamic programming based algorithm to find a Nash equilibrium of the game in a pseudo polynomial time.

Given a two-person knapsack game with $\Delta_1 \Delta_2 \geq 0$, let $\Phi(S_1, S_2)$ be a potential function as defined in Lemmas 3.2-3.5.

**Definition 4.1.** $\Phi(S_1, S_2)$ is separable if $\Phi(S_1, S_2) = \Phi(S_{11}, S_{21}) + \Phi(S_{12}, S_{22})$ for any given states $(S_{11}, S_{21})$ and $(S_{12}, S_{22})$ with $S_1 = S_{11} \cup S_{12}$, $S_2 = S_{21} \cup S_{22}$ and $S_{1i} \cap S_{j2} = \emptyset$ for $i, j = 1, 2$.

It can be verified that the potential functions $\Phi_1$, $\Phi_2$ ($\Phi'_2$), $\Phi_3$ and $\Phi_4$ as defined in Lemmas 3.2-3.5 are all separable. Our proposed algorithm will find a feasible state that achieves the maximum value of its potential function, which is a Nash equilibrium as assured by Corollary 1.

Given a feasible state $(S_1, S_2)$, we can calculate its potential value $\Phi(S_1, S_2)$, $w(S_1)$ and $w(S_2)$, respectively, and use a quintuple $F(S_1, S_2) = [S_1, S_2, \Phi(S_1, S_2), w(S_1), w(S_2)]$ to record the information of the state. When $\Delta_1 \Delta_2 \geq 0$, the following is a dynamic programming based algorithm for finding a Nash equilibrium solution to a two-person knapsack game (DPKG Algorithm in short):

**DPKG Algorithm:**

Step 1. Start with $M_0 = \{F(\emptyset, \emptyset)\}$, where $\emptyset$ is the empty set.

Step 2. For $k = 1, 2, \ldots, n$, do

(a) Set $M_k = M_{k-1}$.

(b) For each $F(S_1, S_2) \in M_{k-1}$, (i) if $w(S_1) + w_{1k} \leq c_1$, add $F(S_1 \cup \{k\}, S_2)$ to $M_k$; (ii) if $w(S_2) + w_{2k} \leq c_2$, add $F(S_1, S_2 \cup \{k\})$ to $M_k$; (iii) if $w(S_1) + w_{1k} \leq c_1$ and $w(S_2) + w_{2k} \leq c_2$, add $F(S_1 \cup \{k\}, S_2 \cup \{k\})$ to $M_k$.

(c) Check $M_k$ to identify pairs of $F(S_1, S_2)$ and $F(S'_1, S'_2)$ with $w(S_1) = w(S'_1)$ and $w(S_2) = w(S'_2)$. For each such pair, delete the one with a smaller potential value.

Step 3. Check $M_n$ to find one $F(S_1, S_2)$ with the largest potential value. Output the state $(S_1, S_2)$ as solution.

The DPKG algorithm is an extension of the dynamic programming algorithm used to solve the knapsack problem presented by [14]. The enumeration of all feasible states of the two-person knapsack game are embedded in Step 2(a) and (b), with the help of the “principle of optimality” due to the separability of potential functions involved to eliminate unnecessary states in Step 2(c). Since all parameters in the game are of integer value, we know the DPKG algorithm eventually finds one feasible state with the largest potential value in Step 3. Then Corollary 1 assures that it must be a Nash equilibrium solution to the game.

Now we examine the complexity of the DPKG algorithm. Step 1 is a trivial step using $O(1)$ computing time. The main computational effort comes from Step 2. Note that there are $n$ stages. For each stage $k$, $k = 1, 2, \ldots, n$, since there are at most $c_1c_2$ elements in $M_k$, (a)-(c) can be realized in $O(c_1c_2)$ computing time. Consequently, Step 2 needs a total of $O(nc_1c_2)$ computing time. Moreover, Step 3 only needs $O(c_1c_2)$ computing time for comparisons. Therefore, we can conclude with the following result:
Theorem 4.2. For a two-person knapsack game with $\Delta_1 \Delta_2 \geq 0$, a Nash equilibrium of the game can be found by the DPKG algorithm in $O(n_1c_2)$ computing time.

The computational complexity in Theorem 4.2 can be further reduced, when $\Delta_1 = 0$ or $\Delta_2 = 0$.

Theorem 4.3. For a two-person knapsack game with either $\Delta_1 = 0$ or $\Delta_2 = 0$, a Nash equilibrium of the game can be found in $O(nc)$ computing time, where $c = \max\{c_1, c_2\}$.

Proof. Without loss of generality, we may assume that $\Delta_1 = 0$. In this case, the objective function of player 1 at the state $(S_1, S_2)$ becomes $\beta_1 p(S_1)$, instead of $\beta_1 p(S_1) - \beta_{12} p(S_2)$. If the feasible state $(S_1^*, S_2^*)$ is a Nash equilibrium, then $p(S_1^*) \geq p(S_1)$ for any feasible state $(S_1, S_2)$. Consequently, we can find $S_1^*$ by solving a classical knapsack problem.

Now we reset the profit of project $k$ to be $\beta_{22} p_s$, if $k \not\in S_2^*$, and $(\beta_{22} - \Delta_2)p_k$, otherwise. Then maximizing player 2’s objective $\beta_{22} p(S_2) - \beta_{21} p(S_1) - \Delta_2 p(S_1 \cap S_2)$ is equivalent to solving a classical knapsack problem with the updated profits. This means that a Nash equilibrium solution can be found by solving two corresponding classical knapsack problems - one for $S_1^*$ and one for $S_2^*$. Hence the total work can be realized in $O(nc)$ computing time, where $c = \max\{c_1, c_2\}$. □

5. Price of anarchy. Even though pure Nash equilibrium solutions indeed exist in a two-person knapsack game when $\Delta_1 \Delta_2 \geq 0$, the noncooperative nature of the game and selfish manner of each investor may prevail at the cost of the total social welfare. In order to study the impact of investment decisions to the whole society led by different objectives, we follow the work of Koutsoupias and Papadimitriou [7] to quantify the Nash equilibrium solutions in a simplified knapsack game in terms of the “price of anarchy.”

In this section, we make the following assumptions to simplify our analysis: (i) The profit is equally shared if both investors select the same project $k$, i.e., $\alpha_1 = \alpha_2 = 1/2$. (ii) In terms of the objectives, each investor can either maximize his own profit, i.e., $\beta_{11} = 1$, $\beta_{1j} = 0$ in (2), or maximize the gap between his profit and the opponent’s, i.e., $\beta_{21} = 1$, $\beta_{2j} = 1$ in (2).

Three quick observations can be made here. First, with Assumption (i), the profit of each player at a feasible state $(S_1, S_2)$ becomes

$$P_1(S_1, S_2) = p(S_1) - \frac{1}{2} p(S_1 \cap S_2),$$

$$P_2(S_1, S_2) = p(S_2) - \frac{1}{2} p(S_1 \cap S_2).$$

(4)

The total profit of the two players becomes

$$p(S_1 \cup S_2) = P_1(S_1, S_2) + P_2(S_1, S_2) = p(S_1) + p(S_2) - p(S_1 \cap S_2).$$

(5)

Second, with symmetry in mind, Assumption (ii) considers three scenarios of the combination of objectives, namely, $(P_1(S_1, S_2), P_2(S_1, S_2))$, $(P_1(S_1, S_2) - P_2(S_1, S_2))$, and $(P_1(S_1, S_2), P_2(S_1, S_2) - P_1(S_1, S_2))$. We call the corresponding knapsack game a “selfish knapsack game,” “competitive knapsack game” and “mixed knapsack game,” respectively. Third, with Assumptions (i) and (ii), for each of three knapsack games, it is easy to verify that $\Delta_i \geq 0$, $i = 1, 2$. Hence a pure Nash equilibrium solution $(S_1^*, S_2^*)$ with a total profit of $z^* = p(S_1^* \cup S_2^*) = p(S_1^*) + p(S_2^*) - p(S_1^* \cap S_2^*)$ exists for each case by Theorem 3.6.

We shall quantify these Nash equilibria by the concept of price of anarchy, which is defined to be the ratio of the worst total profit associated with a Nash equilibrium
to the social optimum. Here the social optimum is defined to be a feasible state \((\bar{S}_1, \bar{S}_2)\) that achieves the maximum total profit \(\bar{z}\) of the two players. Notice that if there is a project belonging to \(S_1 \cap S_2\), because of Assumption (i), we can always remove that project from one player’s selection list to get a new feasible state without changing the maximum profit. Therefore, we may in general assume that \(S_1 \cap S_2 = \emptyset\). Consequently,
\[
\bar{z} = p(\bar{S}_1 \cup \bar{S}_2) = p_1(\bar{S}_1, \bar{S}_2) + p_2(\bar{S}_1, \bar{S}_2) = p(\bar{S}_1) + p(\bar{S}_2).
\]
(6)

What follows is the analysis of the “price of anarchy” for each of the three scenarios:

5.1. **Selfish Knapsack game.**

**Theorem 5.1.** The price of anarchy is 2/3 for the selfish knapsack game.

**Proof.** Let \((S^*_1, S^*_2)\) be a Nash equilibrium and \((\bar{S}_1, \bar{S}_2)\) a social optimum. Since they are feasible, so are \((\bar{S}_1, S^*_2)\) and \((S^*_1, \bar{S}_2)\). From the definition of Nash equilibrium, we know that \(P_1(S^*_1, S^*_2) \geq P_1(S_1, S^*_2)\) and \(P_2(S^*_1, S^*_2) \geq P_2(S^*_1, S_2)\). Moreover, from (4), we have \(P_1(S_1, S^*_2) = p(\bar{S}_1) - \frac{1}{2}p(\bar{S}_1 \cap S^*_2)\) and \(P_2(S^*_1, S_2) = p(S_2) - \frac{1}{2}p(S^*_1 \cap S_2)\). Consequently,
\[
P_1(S^*_1, S^*_2) + P_2(S^*_2, S^*_2) \geq p(\bar{S}_1) + p(\bar{S}_2) - \frac{1}{4}[p(\bar{S}_1 \cap S^*_2) + p(S^*_1 \cap S_2)].
\]
(7)

Since \(\bar{S}_1 \cap \bar{S}_2 = \emptyset\) is generally assumed for social optimum, we know \((\bar{S}_1 \cap S^*_2) \cap (S^*_1 \cap \bar{S}_2) = \emptyset\). Therefore
\[
p(\bar{S}_1 \cap S^*_2) + p(S^*_1 \cap \bar{S}_2) = p((\bar{S}_1 \cap S^*_2) \cup (S^*_1 \cap \bar{S}_2)) \leq p(S^*_1 \cup S^*_2) = P_1(S^*_1, S^*_2) + P_2(S^*_1, S^*_2).
\]
(8)

Combining (7) and (8) results in
\[
\frac{3}{2}[P_1(S^*_1, S^*_2) + P_2(S^*_1, S^*_2)] \geq p(\bar{S}_1) + p(\bar{S}_2).
\]

Consequently, from (6), we have \(z^* \geq \frac{3}{2}\bar{z}\).

Next, we use an instance to show the bound of \(\frac{3}{2}\) is tight.

**Instance 1.** There are two projects and \(c_1 \leq c_2\). For project 1, \(w_{11} = w_{21} = c_1\) and \(p_1 = 2 + \epsilon\) where \(\epsilon > 0\). For project 2, \(w_{21} = w_{22} = c_2\) and \(p_2 = 1\). Then the feasible state \(((1), (1))\) is a Nash equilibrium with \(z^* = 2 + \epsilon\), and the feasible state \(((1), (2))\) achieves the social optimum with \(\bar{z} = 3 + \epsilon\). Since \(z^*/\bar{z} \to 2/3\) when \(\epsilon \to 0\) for this instance, the bound is tight.

5.2. **Competitive Knapsack game.**

**Theorem 5.2.** The price of anarchy is 1/2 for the competitive knapsack game.

**Proof.** Let \((S^*_1, S^*_2)\) be a Nash equilibrium and \((\bar{S}_1, \bar{S}_2)\) a social optimum. Since \((S^*_1, S^*_2)\) is a Nash equilibrium, we have \(P_1(S^*_1, S^*_2) - P_2(S^*_1, S^*_2) \geq p_1(\bar{S}_1, S^*_2) - p_2(S^*_1, S_2)\) and \(P_2(S^*_1, S^*_2) - P_1(S^*_1, S^*_2) \geq p_2(S^*_1, \bar{S}_2) - p_1(S^*_1, S_2)\).

Using (4) for the first inequality, we have \(p(S^*_1) - p(S^*_2) \geq p(\bar{S}_1) - p(S^*_2)\) and, hence, \(p(S^*_1) \geq p(\bar{S}_1)\). Similarly, we have \(p(S^*_2) \geq p(\bar{S}_2)\). Notice that \(p(S^*_1 \cap S^*_2) \leq \frac{1}{2}(p(S^*_1) + p(S^*_2))\). With (6), it follows that
\[
\begin{align*}
z^* &= p(S^*_1) + p(S^*_2) - p(S^*_1 \cap S^*_2) \\
&\geq \frac{1}{2}(p(S^*_1) + p(S^*_2)) \geq \frac{1}{2}(p(\bar{S}_1) + p(\bar{S}_2)) = \frac{1}{2}\bar{z}.
\end{align*}
\]
Next, we use an instance to show the bound of \( \frac{1}{3} \) is tight.

**Instance 2.** There are two projects and \( c_1 \leq c_2 \). For project 1, \( w_{11} = w_{21} = c_1 \) and \( p_1 = 2 + \epsilon \) where \( \epsilon > 0 \). For project 2, \( w_{21} = w_{22} = c_2 \) and \( p_2 = 2 \). Then the feasible state \( \{(1), (1)\} \) is a Nash equilibrium with \( z^* = 2 + \epsilon \), while the feasible state \( \{(1), (2)\} \) achieves the social optimum with \( \bar{z} = 4 + \epsilon \). Since \( z^*/\bar{z} \rightarrow 1/2 \) as \( \epsilon \rightarrow 0 \) for the instance, the bound is tight.

### 5.3. Mixed Knapsack game

Recall that the objective in the mixed knapsack game is \((P_1(S_1, S_2), P_2(S_1, S_2) - P_1(S_1, S_2))\) (or reversely). Because the asymmetric objective structure, this scenario is more complicated. For easy analysis, in this subsection, we further assume that (iii) the cost of selecting project \( k \) is the same for each investor, i.e., \( w_{1k} = w_{2k} = w_k \) for \( k = 1, 2, \ldots, n \). We will divide this scenario into three cases according to the capacity of knapsacks, i.e., \( c_1 < c_2, c_1 = c_2 \) and \( c_1 > c_2 \).

Let us start with a simple lemma.

**Lemma 5.3.** In the mixed knapsack game, if \((S_1^*, S_2^*)\) is a Nash equilibrium and \((\bar{S}_1, \bar{S}_2)\) is a social optimum, then \( P_2(S_1^*, S_2^*) \geq \frac{1}{2}p(\bar{S}_2) \).

**Proof.** Since \((S_1^*, S_2^*)\) is a Nash equilibrium of the game, we know \( P_2(S_1^*, S_2^*) - P_1(S_1^*, \bar{S}_2) \geq P_2(\bar{S}_1, S_2^*) - P_1(S_1^*, \bar{S}_2) \). Using (4), we have \( p(S_2^*) \geq p(\bar{S}_2) \). Noticing that \( P_2(\bar{S}_1, S_2^*) \geq \frac{1}{2}p(\bar{S}_2) \), we have \( P_2(S_1^*, S_2^*) \geq \frac{1}{2}p(\bar{S}_2) \).

### 5.4. When \( c_1 < c_2 \), the price of anarchy is 1/2 for the mixed knapsack game.

**Proof.** Since \((S_1^*, S_2^*)\) is a Nash equilibrium solution, we have \( P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*) \). It is not difficult to see that \( P_1(\bar{S}_1, S_2^*) \geq \frac{1}{2}p(\bar{S}_1) \). Hence \( P_1(S_1^*, S_2^*) \geq \frac{1}{2}p(\bar{S}_1) \).

Together with Lemma 5.3 and (6), we have

\[
\bar{z}^* = P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) \geq \frac{1}{2}(p(\bar{S}_1) + p(\bar{S}_2)) = \frac{1}{2}\bar{z}.
\]

Instance 2 shows that state \( \{(1), (1)\} \) is a Nash equilibrium with \( z^* = 2 + \epsilon \) and state \( \{(1), (2)\} \) achieves the social optimum with \( \bar{z} = 4 + \epsilon \). Since \( z^*/\bar{z} \rightarrow 1/2 \) as \( \epsilon \rightarrow 0 \), the bound is tight.

### 5.5. When \( c_1 = c_2 \), the price of anarchy is 2/3 for the mixed knapsack game.

**Proof.** When \( c_1 = c_2 \), both players have the same budget and states \((S_1^*, S_2^*)\) and \((\bar{S}_2, S_2^*)\) both become feasible. Since \((S_1^*, S_2^*)\) is a Nash equilibrium of the game, we can claim that \( P_2(S_1^*, S_2^*) - P_1(S_1^*, S_2^*) \geq 0 \). Otherwise \( S_1^* \) is a better choice for player 2 to get an improved objective value of \( P_2(S_1^*, S_2^*) - P_1(S_1^*, S_2^*) = 0 \). Noticing that \( P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*) \) and \( P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_2, S_2^*) \), we have

\[
P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*) + P_1(\bar{S}_2, S_2^*) = p(\bar{S}_1) + p(\bar{S}_2) - \frac{1}{2}(p(S_1^* \cap S_2^*) + p(S_2^* \cap S_2^*)).
\]

Since \( S_1^* \cap \bar{S}_2 = \emptyset \) is generally assumed, we have \((\bar{S}_1 \cap S_2^*) \cap (\bar{S}_2 \cap S_2^*) = \emptyset \). Consequently,

\[
p(\bar{S}_1 \cap S_2^*) + p(S_2^* \cap S_2^*) = p((\bar{S}_1 \cap S_2^*) \cup (\bar{S}_2 \cap S_2^*)) \leq p(S_2^*) \leq P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*).
\]

Combining (9) and (10), together with (6), we have

\[
z^* = P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) \geq \frac{2}{3}(p(S_1) + p(\bar{S}_2)) = \frac{2}{3}\bar{z}.
\]
Instance 1 shows that state $\{1\}, \{1\}$ is a Nash equilibrium with $z^* = 2 + \epsilon$, while state $\{(1), (2)\}$ achieves the social optimum with $\bar{z} = 3 + \epsilon$. Since $z^*/\bar{z} \to 2/3$ as $\epsilon \to 0$ for the instance, the bound is tight.

For the case of $c_1 > c_2$, we need to further break it into two subcases, namely, $c_2 < c_1 < 2c_2$ and $c_1 \geq 2c_2$, for discussion.

**Theorem 5.6.** When $c_2 < c_1 < 2c_2$, the price of anarchy is 3/5 for the mixed knapsack game.

**Proof.** Since $(S_1^*, S_2^*)$ is a Nash equilibrium of the game, $P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*)$. Noticing that $p(\bar{S}_1 \cap S_2^*) \leq p(S_2^*) \leq 2P_2(S_1^*, S_2^*)$ and $P_1(\bar{S}_1, S_2^*) = p(\bar{S}_1) - \frac{1}{2}p(\bar{S}_1 \cap S_2^*)$, we have

$$P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) \geq p(\bar{S}_1). \quad (11)$$

When $c_2 < c_1$, we know $(\bar{S}_2, S_2^*)$ is a feasible state. Since $(\bar{S}_1, S_2^*)$ is also a feasible state, we have

$$P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*) = p(\bar{S}_1) - \frac{1}{2}p(\bar{S}_1 \cap S_2^*),$$

$$P_1(S_1^*, S_2^*) \geq P_1(S_2, S_2^*) = p(\bar{S}_2) - \frac{1}{2}p(\bar{S}_2 \cap S_2^*). \quad (12)$$

Remembering that $\bar{S}_1 \cap \bar{S}_2 = \emptyset$ is generally assumed, it is not difficult to verify that

$$p(\bar{S}_1 \cap S_2^*) + p(\bar{S}_2 \cap S_2^*) \leq p(S_2^*) \leq 2P_2(S_1^*, S_2^*). \quad (13)$$

Combining (12) and (13), we have

$$2P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) \geq p(\bar{S}_1) + p(\bar{S}_2). \quad (14)$$

Lemma 5.3 says that

$$P_2(S_1^*, S_2^*) \geq \frac{1}{2}p(\bar{S}_2). \quad (15)$$

Putting (11), (14) and (15) together, we have

$$z^* = P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*) = \frac{1}{2}(P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*)) + \frac{2}{5}(2P_1(S_1^*, S_2^*) + P_2(S_1^*, S_2^*)) = \frac{7}{5}P_2(S_1^*, S_2^*) \geq \frac{1}{2}p(\bar{S}_1) + p(\bar{S}_2) = z^*. \quad (16)$$

The following instance shows the bound is tight.

**Instance 3.** There are three projects. For project 1, $w_1 = c_2 - \epsilon$ and $p_1 = 2 + \epsilon$ where $\epsilon > 0$ is sufficiently small. For project 2, $w_2 = c_1 - c_2 + \epsilon$ and $p_2 = 1$. For project 3, $w_3 = c_2$ and $p_3 = 2$. Then state $\{(1), (2)\}$ is a Nash equilibrium with $z^* = 3 + \epsilon$, while state $\{(1), (2), (3)\}$ achieves the social optimum with $\bar{z} = 5 + \epsilon$. Since $z^*/\bar{z} \to 3/5$ as $\epsilon \to 0$ for the instance, the bound is tight.

**Theorem 5.7.** When $c_1 \geq 2c_2$, the price of anarchy is 2/3 for the mixed knapsack game.

**Proof.** When $c_1 \geq 2c_2$, $(S_2^* \cup \bar{S}_2, S_2^*)$ is a feasible state. Since $(S_1^*, S_2^*)$ is a Nash equilibrium of the game, we have

$$P_1(S_1^*, S_2^*) \geq P_1(\bar{S}_1, S_2^*) = p(\bar{S}_1) - \frac{1}{2}p(\bar{S}_1 \cap S_2^*). \quad (16)$$

Moreover,

$$P_1(S_1^*, S_2^*) \geq P_1(S_2^* \cup \bar{S}_2, S_2^*) = p(S_2^* \cup \bar{S}_2) - \frac{1}{2}p(S_2^*) \quad (17)$$
Notice that $p(S^*_2) ≥ p(\bar{S}_1 \cap S^*_2) + p(\bar{S}_2 \cap S^*_2)$. Combining (16) and (17) results in

$$2P_1(S^*_1, S^*_2) ≥ p(\bar{S}_1) + p(\bar{S}_2) - \frac{1}{2}p(\bar{S}_2 \cap S^*_2).$$

Combining the facts that $p(\bar{S}_1 \cap S^*_2) ≤ p(S^*_2) - p(\bar{S}_2 \cap S^*_2)$, $P_2(S^*_1, S^*_2) ≥ \frac{1}{2}p(S^*_2)$ and (16), we have

$$P_1(S^*_1, S^*_2) ≥ p(\bar{S}_1) - \frac{1}{2}p(\bar{S}_1 \cap S^*_2) ≥ p(\bar{S}_1) - P_2(S^*_1, S^*_2) + \frac{1}{2}p(\bar{S}_2 \cap S^*_2).$$

Adding up (18) and (19), we have

$$3P_1(S^*_1, S^*_2) + P_2(S^*_1, S^*_2) ≥ 2p(\bar{S}_1) + p(\bar{S}_2).$$

Lemma 5.3 says that

$$P_2(S^*_1, S^*_2) ≥ \frac{1}{2}p(\bar{S}_2).$$

Consequently, from (20) and (21), we have

$$z^* = P_1(S^*_1, S^*_2) + P_2(S^*_1, S^*_2) = \frac{1}{3}(3P_1(S^*_1, S^*_2) + P_2(S^*_1, S^*_2)) + \frac{2}{3}P_2(S^*_1, S^*_2) \geq \frac{2}{3}(p(\bar{S}_1) + p(\bar{S}_2)) = \frac{2}{3}\bar{z}.$$

We use the next instance to show the bound is tight.

**Instance 4.** There are two projects. For project 1, $w_1 = c_1$ and $p_1 = 1$. For project 2, $w_2 = c_2$ and $p_2 = 2 + \epsilon$ with $\epsilon > 0$. Then state $\{(2), (2)\}$ is a Nash equilibrium with $z^* = 2 + \epsilon$ while state $\{(1), (2)\}$ achieves the social optimum with $\bar{z} = 3 + \epsilon$. Since $z^*/\bar{z} \to 2/3$ as $\epsilon \to 0$ for the instance, the bound is tight.

**6. Summary of results and concluding remarks.**

6.1. **Summary.** In this paper, we introduced a new model called two-person knapsack game for investment considerations. This model is flexible enough to accommodate some interesting scenarios for real-life applications. When both investors act in a selfish manner with best-response to optimize their own objective functions by choosing portfolios under the budget restriction, we have shown that pure Nash equilibrium exists as long as $\Delta_1\Delta_2 ≥ 0$. In this case, no investor can improve individual’s objective by changing bid unilaterally. A pseudo polynomial-time algorithm was presented for finding pure Nash equilibrium solutions. We also investigated the price of anarchy (the ratio of the worst Nash equilibrium to the social optimum) associated with three simplified two-person knapsack games, say “selfish knapsack game,” “competitive knapsack game” and “mixed knapsack game.” We proved that the price of anarchy is $2/3$ for the selfish knapsack game and it is $1/2$ for the competitive knapsack game. The price of anarchy for the mixed knapsack game is sensitive to the change of players’ budgets, say $c_1$ and $c_2$. The following table (Table 1) summarizes our findings of the price of anarchy for the mixed knapsack game.

<table>
<thead>
<tr>
<th>Cases</th>
<th>$c_1 &lt; c_2$</th>
<th>$c_1 = c_2$</th>
<th>$c_1 &gt; c_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2 &lt; c_1 &lt; 2c_2$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{4}{3}$</td>
</tr>
<tr>
<td>$c_1 ≥ 2c_2$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{4}{3}$</td>
<td>$\frac{4}{3}$</td>
</tr>
</tbody>
</table>

**Table 1.** The price of anarchy in the mixed knapsack game.
6.2. Remarks. 1. In a two-person knapsack game, budget, market power and investment objective are three key factors of an investor. In the model, budget is represented by the knapsack capacity \( c_i \): a richer investor has a larger knapsack. Market power is reflected by the parameter \( \alpha_i \): a smaller \( \alpha_i \) goes with an investor \( i \) with higher market power. The investment objective is characterized by the parameters \( \beta_{ij} \) and \( \beta_{ii} \): an investor \( i \) with the higher ratio of \( \frac{\beta_{ij}}{\beta_{ii}} \) is more aggressive.

2. From an individual’s point of view, every investor acts to achieve the maximum of his objective. Not much can be said about the “behavior” of an investor is good or bad. But from the society’s point of view, some economic behavior is preferred if it leads to a steady state with higher social welfare. Nash equilibrium represents a steady state and the price of anarchy may indicate certain goodness of investors’ behavior.

3. Our finding says that Nash equilibrium exists if \( \Delta_1 \Delta_2 \geq 0 \). Notice that \( \Delta_i = \alpha_i \beta_{ii} - \alpha_j \beta_{ij} \) with \( j \neq i \). Hence \( \Delta_i \) can be considered as a “relative discount” on investor \( i \)’s objective of the projects commonly selected by both investors. When \( \Delta_i > 0 \), any commonly selected project is disadvantageous to player \( i \), so he tends to improve the objective by not selecting a project in the opponent’s list. On the other hand, when \( \Delta_i < 0 \), any commonly selected project is favorable to investor \( i \), so he tends to improve the objective by sticking to the commonly selected projects. When \( \Delta_i = 0 \), selecting or not selecting a commonly selected project makes no difference to investor \( i \). As long as both investors hold the same attitude toward the commonly selected projects, i.e., \( \Delta_1 \Delta_2 \geq 0 \), a mutually agreeable stable state will be reached. When \( \Delta_1 \Delta_2 < 0 \), one player tends to escape from the opponent’s choice while the opponent is chasing for commonly invested projects. Hence there may be no equilibrium that can be reached.

4. If we take the maximum total objective value of the two investors at a feasible state as the social optimum (or system optimum), then our analysis of the price of anarchy says that, in the worst case, (i) two nonaggressive investors (\( \beta_{ij}/\beta_{ii} = 0 \)) in a selfish knapsack game are doing better (or getting closer to social optimum) than two aggressive investors (\( \beta_{ij}/\beta_{ii} = 1 \)) in a competitive knapsack game; (ii) in a mixed knapsack game, if the aggressive investor (\( \beta_{21}/\beta_{22} = 1 \)) has a bigger budget \( c_2 \) than the nonaggressive investor (\( \beta_{12}/\beta_{11} = 0 \)) has \( c_1 \), then they can do as badly as two aggressive investors (\( \beta_{ij}/\beta_{ii} = 1 \)) in a competitive knapsack game; (iii) in a mixed knapsack game, if the aggressive investor and the nonaggressive one have the same budget \( c_1 = c_2 \), then they can do as well as two nonaggressive investors (\( \beta_{ij}/\beta_{ii} = 0 \)) in a selfish game; (iv) in a mixed knapsack game, if the nonaggressive investor (\( \beta_{12}/\beta_{11} = 0 \)) has a bigger budget \( c_1 \) than the aggressive investor (\( \beta_{21}/\beta_{22} = 1 \)) has \( c_2 \), then they are doing better than the case that the aggressive investor has a larger budget. In particular, if the nonaggressive investor’s budget is at least twice as large as that of the aggressive, then they can do as well as two nonaggressive investors in a selfish knapsack game.

5. In this paper, we have only studied the knapsack game with two noncooperative players. A subsequent study concerned a two-group knapsack game in which the investors colligate into two groups [20]. When multiple players (or groups) are involved, it remains a challenge for us to consider.

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