KKT SOLUTION AND CONIC RELAXATION FOR SOLVING QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING PROBLEMS∗

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Abstract. To find a global optimal solution to the quadratically constrained quadratic programming problem, we explore the relationship between its Lagrangian multipliers and related linear conic programming problems. This study leads to a global optimality condition that is more general than the known positive semidefiniteness condition in the literature. Moreover, we propose a computational scheme that provides clues of designing effective algorithms for more solvable quadratically constrained quadratic programming problems.

Key words. quadratically constrained quadratic programming, conic programming, global optimality condition, solvable condition

AMS subject classifications. 49N15, 90C20, 90C26

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1. Introduction. In this paper, we study a quadratically constrained quadratic programming problem (QCQP) of the following form:

\[
\begin{align*}
\min & \quad F(x) = \frac{1}{2} x^T Q x + c^T x, \\
\text{s.t.} & \quad G_i(x) = \frac{1}{2} x^T Q_i x + c_i^T x - b_i \leq 0, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where \( Q \) and \( Q_i \) are \( n \times n \) real symmetric matrices, and \( c \) and \( c_i \) are real vectors in \( \mathbb{R}^n \). For simplicity, we define \( b = [b_1, \ldots, b_m]^T \).

QCQP is one of the fundamental nonlinear programming problems of both theoretical significance and application interests. Many known problems including the trust region method, Max-Cut problem, 0-1 quadratic programming problem and box constrained quadratic programming problem are subclasses of QCQP. It is well known that QCQP is NP-Hard. In other words, the problem cannot be solved in polynomial time unless \( P = NP \) [13].

In case the matrices \( Q, Q_1, \ldots, Q_m \) are all positive semidefinite, QCQP becomes a convex optimization problem. This subclass of problems is solvable in polynomial time (within a given precision level) using the second order cone programming method [17]. However, finding the global optimal solution to a general QCQP is much more difficult. Many interesting papers can be found in the literature. In particular, Ye and Zhang studied the nonconvex quadratic programming problem

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with two quadratic constraints and identified a family of solvable subclasses using the semidefinite programming (SDP) approach [27], and Sturm and Zhang reformulated a nonconvex quadratic programming problem into a linear conic programming problem. Based on the matrix decomposition method and SDP techniques, they identified some polynomial-time solvable subclasses of QCQP in [22]. Fu, Luo, and Ye proposed an approximation algorithm for solving quadratic programs with several ellipsoidal constraints in [8]. Tseng also considered an approximation algorithm for QCQP with special properties using SDP relaxation in [24].

Another conic programming approach is to use the copositive representation of quadratic optimization problems for shedding new insights. For instance, Bomze et al. reformulated the standard quadratic optimization problems into a copositive programming problem in [4], and so did Burer for some nonconvex quadratic programming problems with linear and binary constraints in [5]. Similarly, de Klerk reformulated the stability number of a graph into a copositive programming problem [6]. Notice that there is no known algorithm for directly solving these copositive programming problems, but there are methods that may approximate the copositive cones by computable cones (e.g., see [20, 6]).

Studies on the global optimality conditions for QCQP and its variants are often found in the literature (e.g., see [14, 21, 2, 19]). The canonical duality theory proposed by Gao [9] also leads to some sufficient conditions of global optimization for QCQP and its subclasses (e.g., see [10, 11, 15, 7, 25]). A recent review on this topic can be found in [12]. In the canonical dual approach, a canonical dual function is introduced such that any critical point of this function corresponds to a primal KKT solution. If the critical point also satisfies the so-called “positive semidefiniteness condition” (the condition in Theorem 2 of [12]), then the corresponding KKT solution becomes a global optimal solution of the original problem. Most of the known results essentially employ variants of this positive semidefiniteness condition. Connecting the canonical dual and conic dual problems, we recently extended the solvable conditions for 0-1 quadratic programming problems in [18].

In this paper, we intend to further study the KKT solutions of QCQP and introduce new conic relaxations to uncover some sufficient conditions for KKT solutions to become globally optimal. Based on the sufficient conditions discovered, we develop an algorithm for finding a global optimal solution of QCQP that satisfies a wider than ever solvable condition. Methods for searching KKT solutions of QCQP was also studied by Ye in [26], where he designed a fully polynomial-time approximation scheme to obtain KKT solutions satisfying the second-order necessary conditions of local optimality. Different from Ye’s work, our algorithm generates a KKT solution satisfying a sufficient condition of global optimality for more solvable quadratically constrained quadratic programs.

The rest of this paper is arranged as follows. Section 2 gives a quick review of the KKT necessary conditions of local optimality for QCQP. Then we construct some new conic relaxations for QCQP in section 3, followed in section 4 by a study on the relationship between the KKT system and the conic relaxation problems constructed. An efficient algorithm is designed for solving the KKT system of QCQP, and a solvable condition for this algorithm is proved in section 5. Some numerical examples are presented in section 6 to illustrate the proposed algorithm and to highlight its capability of solving more general quadratically constrained quadratic programming problems.

2. Lagrangian function and KKT system. The following notations are adopted in this paper: \( \mathcal{M}_n \) denotes the set of all \( n \times n \) real symmetric matrices, \( \mathcal{S}_n \) the
set of all $n \times n$ symmetric positive semidefinite matrices, and $\mathcal{N}_n$ the set of all $n \times n$ symmetric matrices with nonnegative elements. Given a vector $x \in \mathbb{R}^n$, $x_i$ or $[x]_i$ denotes the $i$th component of $x$ and \text{Diag}(x) the $n \times n$ diagonal matrix with $x_i$ being its $i$th diagonal element. In particular, for a vector $\lambda \in \mathbb{R}^n$, $\Lambda$ denotes the diagonal matrix \text{Diag}(\lambda). For two vectors $x, y \in \mathbb{R}^n$, $x \circ y$ is a vector in $\mathbb{R}^n$ with $[x]_i[y]_i$ being its $i$th component. For a real symmetric matrix $U$, $U \succeq 0$ means $U$ is positive semidefinite, and $U \succ 0$ means $U$ is positive definite. For two matrices $A$ and $B$, denote $A \cdot B = \text{trace}(A^T B)$. Moreover, for a given optimization problem (\ast), its optimal objective value is denoted by $V(\ast)$.

Now we focus on QCQP whose Lagrangian function is defined by

$$L(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \sum_{i=1}^m \lambda_i \left( \frac{1}{2} x^T Q_i x + c_i^T x - b_i \right)$$

$$= \frac{1}{2} x^T \left( Q + \sum_{i=1}^m \lambda_i Q_i \right) x + \left( c + \sum_{i=1}^m \lambda_i c_i \right)^T x - b^T \lambda,$$

where $\lambda \in \mathbb{R}^m$. 

For a nonconvex QCQP problem, its KKT system provides a necessary condition of local optimality for a feasible solution $x^*$, i.e., there exists a Lagrangian vector $\lambda^* \in \mathbb{R}^m$ such that

$$\nabla_x L(x, \lambda^*)|_{x=x^*} = \left( Q + \sum_{i=1}^m \lambda_i^* Q_i \right) x^* + \left( c + \sum_{i=1}^m \lambda_i^* c_i \right) = 0,$$

(KKT System) $\frac{1}{2} x^*^T Q_i x^* + c_i^T x^* \leq b_i, \quad \lambda_i^* \geq 0,$

$$\lambda_i^* \left( \frac{1}{2} x^*^T Q_i x^* + c_i^T x^* - b_i \right) = 0, \quad i = 1, 2, \ldots, m.$$

For a KKT point $x^*$ with the corresponding Lagrangian vector $\lambda^*$, it is easy to verify that $F(x^*) = L(x^*, \lambda^*)$.

In our study, we denote

$$I(x^*) = \left\{ i = 1, 2, \ldots, m \mid \frac{1}{2} x^*^T Q_i x^* + c_i^T x^* = b_i \right\}$$

as the active constraint set at $x^*$. Moreover, we assume the following condition holds.

\textbf{Condition 1} (linear independence constraint qualification). The vectors of $\nabla G_i(x^*), i \in I(x^*)$, are linearly independent.

The linear independence constraint qualification can be found in Chapter 5 of [1]. Under this condition, there is a unique Lagrangian vector corresponding to the given primal KKT solution $x^*$.

Note that the KKT conditions are merely necessary conditions for local optimality. For a nonconvex QCQP, a KKT solution $x^*$ may not even be locally optimal. The conventional second order sufficient conditions in nonlinear programming can further detect local optimality, but not global optimality. A commonly seen global optimality condition is given as follows.

\textbf{Positive semidefiniteness condition.} Let $x^*$ be a KKT solution of the given QCQP with a corresponding Lagrangian vector $\lambda^*$. If $Q + \sum_{i=1}^m \lambda_i^* Q_i \succeq 0$, then $x^*$ is a global optimal solution of the QCQP.
Notice that the positive semidefiniteness condition is in fact a case of the classical saddle point optimality conditions as described in [1]. Similar conditions can be found in several papers, e.g., Proposition 3.2 of [14], Theorem 2 of [12], and discussions of [21]. One of our objectives is to impose a more general condition (than the positive semidefiniteness condition) on the KKT solutions of a nonconvex QCQP for achieving global optimality.

3. Conic relaxation. Our approach is based on the concept of linear conic programming. To reformulate QCQP as a conic programming problem, we need some definitions. First, let us define the set

\[ F = \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^T Q_i x + c_i^T x \leq b_i \forall i = 1, 2, \ldots, m \right\}, \]

the cone

\[ D_{n+1} = \left\{ U \in M_{n+1} \left| \begin{bmatrix} 1 \\ x \end{bmatrix} U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 \forall x \in F \right. \right\}, \]

and the set of matrices

\[ Z = \left\{ Y \in M_{n+1} \left| Y = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \text{ for some } x \in F \right. \right\}. \]

To avoid triviality in this paper, we assume the set \( F \) is nonempty. In this way, \( D_{n+1} \) is a nonempty closed convex cone.

Now, let \( D_{n+1}^* = \text{Closure}(\text{Cone}(Z)) \) be the closure of the convex cone generated by \( Z \). Then we have the following result.

**Lemma 3.1.** \( D_{n+1} \) is the dual cone of \( D_{n+1}^* \).

**Proof.** For any \( V \in \text{Cone}(Z) \), there exists a positive integer \( r \) with \( x^1, x^2, \ldots, x^r \in F \) and \( \mu_1, \mu_2, \ldots, \mu_r \geq 0 \) such that

\[ V = \mu_1 \begin{bmatrix} 1 \\ x^1 \end{bmatrix} \begin{bmatrix} 1 \\ x^1 \end{bmatrix}^T + \mu_2 \begin{bmatrix} 1 \\ x^2 \end{bmatrix} \begin{bmatrix} 1 \\ x^2 \end{bmatrix}^T + \cdots + \mu_r \begin{bmatrix} 1 \\ x^r \end{bmatrix} \begin{bmatrix} 1 \\ x^r \end{bmatrix}^T. \]

If \( U \in D_{n+1} \), then

\[ U \cdot V = \mu_1 \begin{bmatrix} 1 \\ x^1 \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ x^2 \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^2 \end{bmatrix} + \cdots + \mu_r \begin{bmatrix} 1 \\ x^r \end{bmatrix}^T U \begin{bmatrix} 1 \\ x^r \end{bmatrix} \geq 0. \]

For any \( V_0 \in D_{n+1}^* \), there exists a sequence of \( \{V_k\}_{k=1, 2, \ldots} \) in \( \text{Cone}(Z) \) such that \( \lim_{k \to +\infty} V_k = V_0 \). Hence we have \( U \cdot V_0 = \lim_{k \to +\infty} U \cdot V_k \geq 0 \). This means that \( U \in (D_{n+1}^*)^* \) and, consequently, \( D_{n+1} \subseteq (D_{n+1}^*)^* \).

On the other hand, for any \( x \in F \),

\[ Y = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \in Z \subset D_{n+1}^*. \]

If \( U \in (D_{n+1}^*)^* \), then \( U \cdot Y \geq 0 \). Therefore,

\[ \begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \]

which implies that \( U \in D_{n+1} \) and \((D_{n+1}^*)^* \subseteq D_{n+1} \). This completes the proof. \( \square \)
Notice that since $D_{n+1}$ and $D^*_n$ are closed convex cones, Lemma 3.1 also implies that $D^*_{n+1}$ is the dual cone of $D_{n+1}$.

Now, consider the following reformulation of QCQP:

$$\begin{align*}
\min & \quad \frac{1}{2}Q \cdot X + c^T x \\
\text{s.t.} & \quad \frac{1}{2}Q_i \cdot X + c_i^T x - b_i \leq 0, \quad i = 1, 2, \ldots, m, \\
& \quad \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = Y; \\
& \quad Y \in D^*_{n+1} \\
& \quad \text{rank}(Y) = 1.
\end{align*}$$

(QCQP2)

An immediate result can be stated below.

**Lemma 3.2.** $(x, X)$ is a feasible solution of QCQP2 if and only if $x \in F$ and $X = xx^T$.

**Proof.** If $x \in F$ and $X = xx^T$, then it is only a routine task to verify that $(x, X)$ satisfies all constraints of QCQP2 to be a feasible solution.

On the other hand, if $(x, X)$ is feasible to QCQP2, then the structure of $Y$ and the rank one constraint assure that $X = xx^T$. The constraints of $\frac{1}{2}Q_i \cdot X + c_i^T x - b_i \leq 0$ for all $i$ further implies $x \in F$. \qed

The above lemma indicates that problems QCQP and QCQP2 have an equivalent feasible domain. Since they also have the same objective function, the two problems are actually equivalent.

Note that the rank one requirement of QCQP2 is not a convex constraint. By dropping this constraint, we may relax QCQP2 to the following linear conic programming problem (COP):

$$\begin{align*}
\min & \quad \frac{1}{2}Q \cdot X + c^T x \\
\text{s.t.} & \quad \frac{1}{2}Q_i \cdot X + c_i^T x - b_i \leq 0, \quad i = 1, 2, \ldots, m, \\
& \quad \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = Y; \\
& \quad Y \in D^*_{n+1}.
\end{align*}$$

(COP)

Using the conic duality theory, we may define the dual problem of COP as

$$\begin{align*}
\max & \quad \frac{1}{2}\sigma - b^T \lambda \\
\text{s.t.} & \quad \begin{bmatrix} c + \sum_{i=1}^m \lambda_i c_i & c^T + \sum_{i=1}^m \lambda_i c_i^T \\ Q + \sum_{i=1}^m \lambda_i Q_i \end{bmatrix} \in D_{n+1}, \\
& \quad \lambda \geq 0.
\end{align*}$$

(COD)

The relationship among problems COP, COD, and QCQP is given below.

**Theorem 3.3.** If the problem QCQP has a finite optimal value, then problems COP, COD, and QCQP are all equivalent in the sense that they share the same optimal value.

**Proof.** Since COP is a relaxation of QCQP, we have $V(COP) \leq V(QCQP)$. Besides, the weak duality theorem says that $V(COD) \leq V(COP)$. We would like to
show \( V(COD) = V(QCQP) \). Note that
\[
\begin{bmatrix}
-\sigma \\
c + \sum_{i=1}^{m} \lambda_i c_i \\
c^T + \sum_{i=1}^{m} \lambda_i c_i^T \\
\end{bmatrix}
\begin{bmatrix}
Q + \sum_{i=1}^{m} \lambda_i Q_i \\
\end{bmatrix}
\in D_{n+1}
\]
if and only if
\[
\begin{bmatrix}
1 \\
x
\end{bmatrix}^T
\begin{bmatrix}
-\sigma \\
c + \sum_{i=1}^{m} \lambda_i c_i \\
c^T + \sum_{i=1}^{m} \lambda_i c_i^T \\
\end{bmatrix}
\begin{bmatrix}
Q + \sum_{i=1}^{m} \lambda_i Q_i \\
\end{bmatrix}
\begin{bmatrix}
1 \\
x
\end{bmatrix} \geq 0 \text{ for any } x \in \mathcal{F},
\]
which is equivalent to
\[
\frac{1}{2} \sigma - b^T \lambda \leq L(x, \lambda) \text{ for any } x \in \mathcal{F}.
\]
Let \( \bar{\lambda} = 0 \) and \( \bar{\sigma} = 2V(QCQP) = 2 \min_{x \in \mathcal{F}} F(x) = 2 \min_{x \in \mathcal{F}} L(x, \bar{\lambda}) \), then \((\bar{\sigma}, \bar{\lambda})\) is a feasible solution of COD with \( V(COD) \geq \frac{1}{2} \bar{\sigma} - b^T \bar{\lambda} = V(QCQP) \). Consequently, \( V(COD) = V(COP) = V(QCQP) \).

Remark 1. A similar conic reformulation for quadratic programming problems was discussed by Sturm and Zhang in [22]. For the sake of completeness, we provide a proof of Theorem 3.3 here. Also notice that the first inequality constraint of COP (i.e., \( \frac{1}{2}Q_i \cdot X + c_i^T x - b_i \leq 0 \) for \( i = 1, 2, \ldots, m \)) is embedded in the last cone constraint of COP (i.e., \( Y \in D_{n+1}^* \)). However, if we relax the cone \( D_{n+1}^* \) to some computable cone \( C_{n+1} \supseteq D_{n+1}^* \), then this inequality constraint may no longer be redundant.

It is interesting to note that when \( V(QCQP) \) is finite, we can construct a feasible solution of COD as given in the proof of Theorem 3.3. In case \( V(QCQP) = -\infty \), COD must be infeasible since \( V(COP) \leq V(QCQP) \). Therefore, COD is feasible if and only if QCQP is bounded below. Also note that the gap between COP and COD is always zero in case \( V(QCQP) \) is finite. Hence a primal feasible solution \( (x, X) \) and a dual feasible solution \( (\sigma, \lambda) \) are optimal to COP and COD, respectively, if and only if \( \frac{1}{2}Q_i \cdot X + c_i^T x = \frac{1}{2} \sigma - b_i^T \lambda \). This equation is called the zero gap condition. It is known that the zero gap condition is equivalent to the complementarity condition as stated in the next theorem.

Theorem 3.4. When \( V(QCQP) \) is finite, a primal feasible solution \( (x, X) \) and a dual feasible solution \( (\sigma, \lambda) \) are optimal to COP and COD, respectively, if and only if they satisfy the following complementarity condition:
\[
\begin{bmatrix}
1 \\
x \\
X
\end{bmatrix}^T
\begin{bmatrix}
-\sigma \\
c + \sum_{i=1}^{m} \lambda_i c_i \\
c^T + \sum_{i=1}^{m} \lambda_i c_i^T \\
\end{bmatrix}
\begin{bmatrix}
Q + \sum_{i=1}^{m} \lambda_i Q_i \\
\end{bmatrix} = 0,
\]
(Complementarity Condition) \( \lambda_i \left( \frac{1}{2} Q_i \cdot X + c_i^T x - b_i \right) = 0, \; i = 1, 2, \ldots, m. \)

The above theorem is a classical result of the conic programming theory. More detailed discussions can be found on pages 33–35 of [3].

Since there is no universal computational algorithm for solving a linear conic program over an arbitrary cone, one research direction is to study the underlying structure of \( D_{n+1} \) for constructing a computable cone \( C_{n+1} \) such that \( C_{n+1} \subseteq D_{n+1} \) for COD, or equivalently \( C_{n+1} \supseteq D_{n+1}^* \) for COP, in order to find a lower bound for QCQP. In the following sections, we not only address the issues of getting a lower bound, but also investigate those cases which can be solved in polynomial-time when the cone \( D_{n+1} \) is substituted.

4. KKT system and conic programming. In this section, we discuss the relationship between the KKT system of QCQP and the conic programming problems COP and COD.
For a KKT solution \( x^* \) of QCQP with a corresponding Lagrangian vector \( \lambda^* \), we define a corresponding matrix 

\[
D(x^*, \lambda^*) = \begin{bmatrix}
-2F(x^*) - 2b^T \lambda^* & c^T + \sum_{i=1}^{m} \lambda^*_i c_i^T \\
c + \sum_{i=1}^{m} \lambda^*_i c_i & Q + \sum_{i=1}^{m} \lambda^*_i Q_i
\end{bmatrix}.
\]

Then we have a key result here.

**Theorem 4.1.** If problem QCQP has a KKT solution \( x^* \) with its corresponding Lagrangian vector \( \lambda^* \) such that the corresponding matrix \( D(x^*, \lambda^*) \in D_{n+1} \), then \( (\sigma^*, \lambda^*) \) is an optimal solution of COD and \( x^* \) is a global optimal solution of QCQP, where \( \sigma^* = 2F(x^*) + 2b^T \lambda^* \).

**Proof.** Since \( x^* \in F \), letting \( X^* = x^* x^T \), we know \( (x^*, X^*) \) is a feasible solution of COP. Noticing that

\[
D(x^*, \lambda^*) = \begin{bmatrix}
-2F(x^*) - 2b^T \lambda^* & c^T + \sum_{i=1}^{m} \lambda^*_i c_i^T \\
c + \sum_{i=1}^{m} \lambda^*_i c_i & Q + \sum_{i=1}^{m} \lambda^*_i Q_i
\end{bmatrix} \in D_{n+1},
\]

we know \( (\sigma^*, \lambda^*) \) is a feasible solution of COD. Then, it is easy to verify that

\[
\begin{bmatrix}
1 & x^* T \\
x^* & x^* x^T
\end{bmatrix} \begin{bmatrix}
-\sigma^* & c^T + \sum_{i=1}^{m} \lambda^*_i c_i^T \\
c + \sum_{i=1}^{m} \lambda^*_i c_i & Q + \sum_{i=1}^{m} \lambda^*_i Q_i
\end{bmatrix} = -2F(x^*) + 2L(x^*, \lambda^*) = 0
\]

for the complementarity condition. From the optimality theory of conic programming, we know that \( (x^*, X^*) \) with \( X^* = x^* x^T \) is optimal to COP and \( (\sigma^*, \lambda^*) \) is optimal to COD. By Theorem 3.3, we further know that \( x^* \) is a global optimal solution of QCQP.

The above theorem actually provides a sufficient condition of global optimality for QCQP. Since we shall constantly refer to this condition, we give a name here.

**Condition 2 (extended global optimality condition).** The QCQP problem has a KKT solution \( x^* \) with its Lagrangian vector \( \lambda^* \) such that \( D(x^*, \lambda^*) \in D_{n+1} \).

With this sufficient condition in mind, a challenging problem is to find such a KKT pair of \( (x^*, \lambda^*) \). Theorem 4.1 indicates that the \( \lambda^* \) in the extended global optimality condition together with \( \sigma^* = 2F(x^*) + 2b^T \lambda^* \) must be an optimal solution of COD. However, solving COD may not provide this particular \( \lambda^* \), because COD may have multiple optimal solutions in general. In order to find the desired KKT pair \( (x^*, \lambda^*) \), we need to conduct further analysis on COD and KKT system.

**Lemma 4.2.** Under the extended global optimality condition with \( (x^*, \lambda^*) \) being defined, if \( (\sigma_D, \lambda_D) \) is an optimal solution of problem COD, then \( x^* \) is a global optimal solution of the following optimization problem:

\[
\begin{array}{ll}
\min & L(x, \lambda_D) \\
\text{s.t.} & x \in F.
\end{array}
\]

**Proof.** The feasibility condition of COD implies that

\[
\begin{bmatrix}
-\sigma_D & c^T + \sum_{i=1}^{m} [\lambda_D]_i c_i^T \\
c + \sum_{i=1}^{m} [\lambda_D]_i c_i & Q + \sum_{i=1}^{m} [\lambda_D]_i Q_i
\end{bmatrix} \in D_{n+1}.
\]

Consequently, for any \( x \in F \), we have

\[
\begin{bmatrix}
1 \\
x
\end{bmatrix} ^T \begin{bmatrix}
-\sigma_D & c^T + \sum_{i=1}^{m} [\lambda_D]_i c_i^T \\
c + \sum_{i=1}^{m} [\lambda_D]_i c_i & Q + \sum_{i=1}^{m} [\lambda_D]_i Q_i
\end{bmatrix} \begin{bmatrix}
1 \\
x
\end{bmatrix} \geq 0.
\]
Therefore,
\[ x^T \left( Q + \sum_{i=1}^{m} \lambda_D |Q_i \right) x + 2 \left( c^T + \sum_{i=1}^{m} \lambda_D |c_i \right) x \geq \sigma_D \forall \ x \in \mathcal{F}. \]

Noticing that
\[
\begin{bmatrix}
1 \\
 x^* \end{bmatrix}
\]
is an optimal solution of COP and using the complementarity condition, we have
\[
\begin{bmatrix}
1 \\
 x^* \\
 x^* x^T 
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & x^T & c^T + \sum_{i=1}^{m} \lambda_D |c_i \\\n x^* & x^* x^T & Q + \sum_{i=1}^{m} \lambda_D |Q_i 
\end{bmatrix}
= 0.
\]
Therefore, we have
\[ x^T (Q + \sum_{i=1}^{m} \lambda_D |Q_i \right) x^* + 2(c^T + \sum_{i=1}^{m} \lambda_D |c_i \right) x^* = \sigma_D. \]
This proves that \( x^* \) is indeed a global minimizer of problem MOP.

The optimality condition of MOP leads to the next result.

**Lemma 4.3.** Under the extended global optimality condition with \((x^*, \lambda^*)\) being defined, let \((\sigma_D, \lambda_D)\) be an optimal solution of problem COD. If the linear independence constraint qualification (Condition 1) is also satisfied, then \( \nabla L(x^*, \lambda_D)^T d \geq 0 \) for any \( d \in \mathbb{R}^n \) with \( \nabla G_i(x^*)^T d \leq 0 \forall i \in I(x^*) \).

**Proof.** This is the necessary condition of local optimality for MOP at \( x^* \) under the constraint qualification condition.

**Remark 2.** In the theory of nonlinear programming (see Chapter 5 of [1]), under the linear independence constraint qualification, the cone \( T \) of tangents is equal to the set \( \{ d \in \mathbb{R}^n \mid \nabla G_i(x^*)^T d \leq 0 \forall i \in I(x^*) \} \). A necessary optimality condition for this problem is that \( T \cap \{ d \in \mathbb{R}^n \mid \nabla L(x^*, \lambda_D)^T d < 0 \} = \emptyset \).

The following version of Farkas’ lemma will be used later.

**Lemma 4.4** (Farkas’ lemma). For an \( m \times n \) matrix \( A \) and an \( n \)-dimensional vector \( \alpha \), the system of \("Ax = 0 and \alpha^T x < 0\"\) has a solution if and only if the system \( A^T y - \alpha \) does not have a solution.

Farkas’ lemma is a classical result, whose proof can be founded in Corollary 3 of Theorem 2.4.5 in [1]. Combining Lemmas 4.2, 4.3, and 4.4, we have the following result.

**Lemma 4.5.** Under Conditions 1 and 2 with \((x^*, \lambda^*)\) being defined, if \((\sigma_D, \lambda_D)\) is an optimal solution of problem COD, then \( \lambda_D \leq \lambda^* \).

**Proof.** First consider the components of \( \lambda_D |i \) and \( \lambda^*_i \) with \( i \notin I(x^*) \). Since
\[
L(x^*, \lambda_D) = \frac{1}{2} \sigma_D - b^T \lambda_D = \frac{1}{2} \sigma^* - b^T \lambda^* = F(x^*),
\]
we have
\[
F(x^*) + \sum_{i=1}^{m} [\lambda_D |i] G_i(x^*) = F(x^*).
\]

With each \( [\lambda_D |i \) being nonnegative and \( G_i(x^*) \) nonpositive, we have \( [\lambda_D |i] G_i(x^*) = 0 \) for \( i = 1, 2, \ldots, m \). Therefore, for \( i \notin I(x^*) \), we further have \( [\lambda_D |i = \lambda^*_i = 0 \).

Then consider the components with \( i \in I(x^*) \). Under Condition 1, we know \( \nabla G_i(x^*) \) are linearly independent for \( i \in I(x^*) \). For any \( i \in I(x^*) \), by Lemma 4.4, we can choose a vector \( d \) such that \( \nabla G_i(x^*)^T d < 0 \) and \( \nabla G_{i'}(x^*)^T d = 0 \forall i' \in I(x^*) - \{i\} \).
Lemma 4.3 further implies that
\[ \nabla_x L(x^*, \lambda_D)^T \bar{d} = \nabla F(x^*) \bar{d} + [\lambda_D]_i \nabla G_i(x^*)^T \bar{d} \geq 0. \]

Meanwhile, since \( \nabla_x L(x^*, \lambda^*) = 0 \), we have
\[ \nabla F(x^*) \bar{d} + [\lambda^*]_i \nabla G_i(x^*)^T \bar{d} = 0. \]

Therefore, we have
\[ ([\lambda_D]_i - [\lambda^*]_i) \nabla G_i(x^*)^T \bar{d} \geq 0, \]
which further implies that \([\lambda_D]_i \leq [\lambda^*]_i \). This completes the proof.

Theorem 4.6. Under Conditions 1 and 2 with \((x^*, \lambda^*)\) being defined, if \((\sigma_D, \lambda_D)\) is an optimal solution of problem COD with an optimal objective value \( \nu_d = \frac{1}{2} \sigma_D - b^T \lambda_D \), then \( \lambda^* \) is the unique optimal solution of the following linear conic programming problem:

\[
\begin{align*}
\text{max} & \quad e^T \lambda \\
\text{s.t.} & \quad \begin{bmatrix}
-\sigma \\
-c + \sum_{i=1}^m \lambda_i c_i \\
Q + \frac{1}{2} \sum_{i=1}^m \lambda_i Q_i
\end{bmatrix} \in D_{n+1} \\
& \quad 1 - b^T \lambda = \nu_d, \quad \lambda \geq 0,
\end{align*}
\]

where \( e = [1, \ldots, 1]^T \in \mathbb{R}^m \).

Proof. Notice that any feasible solution \((\sigma, \lambda)\) of problem COD2 is optimal to problem COD with \( \lambda \leq \lambda^* \) implied by Lemma 4.5. Since \((\sigma^*, \lambda^*)\) is feasible to COD2, the linear independence constraint qualification condition further implies that \( \lambda^* \) is the unique optimal solution of problem COD2.

The above theorem says that, under Conditions 1 and 2, the Lagrangian vector corresponding to the global optimal solution of QCQP is the unique optimal solution of COD2. Moreover, if \( Q + \sum_{i=1}^m \lambda^*_i Q_i \) is invertible, then
\[ x_{\lambda^*} = \left( Q + \sum_{i=1}^m \lambda^*_i Q_i \right)^{-1} \left( c - \sum_{i=1}^m \lambda^*_i c_i \right), \]
which is the solution for \( \nabla_x L(x, \lambda^*) = 0 \), is a global optimal solution of QCQP. This information is valuable for our work of algorithm design.

5. Proposed algorithm. Since COP is equivalent to QCQP, which is an NP-Hard problem, there is no polynomial time algorithm for solving COP unless P=NP. Therefore, we consider substituting the cone \( D_{n+1}^* \) in COP by a computable cone \( C_{n+1}^* \supseteq D_{n+1}^* \), or \( C_{n+1} \subseteq D_{n+1} \) on the dual side. Given such a computable cone \( C_{n+1} \), we define the following conic relaxation problem:

\[
\begin{align*}
\text{max} & \quad \frac{1}{2} \sigma - b^T \lambda \\
\text{s.t.} & \quad \begin{bmatrix}
-\sigma \\
-c + \sum_{i=1}^m \lambda_i c_i \\
Q + \frac{1}{2} \sum_{i=1}^m \lambda_i Q_i
\end{bmatrix} \in C_{n+1} \\
& \quad \lambda \geq 0.
\end{align*}
\]

Then we have the next theorem.
Theorem 5.1. The optimal objective value of problem COR1 is a lower bound for that of problem QCQP. Moreover, if there exists a KKT pair \((x^*, \lambda^*)\) such that \(D(x^*, \lambda^*) \in C_{n+1}\), then this lower bound coincides with the optimal objective value of problem QCQP.

Proof. Since \(C_{n+1} \subseteq D_{n+1}\), we have \(V(COR1) \leq V(COD) \leq V(COP) = V(QCQP)\), in which \(V(COD) \leq V(COP)\) is given by the weak duality theorem of conic programming. If there exists a KKT pair \((x^*, \lambda^*)\) such that \(D(x^*, \lambda^*) \in C_{n+1} \subseteq D_{n+1}\), then by Theorem 4.1 its corresponding primal solution \(x^*\) is a global optimal solution of QCQP with the objective value \((COR2)\)

\[
\max \ e^T \lambda \\
\text{s.t.} \quad \frac{1}{2} \sigma^T - b^T \lambda = \nu_d, \quad \lambda \geq 0.
\]

Similar to Theorem 4.6, we have the next result.

Theorem 5.2. If there exists a KKT pair \((x^*, \lambda^*)\) such that \(D(x^*, \lambda^*) \in C_{n+1}\), then \(\nu_d\) is the unique optimal solution of problem COR2.

Proof. From Theorem 5.1, we know the value of \(\nu_d\) in COR2 is equal to \(V(QCQP)\), which is also equal to the \(\nu_d\) defined in COD2. Since \(D(x^*, \lambda^*) \in C_{n+1} \subseteq D_{n+1}\), by Theorem 4.6, \(\lambda^*\) is the optimal solution of COD2. Moreover, because any feasible solution of COR2 is also feasible for COD2, \(\lambda^*\) must be the unique optimal solution of COR2.

With the insights drawn from problems COR1 and COR2, we propose the following algorithm.

Algorithm 1 (QCQP Algorithm).

Step 1: For the given QCQP problem, construct the conic relaxation problem COR1.

Step 2: Solve COR1 to obtain its optimal value \(\nu_d\). If failed, then stop. The problem cannot be solved by the current scheme.

Step 3: Construct the conic programming problem COR2 using \(\nu_d\).

Step 4: Solve COR2 to find its optimal solution \(\lambda^*\).

Step 5: Compute \(x^* = -(Q + \sum_{i=1}^m \lambda_i^* Q_i)^+ (c - \sum_{i=1}^m \lambda_i^* c_i)\), where \((Q + \sum_{i=1}^m \lambda_i^* Q_i)^+\) denotes the Moore–Penrose pseudoinverse of the matrix \((Q + \sum_{i=1}^m \lambda_i^* Q_i)\).

Step 6: If the pair \((x^*, \lambda^*)\) satisfy the KKT system with \(F(x^*) = \nu_d\), then return \(x^*\) as a global optimal solution of QCQP with the objective value \(\nu_d\). Otherwise, return \(\nu_d\) as a lower bound of problem QCQP.

The next theorem validates Algorithm 1.

Theorem 5.3. If Algorithm 1 returns a solution \(x^*\) successfully, then \(x^*\) is a global optimal solution of problem QCQP. If Algorithm 1 returns \(\nu_d\), then it is a lower bound of QCQP.

Proof. In Step 5 of Algorithm 1, if \(x^*\) is returned successfully, then \(D(x^*, \lambda^*) \in C_{n+1} \subseteq D_{n+1}\) and \((x^*, \lambda^*)\) satisfies the KKT system. By Theorem 4.1, \(x^*\) is a global optimal solution of QCQP. If \(\nu_d\) is returned, then, from Theorem 5.1, it is a lower bound of QCQP.
Remark 3. For known methods in the literature, a general solvable condition of nonconvex quadratically constrained quadratic programs is for those satisfying the positive semidefiniteness condition with $Q + \sum_{i=1}^{m} \lambda_i^* Q_i$ being invertible. Using Algorithm 1, we have a solvable condition for those nonconvex quadratically constrained quadratic programs satisfying the condition of Theorem 5.2 with $Q + \sum_{i=1}^{m} \lambda_i^* Q_i$ being invertible. If we choose $C_{n+1} = S_{n+1}$ for Algorithm 1, then these two conditions become equivalent; whereas, the cone $C_{n+1}$ can be better chosen than $S_{n+1}$ only, based on the structure of the feasible domain $\mathcal{F}$, for better performance of Algorithm 1. In this sense, the proposed Algorithm 1 extends the known solvable condition for more general QCQPs.

Remark 4. When Algorithm 1 is applied, if COR1 has a unique optimal solution, then there is no need to construct and solve COR2 in Steps 3 and 4. However, COR1 may have multiple optimal solutions in general and some of them may not provide the desired Lagrangian vector $\lambda^*$. In this case, solving COR2 is needed. Below is a simple example:

$$\begin{align*}
\min & \quad x_1 x_2 + x_1 + x_2 \\
\text{s.t.} & \quad x_1^2 - x_1 \leq 0, \\
& \quad x_2^2 - x_2 \leq 0.
\end{align*}$$

(P1)

Choose $C_3 = S_3 + N_3$, then the corresponding COR1 becomes

$$\begin{align*}
\max & \quad \frac{1}{2} \sigma \\
\text{s.t.} & \quad \begin{bmatrix}
-\sigma & 1 - \lambda_1 & 1 - \lambda_2 \\
1 - \lambda_1 & 2\lambda_1 & 1 \\
1 - \lambda_2 & 1 & 2\lambda_2
\end{bmatrix} \in S_3 + N_3 \\
& \quad \lambda \geq 0.
\end{align*}$$

(P1-COR1)

Notice that the optimal solution of P1 is $x^* = (0, 0)^T$ with the corresponding Lagrangian vector $\lambda^* = (1, 1)^T$. However, solving P1-COR1 directly by SeDuMi [23], we obtain an optimal solution $\bar{\lambda} = (0.2469, 0.2469)^T$, which is not the desired Lagrangian vector.

A similar situation happens to the dual problem of COR1, which is of the following form:

$$\begin{align*}
\min & \quad \frac{1}{2} Q \cdot X + c^T x \\
\text{s.t.} & \quad \frac{1}{2} Q_i \cdot X + c_i^T x - b_i \leq 0, \quad i = 1, 2, \ldots, m, \\
& \quad \begin{bmatrix} 1 \\ x^T \\ x \end{bmatrix} = Y, \\
& \quad Y \in C_{n+1}^n.
\end{align*}$$

(COR1-D)

Since $V(COR1) \leq V(COR1-D) \leq V(QCQP)$, when the gap between COR1 and QCQP is zero, so is the gap between COR1-D and QCQP. However, solving COR1-D may not lead to an optimal solution of QCQP directly. Below is a simple example.

$$\begin{align*}
\min & \quad x_1 x_2 \\
\text{s.t.} & \quad x_1^2 - x_1 \leq 0, \\
& \quad x_2^2 - x_2 \leq 0.
\end{align*}$$

(P2)
Choosing $C_3 = S_3 + N_3$ to solve this problem, we face

$$\begin{align*}
\text{max} & \quad \frac{1}{2} \sigma \\
\text{s.t.} & \quad \begin{bmatrix} -\sigma & -\lambda_1 & -\lambda_2 \\
-\lambda_1 & 2\lambda_1 & 1 \\
-\lambda_2 & 1 & 2\lambda_2 \end{bmatrix} \in S_3 + N_3 \\
& \quad \lambda \geq 0
\end{align*}$$

(P2-COR1)

and

$$\begin{align*}
\text{min} & \quad \frac{1}{2} \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} \cdot X \\
\text{s.t.} & \quad X_{11} \leq x_1, \\
& \quad X_{22} \leq x_2, \\
& \quad \begin{bmatrix} 1 & x^T \\
x & X \end{bmatrix} \in S_3 \cap N_3.
\end{align*}$$

(P2-COR1-D)

Notice that solving P1-COR1-D directly by SeDuMi, we obtain the following optimal solution:

$$\begin{bmatrix} 1 \bar{x}^T \\
\bar{x} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.2930 & 0.2930 \\
0.2930 & 0.2184 & 0.0000 \\
0.2930 & 0.0000 & 0.2184 \end{bmatrix}.$$ 

This solution does not lead us to the optimal solution $x^* = (0, 0)^T$ of P1 (unless we can design a suitable matrix decomposition method).

In contrast, if we follow through Steps 1–6 of Algorithm 1 by solving COR1 and COR2, an optimal solution is obtained successfully for each of the above two examples.

6. Numerical examples. In this section we show how Algorithm 1 works and how it extends the solvable condition for more general QCQPs by two examples.

The first example is a box constrained nonconvex quadratic programming problem.

Example 1.

$$\begin{align*}
\min & \quad \frac{1}{2} x^T Q x + c^T x \\
\text{s.t.} & \quad x \in [0, 1]^n,
\end{align*}$$

where

$$Q = \begin{bmatrix} 123 & 61 & 55 & 230 & -35 & 488 & 691 & -62 & 124 & -92 \\
61 & 114 & 110 & 65 & -61 & 16 & 59 & -15 & -22 & 61 \\
230 & 65 & 115 & 246 & 8 & 364 & 626 & -51 & 76 & -96 \\
-35 & -61 & -21 & 8 & 168 & -37 & 22 & -95 & 7 & 123 \\
488 & 16 & -44 & 364 & -37 & 107 & 477 & 81 & -120 & -116 \\
691 & 59 & -187 & 626 & 22 & 477 & 204 & -7 & -216 & -92 \\
124 & -22 & 189 & 76 & 7 & -120 & -216 & -44 & 218 & 4 \\
and
\[
c = [-62.05, -157.75, -524.75, -105.9, -66, 176.15, 354.2, 58.05, -359.35, -382.85]^T.
\]

Since the box constraint \( x \in [0,1]^n \) can be equivalently written as \( x_i^2 - x_i \leq 0 \) for \( i = 1, 2, \ldots, n \), the Lagrangian function of the problem becomes
\[
L(x, \lambda) = \frac{1}{2} x^T (Q + 2\Lambda) x + (c - \lambda)^T x.
\]
Now we choose \( C_{n+1} = S_{n+1} + N_{n+1} \) as an approximation cone to construct the conic programming problems COR1 and COR2 for applying Algorithm 1. In Step 4, the algorithm generates a solution
\[
\lambda^* = [41, 35, 0, 20, 0, 13, 70, 0, 88, 0]^T.
\]
Then we compute
\[
x^* = -(Q + 2\Lambda^*)^{-1}(c - \lambda^*) = [0, 1, 0.5, 0, 0.75, 0, 0, 0.6, 1, 0.5]^T,
\]
which is indeed a global optimal solution of this example.

Remark 5. For this example, we can verify that \( Q \) is not positive semidefinite, which means we face a nonconvex optimization problem and there is no guarantee that the conventional decent-based methods could find a global optimal solution. Moreover, since \( Q + 2\Lambda^* \) does not satisfy the positive semidefiniteness condition, the methods discussed in [12] are not applicable here.

The next example is a simple nonconvex quadratic programming problem with two quadratic constraints.

Example 2.
\[
\min \ F(x) = \frac{1}{2} x^T Q x + c^T x,
\]
\[\text{s.t.} \ G_1(x) = \frac{1}{2} x^T Q_1 x + c_1^T x - b_1 \leq 0,
\]
\[G_2(x) = \frac{1}{2} x^T Q_2 x + c_2^T x - b_2 \leq 0,
\]
where
\[
Q = \begin{bmatrix} -2 & 10 & 2 \\ 10 & 4 & 1 \\ 2 & 1 & -7 \end{bmatrix}, \quad c = \begin{bmatrix} -12 \\ -6 \\ 56 \end{bmatrix},
\]
\[
Q_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 0 \\ -2 \\ -64 \end{bmatrix},
\]
\[
Q_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 0 \\ -16 \end{bmatrix},
\]
\[b_1 = -256, \ b_2 = -64.
\]

It is easy to verify that the feasible domain of this example is a subset of \( \mathbb{R}_+^3 \). Hence we can choose \( C_4 = S_4 + N_4 \) as an inner approximation of \( D_4 \) for this example. By constructing and solving the conic relaxation problems COR1 and COR2, Algorithm 1 successfully obtains a global optimal solution \( x^* = [0, 0, 8]^T \) and its corresponding Lagrangian vector \( \lambda^* = (1, 2)^T \), with an optimal objective value \( F(x^*) = \)
224. We can further verify that
\[
D(x^*, \lambda^*) = \begin{bmatrix}
320 & -16 & -8 & -40 \\
-16 & 4 & 10 & 2 \\
-8 & 10 & 6 & 1 \\
-40 & 2 & 1 & 5
\end{bmatrix}
\begin{bmatrix}
320 & -16 & -8 & -40 \\
-16 & 4 & 0 & 2 \\
-8 & 0 & 6 & 1 \\
-40 & 2 & 1 & 5
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 \\
0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\in \mathcal{S}_4 + \mathcal{N}_4,
\]
whereas
\[
Q + \lambda_1^* Q_1 + \lambda_2^* Q_2 = \begin{bmatrix}
4 & 10 & 2 \\
10 & 6 & 1 \\
2 & 1 & 5
\end{bmatrix}
\notin \mathcal{S}_3.
\]

It is very interesting to point out that if we approximate the cone \( D_4 \) by \( \mathcal{S}_4 \) to construct the conic relaxation programming, then COR1 becomes the well-known SDP relaxation. In this case, we can use SeDuMi to verify that the optimal value of this SDP relaxation problem is 222.88, which exhibits a positive gap between the original QCQP and its SDP relaxation. Also notice that both \( F(x) \) and \( G_1(x) \) are nonconvex in this example.

Remark 6. Ye and Zhang discussed some solvable subclasses for nonconvex quadratic programming problem with two quadratic constraints in [27]. Since Example 2 exhibits a positive gap for SDP relaxation and the first constraint is nonconvex, we know for sure this example does not belong to the solvable subclasses in [27].

The above two examples demonstrate that Algorithm 1 is applicable for solving more general QCQP problems which cannot be solved by using the reported algorithms in [27] and [12]. Notice that for these two examples, we simply choose the cone \( \mathcal{S}_{n+1} + \mathcal{N}_{n+1} \) as an inner approximation. For more serious computations, based on the underlying structure of the feasible set \( \mathcal{F} \) and the cone \( D_{n+1} \), we may design some tighter computable cones for relaxation in order to further improve the performance of Algorithm 1.

7. Conclusion. In this paper, by exploring the relationship between the Lagrangian multiplies of QCQP and related linear conic programming problems, we have extended our previous work [18] to the general quadratically constrained quadratic programming problem. The global optimality condition obtained in this paper is a generalization of the known positive semidefiniteness condition in the literature. Moreover, the proposed Algorithm 1 provides clues of designing effective algorithms for more solvable quadratically constrained quadratic programming problems.

The relationship between the quadratic programming and conic programming was studied in the literature. For example, Sturm and Zhang reformulated the general nonconvex quadratic programming problem as a conic programming problem in [22], and Burer reformulated the binary and continuous nonconvex quadratic programming problems as a copositive programming problem in [5]. The results they obtained are based on the discussion of the relationship between the solutions of the original quadratic optimization problem and the reformulated conic programming problem. Compared to their works, the main difference of this paper is that we consider the relationship between the Lagrangian multipliers of the quadratic optimization problem
and several corresponding conic programming problems. The new insight provided by the Lagrangian multipliers, which is not easy to see from the original problem and its conic reformulation directly, has led us to discover more solvable quadratically constrained quadratic programming problems.

The relationship between the Lagrangian multipliers and the SDP relaxation method can also be found in the literature, because the Lagrangian dual of a quadratic optimization problem is equivalent to one type of SDP relaxation problems [16]. In this paper, we have refined and extended this relationship to more general cones which include the positive semidefinite cones. In this sense, our work is an extension of the previously known results. Our simple numerical examples clearly indicate that the proposed Algorithm 1 indeed unveils some new solvable QCQP problems that cannot be solved by using previously reported algorithms in the literature. The relationship we identified can also help us verify if a given KKT solution \((\bar{x}, \bar{\lambda})\) (no matter how it is obtained) is globally optimal by checking \(D(\bar{x}, \bar{\lambda}) \in C_{n+1}^{\geq 1}\) for some computable cone \(C_{n+1} \subseteq D_{n+1}\).

The proposed algorithm depends on the choice of the cone \(C_{n+1}\). A tighter inner approximation of the cone \(D_{n+1}\) not only provides a tighter lower bound, but also a higher chance of getting the global optimal solution of QCQP. Some approximation methods of certain cones can be found in the literature. For example, Parrilo attempted to use a sequence of cones to approximate the copositive cone [20] and Zuluaga, Vera, and Peña studied an approximation method for cones of semidefinite form in [28]. We are currently investigating new techniques to approximate the cone \(D_{n+1}\) from the inside.

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