GLOBAL EXTREMAL CONDITIONS FOR MULTI-INTEGER QUADRATIC PROGRAMMING

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(Communicated by Kok Lay Teo)

Abstract. This paper presents a canonical duality approach to solve an integer quadratic programming problem, in which the objective function is quadratic and each variable may assume the value of one of \( p \geq 3 \) integers. We first transform the problem into a \( \{-1, 1\} \) integer quadratic programming problem and then derive its “canonical dual”. It is shown that, under certain conditions, this nonconvex multi-integer programming problem is equivalent to a concave maximization dual problem over a convex feasible domain. A global optimality condition is derived and some computational examples are provided to illustrate this approach.

1. Introduction. This paper considers the following quadratic programming problem:

\[
(Q) : \min \{ P(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in X_a \},
\]

where \( x \) and \( f \) are real \( q \)-dimensional vectors, \( Q \in \mathbb{R}^{q \times q} \) is a symmetric matrix of order \( q \) and \( \langle \cdot, \cdot \rangle \) represents the inner product of two vectors. The feasible domain \( X_a \subset \mathbb{R}^q \) is defined by

\[
X_a = \{ x \in \mathbb{R}^q \mid x_i \in \{ c_1, \cdots, c_p \} \},
\]

with \( c_1, \cdots, c_p \) \( (p \geq 3) \) being different integers.

Integer programming and quadratic programming problems have extensive applications in system science and engineering [2, 13, 19]. It is well known that problem

2000 Mathematics Subject Classification. Primary: 49N15, 49M37, 90C26, 90C20.
Key words and phrases. Global optimization, quadratic programming, duality theory.
$(\mathcal{Q})$ is NP-hard in general, and it remains NP-hard for a simple version of 0-1 quadratic program. Furthermore, the nonconvexity of the objective function makes it complicated to find a global solution (see [14, 16]). Classical dual approaches [17, 18] may suffer from having a potential duality gap.

In this paper, we adopt the recently developed canonical duality approach to study the problem. The canonical duality theory is originally developed for handling general nonconvex and/or nonsmooth systems (see [3]). It is composed mainly of a canonical dual transformation and an associated triality theory. The canonical dual transformation may convert a nonconvex and/or nonsmooth primal problem into a piecewise smooth canonical dual problem without any duality gap in each subregion. In the meanwhile, the triality theory reveals an intrinsic multi-duality pattern in a general nonconvex system that can be used to identify both global and local extremal solutions. The canonical duality theory has shown its potential for some global optimization and nonconvex nonsmooth analysis [1, 4, 5, 7, 8, 9]. Comprehensive reviews of the canonical duality theory and its applications can be found in [6, 10]. Recently, Fang et al. [1] study the 0-1 quadratic programming problem using the canonical dual approach. This paper can be considered as an extension of that work.

The paper is arranged as follows. In Section 2, we transform problem $(\mathcal{Q})$ into a $\{-1, 1\}$ quadratic program. Then we show its canonical dual has zero duality gap under certain conditions. Some extremal conditions for global optimality are presented. In section 3, we consider a special cases of $(\mathcal{Q})$ in which $\{c_j\}$ is an arithmetic series. The special structure significantly simplifies the canonical dual analysis. Some examples are given in Section 4 to illustrate our approach and some extensions and concluding remarks are given in the last section.

2. Canonical duality approach. To develop the canonical duality approach, we first transform the multi-integer quadratic program into a $\{-1, 1\}$ quadratic program.

2.1. Linear transformation. It is known that an integer program with discrete variables can be transformed into a 0-1 integer program using a standard transformation [15]. For problem $(\mathcal{Q})$, the feasible domain $\mathcal{X}_a$ can be written equivalently by using the following transformation:

$$x_i = \sum_{j=1}^{p} c_j y_{(j-1)q+i} \quad \text{for } i = 1, \cdots, q,$$

where the variables $y_{(j-1)q+i}$ satisfy that

$$\begin{align*}
\sum_{j=1}^{p} y_{(j-1)q+i} &= 1 \quad \text{for } i = 1, \cdots, q, \\
y_{(j-1)q+i} &\in \{0, 1\} \quad \text{for } i = 1, \cdots, q \text{ and } j = 1, \cdots, p.
\end{align*}$$

Furthermore, let

$$z_{(j-1)q+i} = 2y_{(j-1)q+i} - 1 \quad \text{for } i = 1, \cdots, q, \ j = 1, \cdots, p,$$

then any feasible solution in $\mathcal{X}_a$ can be written equivalently as

$$x_i = \frac{1}{2} \sum_{j=1}^{p} c_j z_{(j-1)q+i} + \frac{c}{2} \quad \text{for } i = 1, \cdots, q,$$
Let \( z_{(j-1)q+i} = 2 - p \) for \( i = 1, \ldots, q \),
\[ z_{(j-1)q+i} \in \{-1, 1\} \quad \text{for } i = 1, \ldots, q \text{ and } j = 1, \ldots, p, \]
and \( c = \sum_{j=1}^{p} c_j \). Let \( z \) be the vector composed by \( z_t, t = 1, 2, \ldots, pq \). Replacing \( x \) by \( z \) in problem \((Q)\), we have the following equivalent \(\{-1, 1\}\) quadratic program:
\[
(S_1): \min \left\{ \frac{1}{2} \langle z, \bar{Q} z \rangle - \langle z, \bar{f} \rangle + l_1 \mid z \in \{-1, 1\}^{pq} \right\},
\]
where
\[
\bar{Q} = \frac{1}{4} \left( \begin{array}{ccc}
  c_1^2 Q & c_1 c_2 Q & \cdots & c_1 c_p Q \\
  c_1 c_2 Q & c_2^2 Q & \cdots & c_2 c_p Q \\
  \vdots & \vdots & \ddots & \vdots \\
  c_1 c_p Q & c_2 c_p Q & \cdots & c_p^2 Q \\
\end{array} \right),
\]
\[
\bar{f} = \left( \begin{array}{c}
  \frac{1}{4} (2f - cQe) \\
  \frac{1}{4} (2f - cQe) \\
  \vdots \\
  \frac{1}{4} (2f - cQe) \\
\end{array} \right),
\]
\[
l_1 = \frac{1}{8} c^2(e, Qe) - \frac{1}{2} c(f, e) \quad \text{is a constant with } e \text{ being the } q\text{-dimensional vector with } 1 \text{ for all elements, } B = (b_{ij})_{q \times pq} \text{ has } b_{i,(j-1)q+i} = 1 \text{ and } 0 \text{ for other elements, and } b = (2-p)e.
\]

Notice that problem \((S_1)\) is a \(\{-1, 1\}\) quadratic program with \(pq\) variables and \(q\) linear constraints. Let \(m = q\) and \(n = pq\). Ignoring the constant \(l_1\), we consider a general case:
\[
(P_b): \min \{P_b(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in X_b\}
\]
where \(x\) and \(f\) are real \(n\)-dimensional vectors, \(Q \in \mathbb{R}^{n \times n}\) is a symmetric matrix of order \(n\), and \(X_b \subset \mathbb{R}^n\) is defined by
\[
X_b = \{x \in \mathbb{R}^n \mid Bx = b, \ x \in \{-1, 1\}^n\}
\]
with \(b\) being a real \(m\)-dimensional vector and \(B \in \mathbb{R}^{m \times n}\).

2.2. Canonical duality. We first consider the following relaxation problem of \((P_b)\)
\[
\min \{W(x) + \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in \mathbb{R}^n\},
\]
where \(W(x)\) is defined by
\[
W(x) \triangleq \begin{cases}
0 & \text{if } Bx = b \text{ and } x^2_i \leq 1, \ i = 1, \ldots, n, \\
+\infty & \text{otherwise}.
\end{cases}
\]
By the canonical dual transformation (see [3, 5, 7]), we introduce a so-called geometrical measure
\[
\psi \triangleq \Lambda(x) = (\epsilon(x), \rho(x)) : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n, \ \epsilon = Bx, \ \rho = x \circ x - e,
\]
where \(e\) denotes an \(n\)-dimensional vector with 1 for all elements and the notation \(a \circ b \triangleq (a_1b_1, a_2b_2, \ldots, a_nb_n)^T\) for \(a = (a_1, a_2, \ldots, a_n)^T\) and \(b = (b_1, b_2, \ldots, b_n)^T\). Let
\[
\mathcal{E}_a = \{ (\epsilon, \rho) \in \mathbb{R}^m \times \mathbb{R}^n \mid \epsilon = b, \ \rho \leq 0 \}
\]
and
\[
V(\psi) = \begin{cases}
0 & \text{if } \psi \in \mathcal{E}_a \\
+\infty & \text{otherwise}.
\end{cases}
\]
It is easy to see that $W(x) = V(\Lambda(x))$. Thus, the problem (6) can be written in the following canonical form

$$
\min \left\{ V(\Lambda(x)) + \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in \mathbb{R}^n \right\}.
$$

(8)

Note that the canonical dual variable $\psi^* = (\epsilon^*, \rho^*)$ is a vector defined on $\mathcal{E}^* (= \mathcal{E} = \mathbb{R}^m \times \mathbb{R}^n)$ and a bilinear form $\langle *, * \rangle : \mathcal{E} \times \mathcal{E}^* \to \mathbb{R}$ can be defined as $\langle \psi, \psi^* \rangle = \langle \epsilon, \epsilon^* \rangle + \langle \rho, \rho^* \rangle$. Hence, the Fenchel sup-conjugate $V^\sharp$ of the function $V(\psi)$ becomes

$$
V^\sharp(\psi) \triangleq \sup_{\psi \in \mathbb{R}^n \times \mathbb{R}^n} \{ \langle \psi, \psi^\ast \rangle - V(\psi) \} = \left\{ \begin{array}{ll}
\langle b, \epsilon^* \rangle + \infty & \text{if } \rho^* \geq 0 \\
+\infty & \text{otherwise}.
\end{array} \right.
$$

Now define the dual domain as

$$
\mathcal{S} = \{ (\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho_1^* \geq 0 \}.
$$

Similar to the idea in [11], we replace $V(\Lambda(x))$ in (8) by using the Fenchel-Young equality $V(\Lambda(x)) = \langle \Lambda(x), \psi^* \rangle - V^\sharp(\psi^*)$. Then the so-called total complementary function $\Xi(x, y^*) : \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ associated with problem $(P_b)$ can be defined as

$$
\Xi(x, \epsilon^*, \rho^*) = V(\Lambda(x)) + \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle
$$

$$
= \langle \Lambda(x), \psi^* \rangle - \langle b, \epsilon^* \rangle + \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle
$$

$$
= \frac{1}{2} \langle x, (Q + 2 \text{Diag}(\rho^*))x \rangle - \langle x, f - B^T \epsilon^* \rangle - \langle b, \epsilon^* \rangle - \langle \rho^*, \epsilon \rangle,
$$

where the notation $\text{Diag}(\rho^*)$ stands for a diagonal matrix with $\rho_i^*$, $i = 1, 2, ..., n$, being its diagonal elements. Let $Q_d(\rho^*) = Q + 2 \text{Diag}(\rho^*)$ and $F(\epsilon^*) = f - B^T \epsilon^*$, then

$$
\Xi(x, \epsilon^*, \rho^*) = \frac{1}{2} \langle x, Q_d(\rho^*)x \rangle - \langle x, F(\epsilon^*) \rangle - \langle b, \epsilon^* \rangle - \langle \rho^*, \epsilon \rangle.
$$

For any given $(\epsilon^*, \rho^*) \in \mathcal{S}$, the canonical dual function $P^d_b(\epsilon^*, \rho^*)$ can be defined as

$$
\Xi(\bar{x}, \epsilon^*, \rho^*)
$$

with $\bar{x}$ being a critical (stationary) point of $\Xi(x, \epsilon^*, \rho^*)$ with respect to $x \in \mathcal{X}$.

Notice that for any given $(\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n$, the total complementary function $\Xi(x, \epsilon^*, \rho^*)$ is a quadratic function of $x \in \mathcal{X}$. A critical point $\bar{x}$ of $\Xi(x, \epsilon^*, \rho^*)$ satisfies

$$
\frac{\partial \Xi(x, \epsilon^*, \rho^*)}{\partial x} \bigg|_{x=\bar{x}} = Q_d(\rho^*)\bar{x} - F(\epsilon^*) = 0.
$$

When $\|Q_d(\rho^*)\| \neq 0$, we have a unique $\bar{x} = Q_d^{-1}(\rho^*)F(\epsilon^*)$, where $\|Q_d(\rho^*)\|$ represents the determinant of $Q_d(\rho^*)$.

We particularly define $\mathcal{S}_b \subset \mathcal{S}$ as

$$
\mathcal{S}_b = \{ (\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ |Q_d(\rho^*)| \neq 0 \}.
$$

Then the canonical dual function can be eventually written as

$$
P^d_b(\epsilon^*, \rho^*) = \frac{1}{2} \langle F(\epsilon^*), Q_d^{-1}(\rho^*)F(\epsilon^*) \rangle - \langle b, \epsilon^* \rangle - \langle \rho^*, \epsilon \rangle.
$$

(9)

Let the notation $\text{sta} \{ P^d_b(\epsilon^*, \rho^*) \mid (\epsilon^*, \rho^*) \in \mathcal{S}_b \}$ represent finding all critical points of $P^d_b(\epsilon^*, \rho^*)$ over $\mathcal{S}_b$. The canonical dual for our primal problem $(P_b)$ can be proposed as

$$(P_b^d) : \text{sta} \{ P^d_b(\epsilon^*, \rho^*) \mid (\epsilon^*, \rho^*) \in \mathcal{S}_b \}.
$$

(10)
Lemma 2.1. If $|Q_d(\rho^*)| \neq 0$ at a given vector $\rho^* \geq 0$ and $x = Q^{-1}_d(\rho^*)F(\epsilon^*)$ then
\[
\frac{\partial P^d_b(\epsilon^*, \rho^*)}{\partial \epsilon^*} = Bx - b, \quad (11)
\]
\[
\frac{\partial P^d_b(\epsilon^*, \rho^*)}{\partial \rho^*} = x \circ x - e. \quad (12)
\]
Moreover, the Hessian matrix of $P^d_b(\epsilon^*, \rho^*)$ is negative semi-definite when $Q_d(\rho^*)$ is positive definite.

Proof. Since
\[
P^d_b(\epsilon^*, \rho^*) = -\frac{1}{2}(\epsilon^*, BQ^{-1}_d(\rho^*)B^T\epsilon^*) + (BQ^{-1}_d(\rho^*)f, \epsilon^*) - \frac{1}{2}(f, Q^{-1}_d(\rho^*)f) - (b, \epsilon^*) - (\rho^*, e),
\]
we have
\[
\frac{\partial P^d_b(\epsilon^*, \rho^*)}{\partial \epsilon^*} = -BQ^{-1}_d(\rho^*)B^T\epsilon^* + BQ^{-1}(\rho^*)f - b = BQ^{-1}_d F(\epsilon^*) - b = Bx - b.
\]
Consequently, we have (11).

Since $Q_d(\rho^*)x = F(\epsilon^*)$, we have
\[
\frac{\partial (Q_d(\rho^*)x)}{\partial \rho^*} = \frac{\partial (Qx + 2\text{Diag}(\rho)x)}{\partial \rho^*} = Q\frac{\partial x}{\partial \rho^*} + 2\text{Diag}(x) + 2\text{Diag}(\rho^*)\frac{\partial x}{\partial \rho^*} = 0.
\]
Hence
\[
\frac{\partial x}{\partial \rho^*} = -2Q^{-1}(\rho^*)\text{Diag}(x).
\]
In this way,
\[
\frac{\partial P^d_b(\epsilon^*, \rho^*)}{\partial \rho^*} = -\frac{1}{2} \frac{\partial (F(\epsilon^*), x)}{\partial \rho^*} - \frac{\partial (\rho^*, e)}{\partial \rho^*}
= -\frac{1}{2} \left( \frac{\partial x}{\partial \rho^*} \right)^T F(\epsilon^*) - e
= \text{Diag}(x)Q^{-1}_d(\rho^*)F(\epsilon^*) - e
= x \circ x - e.
\]
This implies (12).

Similarly, it is easy to verify that
\[
\frac{\partial^2 P^d_b(\epsilon^*, \rho^*)}{\partial (\epsilon^*)^2} = \frac{\partial (Bx - b)}{\partial \epsilon^*} = -BQ^{-1}_d(\rho^*)B^T, \quad (13)
\]
\[
\frac{\partial^2 P^d_b(\epsilon^*, \rho^*)}{\partial (\rho^*)^2} = \frac{\partial (x \circ x - e)}{\partial \rho^*} = 2\text{Diag}(x)\frac{\partial x}{\partial \rho^*}
= -4\text{Diag}(x)Q^{-1}_d(\rho^*)\text{Diag}(x), \quad (14)
\]
\[
\frac{\partial^2 P^d_b(\epsilon^*, \rho^*)}{\partial \rho^* \partial \epsilon^*} = \frac{\partial (Bx - b)}{\partial \rho^*} = B\frac{\partial x}{\partial \rho^*} = -2BQ^{-1}_d(\rho^*)\text{Diag}(x), \quad (15)
\]
\[
\frac{\partial^2 P^d_b(\epsilon^*, \rho^*)}{\partial \epsilon^* \partial \rho^*} = \frac{\partial (x \circ x - e)}{\partial \epsilon^*} = 2\text{Diag}(x)\frac{\partial x}{\partial \epsilon^*} = -2\text{Diag}(x)Q^{-1}_d(\rho^*)B^T. \quad (16)
\]
Combining (13)-(16), we can write the Hessian matrix as
\[ \nabla^2 P_b^d(\epsilon^*, \rho^*) = - \left( \frac{B}{2 \text{Diag}(x)} \right) Q_d^{-1}(\rho^*) \left( B^T \text{Diag}(x) \right), \]
which is a negative semi-definite matrix when $Q_d^{-1}(\rho^*)$ is positive definite. \hfill \square

From Lemma 2.1, a critical point $(\bar{\epsilon}^*, \bar{\rho}^*)$ of $P_b^d(\epsilon^*, \rho^*)$ over $S_b$ is given by
\[
\frac{\partial P_b^d(\epsilon^*, \rho^*)}{\partial \epsilon^*} \bigg|_{\epsilon^*=\bar{\epsilon}^*, \rho^*=\bar{\rho}^*} = B\bar{x} - b = 0,
\]
(17)
\[
\frac{\partial P_b^d(\epsilon^*, \rho^*)}{\partial \rho^*} \bigg|_{\epsilon^*=\bar{\epsilon}^*, \rho^*=\bar{\rho}^*} = \bar{x} \circ \bar{x} - e = 0,
\]
(18)
where $\bar{x} = Q_d^{-1}(\bar{\rho}^*)F(\bar{\epsilon}^*)$. Therefore, each critical point of $P_b^d(\epsilon^*, \rho^*)$ is corresponding to a feasible solution of the primal problem $(P_b)$. Moreover, the following theorem characterizes the key primal-dual relationship.

**Theorem 2.2.** The canonical dual problem $(P_b^*)$ is perfectly dual to the primal problem $(P_b)$ in the sense that if $(\bar{\epsilon}^*, \bar{\rho}^*) \in S_b$ is a critical point of $P_b^d(\epsilon^*, \rho^*)$, then the vector $\bar{x}(\bar{\epsilon}^*, \bar{\rho}^*) = Q_d^{-1}(\bar{\rho}^*)F(\bar{\epsilon}^*)$ is a KKT point of $(P_b)$ and
\[ P_b(\bar{x}(\bar{\epsilon}^*, \bar{\rho}^*)) = P_b^d(\bar{\epsilon}^*, \bar{\rho}^*). \]
(19)

**Proof.** $(P_b)$ can be written in a standard form
\[
\min_{\substack{x \in \mathbb{R}^n \ \text{s.t.} \ Bx = b \ \text{and} \ x \circ x = e}} P_b(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle.
\]
Its Lagrangian function is
\[ L(x, \epsilon^*, \rho^*) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle + \langle Bx - b, \epsilon^* \rangle + \langle x \circ x - e, \rho^* \rangle. \]

Then the KKT conditions of $(P_b)$ become
\[
\frac{\partial L}{\partial x} = Q_d(\rho^*)x - F(\epsilon^*) = 0,
\]
(20)
\[
\frac{\partial L}{\partial \epsilon^*} = Bx - b = 0,
\]
(21)
\[
\frac{\partial L}{\partial \rho^*} = x \circ x - e = 0.
\]
(22)
From Lemma 2.1 and the definition of $\bar{x}(\bar{\epsilon}^*, \bar{\rho}^*)$, it is obvious that $\bar{x}(\bar{\epsilon}^*, \bar{\rho}^*)$ satisfies the KKT conditions if $(\bar{\epsilon}^*, \bar{\rho}^*)$ is a critical point of $P_b^d(\epsilon^*, \rho^*)$.

Using Lemma 2.1, again, $\bar{x}(\bar{\epsilon}^*, \bar{\rho}^*)$ is a feasible solution of $(P_b)$, then
\[
P_b(\bar{x}) = \frac{1}{2} \langle \bar{x}, Q\bar{x} \rangle - \langle \bar{x}, f \rangle
\]
\[
= \frac{1}{2} \langle \bar{x}, (Q + 2 \text{Diag}(\bar{\rho}^*))\bar{x} \rangle - \langle \bar{x}, F(\bar{\epsilon}^*) + B^T \bar{\epsilon}^* \rangle - \langle \bar{x}, \text{Diag}(\bar{\rho}^*)\bar{x} \rangle
\]
\[
= - \frac{1}{2} \langle \bar{x}, F(\bar{\epsilon}^*) \rangle - \langle B\bar{x}, \bar{\epsilon}^* \rangle - \langle \bar{x}, \text{Diag}(\bar{\rho}^*)\bar{x} \rangle
\]
\[
= - \frac{1}{2} \langle \bar{x}, F(\bar{\epsilon}^*) \rangle - \langle b, \bar{\epsilon}^* \rangle - \langle \bar{\rho}^*, e \rangle
\]
\[
= P_b^d(\bar{\epsilon}^*, \bar{\rho}^*).
\]
The result follows. \hfill \square
Theorem 1 shows that by using the canonical dual transformation, the nonconvex multi-integer quadratic programming problem can be converted into a piecewise continuous canonical dual (with zero duality gap) over \( S_b \). It is known that the KKT conditions are only necessary conditions for local minimizers to satisfy for a nonconvex programming problem. In order to present sufficient conditions for global optimal solutions, we introduce the following subset of the canonical dual space \( S_b^+ \):

\[
S_b^+ = \{ (\varepsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ Q_d(\rho^*) \geq 0 \},
\]

where \( Q_d(\rho^*) \geq 0 \) means that \( Q_d(\rho^*) \) is positive definite. Then we have the following theorem.

**Theorem 2.3.** For any given symmetric matrix \( Q \) and a vector \( f \in \mathbb{R}^m \), assume that \((\varepsilon^*, \rho^*)\) is a critical point of \( P^d_b(\varepsilon^*, \rho^*) \) and \( \bar{x}(\varepsilon^*, \rho^*) = Q^{-1}_d(\rho^*)F(\varepsilon^*) \). If \((\varepsilon^*, \rho^*) \in S_b^+ \), then \( \bar{x} \) is a global minimizer of \( P_b(x) \) over \( X_b \) and \((\varepsilon^*, \rho^*)\) is a global maximizer of \( P^d_b(\varepsilon^*, \rho^*) \) over \( S_b^+ \) with

\[
P_b(\bar{x}) = \min_{x \in X_b} P_b(x) = \max_{(\varepsilon^*, \rho^*) \in S_b^+} P^d_b(\varepsilon^*, \rho^*) = P^d_b(\varepsilon^*, \rho^*)
\]

**Proof.** From Lemma 2.1, we know that the canonical dual function \( P^d_b(\varepsilon^*, \rho^*) \) is concave on \( S_b^+ \). Therefore a critical point \((\varepsilon^*, \rho^*) \in S_b^+ \) must be a global maximizer of \( P^d_b(\varepsilon^*, \rho^*) \) on \( S_b^+ \). For the given critical point \((\varepsilon^*, \rho^*) \in S_b^+ \), we have

\[
P^d_b(\varepsilon^*, \rho^*) = P_b(\bar{x}(\varepsilon^*, \rho^*)).
\]

Assured by Theorem 2.2. When \( Q + 2\text{Diag}(\rho^*) \) is positive definite, then

\[
P_b(\bar{x}(\varepsilon^*, \rho^*)) = \frac{1}{2} \langle \bar{x}, Q\bar{x} \rangle - \langle \bar{x}, f \rangle + \langle \varepsilon^*, B\bar{x} - b \rangle + \langle \rho^*, \bar{x} \circ \bar{x} - e \rangle \leq \min_{x \in \mathbb{R}^n} \frac{1}{2} \{ \langle x, (Q + 2\text{Diag}(\rho^*))x \rangle - \langle x, f - B^T\varepsilon^* \rangle - \langle b, \varepsilon^* \rangle - \langle \rho^*, e \rangle \} \leq \min_{\{x : Bx = b, x_0 = e\}} \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle.
\]

Hence \( \bar{x} \) is a global minimizer of \( P_b(x) \) over \( X_b \). \( \square \)

3. **Multiple-integer programming over an arithmetic series.** In this section, we consider a special case of problem \((Q)\) in which \( \{c_j\} \) forms an arithmetic series, i.e., \( c_{j+1} - c_j \) remains a constant for \( j = 1, \cdots, p - 1 \).

3.1. **Linear transformation.** Assume that \( c_{j+1} - c_j = d \) for \( j = 1, \cdots, p - 1 \) \((p \geq 3)\), where \( d > 0 \) is a constant. Let \( y_i = \frac{x_i - d}{d} \) in \((Q)\), we can get an equivalent integer quadratic program. Without loss of generality, we may assume that \( c_1 = 0 \) and \( d = 1 \). Rewrite the problem as

\[
(Q) : \min \{ P(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in X_a \},
\]

where the feasible domain

\[
X_a = \{ x \in \mathbb{R}^n \mid x_i \text{ is an integer and } 0 \leq x_i \leq p - 1 \}.
\]

Let \( t = \lfloor \log \ p \rfloor \), then \( t \geq 1 \) and \( p \geq 2^t \). Let \( y \) be a \((t+1)q\)-dimensional vector and

\[
\mathcal{Y} = \{ y \in \mathbb{R}^{(t+1)q} \mid y \in \{-1, 1\}^{(t+1)q} \}.
\]

We can prove the following theorem:
Theorem 3.1. The transformation \( x = h(y) \) defined by

\[
x_i(y) = \sum_{j=1}^t 2^{j-2}y_{(j-1)q+i} + \frac{1}{2}(p-2^t)y_{q+i} + \frac{p-1}{2},
\]

is a linear and full mapping from \( \mathcal{Y} \) to \( \mathcal{X}_a \).

Proof. First, we show that \( x = h(y) \) is a mapping from \( \mathcal{Y} \) to \( \mathcal{X} \) by proving that the \( x \) defined by (28) belongs to \( \mathcal{X}_a \) for any \( y \in \mathcal{Y} \).

Notice that there are only three terms that are possibly fractional for \( x_i \) as defined by (28), i.e., the term of \( \frac{1}{2}y_i \), \( \frac{1}{2}(p-2^t)y_{q+i} \) and \( \frac{p-1}{2} \). However, the sum of these terms, \( \frac{1}{2}(y_i + y_{q+i}) + (p-1)(y_{q+i}+1) - 2^ty_{q+i} \), is an integer as \( y_i, y_{q+i} \in \{-1, 1\} \) and \( p \geq 3, t \geq 1 \) are integers.

For \( i = 1, \cdots, q \), since

\[
\max\{x_i\} = \sum_{j=1}^t 2^{j-2} + \frac{1}{2}(p-2^t) + \frac{p-1}{2} = p-1,
\]

we know \( x_i \) is an integer between 0 and \( p-1 \). Therefore, the \( x \) defined by (28) belongs to \( \mathcal{X}_a \).

Secondly, we show that \( x = h(y) \) is a full mapping, i.e., there exists \( y_{(j-1)q+i} \in \{-1, 1\} \) for \( j = 1, \cdots, t+1 \) such that (28) holds for an arbitrary integer \( 0 \leq x_i \leq p-1 \).

If \( 0 \leq x_i \leq 2^t-1 \), then \( x_i \) can be described as \( a_{t-1}a_{t-2}\cdots a_0 \) in the binary code, where \( a_j \in \{0, 1\} \) for \( j = 0, \cdots, t-1 \). Let \( y_{j_{q+i}} = 2a_j - 1 \) for \( j = 0, \cdots, t-1 \) and \( y_{q+i} = -1 \), then \( y_{(j-1)q+i} \in \{-1, 1\} \) for \( j = 1, \cdots, t+1 \). Consequently,

\[
\sum_{j=1}^t 2^{j-2}y_{(j-1)q+i} + \frac{1}{2}(p-2^t)y_{q+i} + \frac{p-1}{2} = \sum_{j=0}^{t-1} 2^ja_j = x_i.
\]

If \( 2^t-1 < x_i \leq p-1 \), then \( 0 \leq x_i + 2^t-p \leq 2^t-1 \) as \( 2^t \leq x_i \) and \( p \leq 2^{t+1} \). We write \( x_i+2^t-p \) as \( a_{t-1}\cdots a_0 \) in the binary code, where \( a_j \in \{0, 1\} \) for \( j = 0, \cdots, t-1 \). Let \( y_{j_{q+i}} = 2a_j - 1 \) for \( j = 0, \cdots, t-1 \) and \( y_{q+i} = 1 \), then \( y_{(j-1)q+i} \in \{-1, 1\} \) for \( j = 1, \cdots, t+1 \). Moreover,

\[
\sum_{j=1}^t 2^{j-2}y_{(j-1)q+i} + \frac{1}{2}(p-2^t)y_{q+i} + \frac{p-1}{2} = \sum_{j=0}^{t-1} 2^ja_j + p-2^t = x_i.
\]

This completes the proof. \( \square \)

Now, replacing \( x \) by \( y \) in problem (Q), we have the following problem:

\[
(S_2): \min \left\{ \frac{1}{2} \langle y, Qy \rangle - \langle y, f \rangle + t_2 \mid y \in \{-1, 1\}^{(t+1)q} \right\},
\]

where

\[
Q = \begin{pmatrix}
\frac{1}{2}Q & \frac{1}{2}Q & \cdots & 2^{-3}Q & \frac{1}{2}(p-2)Q \\
\frac{1}{2}Q & Q & \cdots & 2^{-2}Q & \frac{1}{2}(p-2)Q \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2^{-3}Q & 2^{-2}Q & \cdots & 2^{-1}Q & 2^{-3}(p-2)Q \\
\frac{1}{2}(p-2)Q & \frac{1}{2}(p-2)Q & \cdots & 2^{-3}(p-2)Q & \frac{1}{2}(p-2)^2Q
\end{pmatrix},
\]
comes

2.1

2.2

and noticing that

Proof. Assume that \( x^* \) is an optimal solution of problem (Q), Theorem 3.1 comes with a feasible solution \( y^* \) of \((S_2)\) such that they have the same objective value. Therefore, the optimal value of \((S_2)\) is no more than that of \((Q)\).

Similarly, we know that the optimal value of \((S_2)\) is no less than that of \((Q)\). Hence problems \((Q)\) and \((S_2)\) have the same optimal value. \(\square\)

Comparing with the transformation presented in section 2.1, we see that the primal problem \((Q)\) with \(\{c_j\}\) being an arithmetic series is transformed into a \(\{-1,1\}\) integer quadratic programming problem with only \(q(|\log p| + 1)\) variables and free of linear constraints.

Let \(r = (t + 1)q\). The optimal solution remains the same when we ignore the constant \(l_2\) in \((S_2)\). Therefore, we may focus on

\[
(P_c) : \min \{ P_c(x) = \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in \mathcal{X}_c \}
\]

where \(x\) and \(f\) are real \(r\)-dimensional vectors and \(Q \in \mathbb{R}^{r \times r}\) is a symmetric matrix of order \(r\). The feasible domain is defined by

\[
\mathcal{X}_c = \{ x \in \mathbb{R}^r \mid x \in \{-1,1\}^r \}. \quad (33)
\]

3.2. Canonical duality. Similar to the derivation in Section 2.2 and noticing that there is no linear constraint in \(\mathcal{X}_c\), we can get a corresponding canonical dual problem of \((P_c)\) by setting \(B = 0\) and \(b = 0\):

\[
(P^d_c) : \mbox{sta} \left\{ P^d_c(\rho^*) = \frac{1}{2} \langle f, Q_d^{-1}(\rho^*)f \rangle - \langle \rho^*, e \rangle \mid \rho^* \in \mathcal{S}_c \right\}, \quad (34)
\]

where \(\rho^*\) is the canonical dual vector in a dual feasible domain

\[
\mathcal{S}_c = \{ \rho^* \in \mathbb{R}^r \mid \rho^* \geq 0, \ |Q_d(\rho^*)| \neq 0 \}. \quad (35)
\]

With a similar discussion in Section 2.2, we have the following theorems that characterize the primal-dual relationship.

**Theorem 3.2.** The canonical dual problem \((P^d_c)\) is perfectly dual to the primal problem \((P_c)\) in the sense that if \(\tilde{\rho}^* \geq 0\) is a critical point of \(P^d_c(\rho^*)\), then the vector \(\bar{x}(\tilde{\rho}^*) = Q_d^{-1}(\tilde{\rho}^*)f\) is a KKT point of \((P_c)\) and

\[
P_c(\bar{x}) = P^d_c(\rho^*). \quad (35)
\]

Similar to (23), we define

\[
\mathcal{S}_c^+ = \{ \rho^* \in \mathbb{R}^r \mid \rho^* \geq 0, \ Q_d(\rho^*) > 0 \}. \quad (36)
\]

Then we have the following theorem.

Corollary 1. Solving problem \((Q)\) is equivalent to solving problem \((S_2)\).
Theorem 3.3. For a given symmetric matrix $Q$ and a vector $f \in \mathbb{R}^n$, assume that $\bar{\rho}^*$ is a critical point of $P^d_c(\bar{\rho}^*)$ and $\bar{x}(\bar{\rho}^*) = Q^{-1}(\bar{\rho}^*)f$. If $\bar{\rho}^* \in S^+_c$, then $\bar{x}$ is a global minimizer of $P_c(x)$ over $\mathcal{X}_c$ and $\bar{\rho}^*$ is a global maximizer of $P^d_c(\bar{\rho}^*)$ over $S^+_c$ with

$$P_c(\bar{x}) = \min_{x \in \mathcal{X}_c} P_c(x) = \max_{\rho^* \in S^+_c} P^d_c(\rho^*).$$

(37)

4. Examples. Consider the following tri-integer quadratic programming problem:

$$\min \left\{ \frac{1}{2} \langle x, Qx \rangle - \langle x, f \rangle \mid x \in \mathcal{X} \right\},$$

(38)

where $x$ and $f$ are in $\mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix of order $n$. The feasible domain $\mathcal{X} \in \mathbb{R}^n$ is defined by

$$\mathcal{X} = \{ x \in \mathbb{R}^n \mid x \in \{0, 1, 2\}^n \}.$$ 

(39)

The linear transformation introduced in (28) is $x_i = \frac{1}{2}y_i + \frac{1}{2}y_{n+i} + 1$ for $i = 1, \ldots, n$, where $y_i \in \{-1, 1\}$. Let $y$ be the vector composed by $y_j$, $j = 1, 2, \ldots, 2n$. We have an equivalent problem:

$$\min \left\{ P(y) = \frac{1}{2} \langle y, \bar{Q}y \rangle - \langle y, \bar{f} \rangle + l \mid y \in \{-1, 1\}^{2n} \right\},$$

(40)

where

$$\bar{Q} = \frac{1}{4} \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \quad \bar{f} = \frac{1}{2} \begin{pmatrix} f - Qe \\ f - Qe \end{pmatrix},$$

and $l = \frac{1}{2} \langle e, Qe \rangle - \langle e, f \rangle$. Ignoring the constant $l$, we consider the following problem:

$$(P_c) : \min \left\{ P_c(y) = \frac{1}{2} \langle y, Qy \rangle - \langle y, \bar{f} \rangle \mid y \in \{-1, 1\}^{2n} \right\},$$

(41)

The canonical dual problem of $(P_c)$ becomes

$$(P^d_c) : \text{subj} \left\{ P^d_c(\rho^*) = -\frac{1}{2} \langle \bar{f}, (\bar{Q} + 2\text{Diag}(\rho^*))^{-1} \bar{f} \rangle - \langle \rho^*, e \rangle \mid \rho^* \in S_c \right\},$$

(42)

where the dual feasible domain is defined by

$$S_c = \{ \rho^* \in \mathbb{R}^{2n} \mid \rho^* \geq 0, \ |Q_d(\rho^*)| \neq 0 \}.$$

Define

$$S^+_c = \{ \rho^* \in \mathbb{R}^{2n} \mid \rho^* \geq 0, \ Q + 2\text{Diag}(\rho^*) \succ 0 \}.$$ 

It is easy to check that the canonical dual function $P^d_c(\rho^*)$ is concave over $S^+_c$. If $\bar{\rho}^* \in S^+_c$ is a critical point of $P^d_c(\bar{\rho}^*)$, then Theorem 3.3 implies that $\bar{x}(\bar{\rho}^*) = (\bar{Q} + 2\text{Diag}(\rho^*))^{-1}\bar{f}$ is a global minimizer of problem $(P_c)$.

Case 1: we consider a 5-dimensional problem ($n = 5$) with randomly selected matrix $Q$ and $f$

$$Q = \begin{pmatrix} 20 & 4 & -4 & 4 & -8 \\ 4 & -12 & 0 & -4 & -4 \\ -4 & 0 & 20 & 8 & -8 \\ 4 & -4 & 8 & -16 & 0 \\ -8 & -4 & -8 & 0 & -36 \end{pmatrix}, \quad f = \begin{pmatrix} -56.6 \\ -15.73 \\ 46.6 \\ -48 \\ 20 \end{pmatrix}.$$

The equivalent primal problem $(P_c)$ is a 10 ($= 2 \times 5$) dimensional quadratic integer minimization problem over $\{-1, 1\}^{10}$. In this case, the canonical dual problem $(P^d_c)$
has a unique global maximizer $\tilde{\rho}^* = (10.1, 4.1, 5.6, 14.0, 28.9, 10.1, 4.1, 5.6, 14.0, 28.9)^T$ with a corresponding primal solution

$$\tilde{y} = (\bar{Q} + 2\text{Diag}(\tilde{\rho}^*))^{-1} \bar{f} = (-1, 1, 1, -1, -1, 1, 1, -1, 1, 1)^T.$$ 

By the fact of $\tilde{\rho}^* \in S^+_c$, we know that $\tilde{y}$ is a global minimizer of $(P_c)$ and $P_c(\tilde{y}) = -235.47 \simeq -235.41 = P_c(\tilde{\rho}^*)$. Here $l = 29.73$. The optimal solution for the trintermeger programming problem (38) is $x = (0, 2, 2, 0, 2)^T$ and the optimal value is $-205.74$.

Case 2: we let $n = 20$ and randomly choose $f$ and $Q$ with

$$f = (42.2, -79.8, -225, -93.6, -129, -25.6, -22.304, -56, 70.4, 54.6, -175.8, -126.8, 36.6, 171.4, -145.6, -104.2, 122, -64.8, 75.2, 146.8)^T$$

and $Q =$

$$\begin{pmatrix}
-8 & 0 & -4 & 4 & -4 & 0 & 4 & -4 & 0 & 0 & -8 & 0 & 4 & 0 & 0 & -8 & 0 & 8 & -4 \\
0 & 8 & 0 & 0 & 4 & 8 & -8 & -4 & 0 & 4 & 4 & 4 & 0 & 4 -4 & 4 & -4 & 0 & 4 \\
-4 & 0 & 0 & 4 & 0 & 4 & -4 & -4 & -4 & -8 & 0 & -8 & 0 & -8 & 0 & -8 & -4 & -8 \\
4 & 0 & 4 & -8 & 4 & -4 & -8 & 0 & 0 & 8 & 4 & 8 & 0 & 4 & -4 & -4 & -8 & -4 & 0 \\
-4 & 4 & 0 & 4 & 8 & -8 & -8 & 0 & 0 & -4 & 4 & 4 & 8 & 4 & 4 & -4 & 8 & 0 & 8 & 0 \\
0 & 8 & 4 & -8 & -8 & 0 & 4 & -4 & 4 & 8 & 4 & 0 & 8 & 8 & 0 & 4 & 0 & -8 & -8 & 4 \\
-4 & -8 & -4 & -8 & -8 & 0 & 0 & -4 & 0 & 8 & -8 & 0 & 0 & 8 & 8 & -8 & -4 & 0 & 0 & 0 \\
-4 & -4 & -4 & 0 & 0 & 4 & -4 & -8 & -4 & 0 & 0 & 4 & 8 & 4 & -4 & -4 & 0 & 4 & -8 \\
0 & 0 & -4 & 0 & 0 & -4 & 0 & -4 & 8 & 4 & 0 & 4 & 4 & 4 & 0 & 0 & 4 & 0 & 4 & -4 \\
0 & 4 & -4 & 8 & 4 & 8 & 8 & 0 & 4 & -8 & 0 & -8 & -4 & -4 & -4 & 0 & -4 & 0 & 0 & -4 \\
-8 & -4 & -8 & 4 & -4 & -4 & -8 & 0 & 0 & 0 & 0 & -4 & -8 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 8 & 4 & 8 & 0 & 4 & 4 & -8 & -4 & 8 & 0 & 4 & 0 & 0 & -8 & -8 & -8 & -4 \\
4 & 4 & -8 & 0 & 8 & 4 & 0 & 8 & 4 & -4 & -8 & 0 & 8 & 8 & 0 & 4 & -8 & 8 & 4 \\
0 & 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 & -4 & 0 & 4 & 0 & 0 & 0 & 0 & 8 & 0 & 4 \\
0 & 4 & -8 & -4 & 4 & 8 & 8 & 4 & 0 & -4 & 0 & 8 & 0 & -8 & 0 & 0 & -4 & 0 & 8 \\
-8 & -4 & 0 & -4 & -4 & 8 & 8 & -4 & 0 & -4 & 0 & 0 & 4 & 0 & 0 & 8 & -8 & 0 & 8 & -8 \\
0 & 4 & -8 & -8 & 8 & 4 & -8 & 4 & 4 & 0 & 0 & -8 & 0 & 0 & -8 & 8 & 0 & 0 & 0 & 0 \\
0 & -4 & -4 & -4 & 0 & 0 & -4 & 0 & 0 & 0 & -8 & -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & -8 & -8 & 0 & 8 & 4 & 0 & -4 & -4 & 0 & 0 & 8 & 8 & 0 & 0 & 8 & 0 & 8 & -4 \\
-4 & -4 & -8 & 4 & -8 & 0 & 8 & 4 & -4 & 0 & 4 & 8 & 4 & -8 & 8 & 0 & 0 & -4 & 0 \\
\end{pmatrix}$$

In this case, the dimension of the equivalent primal problem $(P_c)$ is 40. Solving such a problem may need 40 times of enumeration! However, the canonical dual problem can be solved in a few Newton iterations to obtain the global maximizer

$$\tilde{\rho}^* = (10.6, 27.9, 34.3, 29.4, 46.1, 10.4, 7.6, 20.0, 13.6, 21.6, 35.9, 21.7, 5.2, 40.8, 44.4, 18.1, 24.5, 18.2, 10.8, 36.7, 10.6, 27.9, 34.3, 29.4, 46.1, 10.4, 7.6, 20.0, 13.6, 21.6, 35.9, 21.7, 5.2, 40.8, 44.4, 18.1, 24.5, 18.2, 10.8, 36.7)^T.$$ 

The corresponding global minimizer of the primal problem $(P_c)$ then becomes

$$\tilde{y} = (1, -1, -1, -1, -1, -1, -1, -1, 1, 1, -1, -1, -1, -1, -1, -1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1)^T$$

and $P_c(\tilde{y}) = -1899.7 \approx -1900 = P_c(\tilde{\rho}^*)$. Here $l = 525.3$. The optimal solution for the original problem is $x = (2, 0, 0, 0, 0, 0, 0, 2, 2, 0, 0, 2, 2, 0, 2, 0, 2, 2)^T$ and the optimal value is $-1374.4$. 

5. Extension and conclusion. We have developed a canonical duality approach for solving multi-integer quadratic programming problems. By introducing a linear transformation, a multi-integer quadratic programming problem can be transformed into an equivalent \{-1, 1\} quadratic programming problem, and then a canonical dual problem is formulated. The critical points of the canonical dual problem lead to the KKT points of the original problem. There is no duality gap between the primal problem and its canonical dual under certain condition. Sufficient conditions for global optimality are also presented.

Notice that all of our previous results are derived under the assumption that \(|Q_d(\rho^*)| \neq 0\). In case \(|Q_d(\rho^*)| = 0\), the following observations can extent our findings to more general results.

**Observation 1.** In Section 2.2, for any given \((\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n\) with \(F(\epsilon^*) \in \mathcal{C}(Q_d(\rho^*))\), the column space of \(Q_d(\rho^*)\), the canonical dual function \(P^d_b(\epsilon^*, \rho^*)\) remains well defined as \(\Xi(\bar{x}, \epsilon^*, \rho^*)\) with \(\bar{x} = Q_d(\rho^*)^{-1} F(\epsilon^*)\), where \(Q_d(\rho^*)^{-1}\) denotes the Moore-Penrose generalized inverse of \(Q_d(\rho^*)\) [12]. In this way, the dual feasible domain \(\tilde{S}_b \subset \mathcal{S}\) can be expanded as

\[
\tilde{S}_b = \{(\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ F(\epsilon^*) \in \mathcal{C}(Q_d(\rho^*))\}.
\]

Following a similar argument, we can see that Theorem 2.2 holds over \(\tilde{S}_b\) and Theorem 2.3 holds over the following domain:

\[
\tilde{S}_b^+ = \{(\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ Q_d(\rho^*) \succeq 0, \ F(\epsilon^*) \in \mathcal{C}(Q_d(\rho^*))\}.
\]

Replacing \(F(\epsilon^*)\) by \(f\) in Section 3.2, corresponding results can be obtained for Theorem 3.2 and Theorem 3.3.

**Observation 2.** For any given \((\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n\) with \(\lim_{\{\gamma : |Q_d(\gamma)| \neq 0\} \to \rho^*} Q_d^{-1}(\gamma) F(\epsilon^*)\) being existent and finite, we can define

\[
\bar{x}(\epsilon^*, \rho^*) = \lim_{\{\gamma : |Q_d(\gamma)| \neq 0\} \to \rho^*} Q_d^{-1}(\gamma) F(\epsilon^*), \tag{43}
\]

then the canonical dual function \(P^d_b(\epsilon^*, \rho^*)\) remains well defined as \(\Xi(\bar{x}, \epsilon^*, \rho^*)\) with \(\bar{x}\) being defined in (43). Let

\[
\mathcal{D} = \{(\epsilon^*, \rho^*) \mid \lim_{\{\gamma : |Q_d(\gamma)| \neq 0\} \to \rho^*} Q_d^{-1}(\gamma) F(\epsilon^*)\text{ exists and is finite}\}.
\]

Then the dual feasible domain \(\tilde{S}_b \subset \mathcal{S}\) can be expanded as

\[
\tilde{S}_b = \{(\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ (\epsilon^*, \rho^*) \in \mathcal{D}\}.
\]

Following a similar argument, we can see that Theorem 2.2 holds over this \(\tilde{S}_b\), and Theorem 2.3 holds over the following domain:

\[
\tilde{S}_b^+ = \{(\epsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}^n \mid \rho^* \geq 0, \ Q_d(\rho^*) \succeq 0, \ (\epsilon^*, \rho^*) \in \mathcal{D}\}.
\]

Replacing \(F(\epsilon^*)\) by \(f\) in Section 3.2, corresponding results can be obtained for Theorem 3.2 and Theorem 3.3.

As pointed out in [1], for certain given \(Q\) and \(f\), the canonical dual problem may not have a global maximizer in \(\tilde{S}_b^+\). A fundamental question is how to identify the instances that can be solved by the canonical duality theory. It certainly needs further study.
Acknowledgements. Wang and Xing’s research is supported by Tsinghua Basic Research Foundation # 052201070. Fang’s Research is supported by US NSF Grant # DMI-0553310 and David Gao’s research is supported by US NSF Grant # CCF-0514768.

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