OPTIMAL INVESTMENT, CONSUMPTION AND LIFE INSURANCE UNDER STOCHASTIC INTEREST RATE AND VOLATILITY

Zongxia Liang and Xiaoyang Zhao
Department of Mathematical Sciences, Tsinghua University, Beijing100084, China

Abstract. In this paper, we consider the problem of optimal investment, consumption and life insurance for a wage earner who has constant relative risk aversion (CRRA) preferences. The wage earner can invest in zero-coupon bond, stock and life insurance, and can make consumption decision. The interest rate and the volatility of the stock are stochastic, which results in incomplete market. Besides, the labor income is also stochastic, its increasing rate and the interest rate are cointegrated. We derive the optimal strategies of the problem by dynamic programming method and solving the associated HJB equations. We also present a sensitivity analysis to explore the impact of economical parameters on the optimal strategies.

JEL Classifications: G11, G22, G32, C61, D52, D53.


Keywords: Portfolio management; Life insurance; Stochastic interest rate; Stochastic volatility; Stochastic labor income; Co-integration; Dynamic programming; Incomplete market.

1. Introduction

The optimal investment and consumption problem has attracted much attention since it was considered by Merton[19](1969) and [20](1971). The assets in Merton’s model include a risk-free asset and some risky assets and the main goal of these papers is to maximize the utility function of consumption and terminal wealth. Merton used dynamic programming principle to derive an explicit solution of the optimal problem. Richard in his famous paper[25](1975) developed Merton’s work by firstly combining life insurance with investment and consumption. He assumed that the insurer’s death time is arbitrary but its distribution is known, and the insurer can not purchase life insurance at the end of his life time.

Email: zliang@math.tsinghua.edu.cn(Z.Liang), zhao-xy13@mails.tsinghua.edu.cn(X.Zhao)
Since then, Campbell[2](1980), Lewis[17](1989) and Hurd[13](1989) further extended Richard’s work of life insurance from different perspectives and give us many valuable insights into demand for life insurance.

Recently, Pliska and Ye[23](2007) firstly studied the optimal life insurance, consumption and investment problem. They assumed that the wage earner’s lifetime is independent of the financial market, and converted the problem of uncertain life time to the situation of a fixed life time. Nielsen and Steffensen [22](2008) investigated the optimal investment and life insurances under minimum and maximum constraints. They found that the wealth consists of three parts and two of them are similar to the European put options and call options. Kwak et al.[16](2011) looked into the optimal investment, life insurance and consumption decision of a family. They considered the consumption of child after death of parents and successfully got the explicit solution by martingale method. Bruhn and Steffensen[1](2011) solved the problem of a two-person household consumption, investment and life insurance. Duarte et al.[8](2012) extended the framework of one risky asset to multiple risky assets. Kronborg and Steffensen[15](2013) booted up the problem of optimal consumption, investment and life insurance with surrender option guarantee, where the investor with deterministic labor income can invest in a financial market and buy life insurance or annuities, and is assumed to fulfill an American capital guarantee.

However, to the best of our knowledge, there is very little literatures to study the optimal consumption, life insurance and investment problem under the framework that interest rate and volatility of stock are stochastic. Because the investment of life insurance often lasts for a long time, it is necessary to take the risk of interest rate into account. Besides, the assumption of stochastic volatility of stocks is more practical since the price of stock may have different features in the real world, while most of the researches assume that the price is driven by a geometric Brownian motion. Deelstra[7](2000) studied the optimal consumption and investment problem in a CIR framework and derived the closed-form solution. Chang and Rong [3](2013) considered the optimal consumption and investment problem with CIR interest rate and stochastic volatility but with no life insurance. Pirvu and Zhang [24](2012)
investigated the optimal investment, consumption and life insurance under mean-reverting returns’ framework, where the return of stock is stochastic and the labor income of wage earner follows a geometric Brownian motion. The interested reader can refer to recent work Guan and Liang\cite{GuanLiang9, GuanLiang10}(2014) and references therein. Shen and Wei\cite{ShenWei26}(2014) got the solution of optimal investment-consumption-insurance with random parameters, in which they assumed that the parameters are all random processes adapted to the filtration generated by Brownian motion.

Moreover, it is natural to assume that the life insurer receives a labor income stream, therefore choosing what models of labor income is of crucial important to model real asset motion. In Pliska and Ye\cite{PliskaYe23}(2007) the labor income is constant. Pirvu and Zhang\cite{PirvuZhang24}(2012) assumed that the labor income is driven by a geometric Brownian motion but has no relation with interest rate. Huang and Milevsky\cite{HuangMilevsky26}(2008) studied a portfolio choice problem that included life insurance and labor income. They focused on the correlation between the dynamics of human capital and financial capital. Munk and Sørensen\cite{MunkSorensen21}(2010) firstly established the problem of optimal investment and consumption problem under the assumption of the labor income has a co-integration relationship with interest rate. The co-integration technique to examining asset movement was firstly discovered by Granger\cite{Granger11}(1981), and Granger was thus granted the Nobel Prize in Economics in 2003.

In this paper, we study the optimal investment, consumption and life insurance problem for a wage earner with constant relative risk aversion (CRRA) preferences. The wage earner can invest in zero-coupon bond, stock and life insurance, and can make consumption decision. The risk-free interest rate is driven by Vasicek model. Meanwhile, the volatility of stock is described by Heston’s stochastic volatility model. We also assume that the labor income is stochastic, with its increasing rate and the interest rate being co-integrated. We contribute to the existing literatures in at least three aspects. Firstly, we extend the stochastic parameters situation to life insurance framework. Secondly, we use the model that the labor income has a co-integration relationship with interest rate, which is more practical because the labor income usually increases more frequently in booming periods. Thirdly, we extend the method of converting nonlinear differential equations to linear differential equations from constant-non-linear-term situation to
function-non-linear-term situation. We maximize the expected CRRA utility function of the wealth process of wage earner, and then use dynamic programming method to get closed-form solution of the optimal problem, i.e., closed-forms of the optimal strategy and value function of the optimal problem, and finally give their numerical analysis.

The rest of this paper is organized as follows: In Section 2, we describe our model and establish the optimization problem. In Section 3, we first focus on the problem without labor income and then derive the explicit solution of original optimal problem with labor income by dynamic programming method. We also analyze each term of the optimal feedback function in Section 3. Numerical analysis is given in Section 4. Section 5 concludes this paper. The appendices contain rigorous proofs of main results in this paper.

2. Description of the Model

We assume that a wage earner receives a stochastic stream of labor income, and he or she has to make decisions about consumption, investment in stock, bond and life insurance. We let $T > 0$ be a fixed finite time horizon, and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions (cf. [14]). All the processes on $(\Omega, \mathcal{F}, \mathbb{P})$ below are adapted to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$.

2.1. The Financial Market. The financial market consists of a risk-free asset (i.e., cash), a risky asset (i.e., stock) and a zero-coupon bond. The price of risk-free asset $S_0(t)$ is

$$
\frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S_0(0) = 1. \tag{2.1}
$$

Besides, zero-coupon bonds are available in the financial market. We assume that the short-term interest rate follows the Vasicek model (cf. [6, 9, 10]):

$$
dr(t) = a(b - r(t))dt - \sigma_r dW_r(t), \quad r(0) = r_0, \tag{2.2}
$$

where $a$, $b$ and $\sigma_r$ are positive constants, $\{W_r(t)\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$. The market price $\lambda_r$ of interest rate risk is a constant. Then $d\tilde{W}_r(t) = dW_r(t) + \lambda_r dt$ is a Brownian motion under the measure $\tilde{P}$ defined by

$$
\frac{d\tilde{P}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp \left\{ \int_0^t \lambda_r(s)dW_r(s) - \frac{1}{2} \int_0^t \lambda_r^2(s)ds \right\}.
$$

So the price $\{B(t,T)\}$ of a zero-coupon bond which delivers a payment
of one at maturity $T$ is such that $\{\mathbb{D}(t)B(t, T)\}$ is a martingale under $\mathbb{P}$, where $\mathbb{D}(t) = \exp[-\int_t^T r(s)ds]$.

Applying Itô formula to $\{B(t, T)\}$, we have

$$
\frac{dB(t, T)}{B(t, T)} = \frac{1}{B(t, T)}[\partial B(t, T)/\partial t + \partial B(t, T)/\partial r(a(b - r(t)))
+ \frac{1}{2}\sigma^2 \frac{\partial^2 B(t, T)}{\partial r^2}]dt
- \frac{\partial B(t, T)}{\partial r} \sigma_r dW_r(t)
\triangleq \mu_B(t, T)dt + \sigma_B(t, T)dW_r(t).
$$

Because $\{\mathbb{D}(t)B(t, T)\}$ is a martingale under $\mathbb{P}$,

$$
\frac{d(\mathbb{D}(t)B(t, T))}{\mathbb{D}(t)B(t, T)} = \sigma_B(t, T)(dW_r(t) + \frac{\mu_B(t, T) - r(t)}{\sigma_B(t, T)}dt)
= \sigma_B(t, T)d\tilde{W}_r(t),
$$

i.e., $\lambda_r = \frac{\mu_B(t, T) - r(t)}{\sigma_B(t, T)}$.

Substituting (2.3) into (2.4), we get the following BSDE:

$$
\begin{align*}
\frac{\partial B(t, T)}{\partial t} + [a(b - r(t)) + \lambda_r \sigma_r] \frac{\partial B(t, T)}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 B(t, T)}{\partial r^2} &= \mu_B(t, T)B(t, T), \\
B(T, T) &= 1.
\end{align*}
$$

Solving the BSDE, we obtain

$$
B(t, T) = \exp[-e(t, T) - f(t, T)r(t)],
$$

where

$$
\begin{align*}
e(t, T) &= (b + \frac{\sigma_r \lambda_r}{a} - \frac{\sigma^2}{2a^2})(T - t) - f(t, T) + \frac{\sigma^2}{4a}f(t, T)^2, \\
f(t, T) &= \frac{1}{a}(1 - e^{-a(T-t)}).
\end{align*}
$$

Now, for simplicity, fixing the maturity $T$ and denoting $B(t, T)$ by $B(t)$, we have

$$
\frac{dB(t)}{B(t)} = r(t)dt + \sigma_B(t)(dW_r(t) + \lambda_r dt),
$$

where $\sigma_B(t) = -f(t, T)\sigma_r$.

In addition to bond $\{B(t)\}$, there is a risky asset $\{S_1(t)\}$, that is, a stock in the market, and its price satisfies the Heston’s model:

$$
\begin{align*}
\frac{dS_1(t)}{S_1(t)} &= r(t)dt + \sigma_S(t)\sigma_B(t)(\lambda_r dt + dW_r(t))
+ (mL(t)dt + \sqrt{L(t)dW_S(t)}), \\
dL(t) &= (g - hL(t))dt + \sigma_L \sqrt{L(t)dW_L(t)},
\end{align*}
$$

(2.7)
where $S_1(0) = s_1$, $m$, $g$, $h$ and $\sigma_L$ are positive constants, and the condition $2g > \sigma_L^2$ ensures $L(t) > 0$. Furthermore, $\Lambda(t) = (\lambda_r, \lambda_S(t))^T$ is the market price of risk vector $(r, S_1)^T$ (here $(x, y)^T$ means the transpose of $(x, y)$), where $\lambda_S(t) = m\sqrt{L(t)}$ is the market price of the stock. However, in the model above the market price of risk of $W_L(t)$ can be arbitrarily selected, so we can not get an unique risk neutral measure and the market is incomplete. Moreover, $W_r(t)$ is independent of $W_S(t)$ and $W_L(t)$. We assume that $W_S(t)$ and $W_L(t)$ are standard Brownian motions with $\text{Cov}(W_S(t), W_L(t)) = \rho_{SL}t$. To get the explicit solution, we need to let $\rho_{SL} = 1$, which will be explained in the proof of Theorem 3.5 below.

By the dynamics of $B(t)$ and $S_1(t)$, we can simplify expression of the vector $P(t) = (B(t), S_1(t))^T$ as follows:

$$dP(t) = \begin{pmatrix} B(t) & 0 \\ 0 & S_1(t) \end{pmatrix} [(r(t)\cdot 1dt + A(t)(\Lambda(t)dt + dW(t))],$$

where $W(t) = (W_r(t), W_S(t))^T$ and $A(t) = \begin{pmatrix} \sigma_B(t) & 0 \\ \sigma_S(t)\sigma_B(t) & \sqrt{L(t)} \end{pmatrix}$.

2.2. The Insurance Market. We suppose that the wage earner is alive at time $t = 0$ and has a lifetime $\tau$, which is a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ and is independent of the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$, and the wage earner has a instantaneous death rate $\lambda(t)$ defined by

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t \leq \tau < t + \Delta t | \tau \geq t)}{\Delta t}.$$ 

We define the conditional probability survival function by $F(s, t) \triangleq P(\tau > s | \tau > t)$. It is easy to see from the definition of $\{\lambda(t)\}$ that

$$F(s, t) = \exp\{-\int_t^s \lambda(u)du\}. \quad (2.8)$$

Denote by $\tilde{F}(s, t)$ the conditional probability density for death at time $s$ conditional upon the wage earner being alive at time $t \leq s$, hence

$$\tilde{F}(s, t) = \lambda(s)\exp\{-\int_t^s \lambda(u)du\}. \quad (2.9)$$

This instantaneous death rate model was considered by Pliska and Ye[23](2007), Pirvu and Zhang[24](2012) and Guan and Liang[9, 10](2014).

Back to the life insurance market, in our model the wage earner can purchase life insurance continuously by paying premiums at the rate $p(t)$ at time $t$. If the wage earner dies at time $t$ with the premium
payment rate \( p(t) \), the insurance company needs to pay an insurance amount \( \frac{p(t)}{\eta(t)} \), where \( \eta(t) : [0, T] \rightarrow \mathbb{R}^+ \) is a continuous, deterministic and specified function, that is, the so called \textit{the insurance premium-payout ratio}. In general, \( \eta(t) > \lambda(t) \), but for the simplicity we assume that the market is frictionless and \( \eta(t) = \lambda(t) \).

The wage earner receives a continuous non-negative income stream up to time \( \min\{T, \tau\} \), which means the income will be terminated by the death or retirement of the wage earner. Denote the labor income rate at time \( t \) by \( D(t) \), and it is driven by the following process:

\[
\frac{dD(t)}{D(t)} = (\eta_0 + \eta_1 r(t))dt + \sigma_{D_1}(t)\sigma_B(t)(\lambda_r dt + dW_r(t)) \\
+ \sigma_{D_2}(t)(m\sqrt{L(t)}dt + dW_S(t)).
\]

Moreover, in our model, there is a co-integration relationship between the increasing rate of labor income and the interest rate \( r(t) \), which reflects the influence of interest rate on the labor income: labor income usually increases more frequently in booming periods (high interest rates) than in recessions (low interest rates).

The co-integration was firstly introduced by Granger [11](1981), and it reflects an equilibrium relation between non-stationary economic series. Granger got a Nobel prize because of his contribution to economics on co-integration. Chiu and Wong [4](2011) considered mean-variance portfolio selection problem in a financial market in which asset prices are co-integrated and derived the explicit solution. Yang and Wang [27](2013) found the co-integration relation between multi-country mortalities. Also Chiu and Wong [5](2013) used BSDE theory to study the continuous-time mean variance asset-liability management problem of co-integration risk assets and an uncontrollable random liability.

We suppose that the labor income is only sensitive to the risk \( \{W_r(t)\} \) and \( \{W_S(t)\} \), which ensures that it can be replicated by the tradable assets though the market is incomplete. Under this assumption, it can be valued as a tradable asset and the problem can be converted to the situation without labor income. However, if we introduce a new risk \( \{W_D(t)\} \) that also drives the labor income and is independent with \( \{W_r(t)\} \) and \( \{W_S(t)\} \), although this case is more realistic, we can not find a tradable asset in the market sensitive to this risk, so the labor income can not be replicated, meaning the value of it can be different,
which directly leads to the non-uniqueness of the optimal strategy. Thus in this paper we assume the labor income is only sensitive to the risk \{W_r(t)\} and \{W_S(t)\}.

2.3. The Optimization Problem. The wealth of the wage earner at time \(t\) is denoted by \(X(t)\) and the initial wealth at time \(t = 0\) is \(X_0\). The strategy at time \(t\) consists of the following four decisions: the consumption rate \(c(t) \geq 0\), the premium payment rate \(p(t) \geq 0\), the amount \(\pi_B(t)\) invested in the bond and the amount \(\pi_S(t)\) invested in the stock. We denote the strategy at time \(t\) by \(\alpha(t) = \{c(t), p(t), \pi_B(t), \pi_S(t)\}\). So the amount invested in the bank account is \(X(t) = \pi_B(t) - \pi_S(t)\) and

\[
dX(t) = X(t)r(t)dt + (D(t) - c(t) - p(t))dt + (\pi_B(t), \pi_S(t))\mathbf{A}(t)(\Lambda(t)dt + dW(t)).
\]

(2.11)

If the wage earner dies at time \(t\), then his/her total legacy \(I(t)\) is the wealth plus the insurance amount, i.e.,

\[
I(t) = X(t) + \frac{p(t)}{\lambda(t)}.
\]

(2.12)

An investment and consumption strategy \(\alpha(t)\) is said to be admissible if:

(i) \(\{\alpha(t)\}\) is progressively measurable w.r.t. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbf{P})\);

(ii) \(\mathbf{E}[\int^T_0 (\pi_B(t)\sigma_B(t) + \pi_S(t)\sigma_S(t)\sigma_B(t))^2 + \pi_S(t)^2L(t)dt] < +\infty\);

(iii) For any given initial data \((t_0, r_0, L(0), X_0) \in [0, T] \times (0, +\infty)^3\), the SDE(2.11) has a unique strong solution.

Let \(\mathcal{A}\) be the set of all admissible strategies. The wage earner needs to choose strategies to maximize his/her utility function, and the maximum expected utility is expressed by

\[
V(x) \triangleq \sup_{\alpha(t) \in \mathcal{A}} \mathbf{E}_0\left[ \int^T_0 e^{-\delta s}U(c(s))ds + \beta_1 e^{-\delta T}U(I(T))1_{\{\tau \leq T\}}
+ e^{-\delta T}U(X(T))1_{\{\tau > T\}} \right]
\]

(2.13)

\[
\Delta \sup_{\alpha(t) \in \mathcal{A}} \{J(t_0, r_0, L(0), D(0), X_0; \alpha)\},
\]

where \(U(x)\) is a utility function of the wage earner, \(\mathbf{E}_0[\cdot]\) means conditional expectation given the initial value \((t_0, r_0, L(0), D(0), X_0)\) at time 0. The positive constant \(\beta_1\) is the weight on the wage earner legacy when he dies before retirement. The positive constant \(\delta\) stands for the time preference. We aim at finding closed-forms of the optimal strategy \(\alpha^*(t_0, \cdot) = \{\alpha^*(t_0, s) : s \geq t_0\} \in \mathcal{A}\) and the value function \(V(x)\) such
that

\[ J(t_0, r_0, L(0), D(0), X_0; \alpha^*(t_0, \cdot)) = V(x). \quad (2.14) \]

In this paper we consider the CRRA utility function, i.e., \( U(x) = \frac{x^{1-\gamma}}{1-\gamma} \) with \( \gamma > 0 \) and \( \gamma \neq 1 \). Here \( \gamma \) stands for the coefficient of relative risk aversion.

### 3. Solution of the optimization problem

In this section we will use dynamic programming method to derive the HJB equation and from which we get the optimal feedback control. We firstly rewrite (2.13) in a dynamic programming form. For any strategy \( \alpha = (c, p, \pi_B, \pi_S) \), we define

\[
J(t, x, r, l, d; \alpha) \triangleq \mathbb{E}_{t,x,r,l,d} \left[ \int_t^{T \wedge \tau} e^{-\delta(s-t)} U(c(s)) ds + \beta_1 e^{-\delta(T-t)} U(I(T)) \mathbf{1}_{\tau \leq T} \right. \\
\left. + e^{-\delta(T-t)} U(X(T)) \mathbf{1}_{\tau > T} \right],
\]

(3.1)

where \( \mathbb{E}_{t,x,r,l,d}[\cdot] \) means conditional expectation given the initial value \( X(t) = x, r(t) = r, L(t) = l, D(t) = d \) at time \( t \). Define

\[
\tilde{V}(t, x, r, l, d) \triangleq \sup_{\alpha(t, x, r, l, d) \in \mathcal{A}(t)} \{ J(t, x, r, l, d; \alpha) \}.
\]

(3.2)

The definition of \( \mathcal{A}(t) \) is similar to \( \mathcal{A} \), i.e., the initial condition \( (X(0), r(0), L(0), D(0)) \) is replaced by \( (X(t), r(t), L(t), D(t)) \).

Since the death time \( \tau \) is independent of the filtration \( \{ \mathcal{F}_t \}_{t \in [0,T]} \), the following lemma tells us that the expectation with random time horizon \( T \wedge \tau \) can be changed into a deterministic planning horizon, see [28](2006) for details.

**Theorem 3.1.** The equation (3.1) can be rewritten as follows:

\[
J(t, x, r, l, d; \alpha) = \mathbb{E}_{t,x,r,l,d} \left[ \int_t^T F(s, t) e^{-\delta(s-t)} U(c(s)) ds \right. \\
\left. + \beta_1 \int_t^T \tilde{F}(s, t) e^{-\delta(s-t)} U(I(s)) ds \right. \\
\left. + F(T, t) e^{-\delta(T-t)} U(X(T)) \mathbf{1}_{\tau > t} \right],
\]

(3.3)

where \( F(s, t) \) and \( \tilde{F}(s, t) \) are given by (2.8) and (2.9), respectively.

The associated HJB equation with the control problem (3.2) is given by the following.
Theorem 3.2. The $\tilde{V}(t, x, r, l, d)$ satisfies the following HJB equation:

$$
\sup_{\alpha(t, x, r, l, d) \in \mathcal{A}(t)} \{ \tilde{V}_t - (\lambda(t) + \delta)\tilde{V} + \tilde{V}_x [r x + \pi_B(t) \sigma_B(t) \lambda_r + \pi_B(t) \sigma_B(t) \lambda_r + ml] - c(t) - p(t) + d \}$

$$
+ \tilde{V}_t(\phi - ar) + \tilde{V}_t(g - hl) + \frac{1}{2}\tilde{V}_{xx} [\pi_B(t) \sigma_B(t) + \pi_B(t) \sigma_B(t) \sigma_B(t)]^2 + \frac{1}{2}\tilde{V}_{ll} l^2 + \frac{1}{2}\tilde{V}_{rr} r^2 - \tilde{V}_{rr} r \sigma_r \pi_B(t) \sigma_B(t) + \pi_B(t) \sigma_B(t) \sigma_B(t) (t) \}

$$

+ \frac{1}{2} \tilde{V}_{dd} [\pi_D(t) \sigma_B(t)^2 + \pi_B(t) \sigma_D(t) \sigma_B(t) + \pi_B(t) \sigma_D(t) \sigma_B(t)] + \tilde{V}_{dd} [\pi_D(t) \sigma_B(t) + \pi_B(t) \sigma_B(t) \sigma_B(t)] \sigma_B(t) + \pi_B(t) \sigma_D(t) \sigma_B(t) \}

$$
+ \frac{c(t)^{1-\gamma}}{1-\gamma} + \beta \lambda(t) \frac{(x + \hat{p}(t))^{1-\gamma}}{1-\gamma} = 0.

(3.4)

Proof. The proof is simple and we omit it here. \qed

Before solving the associated HJB equation (3.4), we first use the idea of Munk and Sørensen [21](2010) to discount the future labor income stream and convert the problem to the situation without labor income. Because labor income is only sensitive to the risk $\{W_r(t)\}$ and $\{W_S(t)\}$, as we have just discussed above, we can replicate the labor income stream by cash, stock and bond although the market is incomplete. Since the discounted future labor income stream over the time period $[t, T]$, which also stands for the wage earner’s human capital, can be valued. Denote the value of the wage earner’s human capital at time $t$ by $H(t)$. In other words, we can suppose that the wage earner sells the remaining labor income stream (his/her human capital) for the amount $H(t)$.

Thanks to it, we can change the problem to the situation without labor income but with a financial wealth of $X(t) + H(t)$ at time $t$. The following proposition gives an explicit expression of $H(t)$.

Proposition 3.3. The human capital of the wage earner at time $t$, defined by

$$
H(t, r, l, d) \triangleq E_{t, r, l, d}[\int_t^T D(s) e^{-\int_t^s (r(u) + \lambda(u)) du} ds]
$$

equals to

$$
H(t, r, l, d) = d \int_t^T h(t, s)(B(t, s))^{1-n_1} ds,
$$

(3.5)
where $Q$ is the unique risk-neutral probability measure, under which the $\Lambda(t)dt + dW(t)$ is a Brownian motion, and the $h(t, s)$ is

$$h(t, s) = \exp\left\{ \int_t^s \left[ \eta_0 - \lambda(u) + \frac{1}{2}(\eta_1 - 1)^2 \sigma_r^2 f(u, s)^2 
- (\eta_1 - 1)\sigma_r f(u, s)\sigma_{D_1}(u)\sigma_B(u) \right] du 
- (\eta_1 - 1)\left[ \frac{\sigma_r^2}{2a^2}(s - t - f(t, s)) - \frac{\sigma_r^2 f(t, s)^2}{4a} \right] \right\}. \quad (3.6)$$

Proof. See Appendix. \qed

From (3.5) we find that the initial value $l = L(t)$ does not appear in the expression of $H(t, r, l, d)$ due to $\{\int_0^t m\sqrt{L(s)}ds + W_S(t)\}$ in (2.10) is a Brownian motion under $Q$. Thus under the risk-neutral measure $Q$, $L(t)$ has noting to do with the labor income, which will be shown in (6.2) below. So stochastic volatility has nothing to do with the human capital $H(t, r, l, d)$, and this seems to be very natural: no matter how sharp the market swings, the human capital only depends on the labor income and the interest rate.

As discussed, we have converted the problem to the situation without labor income but with a financial wealth of $X(t) + H(t)$ at time $t$. Let the value function without labor income be $V(t, x, r, l)$, then the value function of original problem with labor income is

$$\tilde{V}(t, x, r, l, d) = V(t, x + H(t, r, l, d), r, l).$$

So we first consider the problem without labor income. The wealth process becomes

$$dX(t) = X(t)r(t)dt - (c(t) + p(t))dt 
+ (\pi_B(t), \pi_S(t))A(t)(\Lambda(t)dt + dW(t)). \quad (3.7)$$

**Theorem 3.4.** The value function $V(t, x, r, l)$ satisfies the following HJB equation:

$$\sup_{\alpha(t, x, r, l) \in A(t)} \left\{ V_t - (\lambda(t) + \delta)V 
+ V_x^r [x + \pi_B(t)\sigma_B(t)\lambda_r + \pi_S(t)(\sigma_B(t)\sigma_S(t)\lambda_r + ml) - c(t) - p(t)] 
+ V_x(\phi - ar) + V_l(g - hl) + \frac{1}{2} V_{xx} [\pi_B(t)\sigma_B(t) + \pi_S(t)\sigma_B(t)\sigma_S(t)]^2 + \pi_B^2] 
+ \frac{1}{2} V_{ll}\sigma_l^2 + \frac{1}{2} V_{rr}\sigma_r^2 
- V_{ll}\sigma_r^2 \pi_B(t)\sigma_B(t) + \pi_S(t)\sigma_B(t)\sigma_S(t) 
+ V_{lx}\pi_T(t)\sigma_L \rho_{SL} 
+ c(t)^{1-\gamma} + \beta_1 \lambda(t) \frac{1}{1-\gamma} \left( x + \frac{\eta_0}{\lambda(t)} \right)^{1-\gamma} \right\} = 0. \quad (3.8)$$
Proof. The proof is standard and we omit it here.

Now we give the closed-forms of value function \( V(t, x, r, l) \) and the optimal feedback function \( \alpha^*(t, x, r, l) = (c^*(t, x, r, l), p^*(t, x, r, l), \pi_B^*(t, x, r, l), \pi_S^*(t, x, r, l)) \) without labor income.

**Theorem 3.5.** Suppose that \( \rho_{SL} = 1 \) and \( \gamma > \frac{m \sigma_L (2h + m \sigma_L)}{h^2 + m \sigma_L (2h + m \sigma_L)} \), then the value function \( V(t, x, r, l) \) is a solution to HJB equation (3.8) and is given by

\[
V(t, x, r, l) = \frac{x^{1-\gamma}}{1 - \gamma} K(t, r, l)^\gamma,
\]

where

\[
K(t, r, l) = e^{\phi_1(t,T)+\phi_2(t,T)r+\phi_3(t,T)l} \int_t^T e^{\phi_1(t,s)+\phi_2(t,s)r+\phi_3(t,s)l} A(s)ds, \tag{3.9}
\]

\[
A(s) = \lambda(s) \beta_1^\frac{1}{\gamma} + 1,
\]

\[
\phi_1(t, s) = \int_t^s \left\{ \phi \phi_2(u,s) + g \phi_3(u,s) + \frac{1}{2} \gamma \sigma_r^2 \phi_2(u,s)^2 \right\} du - \frac{\delta}{\gamma} (s - t),
\]

\[
\phi_2(t, s) = \left[ e^{-a(s-t)} - 1 \right] \frac{\gamma - 1}{a\gamma},
\]

\[
\phi_3(t, s) = \frac{\lambda_1 \lambda_2 \exp\{-\sqrt{\Delta_{\phi_3}}(s-t)\} - \lambda_1 \lambda_2}{\lambda_1 \exp\{-\sqrt{\Delta_{\phi_3}}(s-t)\} - \lambda_2},
\]

and

\[
\begin{cases}
\lambda_{1,2} = \frac{(1 - \gamma) m \sigma_L - \gamma h \pm \gamma \sqrt{\Delta_{\phi_3}}}{\gamma \sigma_L^2 [-\gamma + (\gamma - 1)]}, \\
\Delta_{\phi_3} = h^2 + \frac{1 - \gamma}{\gamma} m \sigma_L (2h + m \sigma_L). \tag{3.10}
\end{cases}
\]

The optimal feedback function \( \alpha^{**}(t, x, r, l) = (c^{**}(t, x, r, l), p^{**}(t, x, r, l), \pi_B^{**}(t, x, r, l), \pi_S^{**}(t, x, r, l)) \) without labor income is given by

\[
\begin{align*}
  c^{**}(t, x, r, l) &= \frac{x}{K(t, r, l)} \\
p^{**}(t, x, r, l) &= \lambda(t) \left[ \frac{x}{K(t, r, l)} (\beta_1)\frac{1}{\gamma} \right] \\
  \pi_S^{**}(t, x, r, l) &= \frac{m x}{\gamma} + \frac{\sigma_L x \frac{\partial k}{\partial t}}{K(t, r, l)} \\
  \pi_B^{**}(t, x, r, l) &= \frac{-m \sigma_S x}{\gamma} - \frac{\sigma_S \sigma_L x \frac{\partial k}{\partial t}}{K(t, r, l)} + \frac{\lambda \sigma}{\gamma \sigma_B} - \frac{\sigma_S \sigma_L x \frac{\partial k}{\partial l}}{K(t, r, l)}.
\end{align*}
\tag{3.11}
\]

Proof. See Appendix.
Now turning back to the situation with labor income, as we have just discussed, the value function \( \tilde{V}(t, x, r, l, d) \) is given by

\[
\tilde{V}(t, x, r, l, d) = V(t, x + H(t, r, l, d), r, l).
\]

The following theorem gives the closed-forms of the value function \( \tilde{V}(t, x, r, l, d) \) and the optimal feedback function \( \alpha^*(t, x, r, l, d) = (c^*(t, x, r, l, d), p^*(t, x, r, l, d), \pi_B^*(t, x, r, l, d), \pi_S^*(t, x, r, l, d)) \) as well as the optimal strategy \( \{\alpha^*(t, s) | s \geq t\} \) of the optimal problem (2.13)-(2.14).

**Theorem 3.6.** Suppose that \( \rho_{SL} = 1 \) and \( \gamma > \frac{m\sigma_r (2h + m\sigma_r)}{m^2 + m\sigma_L (2h + m\sigma_L)} \), then the value function \( \tilde{V}(t, x, r, l, d) \) is a solution of the associated HJB equation (3.4) and is given by

\[
\tilde{V}(t, x, r, l, d) = V(t, x + H(t, r, l, d), r, l) = \frac{(x + H(t, r, l, d))^{1-\gamma}}{1-\gamma} K(t, r, l)^\gamma,
\]

where \( H(t, r, l, d) \) and \( K(t, r, l) \) are given by (3.5) and (3.9), respectively. Moreover, the optimal feedback function \( \alpha^*(t, x, r, l, d) \) is given by

\[
\begin{align*}
    c^*(t, x, r, l, d) &= \frac{x + H(t, r, l, d)}{K(t, r, l)}, \\
p^*(t, x, r, l, d) &= \lambda(t) \left[ \frac{x + H(t, r, l, d)}{K(t, r, l)} \right]^{1/\gamma} - x, \\
\pi_S^*(t, x, r, l, d) &= \frac{m(x + H(t, r, l, d))}{\gamma K(t, r, l)} + \frac{\sigma_L (x + H(t, r, l, d)) \frac{\partial K}{\partial r}}{K(t, r, l)}, \\
\pi_B^*(t, x, r, l, d) &= -\frac{m\sigma_L (x + H(t, r, l, d))}{\gamma K(t, r, l)} - \frac{\sigma_L (x + H(t, r, l, d)) \frac{\partial K}{\partial r}}{K(t, r, l)}, \\
\pi_{SB}(t, x, r, l, d) &= \lambda_r (x + H(t, r, l, d)) + \frac{\sigma_r [K(t, r, l)] \frac{\partial H}{\partial r}}{\gamma \sigma_B} - \frac{H(t, r, l, d) \sigma_D 2 \sigma_S}{l} K(t, r, l), \\
\pi_{SB}(t, x, r, l, d) &= -\frac{\sigma_D 1 H(t, r, l, d) + H(t, r, l, d) \sigma_D 2 \sigma_S}{\sqrt{l}} - \frac{H(t, r, l, d) \sigma_D 1 \sigma_S^2}{\sqrt{l}}.
\end{align*}
\]

The optimal strategy of the optimal problem (2.13) and (2.14) is \( \alpha^* = \{\alpha^*(t, s) : s \geq t\} \), \( \alpha^*(t, s) = \alpha^*(t, X^*(s), r, l, d) \) for \( s \geq t \), where the optimal state/wealth process \( \{X^*(s) : s \geq t\} \) is a unique solution of the
following SDE:
\[
\begin{align*}
    dX^*(s) &= X^*(s)r(s)ds - (c^*(t, X^*(s), r, l, d) + p^*(t, X^*(s), r, l, d))ds \\
    &\quad + (\pi_B^*(t, X^*(s), r, l, d), \pi_S^*((t, X^*(s), r, l, d))A(s)(\Lambda(s)ds + dW(s)),
\end{align*}
\]
\[
X^*(t) = x, r(t) = r, L(t) = l, D(t) = d.
\tag{3.14}
\]

**Proof.** From (3.4) we can derive the first order condition:
\[
\begin{align*}
    c^*(t, x, r, l, d) &= eV_x x, \\
    p^*(t, x, r, l, d) &= \lambda(t)[\left(\frac{V_x}{\beta_1}\right)^{\frac{1}{\gamma}} - x], \\
    \pi_S^*(t, x, r, l, d) &= -mV_x - \frac{\tilde{V}_x \sigma_L \rho_{SL}}{V_{xx}} - \frac{d\tilde{V}_x \sigma_D_1 \sigma_S \sigma_B^2}{V_{xx} l}, \\
    \pi_B^*(t, x, r, l, d) &= mV_x - \frac{\tilde{V}_x \sigma_L \rho_{SL}}{V_{xx}} - \frac{\tilde{V}_x \lambda_r}{V_{xx}} + \frac{\sigma_r \tilde{V}_x}{\sigma_B \tilde{V}_{xx} - d\tilde{V}_x \sigma_D_1} \\
    &\quad + \frac{d\tilde{V}_x \sigma_S \sigma_D_2}{V_{xx} l} - \frac{d\tilde{V}_x \sigma_S \sigma_B^2 \sigma_D_1}{V_{xx} l}.
\end{align*}
\tag{3.15}
\]

Substituting (3.12) into the first order condition, we can get the expression (3.13), and the optimal strategy \(\alpha^*(t, s)\) is the composition of the feedback function \(\alpha(t, \cdot, r, l, d)\) defined by (3.13) and the optimal state/wealth process \(\{X^*(s) : s \geq t\}\) defined by (3.14).

**Consumption and insurance.** The optimal consumption and insurance strategies with labor income are the same as that of no labor income if we replace the financial wealth process \(X(t)\) by total wealth process (financial wealth plus human capital) \(X(t) + H(t)\). Comparing our result with Pirvu and Zhang[24], we find that for optimal consumption and insurance strategies, the stochastic volatility has influence on them only by \(K(t, r, l)\), and the stochastic interest rate has influence on them by \(K(t, r, l)\) and \(H(t, r, l, d)\). What’s more, the influence is same for both the optimal consumption and the optimal insurance strategy. Namely, the legacy/consumption ratio is a constant and independent of the market:
\[
\frac{I^*(t)}{c^*(t)} = \beta_1^{\frac{1}{2}}.
\tag{3.16}
\]
We interpret this natural economical phenomenon as follows: when people invest more money in life insurance, they can consume more and save less because the legacy will be supplemented by the insurance payment.
Moreover, we can get another obviously conclusion: a wage earner with higher weight on legacy or a more risk averse wage earner will consume less for leaving more legacy. Specially, the optimal legacy equals the optimal consumption when $\beta_1 = 1$.

Divided by $x$ on both sides of the expression of $\pi^*_S(t)$ and $\pi^*_B(t)$, it can be inferred that both $\frac{\pi^*_S(t)}{x}$ and $\frac{\pi^*_B(t)}{x}$, which stands for the optimal investment weights, do not depend on current financial wealth and labor income separately but only on the wealth/income ratio.

Investment in the stock. The first term of $\pi^*_S(t)$ is a speculative demand, which is only correlated with the total wealth; The second term is the correction term for volatility risk; The third term is the correction terms for stock-like labor income risk, we can see that if labor income is independent of the risk of stock, that is, $\sigma_{D_2} = 0$, then this term will disappear. Thus the correlation between labor income and the risk of stock has a negative effect on the amount of investment in the stock. This can be explained by that if there is correlation between labor income and the risk of stock, then there will be stock-like income risk. Therefore the labor income can take place of the stock in some sense and the wage earner will invest less in the stock. The last term is also the correction terms for stock-like labor income risk, and it does exist because the stock is also driven by the risk of interest rate.

Investment in the bond. The first term of $\pi^*_B(t)$ is a speculative demand too. The second term is the correction term for volatility risk and we can see that the volatility risk has opposite influence on the investment in stock and bond, i.e., if the wage earner put more money in the stock, he(or she) will have less money to invest in the bond. If the wage earner is risk aversion, when the volatility is increasing(market shocks), he(or she) tends to invest more in the bond market and reduce stock holdings. The third term is a hedge against interest rate risk and it can be observed that if the interest rate risk increases, the amount invested in the bond will increase due to the zero-coupon bond is designed to hedge the interest rate risk. Because there is a co-integration relationship between the labor income and the interest rate($\eta_1 \neq 0$), the forth term is needed to against the interest rate risk of human capital, namely, the influence of interest rate on human capital. This is why there is $\frac{\partial H}{\partial r}$ in this term, which is the interest rate sensitivity of the
human capital. The last three terms are necessary to hedge bond-like income risk and stock-like income risk since there are $W_r$ and $W_S$ in (2.10) involving labor income.

4. Sensitivity analysis

We will use the Monte Carlo Methods to give a sensitivity analysis in this section. We try to explore the influence of parameters in our model on the optimal strategies. Unless otherwise stated, the values of the parameters are presented in the following table. We will focus on

<table>
<thead>
<tr>
<th>Table 1. values of the parameters in our model</th>
</tr>
</thead>
<tbody>
<tr>
<td>parameters in interest rate model</td>
</tr>
<tr>
<td>a</td>
</tr>
<tr>
<td>b</td>
</tr>
<tr>
<td>$\sigma_r$</td>
</tr>
<tr>
<td>$\lambda_r$</td>
</tr>
<tr>
<td>$r_0$</td>
</tr>
<tr>
<td>parameters in labor income model</td>
</tr>
<tr>
<td>$\eta_0$</td>
</tr>
<tr>
<td>$\eta_1$</td>
</tr>
<tr>
<td>$\sigma_D_1$</td>
</tr>
<tr>
<td>$\sigma_D_2$</td>
</tr>
<tr>
<td>$D_0$</td>
</tr>
<tr>
<td>parameters in stock model</td>
</tr>
<tr>
<td>$\sigma_S$</td>
</tr>
<tr>
<td>m</td>
</tr>
<tr>
<td>$\sigma_L$</td>
</tr>
<tr>
<td>$h$</td>
</tr>
<tr>
<td>$g$</td>
</tr>
<tr>
<td>$L_0$</td>
</tr>
<tr>
<td>$\rho_{SL}$</td>
</tr>
<tr>
<td>other parameters</td>
</tr>
<tr>
<td>$t$</td>
</tr>
<tr>
<td>$T$</td>
</tr>
<tr>
<td>$X_0$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>$\lambda(t)$</td>
</tr>
<tr>
<td>$\beta_1$</td>
</tr>
<tr>
<td>$\delta$</td>
</tr>
</tbody>
</table>

the process of strategy when $t$ changes from 35 to 65.
Figure 1 illustrates the movement of optimal consumption and investment strategy when time changes. As we can see, the consumption and investment proportions are smooth in the middle of the time horizon and almost do not change except around the initial time and retirement. The proportions of investment in life insurance and bond are negative. It means a short sale for bond and also stands for buying pension annuities rather than life insurance. When the wage earner approaches his(or her) retirement, he(or she) will put more in the life insurance, because the mortality rate is larger. Meanwhile the investment in bond becomes less, which reflects a fact: the risk of long-term interest rate is more difficult to hedge than short-term interest rate. Therefore it needs less investment in bond to hedge the risk of interest rate at the time around retirement.

The difference between Figure 1 and Figure 2 is that the risk aversion in Figure 1 is $\gamma = 2$, while the risk aversion in Figure 2 is $\gamma = 0.5$. It means the wage earner in Figure 1 is risk averse but he(or she) is risk preference in Figure 2. Comparing these two figures, we find that the proportions of consumption and life insurance are almost the same, i.e., the consumption and life insurance are necessary in the life and can not affected by personal emotion(risk aversion or risk preference) easily compared to investment in financial market. Besides, higher risk
aversion requires to hedge more risk in the market. The risk of interest rate can be hedged by bond, and the risk of stock can not be hedged. Thus in this case, there is a higher investment proportion in the stock since the wage earner has more risk preference. We can see that he(or she) invests about 350% to 400% of his wealth in the stock and it is a high leveraged investment. Moreover, the investment in the bond is less because the wage earner needs to hedge less risk.

Figure 3 is the situation when \( m = 10 \). Since \( L(t) \) is positive, it stands for a larger return of stock and consequently the wage earner will put more money in the stock market. We also find that the other three parts are almost the same as Figure 1, which means the amount of cash is negative and the wage earner loans from bank to invest stock. While Figure 4(\( \sigma_L = 1 \)) means the volatility of market is larger, we see that the wage earner has to adjust his(or her) optimal strategy frequently compared to Figure 1. Since we assume the labor income is stock-like in some sense, larger volatility also drives labor income with sudden shocks. As a result, the consumption and investment on life insurance will have dramatic changes.
Figure 3. $m = 10$

Figure 4. $\sigma_L = 1$

Figure 5 to Figure 7 is the process of the ratio of wealth to total wealth, which reflects the proportion’s change of human capital. The human capital’s proportion is largest at the initial time. In Figure 5, it reaches 30% at the age of 36 and becomes less as time goes. Finally it becomes zero when the wage earner retires. Figure 6 is the situation $\eta_1 = 5$, in which the coefficient of co-integration relation is larger, thus the labor income is more positive closely related with interest rate. It
leads a smaller ratio of wealth to total wealth and the human capital’s proportion deceases more slowly than Figure 5. This phenomenon can be explained as that labor income has a more positive closely relation with interest rate, so it increases faster and the human capital has a larger proportion. Figure 7 is the opposite: the labor income has a negative relation with interest rate (which will almost not happen) and
it decrease as interest rate increases. So the human capital is less than Figure 5. We can see the influence of co-integration between interest rate and the labor income through these figure.

5. Conclusion

In this paper, we consider an optimal investment, consumption and life insurance problem. We assume a wage earner with CRRA utility function receives labor income before retirement and he(or she) can invest in zero-coupon bond, stock, cash, and life insurance. The interest rate is stochastic and follows Vasicek model. Moreover, the risky asset(stock) has stochastic volatility and follows Heston’s model. The labor income stream is also stochastic and has a co-integrated relationship with interest rate. To hedge the risk of accidental death, the wage earner can invest in life insurance. The death time is $\tau$ and the death rate is $\lambda(t)$, and we turn the random time horizon problem to a deterministic time horizon problem. To solve this problem, we first consider the situation without labor income and use dynamic programming to derive the HJB equation and get the explicit solution. Then we prove that the problem with labor income can be turned into the problem without labor income if we replace finical wealth by finical wealth plus the wage earner’s human capital. Finally, we get the explicit solution of
the original problem. Thus the closed-forms of the value function, the optimal feedback function and the optimal strategies are derived. We also illustrate the optimal strategy over the life-cycle and analyze the influence of parameters on the optimal strategies.

6. Appendix

6.1. Proof of Proposition 3.3.

Proof. Since $d\tilde{W}_r(t) = dW_r(t) + \lambda_r dt$, we rewrite (2.2) as

$$dr(t) = (ab + \sigma_r \lambda_r - ar(t))dt - \sigma_r d\tilde{W}_r(t)$$

where $\phi = ab + \sigma_r \lambda_r$. Solving this equation, we have

$$r(u) = e^{-a(u-t)}r(t) + \frac{\phi}{a} (1 - e^{-a(u-t)}) - \int_t^u \sigma_r e^{-a(u-v)}d\tilde{W}_r(v).$$

This implies that

$$\int_t^s r(u)du = (r(t) - \frac{\phi}{a^2})(1 - e^{a(s-t)}) + \frac{\phi}{a} (s - t) - \frac{\sigma_r}{a} \int_t^s (1 - e^{-a(s-u)})d\tilde{W}_r(u).$$

Similarly, solving the following SDE by a technique used in Karatzas and Shreve[14](cf. problem 5.6.1 in[14]):

$$\frac{dD(t)}{D(t)} = (\eta_0 + \eta_1 r(t))dt + \sigma_{D_1}(t)\sigma_B(t)(\lambda_r dt + dW_r(t))$$

$$+ \sigma_{D_2}(t)(m\sqrt{L(t)}dt + dW_S(t))$$

$$= (\eta_0 + \eta_1 r(t))dt + \sigma_{D_1}(t)\sigma_B(t)d\tilde{W}_r(t) + \sigma_{D_2}(t)d\tilde{W}_S(t),$$

we have

$$D(s) = D(t) \exp\left\{ \int_t^s [\eta_0 + \eta_1 r(u) - \frac{1}{2}(\sigma_{D_1}^2(u)\sigma_B^2(u) + \sigma_{D_2}^2(u))]du 
+ \int_t^s \sigma_{D_1}(u)\sigma_B(u)d\tilde{W}_r(u) + \int_t^s \sigma_{D_2}(u)d\tilde{W}_S(u) $$

$$\right\}. \quad (6.3)$$
Combining (6.1) and (6.3), we obtain

\[
D(s) = e^{\int_t^s (r(u) + \lambda(u)) du} D(t) \exp \left\{ \int_t^s \left[ \eta_0 - \frac{1}{2} \left( \sigma^2_{D_1}(u) \sigma^2_B(u) + \sigma^2_{D_2}(u) \right) \right. \\
- \lambda(u) \left. \right] du + (\eta_1 - 1) \left[ (r(t) - \frac{\phi}{a}) f(t, s) + \frac{\phi}{a} (s - t) \right] \\
+ \int_t^s \left[ \sigma_{D_1}(u) \sigma_B(u) - (\eta_1 - 1) \sigma_r f(u, s) \right] d\bar{W}_r(u) \\
+ \int_t^s \sigma_{D_2}(u) d\bar{W}_S(u) \right\}.
\]

Taking expectation at both sides of the last equation, we have

\[
E^Q_t, r, l, d[D(s) e^{-\int_t^s (r(u) + \lambda(u)) du}] = D(t) e^{(\eta_1 - 1) f(t, s) r(t)} e^{F(t, s)},
\]

where

\[
F(t, s) = \int_t^s \left[ \eta_0 - \lambda(u) + \frac{1}{2} (\eta_1 - 1)^2 \sigma^2_r f(u, s)^2 \\
- (\eta_1 - 1) \sigma_r f(u, s) \sigma_{D_1}(u) \sigma_B(u) \right] du \\
+ (\eta_1 - 1) \frac{\phi}{a} [(s - t) - f(t, s)].
\]

Thanks to (2.5), we again represent \( r(t) \) by zero-coupon bond \( B(t, s) \) as follows:

\[
E^Q_t, r, l, d[D(s) e^{-\int_t^s (r(u) + \lambda(u)) du}] = D(t) e^{F(t, s) - (\eta_1 - 1) c(t, s)} (B(t, s))^{1-m}.
\]

Letting \( h(t, s) \triangleq e^{F(t, s) - (\eta_1 - 1) c(t, s)} \) and integrating over \( s \) at both sides of the last equation, we end the proof.

6.2. Proof of Theorem 3.5.

Proof. The first order condition is given by

\[
\begin{align*}
\pi^*_S(t, x, r, l) &= -\frac{m V_x}{V_{xx}} - \frac{V_{xx} \sigma_L \rho_{SL} \sigma_r}{V_{xx}} , \\
\pi^*_B(t, x, r, l) &= \frac{m \sigma_S V_x}{V_{xx}} + \frac{\sigma_S V_{xx} \sigma_L \rho_{SL} \sigma_r}{V_{xx}} - \frac{V_x \lambda_r}{\sigma_B V_{xx}} + \frac{\sigma_r V_{xx}}{\sigma_B V_{xx}} .
\end{align*}
\]

(6.4)
Substituting (6.4) into the HJB equation (3.8), we have
\[ 0 = V_t - V(\lambda(t) + \delta) + V_x(r + \lambda(t))x + V_{xx} \frac{\gamma}{1 - \gamma}(\lambda(t)\beta_1^2 + 1) \]
\[ + V_x(\phi - ar) + V_t(g - hl) + \frac{1}{2}\sigma_r^2V_{rr} + \frac{1}{2}\sigma_L^2V_{ll} \]
\[ - \frac{(V_{rr}\sigma_r - V_x\lambda_r)^2}{2V_{xx}} - l(V_xm + V_{lx}\sigma_{L\rho SL})^2 \] (6.5)
with boundary
\[ V(T, x, r, l) = \frac{x^{1-\gamma}}{1 - \gamma}. \]

So we guess that \( V(t, x, r, l) \) is of the following form:
\[ V(t, x, r, l) = \frac{x^{1-\gamma}}{1 - \gamma} G(t, r, l) \]
with boundary
\[ G(T, r, l) = 1. \]

Substituting this form into (6.5), we get
\[ 0 = G^{\frac{\gamma}{2\gamma}}(\lambda(t)\beta_1^2 + 1) + G_t + G[(1 - \gamma)r - \gamma\lambda(t) - \delta] \]
\[ + G_x(\phi - ar) + G_t(g - hl) + \frac{1}{2}\sigma_r^2G_{rr} + \frac{1}{2}\sigma_L^2G_{ll} \]
\[ + \frac{1 - \gamma}{2G\gamma}(G_x\sigma_r - G\lambda_r)^2 + \frac{(1 - \gamma)}{2G\gamma}(Gm + G_{L\rho SL})^2. \] (6.6)
Suppose the solution of (6.6) is of the following form:
\[ G(t, r, l) = K(t, r, l)^\gamma \] (6.7)
with boundary
\[ K(T, r, l) = 1. \]

Substituting (6.7) into (6.6), after somewhat complicated simplification, we have
\[ 0 = (\lambda(t)\beta_1^2 + 1) + K_t + \left[ \frac{(1 - \gamma)}{\gamma}r - \lambda(t) - \frac{\delta}{\gamma} + \frac{(1 - \gamma)(\lambda^2 + lm^2)}{2\gamma^2} \right]K_t \]
\[ + \left( \phi - ar - \frac{(1 - \gamma)\sigma_r\lambda_r}{\gamma} \right)K_r + \left( g - hl + \frac{m\sigma_{L\rho SL}(1 - \gamma)}{\gamma} \right)K_t \]
\[ + \frac{1}{2}\sigma_r^2K_{rr} + \frac{1}{2}\sigma_L^2K_{ll} + \frac{1}{2}\sigma_L^2(\gamma - 1)(1 - \rho_{SL}^2) \frac{K_t^2}{K}. \] (6.8)

The (6.8) is a nonlinear second-order partial differential equation which is difficult to solve. Since we add consumption and insurance to Merton’s
original problem, the term $A(t) \triangleq (\lambda(t)\beta_1^1 + 1)$ exists compared to the classical investment problem considered by Merton[19, 20](1969,1971). If there is no insurance but only consumption, then the term will be constant instead of a function $A(t)$(cf.[3]). Inspired by Guan and Liang [9, 10](2014) and Liu[18](2007), we first solve a linear situation, that is, without term $A(t)$, and get a solution $\tilde{K}(t)$, and then we represent $K(t, r, l)$ by $\tilde{K}(t, r, l)$.

Assume $\tilde{K}(t, r, l)$ is the solution of the following linear second-order partial differential equation:

$$0 = \tilde{K}_t + \frac{(1 - \gamma)}{\gamma} r \lambda(t) - \frac{\delta}{\gamma} + \frac{(1 - \gamma)(\lambda_r^2 + l \gamma^2)}{2 \gamma^2} \tilde{K}$$

$$+ \left(\phi - ar - \frac{(1 - \gamma)\sigma_r \lambda_r}{\gamma}\right) \tilde{K}_r + \left(g - hl + \frac{m \sigma_L \rho SL (1 - \gamma)}{\gamma}\right) \tilde{K}_l$$

$$+ \frac{1}{2} \sigma_r^2 \tilde{K}_{rr} + \frac{1}{2} l \sigma_L^2 \tilde{K}_{ll} + \frac{1}{2} l \sigma_L^2 (\gamma - 1)(1 - \rho^2) \frac{\tilde{K}_l^2}{\tilde{K}}$$

(6.9)

with the boundary

$$\tilde{K}(T, r, l) = 1.$$

Then we can assume that $\tilde{K}(t, r, l)$ has the following form:

$$\tilde{K}(t, r, l) = \exp\left\{\phi_1(t) + \phi_2(t)r + \phi_3(t)l\right\}$$

(6.10)

with boundary

$$\phi_1(T) = \phi_2(T) = \phi_3(T) = 0.$$

Substituting (6.10) into (6.9) and letting the coefficient of $r$ and $l$ to be zero, we get three equations:

$$\begin{cases}
\phi_1'(t) + \phi_2(t) + g \phi_3(t) + \frac{1}{2} \gamma \sigma_r^2 \phi_2(t)^2 \\
+ \frac{1}{2} \gamma (\sigma_r \phi_2(t) - \frac{\lambda_r}{\gamma})^2 - \lambda(t) - \frac{\delta}{\gamma} = 0,
\phi_1(T) = 0,
\end{cases}$$

(6.11)

$$\begin{cases}
\phi_2'(t) - a \phi_2(t) + \frac{1}{\gamma} = 0, \\
\phi_2(T) = 0
\end{cases}$$

(6.12)
and

\[
\begin{cases}
\phi_3'(t) - h\phi_3(t) + \frac{1}{2} \gamma \sigma_L^2 \phi_3(t)^2 \\
+ \frac{1 - \gamma}{2} \left[ \frac{m}{\gamma} + \sigma_L \rho_{SL} \phi_3(t) \right]^2 = 0, \\
\phi_3(T) = 0.
\end{cases}
\]

(6.13)

Solving the equation (6.12), we get the expression of \(\phi_2(t)\):

\[
\phi_2(t) = \left[ e^{-a(T-t)} - 1 \right] \frac{\gamma - 1}{\sigma^2}.
\]

(6.14)

To solve (6.13), rewrite it by

\[
\phi_3'(t) = \left( -\frac{1}{2} \gamma \sigma_L^2 + \frac{\gamma - 1}{2} \sigma_{L} \rho_{SL} \right) \phi_3(t)^2 \\
- \left( \frac{1 - \gamma}{\gamma} m \sigma_L \rho_{SL} - h \right) \phi_3(t) + \frac{(\gamma - 1) m^2}{2 \gamma^2} = 0,
\]

(6.15)

Let \(\Delta_{\phi_3}\) denote the discriminant of the following quadratic equation:

\[
\left( -\frac{1}{2} \gamma \sigma_L^2 + \frac{\gamma - 1}{2} \sigma_L \rho_{SL} \right) \phi_3(t)^2 \\
- \left( \frac{1 - \gamma}{\gamma} m \sigma_L \rho_{SL} - h \right) \phi_3(t) + \frac{(\gamma - 1) m^2}{2 \gamma^2} = 0,
\]

then

\[
\Delta_{\phi_3} = h^2 + \frac{\gamma - 1}{\gamma} m \sigma_L (2 \rho_{SL} h + m \sigma_L).
\]

Under \(\Delta_{\phi_3} > 0\), that is,

\[
\gamma > \frac{m \sigma_L (2 \rho_{SL} h + m \sigma_L)}{h^2 + m \sigma_L (2 \rho_{SL} h + m \sigma_L)},
\]

we assume that \(\lambda_1\) and \(\lambda_2\) are two real roots of (6.15) which can be expressed by

\[
\lambda_{1,2} = \frac{(1 - \gamma) m \sigma_L \rho_{SL} - \gamma h \pm \gamma \sqrt{\Delta_{\phi_3}}}{\gamma \sigma_L^2 \left[ -\gamma + (\gamma - 1) \rho_{SL}^2 \right]}.
\]

Thus

\[
\frac{1}{\lambda_1 - \lambda_2} \int_t^T \left( \frac{1}{\phi_3(t) - \lambda_1} - \frac{1}{\phi_3(t) - \lambda_2} \right) d\phi_3(t)
= \left( -\frac{1}{2} \gamma \sigma_L^2 + \frac{\gamma - 1}{2} \sigma_L^2 \rho_{SL} \right) (T - t).
\]

Solving the last equation with boundary \(\rho_3(T) = 0\), we derive

\[
\phi_3(t) = \frac{\lambda_1 \lambda_2 \exp \left\{ -\sqrt{\Delta_{\phi_3}} (T - t) \right\} - \lambda_1 \lambda_2}{\lambda_1 \exp \left\{ -\sqrt{\Delta_{\phi_3}} (T - t) \right\} - \lambda_2}.
\]
We derive $\phi_1(t)$ from (6.11):

$$
\phi_1(t) = \int_t^T \{ \phi\phi_2(s) + g\phi_3(s) + \frac{1}{2}\gamma^2\sigma_r^2\phi_2^2(s) + \frac{1}{2}(\sigma_r\phi_2(s) - \frac{\lambda_r}{\gamma})^2 - \lambda(s) \} ds - \frac{\delta}{\gamma}(T - t).
$$

Now we have the expression of

$$
\tilde{K}(t, r, l) = \exp\{\phi_1(t) + \phi_2(t)r + \phi_3(t)l\}.
$$

Our next step is to represent $K(t, r, l)$ by $\tilde{K}(t, r, l)$. We let $T$ in $\phi_i(t)$ ($i = 1, 2, 3$) to be a variable, that is, assume

$$
\begin{align*}
\phi_2(t, s) &= \left[ e^{-(s-t)} - 1 \right] \frac{\gamma - 1}{a\gamma}, \\
\phi_3(t, s) &= \frac{\lambda_1\lambda_2 \exp\{-\sqrt{\Delta\phi_3(s-t)}\} - \lambda_1\lambda_2}{\lambda_1 \exp\{-\sqrt{\Delta\phi_3(s-t)}\} - \lambda_2}, \\
\phi_1(t, s) &= \int_s^t \{ \phi\phi_2(u,s) + g\phi_3(u,s) + \frac{1}{2}\gamma^2\sigma_r^2\phi_2^2(u,s) + \frac{1}{2}(\sigma_r\phi_2(u,s) - \frac{\lambda_r}{\gamma})^2 - \lambda(u) \} du - \frac{\delta}{\gamma}(s - t).
\end{align*}
$$

Then

$$
\tilde{K}(t, r, l) = K(t, T, r, l) = \exp\{\phi_1(t, T) + \phi_2(t, T)r + \phi_3(t, T)l\}
$$

is the solution to (6.9). Denote (6.9) by

$$
\tilde{K}_t + \nabla \tilde{K} = 0, \quad (6.16)
$$

where $\nabla$ is a differential operator defined by

$$
\nabla K \triangleq \left[ \frac{(1 - \gamma)(1 - \rho^2)\phi}{\gamma} - \lambda(t) - \frac{\delta}{\gamma} + \frac{(1 - \gamma)(\lambda_r + \lambda_m^2)}{2\gamma^2} \right] K + \phi - ar - \frac{(1 - \gamma)\sigma_r}{\gamma} \lambda_r K_t + (g - hl + \frac{m^2 \rho S_L l (1 - \gamma)}{2}) K_l + \frac{1}{2}\sigma_r^2 K_{rr} + \frac{1}{2}\sigma_l^2 K_{ll} + \frac{1}{2}\sigma_l^2 (\gamma - 1)(1 - \rho^2) \frac{K_l^2}{K}
$$

(6.17)

for any function $K$. Thus $K(t, r, l)$ satisfies

$$
A(t) + K_t + \nabla K = 0.
$$

Here the operator $\nabla$ is a non-linear operator because of the existing of $\frac{K_l^2}{K}$. To get the explicit solution, we let the operator become linear, that is, $\nabla f + \nabla g = \nabla (f + g)$ for any function $f$ and $g$. Thus we assume
\( \rho_{SL} = 1 \), then coefficient of \( \frac{K^2}{\mathcal{K}} \) becomes 0 and the operator \( \nabla \) becomes linear.

Although letting \( \rho_{SL} = 1 \) is not realistic in some sense because it means the risks of stock and the volatility are the same, we have to make this assumption to get the explicit solution. All the research before have not solve the problem similar to this. That is, there is not only the non-linear term \( A(t) \) in the equation, but also the operator \( \nabla \) is non-linear. We believe there is no explicit solution under this situation using the method existed. The most similar research is [3](2013). Unfortunately, we find some mistake after check their proof carefully. They let the operator \( \nabla \) to be linear although it does be non-linear.

Thus we let \( \rho_{SL} = 1 \) and

\[
K(t, r, l) \triangleq \tilde{K}(t, T, r, l) + \int_t^T \tilde{K}(t, s, r, l) A(s) ds.
\]

We find

\[
K(t, r, l)_t = \tilde{K}(t, T, r, l)_t + \int_t^T \tilde{K}(t, s, r, l)_t A(s) ds - \tilde{K}(t, t, r, l) A(t)
\] (6.18)

and

\[
\nabla K(t, r, l) = \nabla \tilde{K}(t, T, r, l) + \int_t^T \nabla \tilde{K}(t, s, r, l) \cdot A(s) ds.
\] (6.19)

Combining (6.18) and (6.19), and noticing that \( \tilde{K}(t, t, r, l) = 1 \), we finally have

\[
K(t, r, l)_t + \nabla K(t, r, l) = -A(t).
\]

Thus \( K(t, r, l) \) is the solution to (6.8) and substituting it into (6.4), we obtain (3.11).

**Acknowledgements.** The authors would like to thank the participants of the seminar on Stochastic Analysis, Insurance Mathematics and Mathematical Finance at the Department of Mathematical Sciences, Tsinghua University for their feedbacks and useful conversations. This work was supported by the National Natural Science Foundation of China(Grant No.11471183).
REFERENCES


