Optimal financing and dividend control of the insurance company with proportional reinsurance policy

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Received July 2007; received in revised form November 2007; accepted 25 November 2007

Abstract

We consider the optimal control problem of the insurance company with proportional reinsurance policy. The management of the company controls the reinsurance rate, dividends payout as well as the equity issuance processes to maximize the expected present value of the dividends minus the equity issuance until the time of bankruptcy. This is the first time that the financing process in an insurance model has been considered, which is more realistic. To find the solution of the mixed singular-regular control problem, we firstly construct two categories of suboptimal models, one is the classical model without equity issuance, the other never goes bankrupt by equity issuance. Then we identify the value functions and the optimal strategies corresponding to the suboptimal models depending on the relationships between the coefficients.

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MSC: primary 91B28; 91B16; secondary 60H05; 60H10

Keywords: Optimal dividends control; Optimal financing control; Proportional reinsurance; HJB equation; Transaction cost; Regular-singular control

1. Introduction

In this paper, we consider an insurance company in which the dividends payout, equity issuance and the risk exposure are controlled by the management. We assume that the company can only reduce its risk exposure by proportional reinsurance policy for simplicity. Moreover, there exists a minimal reserve requirement. We associate the value of the company with the expected present value of the dividends payout minus the equity issuance. The transaction cost is also taken into consideration in the model.


Unfortunately, there are very few results concerned with the equity issuance of the insurance company. In the real financial market, equity issuance is an important approach for the insurance company to earn profit and reduce risk. Harrison and Taksar (1983) consider the optimal control problem with a lower and an upper reflecting barrier. Their work provides a good idea to solve this kind of problems. Sethi and Taksar (2002) recently considered the model for the company that can control its risk exposure by issuing new equity as well as by paying dividends. Løkka and Zervos (2006) use the above approaches to study the problem with the possibility of bankruptcy. By these innovative ideas, we solve the optimal control problem of the insurance company
effectively. The equity issuance could be considered as the absorbing or reflecting boundary of the reserve process. It turns out that the optimal control problem is associated with different optimal strategies depending on the relationships between the coefficients. Firstly, we study the solutions of two models: one is the classical diffusion control model as in Højgaard and Taksar (1999), the other stands for the model with equity issuance to meet the minimal reserve requirement, so it never goes bankrupt. Then we prove that the value functions and the optimal strategies are the solutions of the two control problems.

The paper is organized as follows: In Section 2, we establish the control model of the insurance company. In Section 3, we give some preliminary mathematical results related to the problem. In Section 4, we construct solutions of two categories of suboptimal models. One is the classical model without equity issuance, the other never goes bankrupt by equity issuance. In Section 5, we prove that the value functions and the optimal strategies are the solutions of the two control problems, respectively. We give the conclusion of this paper in the last section.

2. Control model of the insurance company with proportional reinsurance policy

We consider an insurance company with proportional reinsurance policy. In this case, the company’s management can accommodate the profit and the risk by choosing the amount of equity issuance, dividends payout and the reinsurance rate.

In this paper we will consider the linear Brownian motion model. In this model, if there are no equity issuance and dividends payout to control the risk, then the liquid reserves of the company evolve according to the following stochastic differential equations,

$$dR_t = \mu a(t) dt + \sigma a(t) dW_t,$$

where $W_t$ is a standard Brownian motion, $1 - a(t) \in [0,1]$ is the proportional reinsurance rate.

To give a mathematical foundation of the optimization problem, we fixed a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ and $\{W_t, t \geq 0\}$ is a standard Brownian motion on this probability space, where $\mathcal{F}_t$ represents the information available at time $t$ and any decision is made based on this information. In our model, we denote $L_t$ as the cumulative amount of dividends paid from time 0 to time $t$, $G_t$ denotes the total amount raised by issuing equity from time 0 to time $t$. We assume that the processes $\{L_t, t \geq 0\}$ and $\{G_t, t \geq 0\}$ are $\{\mathcal{F}_t, t \geq 0\}$-adapted, increasing and right-continuous with left limits.

A control policy $\pi$ is described by the three stochastic processes $\{a_\pi, L^\pi, G^\pi\}$. Given a control policy $\pi$, we assume that the liquid reserves of the insurance company are modeled by the following equations,

$$dR^\pi_t = \mu a_\pi(t) dt + \sigma a_\pi(t) dW_t - dL^\pi_t + dG^\pi_t, \quad R^\pi_0 = x.$$  (2.2)

On the one hand, we suppose the liquid reserves of the insurance company must satisfy some minimal reserve requirement. In this case, we assume that the company needs to keep its reserves above $m$, $m > 0$ is the minimal reserve requirement. The company is considered bankrupt as soon as the reserves fall below $m$. We define the time of bankruptcy by $\tau = \inf\{t \geq 0 : R^\pi_t < m\}$. $\tau$ is an $\mathcal{F}_t$-stopping time. If the company issues some equity, then the time of bankruptcy could be infinite.

On the other hand, we also consider the transaction cost in the model. To simplify the problem, we consider the proportional transaction cost. If the company pays $l$ as dividends, then the shareholders can get $\beta_1 l$, $\beta_1 < 1$. In the meanwhile, the shareholders must pay out $\beta_2 g$, $\beta_2 > 1$ to meet the amount of $g$ as new equity of the company.

So the management of the insurance company should maximize the expected present value of the dividends payout minus the equity issuance by control policy $\pi$.

Our main objective is to maximize the expected present value of the dividends payout minus the equity issuance before bankruptcy,

$$J(x, \pi) = E \left[ \left( \int_0^\tau e^{-ct} \beta_1 dL^\pi_s - \int_0^\tau e^{-cs} \beta_2 dG^\pi_s \right) \right].$$  (2.3)

$$V(x) = \sup_{\pi \in \Pi} J(x, \pi),$$  (2.4)

where $\Pi = \{\pi\}$ denotes the set of all admissible policies, $c$ denotes the discount rate. In addition, the minimal reserve requirement asks for $V(x) = 0$, for $\forall x < m$. We aim at finding some conditions on coefficients $(\mu, \sigma, c, \beta_1, \beta_2)$ of Eqs. (2.2) and (2.3) such that $V(x)$ satisfies a type of HJB equations. Then we get $V(x)$ and the optimal strategy $\pi^*$ associated with $V(x)$.

3. Preliminaries of the problem

In this section we will give two lemmas before proving the main results in Sections 4 and 5.

**Lemma 3.1.** Let $x_0 = \frac{(x_1 - m)^2}{\mu^2}$ and $m < x_0$, then there exists a unique $x_1 = x_1(c, \mu, \sigma, \beta_1) > x_0$ satisfying the following equation in $x_1$,

$$\left( \frac{m^{\gamma}}{\gamma x_0^{\gamma-1}} + \frac{1}{d_+} - \frac{x_0}{\gamma} \right) e^{d_-(x_0-x_1)} - \left( \frac{m^{\gamma}}{\gamma x_0^{\gamma-1}} + \frac{1}{d_+} - \frac{x_0}{\gamma} \right) e^{d_+(x_0-x_1)} = 0,$$  (3.1)

where $d_- = -\frac{\mu^2(2\sigma^2\gamma + \mu^2)}{\sigma^2}$, $d_+ = \frac{\mu^2(2\sigma^2\gamma + \mu^2)}{\sigma^2}$, $\gamma = \frac{c}{\mu^2 + \frac{\sigma^2}{2\gamma}}$.

**Proof.** Clearly, $d_- < 0$, $d_+ > 0$. Denote the left-hand side of (3.1) by $k(x_1)$. Differentiating $k(x_1)$ with respect to $x_1$, we have

$$k'(x_1) = \frac{\beta_1 d_+ d_-}{d_+ - d_-} \left[ \left( \frac{m^{\gamma}}{\gamma x_0^{\gamma-1}} + \frac{1}{d_+} - \frac{x_0}{\gamma} \right) e^{d_-(x_0-x_1)} - \left( \frac{m^{\gamma}}{\gamma x_0^{\gamma-1}} + \frac{1}{d_+} - \frac{x_0}{\gamma} \right) e^{d_+(x_0-x_1)} \right].$$
Since \( x_0 \leq x_1, \ e^{d_-(x_0-x_1)} \geq 1 \) and \( e^{d_+(x_0-x_1)} \leq 1 \). Using \( m < x_0 \),
\[
\frac{m'}{y'x_0^{y-1}} - \frac{x_0}{y} < 0,
\]
\[
\frac{m'}{y'x_0^{y-1}} + \frac{1 - x_0}{d_+} > 0.
\]
We claim that \( k'(x_1) < 0 \).

The claim is trivial if \( \frac{m'}{y'x_0^{y-1}} + \frac{1 - x_0}{d_+} > 0 \). It suffices to prove the claim for
\[
\frac{m'}{y'x_0^{y-1}} + \frac{1 - x_0}{d_+} < 0.
\]
Because of
\[
\left| \frac{m'}{y'x_0^{y-1}} + \frac{1 - x_0}{d_+} \right| > \left| \frac{m'}{y'x_0^{y-1}} + \frac{1 - x_0}{d_+} \right|,
\]
k'(x_1) is negative. Therefore \( k'(x_1) \) is always negative, \( k(x_1) \) reaches its maximum at \( x_0 \) on \([x_0, +\infty)\). We deduce from \( k(x_1) \rightarrow -\infty \), as \( x_1 \rightarrow \infty \) and

\[
k(x_0) = \left( \frac{m'}{y'x_0^{y-1}} - \frac{x_0}{y} \right) \beta_1 + \left( \frac{d_+ - d_-}{d_+} \right) \beta_1 \\
= \left( \frac{m'}{y'x_0^{y-1}} - \frac{x_0}{y} + \frac{\mu}{c} \right) \beta_1 \\
= \left( \frac{m'}{y'x_0^{y-1}} + \frac{\mu}{2c} \right) \beta_1 > 0
\]
that Eq. (3.1) has a unique solution \( x_{1a} \) and \( x_{1a} > x_0 \). □

**Lemma 3.2.** Let \( x_0 = \frac{(1-\nu)\nu^2}{\mu} \) and \( m < x_0 \), there exists a unique \( x_{1aa} = x_{1aa}(c, \mu, \nu, \beta_1, \beta_2) > x_0 \) satisfying the following equation in \( x_1 \),
\[
m'_{x_0^{y-1}} - \frac{\beta_1 d_+ - \beta_1 d_-}{d_+ - d_-} e^{d_-(x_0-x_1)} - \frac{\beta_1 d_+ - \beta_1 d_-}{d_+ - d_-} e^{d_+(x_0-x_1)} = \beta_2,
\]
where \( d_-, d_+ \) and \( y \) are the same as in Lemma 3.1.

**Proof.** Denote the left-hand side of Eq. (3.2) by \( h(x_1) \). Differentiating \( h(x_1) \) with respect to \( x_1 \), we get
\[
h'(x_1) = \frac{\beta_1 d_+ - \beta_1 d_-}{d_+ - d_-} e^{d_-(x_0-x_1)} - e^{d_+(x_0-x_1)}.
\]
Since \( x_1 \geq x_0, h'(x_1) > 0, h(x_1) \) is a strictly increasing function of \( x_1 \). It reaches its minimum at \( x_0 \) and
\[
h(x_0) = \frac{m'_{x_0^{y-1}}}{y'x_0^{y-1}} \beta_1 < \beta_2
\]
because of \( m < x_0 \). Thus, using \( h(x_1) \rightarrow +\infty \) as \( x_1 \rightarrow \infty \), we know that Eq. (3.2) has a unique solution \( x_{1aa} > x_0 \). □

4. Two categories of suboptimal solutions

In this section, we consider two categories of suboptimal control problems. \( \pi_p = \{a_p, L_p, 0\} \in \Pi \) stands for the control process for the company in which equity issuance is not permitted. The optimal return function associated with \( \pi_p \) is defined by
\[
V_p(x) = \sup_{\pi_p \in \Pi} J(x, \pi_p), \quad \text{for } x \geq m.
\]
\( \pi_q = \{a_q, L_q, G_q\} \in \Pi \) is the control process for the company with issuance procedures, i.e., it will never go bankrupt. The optimal return function associated with \( \pi_q \) is
\[
V_q(x) = \sup_{\pi_q \in \Pi} J(x, \pi_q), \quad \text{for } x \geq m.
\]
Since \( \pi_p, \pi_q \in \Pi, V(x) \geq \max\{V_p(x), V_q(x)\} \) via (2.4). The two suboptimal solutions will play an important role in constructing the overall optimal control solutions. In the next two subsections, we will establish solutions for each category of the control models.

4.1. The solution to the problem without equity issuance

In this subsection, we consider the company without equity issuance. Our objective is to maximize the expected discounted dividends payout.

**Theorem 4.1.** Assume that the coefficients are such that \( x_{1a} \leq x_{1aa}, \) where \( x_{1a} \) and \( x_{1aa} \) are defined in Lemmas 3.1 and 3.2. Then the function \( f \) defined by
\[
f = \begin{cases} 
 f_1(x) = C_1(x_{1a}) x_{1a} + C_2(x_{1a}) & m \leq x < x_{1a}, \\
 f_2(x) = C_3(x_{1a}) e^{d_-(x-x_{1a})} + C_4(x_{1a}) e^{d_+(x-x_{1a})}, & x_0 \leq x < x_{1aa}, \\
 f_3(x) = \beta_1 (x - x_{1a}) + f_2(x_{1a}), & x \geq x_{1a} 
\end{cases}
\]
satisfies the following HJB equation and the boundary conditions: for \( x \geq m \),
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 \sigma^2 f''(x) + \mu f'(x) - cf(x) \right],
\]
\[
\beta_1 - f'(x) = 0,
\]
\[
f(m) = 0.
\]
Moreover, for \( x \geq m \),
\[
f'(x) \leq \beta_2,
\]
where \( d_-, d_+ \), \( y \), \( x_0 \) are the same as in Lemma 3.1, \( C_1(x_{1a}), C_2(x_{1a}), C_3(x_{1a}) \) and \( C_4(x_{1a}) \) are defined by
\[
C_1(x_{1a}) = \frac{1}{y_{x_0}^{y-1}} \left( \frac{\beta_1 d_+}{d_+ - d_-} e^{d_-(x_{1a}-x_0)} - \frac{\beta_1 d_-}{d_+ - d_-} e^{d_+(x_{1a}-x_0)} \right),
\]
and \( C_2(x_{1a}), C_3(x_{1a}), C_4(x_{1a}) \) are similarly defined.
Continuity of the function \( f' \) and \( f'' \) at point \( x_1 \) implies that

\[
C_3(x_1) = \frac{\beta_1 d_+}{e^{d_- (d_+ - d_-)}},
\]

\[
C_4(x_1) = \frac{\beta_1 d_-}{e^{d_- (d_+ - d_-)}},
\]

Also, since the solutions \( f \) and \( f' \) are continuous at \( x_0 \),

\[
f_1(x_0) = f_2(x_0), \quad f_1' = f_2'(x_0),
\]

which imply that

\[
C_1(x_1) = \frac{1}{\gamma x_0^{\gamma - 1}} \left( \frac{\beta_1 d_+}{e^{d_- (x_0 - x_1)}} - \frac{\beta_1 d_-}{e^{d_- (x_0 - x_1)}} \right) > 0, \quad (4.14)
\]

\[
C_2(x_1) = \frac{1}{\gamma x_0^{\gamma - 1}} \left( \frac{\beta_1 d_+}{e^{d_- (x_0 - x_1)}} - \frac{\beta_1 d_-}{e^{d_- (x_0 - x_1)}} \right) > 0. \quad (4.15)
\]

Putting (4.9), (4.14) and (4.15) together, we have

\[
x_1 = x_{1*} \quad \text{and} \quad f(m) = 0
\]

by Lemma 3.1.

The problem remaining is to prove that the solution \( f \) satisfies (4.2)–(4.4). Noticing that \( \beta_1 < 1, \beta_2 > 1 \), it suffices to prove the following conditions:

\[
f' \leq \beta_1, \quad f'' \geq \beta_2, \quad \text{for} \quad m \leq x < x_0.
\]

\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right] \leq 0,
\]

\[
f' \geq \beta_1, \quad f'' \leq \beta_2, \quad \text{for} \quad x_0 \leq x < x_{1*}.
\]

\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right] \leq 0, \quad \text{for} \quad x \geq x_{1*}.
\]

The proof is as follows.

For \( x \geq x_{1*} \),

\[
f_3 = \beta_1 (x - x_{1*}) + f_2(x_{1*})
\]

\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right] \leq 0,
\]

\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right] \leq 0
\]

because of \( x \geq x_{1*} \) and \( \mu \beta_1 - cf_2(x_{1*}) = 0 \).

Using the same way as in Højgaard and Taksar (1999), it is easy to prove that

\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right] \leq 0
\]

holds for \( x_0 \leq x \leq x_{1*} \) (see Højgaard and Taksar (1999) for details).

Since

\[
f_1'(x) = C_1(x_{1*}) \gamma (y - 1) x^{\gamma - 2} \leq 0,
\]

\[
f_2'(x) = \frac{\beta_1 d_+}{d_+ - d_-} e^{d_-(x-x_{1*})} - e^{d_+(x-x_{1*})} \leq 0,
\]

\( f'(x) \) is a decreasing function on \([m, x_{1*}])\), its maximum and minimum are reached at \( m \) and \( x_{1*} \) respectively. Moreover, \( f'(x_{1*}) = \beta_1 \) and \( f'(x) \geq \beta_1 \) are obvious. So it suffices to
prove that $f'(m) \leq \beta_2$. Since
\[ f'(m) = C_1(x_{1a}) y m^{y-1} = \frac{m^{y-1}}{x_0^{y-1}} \left( \frac{\beta_1 d_+}{d_+ - d_-} e^{c(x_0 - x_{1a})} - \frac{\beta_1 d_-}{d_+ - d_-} e^{d_+(x_0 - x_{1a})} \right) \]
x_{1a} \leq x_{1ss} and $h(x)$ is a strictly increasing function, we have $f'(m) = h(x_{1a}) \leq h(x_{1ss}) = \beta_2$ by Lemma 3.2. Thus the proof has been done. □

4.2. The solution to the problem that never goes bankrupt

In this subsection we consider the problem that aims at maximizing the expected discounted dividends payout minus the expected discounted equity issuance over all reinsurance, dividends payout and equity issuance strategies. This kind of insurance companies will never go bankrupt.

**Theorem 4.2.** Assume that the coefficients are such that $x_{1a} \geq x_{1ss}, x_{1a}, x_{1ss}$ are defined in Lemmas 3.1 and 3.2. Then the function $g$ defined by
\[
g(x) = \begin{cases} 
  g_1(x) = C_1(x_{1ss}) x^y + C_2(x_{1ss}), & m \leq x < x_0, \\
  g_2(x) = C_3(x_{1ss}) e^{d_+ x} + C_4(x_{1ss}) e^{d_- x}, & x_0 \leq x \leq x_{1ss}, \\
  g_3(x) = \beta_1 (x - x_{1ss}) + g_2(x_{1ss}), & x \geq x_{1ss}
\end{cases}
\]
satisfies the following HJB equation and the boundary conditions:
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 g''(x) + \mu a g'(x) - cg(x) \right],
\]
\[
\beta_1 - g'(x), g'(x) - \beta_2 = 0
\]
\[
g(m) \geq 0,
\]
where $\gamma, x_0, d_-$ and $d_+$ are the same as in Theorem 4.1, $C_1(x_{1ss}), C_2(x_{1ss}), C_3(x_{1ss})$ and $C_4(x_{1ss})$ are defined as in Theorem 4.1 and by replacing $x_{1a}$ with $x_{1ss}$.

**Proof.** Considering the time value of money leads us to the conclusion that it is optimal to postpone the new equity issuance as long as possible. We conjecture that it is optimal to issue equity only when the reserves become $m$. One point is that: if we issue equity at the reserve $u$ prior to $m$, $g'(u) = \beta_2$ and $g'(x)$ is a decreasing function, we conjecture that $g''(u)$ must be 0 to meet the requirement $g' \leq \beta_2$. Unfortunately, it is not compatible with $a \in [0,1]$.

Using the same way as in Section 4.1, this strategy is associated with a solution to the HJB equation (4.17). It should be characterized by
\[
g'(m) = \beta_2,
\]
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 g''(x) + \mu a g'(x) - cg(x) \right] = 0,
\]
for $m \leq x < x_1$,
\[
g'(x) = \beta_1, \quad \text{for } x \geq x_1,
\]
\[
g'(x) = 0, \quad \text{for } x \geq x_1
\]
where $x_1$ is an unknown variable and $x_1$ will be specified later.

Doing the same procedures as in Section 4.1, we can prove that the solution $g(x)$ of Eq. (4.19)–(4.22) has the same form as $f(x)$ in (4.9)–(4.15), and $x_1$ is the solution of the following equation
\[
\frac{1}{\gamma x_0^{y-1}} \left( \frac{\beta_1 d_+}{d_+ - d_-} e^{c(x_0 - x_{1b})} - \frac{\beta_1 d_-}{d_+ - d_-} e^{d_+(x_0 - x_{1b})} \right) \gamma m^{y-1} = \beta_2.
\]
By Lemma 3.2, we have
\[
x_1 = x_{1ss} \quad \text{and} \quad x_{1ss} \geq x_0.
\]

The problem remaining is to prove that the solution $g$ satisfies the conditions mentioned in Theorem 4.2. For this, it suffices to prove the following
\[
g' \geq \beta_1, g' \leq \beta_2, \quad \text{for } m \leq x < x_0,
\]
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 g'' + \mu a g' - cg \right] \leq 0,
\]
\[
g' \geq \beta_1, g' \leq \beta_2, \quad \text{for } x_0 \leq x < x_{1ss},
\]
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 g'' + \mu a g' - cg \right] \leq 0, \quad \text{for } x \geq x_{1ss}.
\]
Using the similar approach as in Section 4.1, we can prove the above claims.

Now, we will check the boundary conditions. Based on our assumption, reinvestment is not compulsory and bankruptcy at $m$ is an option, so we need only to prove $g(m) \geq 0$, i.e.,
\[
g(m) = \left( \frac{m^y}{\gamma x_0^{y-1}} + \frac{1}{d_+ - d_-} \left( \frac{\beta_1 d_+}{d_+ - d_-} e^{c(x_0 - x_{1ss})} - \frac{\beta_1 d_-}{d_+ - d_-} e^{d_+(x_0 - x_{1ss})} \right) \right) \gamma m^{y-1} \geq 0
\]
By the proof of Lemma 3.1, $k(x_1) = \left( \frac{m^y}{\gamma x_0^{y-1}} + \frac{1}{d_+ - d_-} \left( \frac{\beta_1 d_+}{d_+ - d_-} e^{c(x_0 - x_{1b})} - \frac{\beta_1 d_-}{d_+ - d_-} e^{d_+(x_0 - x_{1b})} \right) \right)$ is a decreasing function of $x_1$ and $k(x_1) = 0$. Since $x_{1a} \geq x_{1ss}$, $g(m) = k(x_{1ss}) \geq k(x_1) = 0$, the inequality (4.23) holds. Thus we complete the proof. □

5. The solution of the optimal control problem

**Theorem 5.1.** Let $W(x)$ satisfy the following HJB equation and boundary condition,
\[
\max_{a \in [0,1]} \left[ \frac{1}{2} \sigma^2 a^2 W'' + \mu a W' - cW \right],
\]
\[ \beta_1 - W', W' - \beta_2 \geq 0, \quad \text{for } x \geq m, \]  
(5.1)

\[ \max\{-W(m), W'(m) - \beta_2\} = 0. \]  
(5.2)

Then \( W(x) \geq J(x, \pi) \) for any admissible policy \( \pi \).

**Proof.** Fix a policy \( \pi \), let \( A = \{s : L^\pi_s - L^\pi_{s-} \neq L^\pi_s\} \), \( A' = \{s : G^\pi_{s-} \neq G^\pi_s\} \), \( L_i^\pi = \sum_{s \in A_i, s \leq t} (L^\pi_s - L^\pi_{s-}) \) be the discontinuous part of \( L^\pi \) and \( L_{\pi i}^\pi = L^\pi_i - \hat{L}^\pi_{\pi i} \) be the continuous part of \( L^\pi \). Similarly, \( \hat{G} \) and \( G \) stand for discontinuous and continuous parts of \( G^\pi \). Let \( \tau \) be the first time that the corresponding reserves \( R_i \) defined by Eq. (2.2) hit \((-\infty, m)\). Then, by generalized Itô formula,

\[ e^{-c(\tau \wedge t)} W(R_{\pi t}^\pi) = W(x) + \int_0^{\tau \wedge t} e^{-cs} \mathcal{L} W(R_s^\pi) ds + \int_0^{\tau \wedge t} a_\pi \sigma e^{-cs} W'(R_s^\pi) dW_s \]

\[ - \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) dL_s^\pi + \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) dG_s^\pi \]

\[ + \sum_{s \in A_i} \int_0^{\tau \wedge t} e^{-cs} [W(R_s^\pi) - W(R_{s-}^\pi)] d\tau_s \]

\[ = W(x) + \int_0^{\tau \wedge t} e^{-cs} \mathcal{L} W(R_s^\pi) ds + \int_0^{\tau \wedge t} a_\pi \sigma e^{-cs} W'(R_s^\pi) dW_s \]

\[ - \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) dL_s^\pi + \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) dG_s^\pi \]

\[ + \sum_{s \in A_i} \int_0^{\tau \wedge t} e^{-cs} [W(R_s^\pi) - W(R_{s-}^\pi)], \]  
(5.3)

where

\[ \mathcal{L} = \frac{1}{2} a_\pi \sigma^2 \frac{d^2}{dx^2} + a_\mu \frac{d}{dx} - c. \]

In view of (5.1), the second term on the right-hand side is non-positive. Since \( \beta_1 \leq W'(R_s^\pi) \leq \beta_2 \), the third term is a square integrable martingale. Taking expectations on both sides of Eq. (5.3),

\[ \mathbb{E} \left\{ e^{-c(\tau \wedge t)} W(R_{\pi t}^\pi) \right\} \]

\[ \leq W(x) - \mathbb{E} \left\{ \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) d\hat{L}_s^\pi \right\} \]

\[ + \mathbb{E} \left\{ \int_0^{\tau \wedge t} e^{-cs} W'(R_s^\pi) d\hat{G}_s^\pi \right\} \]

\[ + \mathbb{E} \left\{ \sum_{s \in A_j} \int_0^{\tau \wedge t} e^{-cs} [W(R_s^\pi) - W(R_{s-}^\pi)] d\tau_s \right\}. \]  
(5.4)

Since \( \beta_1 \leq W'(R_s^\pi) \leq \beta_2 \),

\[ W(R_s^\pi) - W(R_{s-}^\pi) \leq \beta_2 (G_{s-}^\pi - G_s^\pi) - \beta_1 (L_{s-}^\pi - L_s^\pi), \]

which, together with (5.4), implies that

\[ \mathbb{E} \left\{ e^{-c(\tau \wedge t)} W(R_{\pi t}^\pi) \right\} + \mathbb{E} \left\{ \int_0^{\tau \wedge t} e^{-cs} \beta_1 dL_s^\pi \right\} \]

\[ - \mathbb{E} \left\{ \int_0^{\tau \wedge t} e^{-cs} \beta_2 dG_s^\pi \right\} \leq W(x). \]  
(5.5)

By the definition of \( \tau \) and \( \beta_1 \leq W'(x) \leq \beta_2 \) for \( x \geq m \), it is easy to prove that

\[ \liminf_{t \to \infty} e^{-c(\tau \wedge t)} W(R_{\pi t}^\pi) = e^{-c} W(m) I_{[\tau < \infty]} \]

\[ + \liminf_{t \to \infty} e^{-c} W(R_t) I_{[\tau = \infty]} \geq e^{-c} W(m) I_{[\tau < \infty]} \geq 0. \]  
(5.6)

So, we deduce from (5.5) and (5.6) that

\[ J(x, \pi) = \mathbb{E} \left\{ \left\{ \int_0^{\tau \wedge t} e^{-cs} \beta_1 dL_s^\pi - \int_0^{\tau \wedge t} e^{-cs} \beta_2 dG_s^\pi \right\} \right\} \]

\[ \leq W(x). \]  
(5.7)

Thus the proof has been done. \( \square \)

Let

\[ a(x) = \begin{cases} \frac{\mu x}{\sigma^2(1-\gamma)}, & x < x_0, \\ 1, & x \geq x_0 \end{cases} \]

where \( \gamma = \frac{c}{\sigma^2(1-\gamma)} \) and \( x_0 = \left( \frac{(1-\gamma)\sigma^2}{\mu} \right)^{1/2} \). The main results of this paper are the following.

**Theorem 5.2.** If \( x_{1s} \leq x_{1ss} \), then \( V(x) = f(x) = V_p(x) \). The optimal policy \( \pi^* = (a_\pi^*, L^\pi^*, G^\pi^*) \) satisfies the following

\[ R^\pi^* = x + \int_0^t \mu a(R_s^\pi) ds + \int_0^t \sigma a(R_s^\pi) dW_s - L_t^\pi, \]

\[ R^\pi^* \leq x_{1s}, \]

\[ \int_0^\infty I_{[t < x_{1s}]} (t)dL_t^\pi = 0, \]

\[ G_t^\pi = 0, \]  
(5.8)

where \( a_\pi^*(t) = a(R_t^\pi) \), \( x_{1s} \) is given in Lemma 3.1, \( V(x) \) and \( f(x) \) are defined by (2.4) and (4.1), respectively. \( V_p(x) \) is defined in Section 4.

If \( x_{1s} \geq x_{1ss} \), then \( V(x) = g(x) = V_q(x) \). The optimal policy \( \pi^{**} = (a_\pi^{**}, L^\pi^{**}, G^\pi^{**}) \) satisfies the following

\[ R_t^{**} = x + \int_0^t \mu a(R_s^{**}) ds + \int_0^t \sigma a(R_s^{**}) dW_s - L_t^{**}, \]

\[ - L_t^{**} + G_t^{**}, \quad m \leq R_t^{**} \leq x_{1ss}, \]

\[ \int_0^\infty I_{[t < x_{1ss}]} (t)dL_t^{**} = 0, \]

\[ \int_0^\infty I_{[t > x_{1ss}]} (t)dG_t^{**} = 0, \]  
(5.9)

where \( a_\pi^{**}(t) = a(R_t^{**}), x_{1ss} \) is given in Lemma 3.2, \( V(x) \) and \( g(x) \) are defined by (2.4) and (4.16), respectively. \( V_q(x) \) is defined in Section 4.
Remark that using Theorem 3.1 in Lions and Sznitman (1984) the processes \((R^\pi, L^\pi, G^\pi)\) and \((R^{\pi*}, L^{\pi*}, G^{\pi*})\) are uniquely determined by Eqs. (5.8) and (5.9).

**Proof.** If \(x_{1s} \leq x_{1ss}\), then the function \(f(x)\) satisfies the HJB equation and boundary conditions (4.2)–(4.4). It is not hard to see that \(f(x)\) also satisfies conditions (5.1) and (5.2) in Theorem 5.1. So \(f(x) \geq V(x) \geq V_p(x)\) by Theorem 5.1.

Next, we will prove \(V(x) = f(x)\) corresponding to \(\pi^*\). Applying generalized Itô formula, we deduce from (4.5) and (4.8) that \(\mathcal{L} f(R^\pi_{t,x}) = 0\) and

\[
e^{-c(t \wedge \tau)} f(R^\pi_{t,x}) = f(x) + \int_0^{t \wedge \tau} e^{-cs} \mathcal{L} f(R^\pi_s) ds
\]

where \(\tau^* = \inf\{t \geq 0 : R^\pi_{t,x} < m\}\).

Since \(\lim_{t \to \infty} e^{-c(t \wedge \tau)} f(R^\pi_{t,x}) = e^{-ct} f(m) = 0\), by taking expectations at both sides of (5.10), we get

\[
\mathbb{E} \left[ \limsup_{t \to \infty} \left( \int_0^{t \wedge \tau} e^{-cs} \beta_1 dL^\pi_s \right) \right] = J(x, \pi^*).
\]

So \(f(x)\) is the return function corresponding to \(\pi^*\), and \(x \leq V_p(x)\). Using the results \(f(x) \geq V(x) \geq V_p(x)\), we have \(f(x) = V(x) = V_p(x)\) under the circumstance \(x_{1s} \leq x_{1ss}\).

If \(x_{1s} \geq x_{1ss}\), then \(g(x)\) defined in (4.16) satisfies the HJB equation and boundary conditions (4.17) and (4.18). Thus \(g(x)\) satisfies conditions (5.1) and (5.2) in Theorem 5.1. So \(g(x) \geq V(x) \geq V_q(x)\) by Theorem 5.1.

Next, we will prove \(V(x) = g(x)\) corresponding to \(\pi^{**}\). By generalized Itô formula, we deduce from (4.10) and (4.20) that \(\mathcal{L} g(R^\pi_{t,x}) = 0\) and

\[
e^{-c(t \wedge \tau^{**})} g(R^{\pi^{**}}_{t,x}) = g(x) + \int_0^{t \wedge \tau^{**}} e^{-cs} \mathcal{L} g(R^\pi_s) ds
\]

\[
+ \int_0^{t \wedge \tau^{**}} a(R^{\pi^{**}}_s)\sigma e^{-cs} g'(R^{\pi^{**}}_s) dW_s
\]

\[
- \int_0^{t \wedge \tau^{**}} \beta_1 e^{-cs} g(R^{\pi^{**}}_s) dL^\pi_s
\]

\[
+ \int_0^{t \wedge \tau^{**}} \beta_2 e^{-cs} g(R^{\pi^{**}}_s) dG^{\pi^{**}}_s
\]

\[
+ \sum_{s \in \Lambda \cup \Lambda', s \leq t \wedge \tau^{**}} e^{-cs}[g(R^{\pi^{**}}_{s-}) - g(R^{\pi^{**}}_{s-})]
\]

\[
- g'(R^{\pi^{**}}_{s-}) (R^{\pi^{**}}_{s-} - R^{\pi^{**}}_{s-})
\]

\[
\quad = g(x) - \int_0^{t \wedge \tau^{**}} e^{-cs} \beta_1 dL^\pi_s + \int_0^{t \wedge \tau^{**}} e^{-cs} \beta_2 dG^{\pi^{**}}_s
\]

\[
+ \int_0^{t \wedge \tau^{**}} a(R^{\pi^{**}}_s)\sigma e^{-cs} g'(R^{\pi^{**}}_s) dW_s,
\]

(5.11)

where \(\tau^{**} = \inf\{t \geq 0 : R^{\pi^{**}}_{t,x} < m\}\). Since \(\liminf_{t \to \infty} e^{-c(t \wedge \tau^{**})} g(R^{\pi^{**}}_{t,x}) = \lim_{t \to \infty} e^{-ct} g(R^{\pi^{**}}_{t,x}) = 0\), (see Hojgaard and Taksar (2001) for details), by taking mathematical expectations at both sides of (5.11), we get

\[
g(x) = \mathbb{E} \left[ \left( \int_0^{t \wedge \tau^{**}} e^{-cs} \beta_1 dL^\pi_s \right) \right]
\]

\[
- \left( \int_0^{t \wedge \tau^{**}} e^{-cs} \beta_2 dG^{\pi^{**}}_s \right)
\]

\[
= J(x, \pi^{**}).
\]

So \(g(x)\) is the return function corresponding to \(\pi^{**}\), \(g(x) \leq V_p(x)\). Using the results \(g(x) \geq V(x) \geq V_q(x)\), we have \(g(x) = V(x) = V_q(x)\) under the circumstance \(x_{1s} \geq x_{1ss}\). The proof has been done.

\[
\square
\]

6. Conclusion

In this paper, we consider the optimal control problem of the insurance company with proportional reinsurance policy. The management of the company controls the reinsurance rate, dividends payout and the equity issuance to maximize the expected present value of the dividends payout minus the equity issuance before bankruptcy. To be more realistic, we require the minimal reserve restrictions and also take transaction cost into consideration. This is the first time that the financing process in an insurance model has been considered, and we finally find that it acts as absorbing or reflecting boundary of the reserve process. To find the solution of the mixed singular-regular control problem, we construct two categories of suboptimal models, one is the classical model without equity issuance, the other never goes bankrupt by equity issuance. At last, we identify the value function and the optimal strategy with the corresponding solution in either category of suboptimal models, depending on the relationships between the coefficients.

Acknowledgements

We are very grateful to the referee for bringing the paper Harrison and Taksar (1983) to our attention. We also express our deep thanks to the referee and the editor for their careful reading of the manuscript, correction of errors, improvement of the written language and valuable suggestions which made the main results of this paper much better. This work is supported by Project 10771114 of NSFC, Project 20060003001 of SRFDP, and SRF for ROCS, SEM. We would like to thank the institutions for their generous financial support.
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