Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs

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**Abstract**

We consider the optimal financing and dividend control problem of the insurance company with fixed and proportional transaction costs. The management of the company controls the reinsurance rate, dividends payout as well as the equity issuance process to maximize the expected present value of the dividends payout minus the equity issuance until the time of bankruptcy. This is the first time that the financing process in an insurance model with two kinds of transaction costs, which come from real financial market has been considered. We solve the mixed classical-impulse control problem by constructing two categories of suboptimal models, one is the classical model without equity issuance, the other never goes bankrupt by equity issuance.

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1. Introduction

In this paper, we consider an insurance company with the fixed and the proportional transaction costs. In this model the management controls the dividends payout, equity issuance and the risk exposure by proportional reinsurance policy. We study the value of the company via the expected present value of the dividends payout minus the equity issuance. This is a mixed classical-impulse control on diffusion models. Diffusion models for companies that control their risk exposure by means of dividends payout have attracted a lot of interests recently. We refer the readers to He and Liang (2007) and the references therein. Optimizing dividends payout is a classical problem starting from the early work of Borch (1969, 1967) and Gerber (1972). For some applications of control theory in insurance mathematics, see, Harrison and Taksar (1983), Højgaard and Taksar (1998a,b), Martin-Löf (1983), Asmussen and Taksar (1997) and Cadenillas et al. (2006). A survey can be found in Taksar (2000).

However, there are very few results concerned with the equity issuance of the insurance company. In the real financial market, equity issuance is an important approach for the insurance company to earn profit and reduce risk. Harrison and Taksar (1983) consider the optimal control problem with a lower and an upper reflecting barrier. Sethi and Taksar (2002) recently considered the model for the company that can control its risk exposure by issuing new equity as well as paying dividends. He and Liang (2007) work out the optimal financing and dividend control problem of the insurance company without the fixed transaction costs.

In this paper, we consider both the fixed and the proportional transaction costs incurred by the equity issuance. The amount of money paid by the shareholder is \( K + \beta_2 \xi \), \( \beta_2 > 1 \), to meet the equity issuance of \( \xi \). \( K \) is the fixed transaction costs generated by the advisory and consulting fees, \( \beta_2 \) is the proportional transaction costs generated by the tax. We assume that if the company pays dividends, the shareholder can get \( \beta_1 l \), \( \beta_1 < 1 \), and we can omit the fixed transaction costs in the dividends payout process because the financial system is operated with an ever increasing efficiency and the dividends payout processes seldom generate fixed transaction costs.
costs. We refer the reader to Cadennias et al. (2006), which consider the optimal dividends policy of the insurance company with the fixed and proportional transaction costs, and without the equity issuance.

Motivated by the work of He and Liang (2007), Harrison and Takacs (1983) and Sethi and Takacs (2002), we can consider the equity issuance and dividends payout as the absorbing and reflecting boundaries of the reserve process, respectively. We will deal with the mixed classical-impulse control problem by using the line of He and Liang (2007). We expect our results would be of interest for theory of mixed classical-impulse control.

The paper is organized as follows: In Section 2, we establish the control model of the insurance company with fixed and proportional transaction costs. In Section 3, we present some mathematical results proved by He and Liang (2007) for proving the main results of this paper. In Section 4, we construct solutions of two categories of suboptimal models. One is the classical model without equity issuance, the other never goes bankrupt by equity issuance, and the other solutions of suboptimal models. In Section 5, we identify the value function and the optimal strategy with the corresponding solution in each category of suboptimal models, depending on the relationships between the coefficients. We give the conclusion of this paper in Section 6.

2. Control model of the insurance company with fixed and proportional transaction costs

To give a mathematical foundation of our model, we fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). \((\mathcal{F}_t, t \geq 0)\) is a standard Brownian Motion on this probability space, \(\mathcal{F}_t\) represents the information available at time \(t\), any decision made up to time \(t\) is based on \(\mathcal{F}_t\). In our model, we denote \(L_t\) as the cumulative amount of dividends paid from time 0 to time \(t\). We assume that the process \(\{L_t, t \geq 0\}\) is \(\{\mathcal{F}_t, t \geq 0\}\)-adapted, increasing, right-continuous with left limits and \(L_0 = 0\).

The equity issuance process \(\mathcal{G}_t\) is described by a sequence of increasing stopping times \(\{\tau_i, i = 1, 2, \ldots\}\) and a sequence of random variables \(\{\xi_i, i = 1, 2, \ldots\}\), which are associated with the time and the amount of the equity issuance. A control policy \(\pi\) is described by stochastic processes \(\pi = (a_t; L_t; \mathcal{G}_t) = (a_t; L_t^1, \tau_1, L_t^2, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)\).

For a control policy \(\pi\), we assume that the liquid reserve of the insurance company evolves according to the stochastic equation,

\[
R_t = x + \int_0^t \mu a_s (s) \sigma dW_s + \int_0^t \sigma a_s (s) dW_s - L_t^1 + \sum_{n=1}^{\infty} I_{[\tau_n \leq t]} \xi_n^2 + \int_0^\tau \sigma a_s (s) dW_s - L_t^2,
\]

\[\quad \text{where } 1-a_s(t) \in [0, 1] \text{ is the proportional reinsurancerate.}
\]

The company is considered as bankrupt as soon as the reserves fall below 0. We define the time of bankruptcy as \(\tau = \inf \{t \geq 0 : R_t < 0\}\), \(\tau\) is an \(\mathcal{F}_t\)-stopping time. If the company issues some equity, the time of bankruptcy could be infinite.

Our main objective is to maximize the expected present value of the dividends payout minus the equity issuance before bankruptcy,

\[
J(x, \pi) = E \left[ \int_0^\tau e^{-ct} \beta_1 dL_t^1 - \sum_{n=1}^{\infty} e^{-ct_{n}} \left( K + \beta_2 \xi_n^2 \right) I_{[\tau_n \leq t]} \right],
\]

\[\quad \text{where } 1 \leq \tau_n \in [0, 1] \text{ denotes the set of all admissible control policies,}
\]

\(c\) denotes the discount rate. In the equity issuance process, \(\beta_2 > 1\) is the proportional transaction costs generated by the tax. \(K\) is the fixed transaction costs generated by the advisory and consulting fees. In the dividends payout process, \(\beta_1 < 1\) is the proportional transaction costs generated by the tax. We solve the optimal mixed classical-impulse control problem \((2.2)\) and \((2.3)\) by finding some conditions on \((\mu, \sigma, c, \beta_1, \beta_2, K)\) such that \(V(x)\) satisfies a type of HJB equations. Then we get \(V(x)\) and the optimal strategy \(\pi^*\) associated with \(V(x)\).

3. Preliminary of the problem

In this section we present two lemmas before proving the main results of this paper in Sections 4 and 5. Since the proofs of the lemmas are completely similar to that of He and Liang (2007), we omit them here.

**Lemma 3.1.** Let \(x_0 = \frac{(1-\gamma)\mu^2}{\sqrt{\gamma}}\) and \(m = \frac{\gamma K}{\sqrt{(1-\gamma)\beta_2}} < x_0\). Then there exists a unique \(x_{1*} = x_{1*}(c, \mu, \sigma, \beta_1) > x_0\) satisfying the following equation in \(x_1\),

\[
\left( \frac{1}{d_+} - \frac{x_0}{\gamma} \right) \frac{\beta_1 d_+}{d_+ - d_-} e^{x_0 - x_1} = 0,
\]

\[\quad \text{where } d_- = \frac{-\mu + \sqrt{\mu^2 + 2\gamma^2}}{\gamma}, \quad d_+ = \frac{-\mu + \sqrt{\mu^2 + 2\gamma^2}}{\gamma}, \quad \gamma = \frac{c}{\sqrt{\gamma}}.
\]

**Lemma 3.2.** Let \(x_0 = \frac{(1-\gamma)\mu^2}{\sqrt{\gamma}}\) and \(m = \frac{\gamma K}{\sqrt{(1-\gamma)\beta_2}} < x_0\). Then there exists a unique \(x_{1*} = x_{1*}(c, \mu, \sigma, \beta_1, \beta_2, K) > x_0\) satisfying the following equation in \(x_1\),

\[
\left( \frac{1}{\gamma x_0^2} \frac{\beta_1 d_+}{d_+ - d_-} e^{x_0 - x_1} - \frac{\beta_2 d_-}{d_+ - d_-} e^{x_0 - x_1} \right) = \frac{\beta_2}{\gamma m^{-1}},
\]

\[\quad \text{where } d_-, d_+, \text{ and } \gamma \text{ are the same as in Lemma 3.1.}
\]

4. Two categories of suboptimal solutions

In this section, we consider two categories of suboptimal control problems. \(\pi_{\pi} = \{a_t, L_t, \mathcal{G}_t\} \in \Pi\) stands for the control process for the company in which equity issuance is not permitted. The associated optimal return function is defined by

\[
V_{\pi}(x) = \sup_{\pi \in \Pi} J(x, \pi), \quad \text{for } x \geq 0.
\]

Let \(\pi_q = \{a_t, L_t, \mathcal{G}_t\} \in \Pi\) be the control process for the company with equity issuance procedures, where \(\mathcal{G}_t = (\tau_1^q, \tau_2^q, \ldots, \tau_n^q, \ldots; \xi_1^q, \xi_2^q, \ldots, \xi_n^q, \ldots)\). The associated optimal return function is

\[
V_q(x) = \sup_{\pi \in \Pi} J(x, \pi), \quad \text{for } x \geq 0.
\]

The company will never go bankrupt in this case.

Since \(\pi_p, \pi_q \in \Pi\), it is easy to see that \(V(x) \geq \max\{V_p(x), V_q(x)\}\) via \((2.3)\). We will show that the two suboptimal solutions play an important role in constructing the overall optimal control solutions. In next two subsections, we will establish solutions for each category of the control models.
4.1. The solution to the problem without equity issuance

In this subsection, we consider the company without equity issuance. Our objective is to maximize the expected discounted dividends payout.

**Theorem 4.1.** Assume that the coefficients are such that $x_{1s} \leq x_{1s+}$, where $x_{1s}$ and $x_{1s+}$ are defined in Lemmas 3.1 and 3.2. Then the function $f$ defined by

$$
\begin{align*}
  f &= \begin{cases} 
  f_1(x) = C_1(x_1)x^r + C_2(x_1), & 0 \leq x < x_0, \\
  f_2(x) = C_3(x_1)e^{d_+}x + C_4(x_1)x + C_5(x_1)e^{d_+}x, & x_0 \leq x < x_{1s}, \\
  f_3(x) = \beta_1(x - x_{1s}) + f_2(x), & x \geq x_{1s},
  \end{cases}
\end{align*}
$$

(4.1)

satisfies the following HJB equation and the boundary conditions for $x \geq 0$,

$$
\max_{a \in [0, 1]} \left\{ \frac{1}{2} \sigma^2 a^2 f''(x) + \mu a f'(x) - cf(x), \right\} \beta_1 f'(x) = 0,
$$

(4.2)

and

$$
a(x) = \frac{\mu x}{\sigma^2 (1 - \gamma)}.
$$

(4.8)

The validity of the solution requires $a(x) \in [0, 1]$ means that $0 \leq x \leq x_0 = \left(\frac{(1 - \gamma)x^2}{\sigma^2}\right)^{1/2}$.

On the other hand, if $x_0 \leq x < x_1$, we have $a(x) = 1$ and the first equation in (4.5) becomes

$$
\frac{1}{2} \sigma^2 f''(x) + \mu f'(x) - cf(x) = 0,
$$

(4.9)

so the solution of Eq. (4.9) is

$$
f_2(x) = C_3(x_1)e^{d_+}x + C_4(x_1)x + C_5(x_1)e^{d_+}x \quad \text{for } x_0 \leq x < x_1.
$$

(4.10)

For $x \geq x_1$, the solution of (4.5) has the following form,

$$
f_3(x) = \beta_1(x - x_{1s}) + f_2(x),
$$

(4.11)

Continuity of the function $f'$ and $f''$ at point $x_1$, the second quality and the third equality of (4.5) imply that

$$
C_3(x_1)d_+e^{d_+}x + C_4(x_1)d_+e^{d_+}x_1 = \beta_1,
$$

$$
C_5(x_1)d_+e^{d_+}x_1 + C_4(x_1)d_+e^{d_+}x_1 = 0,
$$

(4.12)

i.e.,

$$
C_3(x_1) = \frac{\beta_1d_+}{e^{d_-}x_1(d_+ - d_-)} < 0,
$$

$$
C_4(x_1) = \frac{\beta_1d_+}{e^{d_-}x_1(d_+ - d_-)} > 0.
$$

(4.13)

Also, since the solution $f$ and $f'$ are continuous at $x_0$, $f_1(x_0) = f_2(x_0)$,

$$
f_1(x_0) = f_2(x_0),
$$

(4.14)

which imply that

$$
C_1(x_1) = \frac{1}{\gamma x_0^{x_0-1}} \left( \frac{\beta_1d_+}{d_+ - d_-}e^{d_-(x_0 - x_1)} - \frac{\beta_1d_+}{d_+ - d_-}e^{d_+(x_0 - x_1)} \right) > 0,
$$

(4.15)

$$
C_2(x_1) = \frac{1}{\gamma x_0^{x_0-1}} \left( \frac{x_0}{d_+ - d_-} \frac{\beta_1d_+}{d_+ - d_-}e^{d_-(x_0 - x_1)} - \frac{x_0}{d_+ - d_-} \frac{\beta_1d_+}{d_+ - d_-}e^{d_+(x_0 - x_1)} \right).
$$

(4.16)

By (4.7) and (4.13), if $f(0) = 0$, then we have

$$
f_1(0) = C_2(x_1) = 0,
$$

(4.17)

which implies that $x_1$ is a solution of Eq. (3.1). Using Lemma 3.1, we have $x_1 = x_{1s}$. Similarly, if $x_1 = x_{1s}$, then $f(0) = 0$. So

$$
f(0) = 0 \Longleftrightarrow x_1 = x_{1s}.
$$

(4.18)

The problem remained is to prove that the solution $f$ satisfies (4.2)-(4.4). Noticing that $\beta_1 < 1$, $\beta_2 > 1$, it suffices to prove the following conditions:

$$
f'(x) \geq \beta_1, \quad \text{for } 0 \leq x < x_0,
$$

$$
\max_{a \in [0, 1]} \left\{ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right\} \leq 0, \quad \text{for } x_0 \leq x < x_{1s},
$$

$$
\max_{a \in [0, 1]} \left\{ \frac{1}{2} \sigma^2 a^2 f'' + \mu a f' - cf \right\} \leq 0, \quad \text{for } x \geq x_{1s},
$$

and there exists $x_2 \in (0, x_{1s})$ such that $f'(x_2) = \beta_2$ and

$$
f(x_2) - f(0) \leq K + \beta_2x_2.
$$

(4.19)
The proof is as follows: for \( x \geq x_{1+} \), \( f_3 = \beta_1(x-x_{1+}) + f_2(x_{1+}) \), by (4.15) we have

\[
\mu_2 - c f_2(x_{1+}) = 0.
\]

Therefore

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 f''_\alpha + \mu \alpha f'_\alpha - c f_2 \right] = \max_{\alpha \in [0,1]} \left\{ \mu \alpha \beta_1 - c(\beta_1(x-x_{1+}) + f_2(x_{1+})) \right\} = \mu_2 - c f_2(x_{1+}) - c \beta_1(x - x_{1+}) \leq 0.
\]

By the same way as in Højgaard and Taksar (1999), it is easy to prove that

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 f''_\alpha + \mu \alpha f'_\alpha - c f_2 \right] \leq 0
\]

holds for \( x_0 \leq x \leq x_{1+} \). See Højgaard and Taksar (1999) for details.

Since

\[
f''_\alpha(x) = C_1(x_{1+}) \gamma' (\gamma' - 1) x^{\gamma - 2} \leq 0,
\]

\[
f'_\alpha(x) = \frac{\beta_1 d_x d_y}{d_x - d_y} (e^{d_x (x - x_{1+})} - e^{d_y (x - x_{1+})}) \leq 0,
\]

\( f'(x) \) is a decreasing function on \([0, x_{1+}]\). \( f'(x_{1+}) = \beta_1 \) and \( f'(x) \geq \beta_1 \) on \([0, x_{1+}]\) are obvious.

Since \( f'(x) \to \infty \) as \( x \to 0+ \), there exists \( x_2 \in (0, x_{1+}) \) such that \( f'(x_2) = \beta_2 \). The proof of (4.14) can be reduced to proving the following

\[
C_1(x_{1+}) x_2^\gamma \leq K + \beta_2 x_2,
\]

where \( C_1(x_{1+}) \gamma x_2^{\gamma - 1} = \beta_2 \).

\[
C_1(x_{1+}) x_2^\gamma - \beta_2 x_2 = C_1(x_{1+}) (1 - \gamma) x_2^{\gamma - 1} \leq \frac{(1 - \gamma) \beta_2 x_2}{\gamma}.
\]

Let \( h(x) = \frac{1}{\gamma} \left( \frac{\beta_1 d_x d_y}{d_x - d_y} e^{d_x (x - x_{1+})} - \frac{\beta_1 d_x d_y}{d_x - d_y} e^{d_y (x - x_{1+})} \right) \). Then \( h(x) \) is an increasing function of \( x \) (see He and Liang (2007) for details).

Using (4.12), Lemma 3.2 and \( x_{1+} \leq x \leq x_{1+} \), we have \( \frac{\partial h}{\partial x} = C_1(x_{1+}) \leq h(x_{1+}) = \frac{\beta_2}{\gamma} \). So \( x_2 \leq \frac{\gamma K}{(1 - \gamma) \beta_2} \), which, together with (4.15), implies that \( C_1(x_{1+}) x_2^\gamma \leq K + \beta_2 x_2 \). Thus, we complete the proof. \( \square \)

**Remark 4.1.** We have used the assumption \( m = \frac{\gamma K}{(1 - \gamma) \beta_2} < x_0 \) in the proof of Theorem 4.1. Indeed, the fixed transaction costs \( K \) in the equity issuance process is often small. To simplify the problem, we assume this relationship is always reasonable.

4.2. The solution to the problem with equity issuance

In this subsection we aim at maximizing the expected discounted dividends payout minus the expected discounted equity issuance. The insurance company will never go bankrupt by equity issuance.

**Theorem 4.2.** Assume that the coefficients are such that \( x_{1+} \geq x_{1+} \). \( x_{1+} \) is defined in Lemma 3.1. Then the function \( g \) defined by

\[
g = \begin{cases} 
    g_1(x) = C_1(x_{1+}) x^\gamma + C_2(x_{1+}), & 0 \leq x < x_0, \\
    g_2(x) = C_3(x_{1+}) e^{d_x x} + C_4(x_{1+}) e^{d_y x}, & x_0 \leq x < x_{1+}, \\
    g_3(x) = \beta_1 (x - x_{1+}) + g_2(x_{1+}), & x \geq x_{1+}
\end{cases}
\]

satisfies the following HJB equation and the boundary condition; for \( x_2 \) satisfying \( g'(x_2) = \beta_2 \),

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 g'' \left( x \right) + \mu \alpha g' \left( x \right) - c g \left( x \right) \right] \cdot \beta_1 - g' \left( x \right),
\]

\[
g(x) - g(0) - \beta_2 x_2 - K = 0,
\]

(4.17)

\[
g(0) \geq 0,
\]

(4.18)

where \( \gamma, x_0, d_x \) and \( d_y \) are the same as in Theorem 4.1. \( C_1(x_{1+}), C_2(x_{1+}), C_3(x_{1+}) \) and \( C_4(x_{1+}) \) are defined as same as in Theorem 4.1 by replacing \( x_1 \) with \( x_{1+} \).

**Proof.** Considering the time value of money leads us to the conclusion that it is optimal to postpone the new equity issue as long as possible. We conjecture that it is optimal to issue equity only when the reserves become 0. Using the same way as in Section 4.1, this strategy is associated with a solution to the HJB equations (4.17) and (4.18). It should be characterized by

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 g'' \left( x \right) + \mu \alpha g' \left( x \right) - c g \left( x \right) \right] = 0,
\]

for \( 0 \leq x < x_1 \),

\[
g' \left( x \right) = \beta_1, \quad \text{for} \ x \geq x_1 \,
\]

\[
g'' \left( x \right) = 0, \quad \text{for} \ x \geq x_1 \,
\]

where \( x_1 \) is an unknown variable and \( x_1 \) will be specified later, and for \( x_2 \) satisfying \( g'(x_2) = \beta_2 \),

\[
g(x_2) - g(0) = \beta_2 x_2 + K.
\]

(4.22)

Doing the same procedures as in Section 4.1, we can prove the solution \( g(x) \) of Eqs. (4.19)-(4.21) has the same form as \( f(x) \) in Eqs. (4.7)-(4.13), and \( x_1 \) is the solution of the following equation

\[
\frac{1}{\gamma} x_1^{\gamma - 1} \left( \frac{\beta_1 d_x d_y}{d_x - d_y} e^{d_x (x_{1+} - x_{1+})} - \frac{\beta_1 d_x d_y}{d_x - d_y} e^{d_y (x_{1+} - x_{1+})} \right) \frac{\beta_2}{\gamma} = \frac{\beta_2}{\gamma} e^{d_y (x_{1+} - x_{1+})},
\]

where \( m = \frac{\gamma K}{(1 - \gamma) \beta_2} \).

By Lemma 3.2, we have

\[
x_1 = x_{1+} \quad \text{and} \quad x_{1+} \geq x_0.
\]

The problem remains is to prove that the solution \( g \) satisfies the conditions mentioned in Theorem 4.2. For this, it suffices to prove the following

\[
g' \geq \beta_1 \quad \text{for} \ 0 \leq x < x_0,
\]

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 g'' \left( x \right) + \mu \alpha g' \left( x \right) - c g \left( x \right) \right] \leq 0, \quad g' \geq \beta_1,
\]

for \( x_0 \leq x < x_{1+} \),

\[
\max_{\alpha \in [0,1]} \left[ \frac{1}{2} \sigma^2 \alpha^2 g'' \left( x \right) + \mu \alpha g' \left( x \right) - c g \left( x \right) \right] \leq 0, \quad \text{for} \ x \geq x_{1+}.
\]

Using the similar procedures as in Section 4.1, we can prove the above claims.

Now, we will check the boundary conditions. Based on our assumption, equity issue is not compulsory and bankrupt at 0 is an option, so we need to prove \( g(0) \geq 0 \), i.e.,

\[
g(0) = \left( \frac{1}{d_x - d_y} \frac{x_0}{x_{1+}} \beta_1 d_x d_y e^{d_x (x_{1+})} \right) \frac{x_0}{x_{1+}} \beta_1 d_x d_y e^{d_y (x_{1+})} - \left( \frac{1}{d_x - d_y} \frac{x_0}{x_{1+}} \beta_1 d_x d_y e^{d_x (x_{1+})} \right) = k(x_{1+}) \geq k(x_{1+}) = 0
\]

(4.23)
due to the decreasing properties of \( k(x_1) \) on \( x_1 \). Here \( k(x) = \left( \frac{1}{a} - \frac{m}{\gamma} \right) \frac{b \gamma d}{a \delta - \delta \gamma} x^2 + \frac{b \gamma d}{a \delta - \delta \gamma} \left( \gamma k_0 - \gamma k_1 \right) \). Thus we complete the proof. \( \square \)

5. The solution of the optimal control problem

**Theorem 5.1.** Let concave function \( W(x) \in \mathbb{C}^2 \) satisfy the following HJB equation and boundary condition: for \( x_2 \) satisfying \( W'(x_2) = \beta_2 \), and any \( x \geq 0 \),

\[
\max_{\alpha \in [0,1]} \left\{ \frac{1}{2} \sigma^2 \alpha^2 W'' + \mu \alpha W' - cW \right\}, \beta_1 - W' = 0 \tag{5.1}
\]

max\([-W(0), W(x_2) - W(0) - \beta_2 x_2 - K] = 0 \tag{5.2}
\]

Then \( W(x) \geq f(x, \pi) \) for any admissible policy \( \pi \).

**Proof.** Since \( W(x) \) is a concave, increasing and continuous function on \([0, \infty)\), for \( x_2 \) satisfying \( W'(x_2) = \beta_2 \), \( W(x_2) - W(0) - \beta_2 x_2 - K \leq 0 \) is equivalent to following condition:

\[
W(x + \xi) - W(x) \leq K + \beta_2 \xi, \quad \forall x \geq 0, \quad \xi \geq 0 \tag{5.3}
\]

Because this is trivial, we omit the proof here.

For a policy \( \pi \), let \( A = \{s : L^n_s \neq L^0 \} \), \( L^n_s = \bigcup_{s \notin A, t \leq t^*} (L^n_t - L^0_t) \) be the discontinuous part of \( L^n_t \) and \( L^0_t = L^n_t - L^0_L \) be the continuous part of \( L^n_t \). Let \( A^n_t \equiv \bigcup_{s \notin A, t \leq t^*} (L^n_t - L^0_t) \). Let \( I_t = \mathbb{I} \left[ \sigma e^{-c t} W''(R^{x,t}_{\pi}) d\omega_1 - \int_0^t e^{-c s} W'(R^{x,t}_{\pi}) d\omega_2 \right] + \sum_{s \in A \cup A', t \leq t^*} e^{-c s} \left[ W(R^{x,t}_{\pi}) - W(R^{x,t}_{\pi'}) \right] \tag{5.4}
\]

where

\[
L = \frac{1}{2} \sigma^2 \alpha^2 \frac{d}{dr} + \mu a \frac{dr}{dr} - c.
\]

Noticing that \( s \in A \cup A', s \leq t \wedge t^* \),

\[
W(R^n_t) - W(R^0_t) = \left[ W(R^n_t) - W(R^0_t) \right] I_t(s) + \sum_{s \in A \cup A', s \leq t^*} \left[ W(R^n_t) - W(R^0_t) \right] I_{(s \notin A, t \leq t^*)}(s),
\]

\( a_1(s) + a_2(s) \). Using \( W \) is increasing and (5.3),

\[
a_1(s) \leq \sum_{s \in A} (K + \beta_2 e_n) I_{(s \notin A, t \leq t^*)}, \tag{5.6}
\]

\[
a_2(s) \leq -\beta_1 (L^n_s - L^0_s) I_{(s \notin A, t \leq t^*)}, \tag{5.7}
\]

Putting (5.4)–(5.7) together, we have

\[
E \left[ e^{-c (t \wedge t^*)} W(R^{x,t}_{\pi}) \right] + E \left[ \int_0^{t \wedge t^*} e^{-cs} \beta_1 d\omega_2 \right] \leq W(x). \tag{5.8}
\]

Letting \( \varepsilon \to 0 \) in (5.8), by the definition of \( \tau \) and \( \beta_1 \leq W'(x) \), it is easy to prove that

\[
\lim_{t \to \infty} e^{-c (t \wedge t^*)} W(R^{x,t}_{\pi}) = e^{-c t} W(0) I_{\tau < \infty} + \lim_{t \to \infty} e^{-c t} W(R_t) I_{\tau = \infty} \geq e^{-c t} W(0) I_{\tau < \infty} = 0 \tag{5.9}
\]

So, we deduce from (5.8) and (5.9) that

\[
J(x, \pi) = E \left[ \left( \int_0^t e^{-c s} \beta_1 d\omega_2 - \sum_{n=1}^{\infty} e^{-c t_n} (K + \beta_2 e_n) I_{t_n < \tau} \right) \right] \leq W(x).
\]

Thus the proof has been done. \( \square \)

Let

\[
a(x) = \begin{cases} \frac{\mu \lambda X}{\sigma^2 (1 - \gamma)^2}, & x < x_0, \\ 1, & x \geq x_0 \end{cases}
\]

where \( \gamma = \frac{c_1}{c + \pi} \) and \( x_0 = \frac{(1 - \gamma) \mu \lambda^2}{\mu \lambda} \).

The main results of this paper are the following.

**Theorem 5.2.** If \( x_{1*} \leq x_{1**} \), then \( V(x) = f(x) = V^0(x) \). The optimal policy \( \pi^* = (a^*_x, L^*, G^*) \) satisfies the following

\[
\left\{ \begin{array}{l}
R^{x^*}_{t} = x + \int_0^t \mu a(R^{x^*}_{t}) d\omega_1 + \int_0^t \sigma a(R^{x^*}_{t}) d\omega_2 - L^{x^*}_{t}, \\
0 \leq R^{x^*}_{t} \leq x_{1*}, \\
\int_{t_{R^{x^*}_{t}}} I_{\tau^* < x_{1*}}(t) dL^{x^*}_{t} = 0, \\
G^{x^*}_{t} = 0,
\end{array} \right.
\]

where \( a^*_x(t) = \alpha(R^{x^*}_{t}) \), \( x_{1*} \) is given in Lemma 3.1, \( V(x) \) and \( f(x) \) are defined by (2.3) and (4.1), respectively. \( V^0(x) \) is defined in Section 4.

If \( x_{1*} \geq x_{1**} \), then \( V(x) = g(x) = V^0(x) \). The optimal policy \( \pi^{**} = (a^{**}_x, L^{**}, G^{**}) \) satisfies the following

\[
\left\{ \begin{array}{l}
R^{x^{**}}_{t} = x + \int_0^t \mu a(R^{x^{**}}_{t}) d\omega_1 + \int_0^t \sigma a(R^{x^{**}}_{t}) d\omega_2 - L^{x^{**}}_{t}, \\
0 \leq R^{x^{**}}_{t} \leq x_{1**}, \\
\int_{t_{R^{x^{**}}_{t}}} I_{\tau^{**} < x_{1**}}(t) dL^{x^{**}}_{t} = 0, \\
0 \leq \xi^{x^{**}}_{t} \leq x_{1**},
\end{array} \right.
\]

where \( \tau^{**} = \inf\{t > 0 : R^{x^{**}}_{t} = 0\} \), \( \tau^{**} = \inf\{t > t_{n_{R^{x^{**}}_{t}}} : g(R^{x^{**}}_{t}) - g(R^{x_{1**}}_{t}) = K + \beta_2 (R^{x^*}_{t} - R^{x^{**}}_{t}) \} \) and \( \xi^{x^{**}}_{t} = R^{x^{**}}_{t} - R^{x_{1**}}_{t} \). Moreover, \( a^{**}_x(t) = \alpha(R^{x^{**}}_{t}) \), \( x_{1**} \) is given in Lemma 3.2, \( V(x) \) and \( g(x) \) are defined by (2.3) and (4.16), respectively. \( V^0(x) \) is defined in Section 4.

In fact, based on the construction procedures of \( g(x), \xi^{x^{**}}_{t} \) and \( \xi^{x^{**}}_{t} \), we have \( R^{x_{1**}}_{t} = 0, \xi^{x_{1**}}_{t} = m = \frac{K}{\mu \lambda} \). In this circumstance, the insurance company issues equity whenever the reserves reach 0 and the amount should always be m.

**Remark** that using Theorem 3.1 in Lions and Sznitman (1984) the processes \( (R^{x^*}_{t}, L^{x^*}_{t}, G^{x^*}_{t}) \) and \( (R^{x^{**}}_{t}, L^{x^{**}}_{t}, G^{x^{**}}_{t}) \) are uniquely determined by Eqs. (5.10) and (5.11).

**Proof.** If \( x_{1*} \leq x_{1**} \), the function \( f(x) \) satisfies the HJB equation and boundary conditions (4.2)–(4.4). It is easy to see that \( f(x) \) also satisfies conditions (5.1) and (5.2) in Theorem 5.1. So \( f(x) = V(x) \).
Next, we will prove $V(x) = f(x)$ corresponding to $\pi^*$. Applying generalized Itô formula (see Cont and Tankov (2003)), we deduce from (4.5) and (4.6) that $\mathcal{L}f(R_{t_1}^\tau) = 0$ and

$$e^{-c(x,x^*)}f(R_{t_1}^\tau) = f(x) - \int_0^{\tau(x,x^*)} e^{-\alpha} \beta_1 \mathcal{d}t_1^\tau + \int_0^{\tau(x,x^*)} a(R_{t_1}^\tau) \sigma e^{-c(x,x^*)} \mathcal{d}W_1,$$

(5.12)

where $t^* = \inf\{t \geq 0 : R_t^\tau < 0\}$. Since $\lim_{t \to \infty} e^{-c(x,x^*)} f(R_{t_1}^\tau) = e^{-c(x,x^*)} f(0) = 0$, taking expectations at both sides of (5.12), we get

$$f(x) = E \left[ \int_0^{x^*} e^{-\alpha} \beta_1 \mathcal{d}t_1^\tau \right] = f(x, \pi^*).$$

So $f(x)$ is the return function corresponding to $\pi^*$, and $f(x) \leq V_p(x)$. Using the results $f(x) \geq V(x) \geq V_p(x)$, we have $f(x) = V(x) = V_p(x)$ under the circumstance $\tau(x,x^*) \leq \tau^*$. If $\tau(x,x^*) \geq \tau^*$, then $g(x)$ defined in (4.16) satisfies the HJB equation and boundary conditions (4.17) and (4.18). Thus $g(x)$ satisfies conditions (5.1) and (5.2) in Theorem 5.1. So $g(x) \geq V(x) \geq V_p(x)$ by Theorem 5.1.

Next, we will prove $V(x) = g(x)$ corresponding to $\pi^{**}$. Applying generalized Itô formula (see Cont and Tankov (2003)), we deduce from (4.8) (4.19) and (4.20) that $\mathcal{L}g(R_{t_1}^\tau) = 0$ and

$$e^{-c(x,x^{**})}g(R_{t_1}^{\tau(x,x^{**})}) = g(x) + \int_0^{\tau(x,x^{**})} e^{-\alpha} \beta_1 \mathcal{d}t_1^{\tau(x,x^{**})} + \int_0^{\tau(x,x^{**})} a(R_{t_1}^{\tau(x,x^{**})}) \sigma e^{-c(x,x^{**})} \mathcal{d}W_1 - \int_0^{\tau(x,x^{**})} e^{-\alpha} g(R_{t_1}^{\tau(x,x^{**})}) \mathcal{d}t_1^{\tau(x,x^{**})}$$

$$+ \sum_{\eta \in \Delta, \eta \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4} e^{-\alpha}[g(R_{t_1}^{\tau(x,x^{**})}) - g(R_{t_1}^{\tau_i(x,x^{**})})]$$

$$\geq g(x) - \int_0^{\tau(x,x^{**})} e^{-\alpha} \beta_1 \mathcal{d}t_1^{\tau(x,x^{**})} + \int_0^{\tau(x,x^{**})} a(R_{t_1}^{\tau(x,x^{**})}) \sigma e^{-c(x,x^{**})} \mathcal{d}W_1$$

$$+ \sum_{\eta \in \Delta, \eta \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4} e^{-\alpha}[g(R_{t_1}^{\tau(x,x^{**})}) - g(R_{t_1}^{\tau_i(x,x^{**})})],$$

(5.13)

where $\tau^{**} = \inf\{t \geq 0 : R_t^{\tau} \leq 0\}$. Using the same way as in (5.5) and (5.6), we have by (5.2) and the definition of $\tau^{**}$,

$$\sum_{\eta \in \Delta, \eta \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4} e^{-\alpha}[g(R_{t_1}^{\tau(x,x^{**})}) - g(R_{t_1}^{\tau_i(x,x^{**})})]$$

$$= \sum_{n=1}^{\infty} e^{-c(x,x^{**})} (K + \beta_2 z_n^{\tau(x,x^{**})}) I_{\{t_n^{\tau(x,x^{**})} < \tau^{**}\}}.$$ 

(5.14)

Since $\lim_{t \to \infty} e^{-c(x,x^{**})} g(R_{t_1}^{\tau(x,x^{**})}) = \lim_{t \to \infty} e^{-c(x,x^{**})} g(R_{t_1}^{\tau}) = 0$ (see Haigard and Taksar (2001) for details), taking mathematical expectations at both sides of (5.13) and using (5.14), we get

$$g(x) = E \left[ \int_0^{x^{**}} e^{-\alpha} \beta_1 \mathcal{d}t_1^{x^{**}} - \sum_{n=1}^{\infty} e^{-c(x,x^{**})} (K + \beta_2 z_n^{\tau(x,x^{**})}) I_{\{t_n^{\tau(x,x^{**})} < \tau^{**}\}} \right]$$

$$= f(x, \pi^{**}).$$

So $g(x)$ is the return function corresponding to $\pi^{**}$, and $g(x) \leq V_p(x)$. Using the results $g(x) \geq V(x) \geq V_p(x)$, we have $g(x) = V(x) = V_p(x)$ under the circumstance $\tau^{**} \geq \tau^{**}$. The proof has been done.

Remark 5.1. In real market the bankruptcy time $\tau$ is the right time to decide whether we should go bankrupt or issue equity to make the company survive long. The main results of this paper explain under what conditions we should go bankrupt directly and under what conditions we should issue equity to make the company survive long. Based on existences of $\tau_{1s}$ and $\tau_{1s}$, in Lemmas 3.1 and 3.2, we divide the conditions into two categories: $\tau_{1s} \leq \tau_{1s}$ and $\tau_{1s} \geq \tau_{1s}$. For $\tau_{1s} \leq \tau_{1s}$, the management should let the insurance company go bankrupt directly once its reserves reach 0. For $\tau_{1s} \geq \tau_{1s}$, the management should issue equity to avoid bankruptcy. Theorem 4.1–4.2 and Theorem 5.2 state that if one of the above conditions is satisfied, then the management can take the optimal policy corresponding to that condition.

6. Conclusion

In this paper, we consider the optimal control problem of the insurance company with both the fixed and proportional transaction costs. The management of the company controls the reinsurance rate, dividends payout and the equity issuance to maximize the expected present value of the dividends payout minus the equity issuance before bankruptcy. To be more realistic, we assume the equity issuance processes incur both the fixed and the proportional transaction costs. The former are generated by the advisory and consulting fees as well as the latter are generated by the tax. This is the first time that the financing process in an insurance model with the fixed transaction costs has been considered, and we finally find that it acts as absorbing or reflecting boundary of the reserve process. In order to find the solution of mixed classical-impulse control problem. We construct two categories of suboptimal models, one is the classical model without equity issuance, the other never goes bankrupt by equity issuance. At last, we identify the value function and the optimal strategy with the corresponding solution in either category of suboptimal models, depending on the relationships between the coefficients.

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