VALUATION OF INFINITE MATURITY STOCK LOANS WITH GEOMETRIC LÉVY MODEL IN A RISK-NEUTRAL FRAMEWORK

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ABSTRACT. This paper deals with the pricing problem of infinite maturity stock loan where the underlying stock price follows the geometric Lévy model for incorporating more empirical features in a risk-neutral framework. Since the way of dividends distribution has a great influence on pricing, we aim at deriving the closed-form solutions and optimal strategy of the pricing problem subject to three specified ways of dividend distribution by variational inequalities approach. The relationships among the parameters, such as loan sizes, interest rates and service fees, are also discussed. Numerical examples are included to illustrate the results.

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1. Introduction

A stock loan is a simple economy where a borrower, who owns one share of a stock, borrows a loan of amount $q$ from a lender (bank or other financial institution) with the share as collateral. The lender charges an amount $c$ ($0 \leq c \leq q$) from the borrower as the service fee. The borrower has the right to redeem the stock at any time before or on the loan maturity by repaying principal and interest to the bank, or alternatively surrender the stock. Here, for analytical tractability of the stock loan, we take the

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maturity as infinity, i.e., we mainly deal with infinite maturity stock loan throughout this paper.

The stock loan is a currently popular financial derivative because firstly it can create liquidity by overcoming the barrier of large block sales, such as triggering tax events or control restrictions on sales of stocks; It is also able to serve as a hedge: if the stock price goes down, the borrower may forfeit the loan at initial time and does not repay the loan; if however the stock price goes up, the client keeps all the benefits upside by repaying the principal and interest. In other words, a stock loan can help high-net-worth investors with large equity positions to achieve a variety of objectives.

Xia and Zhou [23](2007) initiated the study of above pricing problems of infinite maturity stock loan under the classical Black-Scholes framework (see [5]), i.e., the underlying stock price follows the geometric Brownian motion. They used the pure probability approach to obtain the value function as well as the optimal redeeming strategy. They also mentioned that the variational inequalities approach can’t be directly applied here as in perpetual American option pricing(see [12, 13] and references therein). Liu and Xu [17](2010) also got valuation of infinite maturity capped stock loans by this approach. However, Liang, Wu and Jiang [15, 16](2010), Zhang and Zhou[25](2009) showed that the reformed variational inequalities approach can achieve the same results, respectively. Zhang and Zhou [25](2009) extended it to a regime switching market, Liang, Wu and Jiang [15, 16](2010) even covered a more complicated case, i.e., with automatic termination clause, cap and margin. On the other hand, Dai and Xu [9] (2010) firstly studied the finite maturity stock loan and showed that the way how dividends is distributed really has a significant effect on pricing and optimal strategy though closed-form price formulas of this kind of stock loans are generally not available. Yam, Yung and Zhou [24](2011) established explicitly the value function of the perpetual Dynkin game associated with the value of a callable stock loan.

However, all the papers above used the classical Black-Scholes model (cf.[5]). A successful model as it is, still has weaknesses to match the reality. As stated in Cont and Tankov [8](2004), the evolution of a stock price
shows evident discontinuity and scale variance, neither of which the classical Black-Scholes model has. What’s more, the empirical distribution of increments of the log-price\(^1\) results in heavier tails than the normal distribution. In addition, in real stock market, perfect hedging is not possible and options enable market participants to hedge risks that can’t be hedged by trading in the underlying only. Whereas, in the market derived by the classical Black-Scholes, perfect hedging is possible and options can be replicated by a self-financing strategy involving the underlying and cash. That is to say, options are just portfolios with stock and cash. This contradicts with our intuitions that the market of options exists for the reason that it can provide something that is not available in the market consisted of stock and cash. Stochastic volatility model such as the geometric Brownian motion with regime switching is a choice of refining the Black-Scholes model, and Zhang and Zhou [25](2009) considered the pricing problem of stock loan in this refined model. Moreover, it is well-known that the geometric Lévy model is usually regarded as an improvement, not only because it remains some attractive features of Black-Scholes model but also it satisfies the properties of the real market as mentioned above.

The market under both stochastic volatility model and geometric Lévy model is incomplete. Therefore, there is no unique price of any option and one price should be chosen. However, this difficulty can be handled by working with the risk-neutral measure directly instead of trying to find one risk-neutral measure from the original probability. This procedure is called risk-neutral modeling and it is reasonable because the stock process under all the risk-neutral measures remains to be a geometric Lévy martingale with different parameters. There are two basic ways to obtain the parameters. The first one is that man-made criterions can be set to obtain it if one holds the belief that there should be a internal relationship regardless of the movement of the stock price. Such criterions include relative entropy by Fujiwara and Miyahara[10](2003) and quadratic distance by Schweizer[21](1999). The second is that all the parameters can be calibrated directly from the price of options. The following papers [1, 3, 7] are on this topic. Of course, those two ways can be combined

\(^1\)The increments of the log-price is also called returns.
together to generate new ways as stated in [8]. By the way, Grasselli and Gómez[11](2012) is focusing on a different kind of incomplete market. Its incompleteness comes from the fiction of environment such as the transaction fees as well as unavailable continuous trading while the incompleteness mentioned above is from the model. Although such a contract can not be traded in the market, it is still meaningful to price it with risk-neutral measure for the reason that the bank who provides the contract can use relative strategy to hedge risks generated by the contract.

This paper will focus on extending the analytical tractability of the pricing problem of stock loan with the classical geometric Brownian motion to alternative models with jumps. In particular, the stock loan with geometric Lévy model will be chosen as main object of the present paper. With the Lévy jump part, however, it becomes very difficult to derive analytical solutions for the pricing problem. We are informed that Cai and Sun[6](2009) showed it in an double exponential jump-diffusion model which is a good example of Lévy process. Moreover, following Dai and Xu[9](2010), we also take the ways of dividends distribution into consideration. To the best of our knowledge, this is the first systematic presentation of the topic dealing the pricing problem of such stock loan with the geometric Lévy model. By variational inequalities(VI) approach, explicit value functions and optimal strategy of stock loan are derived here with three different ways of dividends distribution, and reasonable values of critical parameters, such as loan sizes, loan rates and service fees in terms of certain algebraic equations are also given. Our results also state that the variational inequalities approach can be extended to treat the valuation problem in more general setting involving the stock loan models with jumps. The main difficulty associated with the VI approach is that the corresponding VIs need to establish necessary conditions to find a certain of the functions of HJB is luckily equal to the value function. So, to overcome the difficult, a delicate analysis will be carried out by Cramér’s estimation of ruin for Lévy process, which is proved by using the excursion theory of Lévy process (cf. [4, 14]).

The rest of the paper is organized as follows. In Section 2, the mathematical description of infinite maturity stock loan with geometric Lévy
model is given. Section 3 is devoted to calculating the value function. In Section 4, we give the relationships among the parameters. In Sections 5 and 6, we investigate the same pricing problem with different dividends distributions. Finally, numerical studies are presented in Section 7.

2. Optimal stopping problems with geometric Lévy process

As mentioned above, one risk-neutral measure is chosen directly by making the discounted stock price, denoted as \( \tilde{S}_t \), a geometric Lévy martingale process. The parameters are \( \{\sigma, \nu(dz)\} \) where \( \sigma > 0 \) is volatility and \( \nu(dz) \) is the Lévy measure of \( \tilde{N}(dz, dt) \):

\[
\frac{d\tilde{S}_t}{\tilde{S}_{t^-}} = \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt),
\]

By the way, since the right side is a martingale, \( \tilde{S}_t \) is a martingale satisfying the requirements of one risk-neutral measure.

As mentioned above, the distribution of dividends matters, thus it should be specified before pricing. Since the main idea with different distributions remains invariant, one basic distribution mechanism, namely the dividends gained by the lender before redemption, is examined in detail here while the others are sketched in Section 5 and Section 6 below. Under this distribution, the dynamics of stock price \( S_t = e^{(r-\delta)t}\tilde{S}_t \) is as following by Itô’s formula.

\[
\frac{dS_t}{S_{t^-}} = (r-\delta)dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt),
\]

Noticing that once the distribution of dividends is given, \( \tilde{S}_t \) and \( S_t \) are equivalent. Since the latter is much more convenient to use, we begin with it instead of \( \tilde{S}_t \) below.

We now formulate the problem in a rigorous way: the uncertainty is described by a Lévy process \( \{X_t, t \geq 0\} \) on a probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \), where \( \{\mathcal{F}_t\}_{t \geq 0} \) is the filtration generated by \( X \), \( \mathcal{F}_0 = \sigma[\Omega, \emptyset] \) and \( \mathcal{F} = \sigma[\bigcup_{t \geq 0} \mathcal{F}_t] \). The stock price \( S \) follows the following geometric Lévy process,

\[
\frac{dS_t}{S_{t^-}} = dX_t = (r-\delta)dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt),
\]

\( \text{(2.1)} \)

\( ^2 \text{We denote this as the basic stock loan below and latter we use it to derive the pricing in other situations.} \)
where \( r \) is risk-less interest rate; \( \delta \geq 0 \) is dividend yield and \( \sigma > 0 \) is volatility; \( W_t \) and \( \tilde{N}(dz,dt) \) are, respectively, the Brownian motion and compensated Poisson measure with respect to the probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P) \), which will be proved to be the risk-neutral probability space of \( S_t \) in the appendix below. In order to highlight the main idea of Lévy process, a restriction \( \int_1^{+\infty} z \nu(dz) < +\infty \) is added here, under which condition the third term on the right hand side of Eq.(2.1) is well-defined. This restriction is reasonable since it illustrates that the effect of upward jumps isn’t infinity accorded with the reality.

Using Itô’s formula and following the method used in Chapter 1 of Øksendal and Sulem[18](2005), we can derive the following stochastic dynamic system,

\[
S_t = x \exp \left\{ \left[ r - \delta - \frac{1}{2} \sigma^2 + \int_{-1}^{+\infty} (\ln(1 + z) - z) \nu(dz) \right] t + \sigma W_t \right\} + \int_0^t \int_{-1}^{+\infty} \ln(1 + z) \tilde{N}(dz, ds),
\]

where \( x = S_0 \) is the initial stock price.

The payoff process of the basic stock loan is modeled as the following,

\[ P(t) = e^{-rt}(S_t - qe^{\gamma t})_+, \]

where \( \gamma \) is continuously compounding stock loan interest rate larger than risk-less rate, \( r \). According to theory of perpetual American options referring to [22], the initial value of this basic stock loan is

\[
V(x) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}[e^{-r\tau}(S_\tau - qe^{\gamma \tau})_+1_{(\tau < +\infty)}],
\]

where \( \tilde{r} = r - \gamma, \tilde{S}_t = e^{-\gamma t}S_t \), and \( \mathcal{T}_0 \) denotes all \( \{\mathcal{F}_t\}_{t \geq 0}\)-stopping times. We’d like to simplify the expression above by adding a natural definition

\[
e^{-\tilde{r}\tau}(S_\tau - qe^{\gamma \tau})_+1_{(\tau < +\infty)} = \lim_{\tau \to +\infty} e^{-r\tau}(S_\tau - qe^{\gamma \tau})_+,
\]

under which

\[
\lim_{n \to +\infty} e^{-r\tau_n}(S_{\tau_n} - qe^{\gamma \tau_n})_+ = e^{-\tilde{r}\tau}(S_\tau - qe^{\gamma \tau})_+,
\]

holds almost surely. Let

\[
Y_t = \ln \frac{e^{-\tilde{r}t}S_t}{x},
\]

(2.4)
then $Y_t$ is a Lévy process with $\mathbb{E}Y_1 = -\delta - \frac{1}{2} \sigma^2 + \int_{-\infty}^{+\infty} (\ln(1+z)-z)\nu(dz) < 0$. By Theorem 7.2 of Kyprianou[14](2006),

$$\lim_{t \to +\infty} Y_t = -\infty,$$

thus

$$\lim_{t \to +\infty} e^{-rt}S_t = \lim_{t \to +\infty} xe^Y_t = 0.$$

Actually, we have $0 < e^{-rt}(S_t - qe^{\gamma t})_+ \leq e^{-rt}S_t$, which implies that

$$e^{-rt}(S_t - qe^{\gamma t})_+1_{\{t=+\infty\}} = 0.$$

Then we can rewrite (2.3) as

$$V(x) = \sup_{\tau \in T_{t}} \mathbb{E}[e^{-\tilde{r}\tau}(S_\tau - qe^{\gamma \tau})_+].$$

The value process of this stock loan is

$$V_t(x) = \sup_{\tau \in T_{t}} \mathbb{E}[e^{-\tilde{r}(\tau-t)}(S_\tau - qe^{\gamma \tau})_+|\mathcal{F}_t],$$

i.e.,

$$e^{-rt}V_t(x) = \sup_{\tau \in T_{t}} \mathbb{E}[e^{-\tilde{r}\tau}(\tilde{S}_\tau - q)_+|\mathcal{F}_t],$$

where $T_{t}$ denotes all $\{\mathcal{F}_\tau\}_{t \geq 0}$-stopping times $\tau$ with $\tau \geq t$.

To avoid arbitrage, the fair values of $q, \gamma, c$ must satisfy

$$V(x) = x - q + c.$$  \hfill (2.7)

To determine the range of the fair values of the parameters $(q, \gamma, c)$, it suffices to calculate $V(x)$. Because $\tilde{r} = r - \gamma \leq 0$, the problem we concern with is essentially to calculate the initial value of a conventional perpetual American call option with a negative interest rate. This problem poses significant challenges, largely due to the negative interest rate. We firstly make a study of some properties of value function, which helps a lot in understanding the conditions given in Section 3.

**Proposition 2.1.** $V(x)$ is continuous and nondecreasing on $(0, +\infty)$ with $V(0) = 0$.

**Proof.** Rewrite (2.2) as the following,

$$V(x) = \sup_{\tau \in T_{t}} \mathbb{E}[g(x, \tau(\omega), \omega)].$$

For fixed $\omega$, the $g$ as a function of $x$ is clearly a convex function, by the linear property that expectation has, we know that $\mathbb{E}[g(x, \tau(\omega), \omega)]$ is also
a convex function. Following the same procedure as in Proposition 8.1 given in the appendix below, we know that function $V$ is convex, so it's continuous. And the nondecreasing property of $V$ follows directly from its definition. Noting that $x = 0$ implies $S_t \equiv 0$, so $V(0) = 0$ is directly from (2.5).

**Proposition 2.2.** $(x - q)_+ \leq V(x) \leq x$ for $x \geq 0$.

**Proof.** If we take $\tau = 0$ then the inequality $(x - q)_+ \leq V(x)$ easily follows. Since $e^{-r\tau}(S_\tau - qe^{r\tau})_+ \leq e^{-r\tau}S_\tau \leq e^{-(r-\delta)\tau}S_\tau$ and \{e^{-(r-\delta)\tau}S_\tau, $t \geq 0$\} is a martingale, by applying the optional stopping theorem and the Fatou’s lemma we have

$$V(x) \leq \sup_{\tau \in T_0} \mathbb{E}\left[e^{-(r-\delta)\tau}S_\tau\right]$$

$$= \sup_{\tau \in T_0} \mathbb{E}\left[\lim_{n \to +\infty} e^{-(r-\delta)\tau \wedge n}S_{\tau \wedge n}\right]$$

$$\leq \sup_{\tau \in T_0} \liminf_{n \to +\infty} \mathbb{E}\left[e^{-(r-\delta)\tau \wedge n}S_{\tau \wedge n}\right]$$

$$= \sup_{\tau \in T_0} \lim_{n \to +\infty} x = x.$$  

**Proposition 2.3.** (Xia and Zhou[23]) Let $k = \inf\{x > 0; x - q = V(x)\}$, where $\inf \emptyset \equiv +\infty$. Then $k \geq q$ and \{x > 0; x - q = V(x)\} = [k, +\infty).

**Proof.** It’s trivial for the case $k = +\infty$. By Proposition 2.2, $V(x) \geq (x - q)_+ > (x - q)$ when $x < q$, so $k \geq q$. If there is a $k_0 > k$ such that $V(k_0) \neq k_0 - q$ then $V(k_0) > k_0 - q$ from which we know $\beta \equiv \frac{V(k_0) - V(k)}{k_0 - k} > 1$.

By Proposition 2.1, $V(x)$ is convex, and so

$$\frac{V(x) - V(k)}{x - k} \geq \frac{V(k_0) - V(k)}{k_0 - k} = \beta, \quad \forall x \geq k_0,$$

or

$$V(x) \geq \beta x - k\beta + k - q, \quad \forall x \geq k_0,$$

which implies that $V(x) > x$ for sufficient large $x$. Thus we arrive at a contradiction with Proposition 2.2.  

**3. Valuations of basic stock loans by variational inequalities approach**

In this section we present the main result of this paper whose proof follows the following three steps. Firstly, a pure variational inequalities approach is
developed to prove that a function satisfying certain conditions including HJB equation must be sightly larger than the value function. Secondly, we try to find a certain kind of the functions in step one which is luckily equal to the value function. Finally, the explicit expression of the function mentioned in step two is derived.

Now we state the main result of this paper as follows.

**Theorem 3.1.** If $\delta > 0$, then the value function defined by (2.3) is

$$ V(x) = \begin{cases} \frac{(\gamma-1)^{l-1}}{(l^*)^{l-1}}q^{l^*-1}x^{l^*}, & 0 \leq x < x^*, \\ x-q, & x \geq x^*, \end{cases} $$

where $x^* = \frac{r}{r-1}q$ and $l^* > 1$ is the solution of the following equation,

$$ h(l) \triangleq -\tilde{r} + \tilde{\mu}l + \frac{1}{2}\sigma^2l(l-1) + \int_{-1}^{+\infty} ((1+z)l - 1 - lz) \nu(dz) = 0, $$

and $\tilde{\mu} = r - \gamma - \delta$.

**Remark 3.1.** As a matter of fact, $\int_{1}^{+\infty} z \nu(dz) < +\infty$ can also be regarded as a fundamental assumption we need throughout this paper.

In order to state the proof of Theorem 3.1, we first introduce the following infinitesimal generator $\mathcal{L}$ defined by

$$ \mathcal{L} f(x) = -\tilde{r} f(x) + \tilde{\mu} x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) + \int_{-1}^{+\infty} (f(x(1+z)) - f(x) - f'(x)z) \nu(dz). $$

For more information about the infinitesimal generator $\mathcal{L}$, we refer to the reader to Revuz and Yor[20]. Next, we try to find a function slightly larger than the value function.

**Theorem 3.2.** The function $f(x)$ is larger than value function, i.e., $f(x) \geq V(x)$, if $f(x)$ satisfies the following conditions:

1) $f(0) = 0, f(x) \leq x$;
2) there is an $x^* > 0$ such that $f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^*\})$;
3) $\max \{\mathcal{L} f(x), (x-q)_+ - f(x)\} = 0$.

**Proof.** From the Itô-Tanaka formula, we know that the Meyer-Itô formula (see Protter[19]) holds for the functions $f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^*\})$ which

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3 This is exactly the HJB equation.
Using (2.1) and Itô formula we have
\[ Y_{t} \]

Recall that by using Itô formula again
\[ C \]
are not necessarily in \( \mathbb{C}^2 \). For details we refer to Chapter 6 of Revuz and Yor[20].

Using (2.1) and Itô formula we have
\[
d\tilde{S}_t = d\exp(-\gamma t)S_t = -\gamma e^{-\gamma t}S_t dt + \sigma e^{-\gamma t}S_t dW_t \]
\[
= e^{-\gamma t}S_t \left( (r - \delta) dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right) - \gamma e^{-\gamma t}S_t dt \\
= e^{-\gamma t}S_t \left( (r - \gamma - \delta) dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right) \\
= \tilde{S}_t \left( \tilde{\mu} dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right).
\]

Hence by the Meyer-Itô formula
\[
d f(\tilde{S}_t) = f'(\tilde{S}_t) d\tilde{S}_t + \frac{1}{2} f''(\tilde{S}_t) d[\tilde{S}]_t \\
+ d\left( \sum_{\Delta \geq 0} \left[ f(\tilde{S}_t) - f(\tilde{S}_{t^-}) - f'(\tilde{S}_{t^-}) \Delta \tilde{S}_t \right] \right) \\
= f'(\tilde{S}_t) \tilde{S}_t \left( \tilde{\mu} dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right) + \frac{1}{2} f''(\tilde{S}_t) \tilde{S}_t^2 \sigma^2 dt \\
+ \int_{-1}^{+\infty} \left( f(\tilde{S}_t(1 + z)) - f(\tilde{S}_t) - f'(\tilde{S}_t) z\tilde{S}_t \right) \tilde{N}(dz, dt) \\
+ \int_{-1}^{+\infty} \left( f(\tilde{S}_t(1 + z)) - f(\tilde{S}_t) - f'(\tilde{S}_t) z\tilde{S}_t \right) \nu(dz) dt.
\]

So by using Itô formula again \(^4\) we obtain
\[
d \left( \exp(-\gamma t) f(\tilde{S}_t) \right) \\
= -\gamma \exp(-\gamma t) f(\tilde{S}_t) dt + \exp(-\gamma t) df(\tilde{S}_t) \\
= \exp(-\gamma t) \left( -\gamma f(\tilde{S}_t) + \tilde{\mu} \tilde{S}_t f'(\tilde{S}_t) + \frac{1}{2} \sigma^2 \tilde{S}_t^2 f''(\tilde{S}_t) \right) dt \\
+ \exp(-\gamma t) \int_{-1}^{+\infty} \left( f(\tilde{S}_t(1 + z)) - f(\tilde{S}_t) - f'(\tilde{S}_t) z\tilde{S}_t \right) \nu(dz) dt \\
+ \exp(-\gamma t) \int_{-1}^{+\infty} \left( f(\tilde{S}_t(1 + z)) - f(\tilde{S}_t) - f'(\tilde{S}_t) z\tilde{S}_t \right) \tilde{N}(dz, dt) \\
= \exp(-\gamma t) \mathcal{L} f(\tilde{S}_t) dt \\
+ \exp(-\gamma t) \left( \tilde{S}_t f'(\tilde{S}_t) \sigma dW_t + \int_{-1}^{+\infty} \left( f(\tilde{S}_t(1 + z)) - f(\tilde{S}_t) \right) \tilde{N}(dz, dt) \right).
\]

\(^4\)Since \( \tilde{S}(t) \) only has countable jumps, the equality holds almost everywhere with respect to Lebesgue measure on \( \mathbb{R}_+ \).
Since the second term on the right hand side (RHS) of Eq.(3.1) is a martingale whose expectation is zero, taking expectations in Eq.(3.1) we have

$$\mathbb{E}\left[e^{-\tilde{r}t} f(\tilde{S}_t) \right] - f(x) = \mathbb{E} \int_0^t e^{-\tilde{r}u} \left( L f(\tilde{S}_{u-}) \right) du. \tag{3.2}$$

By the condition 3), we know $L f(x) \leq 0$, from which we deduce that $\mathbb{E}\left[e^{-\tilde{r}t} f(\tilde{S}_t) \right] \leq f(x)$ holds for all $t \geq 0$. In addition, by the same proof as in Remark 3.3 below, the inequality holds for any bounded stopping time. In fact, by using the method of truncation as follows,

$$\mathbb{E}\left[e^{-\tilde{r}\tau} f(\tilde{S}_{\tau}) \right] = \mathbb{E}\left[\lim_{n \to +\infty} e^{-\tilde{r}\tau\land n} f(\tilde{S}_{\tau\land n}) \right] \leq \liminf_{n \to +\infty} \mathbb{E}\left[e^{-\tilde{r}\tau\land n} f(\tilde{S}_{\tau\land n}) \right] \leq f(x).$$

Thus the inequality holds for any stopping time $\tau$. Moreover, by condition 3), we have $(x - q) \leq f(x)$, so

$$\mathbb{E}\left[e^{-\tilde{r}\tau}(\tilde{S}_{\tau} - q) \right] \leq \mathbb{E}\left[e^{-\tilde{r}\tau} f(\tilde{S}_{\tau}) \right] \leq f(x).$$

Taking $\sup_{\tau \in \mathcal{T}_0}$ in the last inequality, we get $V(x) \leq f(x)$ and the proof is complete. \hfill \Box

**Remark 3.2.** Keep in mind that our main goal is to find the explicit expression of value function $V(x)$. Theorem 3.2 focuses on finding $f(x)$, which is close to $V(x)$ instead of far larger ones, so $f(x)$ is expected to share some properties with $V(x)$. As a matter of fact, Propositions 2.1, 2.2 and 2.3 provide some clues to all the conditions here.

**Remark 3.3.** To emphasize the main idea of the proof on Theorem 3.2, Eq.(3.2) is presented with determined time $t$ instead of bounded stopping time, such as $\tau \land n$. The proof for $\tau \land n$ follows the next steps. Firstly, notice that (3.1) is same for both cases. Secondly, by optional stopping theorem, the integral in the second term on RHS of Eq.(3.1) with upper limit $\tau \land n$ is still a martingale from which its expectation is zero.

In what follows, we try to find the specific function belongs to the class of functions $f$ in Theorem 3.2 and is equal to the value function.

**Theorem 3.3.** Assume that $\delta > 0$ and an increasing function $f(x)$ satisfies the following conditions:

1) $f(0) = 0, f(x) \leq x$;
2) there is an $x^* > q$ such that $f \in C^1(R_+) \cap C^2(R_+ \setminus \{x^*\})$;
3) $L f(x) = 0, (x - q)_+ < f(x), \forall x < x^*$.
4) \( \mathcal{L}f(x) < 0, \ x - q = f(x), \ \forall x \geq x^* \).

Then \( f(x) \) must be equal to the value function \( V(x) \). Moreover, the optimal stopping time is \( \tau^* = \text{inf} \{ t : \hat{S}_{t-} > x^* \} \).

In order to prove this theorem, let us firstly introduce Cramér’s estimation of ruin for Lévy process as a powerful tool. It’s proved by using the excursion theory of Lévy process, which is explained well in Kyprianou[14] (2006) and Bertoin and Doney[4](1994), so the following lemma is stated without proof.

**Lemma 3.1.** Assume that \( X \) is a Lévy process without monotone paths. If the conditions below hold,

i) \( \lim_{t \to +\infty} X_t = -\infty \),

ii) there exists a \( \xi \in (0, +\infty) \) such that \( \psi(\xi) = 0 \), where \( \psi(\theta) = \log \mathbb{E}(\exp{\theta X_1}) \) is the Laplace exponent of \( X \),

then

\[
\lim_{x \to +\infty} e^{\xi x} P(\tau_x^+ < +\infty) < +\infty,
\]

where \( \tau_x^+ = \text{inf}\{ t : X_t > x \} \). And, as a direct consequence, there exists a constant \( C \) such that the following inequality holds for any \( x > 0 \)

\[
e^{\xi x} P(\tau_x^+ < +\infty) < C. \tag{3.3}
\]

Then, combining Lemma 3.1 and the definition of \( Y_t \) from (2.4), we now prove the integrability of \( \sup_{0 \leq t \leq +\infty} e^{Y_t} \). It’s of great importance because \( \sup_{0 \leq t \leq +\infty} e^{Y_t} \) can be used as a dominated function for \( e^{-r(\tau - q)} \), here \( \tau \) is a stopping time.

**Corollary 3.1.** If \( \delta > 0 \), then \( \mathbb{E}\left[ \sup_{0 \leq t \leq +\infty} e^{Y_t} \right] < +\infty. \)

**Proof.** Applying a approach being somewhat like that of Lévy -Itô decomposition in Applebaum[2] we have

\[
\phi(\theta) = \left( -\delta - \frac{1}{2} \sigma^2 + \int_{-1}^{+\infty} (\ln(1 + z) - z) \nu(dz) \right) \theta + \frac{1}{2} \sigma^2 \theta^2 \\
+ \int_{-1}^{+\infty} \left( (1 + z)^\theta - 1 - \theta \ln(1 + z) \right) \nu(dz)
\]

\[
= (-\delta - \frac{1}{2} \sigma^2) \theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{-1}^{+\infty} \left( (1 + z)^\theta - 1 - \theta z \right) \nu(dz). \tag{3.4}
\]

If the third term on RHS of the last equation of (3.4) is finite, we denote it by \( I(\theta) \). Note that, for \( z > -1, (1 + z)^\theta - 1 - \theta z \) is increasing with respect to
\( \theta \), thus by monotone convergence theorem \( I(\theta) \) is continuous with respect to \( \theta \). On the other hand, since \((1 + z)^{\theta} - 1 - \theta z \geq 0, \forall z > -1 \) and \( \theta > 1 \), \( I(\theta) \) is either finite for all \( \theta > 1 \) or positive infinite for some \( \theta > 1 \). For the latter case, using the continuity of \( I(\theta) \) we know that there is a minimal number \( \theta_0 > 1 \) such that \( \forall \theta < \theta_0, I(\theta) < +\infty \) and \( I(\theta_0) = +\infty \). We’d like to study the two cases, respectively.

For the former case, \( \phi(\theta) \) is a quadratic form plus a finite increasing non-negative function. Since the coefficient of \( \theta^2 \) is positive, \( \lim_{\theta \to +\infty} \phi(\theta) = +\infty \), so using \( \phi(1) = -\delta < 0 \) we know that there exists a number \( \xi > 1 \) such that \( \phi(\xi) = 0 \).

For the latter case, by \( \lim_{\theta \to \theta_0} I(\theta) = +\infty \), we know \( \lim_{\theta \to \theta_0} \phi(\theta) = +\infty \), which also implies that there exists a number \( \xi > 1 \) such that \( \phi(\xi) = 0 \) due to \( \phi(1) = -\delta < 0 \).

So we have just proved that condition 2) of Lemma 3.1 holds and condition 1) has also been proved in Section 2 above, so by Lemma 3.1

\[ e^{\xi x} P(\tau_x^* < +\infty) < C. \]

Therefore we deduce directly from the last inequality and definitions of \( Y_t \) and \( \tau_x^* \) that

\[ P\left( \sup_{0 \leq t < +\infty} e^{Y_t} > Q \right) = P\left( \sup_{0 \leq t < +\infty} Y_t > \ln Q \right) = P(\tau^*_Q < +\infty) < CQ^{-\xi}, \]

where \( Q > 0 \). Furthermore, since \( \xi > 1 \) we have

\[ \mathbb{E}\left[ \sup_{0 \leq t < +\infty} e^{Y_t} \right] = \int_0^{+\infty} P(\sup_{0 \leq t < +\infty} e^{Y_t} > Q) dQ \leq \int_0^{+\infty} CQ^{-\xi} dQ < +\infty. \]

Thus the proof is complete. \( \square \)

We now return to the proof of Theorem 3.3.

**Proof.** It’s obvious that \( f(x) \) here satisfies all the conditions in Theorem 3.2, thus \( f(x) \geq V(x) \), so we only need to prove the converse inequality: \( f(x) \leq V(x) \) also holds. By the definition of \( V(x) \) defined by (2.3), \( f(x) \leq V(x) \) can be reduced to proving existence of a stopping time \( \tau^* \) such that \( f(x) = \mathbb{E}\left[ e^{-\tau^*} (\bar{S}_{\tau^*} - q)^+ \right] \).
By (3.2) and inspecting the proof used therein

\[
\mathbb{E}\left[ e^{-r\tau^+}f(\bar{S}_{\tau^+}) \right] - f(x) = \mathbb{E} \int_0^{\tau^+} e^{-ru} \left( Lf(\bar{S}_u) \right) du
\]

\[
= \mathbb{E} \int_0^{\tau^+} e^{-ru} \left( \bar{r}q - \delta \bar{S}_u \right) 1_{\bar{S}_u > x} du
\]

\[
= \mathbb{E} \int_0^{\tau^+} e^{-ru} \left( \bar{r}q - \delta \bar{S}_u \right) 1_{u > \tau^+} du
\]

\[
= 0.
\]

Since

\[
e^{-r\tau^+}f(\bar{S}_{\tau^+}) \leq e^{-r\tau^+}\bar{S}_{\tau^+} = e^{-r\tau^+}S_{\tau^+} \leq x \sup_{0 \leq t < +\infty} e^{Y_t},
\]

by the Dominated convergence theorem we have

\[
f(x) = \mathbb{E}\left[ e^{-r\tau^+}f(\bar{S}_{\tau^+}) \right] = \lim_{n \to +\infty} \mathbb{E}\left[ e^{-r\tau^+}f(\bar{S}_{\tau^+}) \right]
\]

\[
= \mathbb{E}\left[ \lim_{n \to +\infty} e^{-r\tau^+}f(\bar{S}_{\tau^+}) \right] = \mathbb{E}\left[ e^{-r\tau^+}f(\bar{S}_{\tau^+}) \right].
\]

By definition of $\tau^*$, $\bar{S}_{\tau^*} \geq x^*$, so $f(\bar{S}_{\tau^*}) = (\bar{S}_{\tau^*} - q)_+$, which, together with the last equation of (3.6), ends the proof. \qed

**Remark 3.4.** We emphasize that $x^*$ is predetermined, i.e. has nothing to do with $x$. However, $\tau^* \equiv \inf \left\{ t : Y_{t-} > \ln \frac{\bar{r}}{x} \right\}$, and it is indeed a function of $x$, so we will denote it as $\tau^*(x)$ if necessary.

Before going any further, we take a look at $f'(0)$, which maybe provides some clues and ideas on deriving the explicit expression of the value function below.

**Proposition 3.1.** If $f(x)$ in Theorem 3.3 exists, then $f'(0) = 0$.

**Proof.** By definition,

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x}
\]

\[
= \lim_{x \to 0} \mathbb{E} \left[ \frac{e^{-r\tau^+}(S_{\tau^+(x)} - qe^{r\tau^+(x)})_+}{x} \right].
\]

Since $\frac{e^{-r\tau^+}(S_{\tau^+(x)} - qe^{r\tau^+(x)})_+}{x}$ is dominated by $\sup_{0 \leq t < +\infty} \frac{e^{-rt}S_t}{x}$ which is independent of $x$, by Corollary 3.1 we know that the latter is integrable. Applying the
Dominated convergence theorem we have
\[
\lim_{x \to 0} \mathbb{E} \left[ e^{-r \tau} (S_{\tau} - q e^{r \tau}) \right] = \mathbb{E} \left[ \lim_{x \to 0} \frac{e^{-r \tau} (S_{\tau} - q e^{r \tau})}{x} \right]
\]
\[
= \mathbb{E} \left[ \lim_{x \to 0} \frac{(e^{-r \tau} S_{\tau} - q e^{r \tau})}{x} \right]
\]
\[
\leq \mathbb{E} \left[ \lim_{x \to 0} \frac{(e^{-r \tau} S_{\tau} - q)}{x} \right]
\]
\[
\leq \mathbb{E} \left[ \lim_{x \to 0} \left( \sup_{0 \leq t < +\infty} e^{-r t} S_t x - q \right) \right].
\]
(3.8)

So \( f'(0) = 0 \) due to \( \sup_{0 \leq t < +\infty} e^{-r t} S_t x \) is independent of \( x \) and \( \lim_{x \to 0} \frac{q}{x} = +\infty \).

The proof is thus complete. \( \Box \)

Finally, we return to the proof of Theorem 3.1. Instead of only checking the function in Theorem 3.1 satisfies all the conditions in Theorem 3.3, we show the following Theorem 3.4, which is more illustrative to show the idea of finding the explicit expression of value function \( V(x) \) and the optimal stopping time.

**Theorem 3.4.** If \( \delta > 0 \), then the following function satisfies all the conditions in Theorem 3.3

\[
V(x) = \begin{cases} 
\frac{(l^*-1)^{l^*-1}}{(l^*)^{l^*-1}} q \, x^{l^*-1}, & \text{if } 0 \leq x < x^*, \\
q, & \text{if } x \geq x^*, 
\end{cases}
\]

where \( l^* > 1 \) is the unique solution of \( h(l) = 0 \) (see Theorem 3.1) and \( x^* = \frac{\bar{S}}{\bar{r} - l} q \). Moreover, the optimal stopping time is \( \tau^* = \inf \{ t : \bar{S}_{t-} > x^* \} \).

**Proof.** Checking the procedure of calculating the value function in Liang, Wu and Jiang [16] on the classical Black-Scholes model we know that if \( x \) smaller than a fixed number \( x^* \), then the value function is a power function. We’d like to try the same idea, i.e., we guess that \( f(x) = C_1 x^l \), where \( C_1 > 0 \), \( l > 0 \) if \( x \) is smaller than \( x^* \), a number to be determined later. Then

\[
\mathcal{L}(x^l) = -\bar{r} x^l + \bar{\mu} l x^l + \frac{1}{2} \sigma^2 (l - 1) x^l + \int^{+\infty}_{-1} \left( x^l (1 + z)^l - x^l - l z x^l \right) \nu(dz)
\]
\[
= x^l \left( -\bar{r} + \bar{\mu} l + \frac{1}{2} \sigma^2 (l - 1) + \int^{+\infty}_{-1} \left( (1 + z)^l - 1 - l z \right) \nu(dz) \right)
\]
\[
= x^l h(l),
\]
(3.9)
so \( \mathcal{L}(x') = 0 \ \forall 0 < x < x^* \) is equal to \( h(l) = 0 \). It’s easy to observe that \( h(l) \) may have solution smaller than 1. However, by Proposition 3.1, we have \( f'(0) = 0 \), therefore only those solutions being larger than 1 are needed. Noting that \( \phi(l) \) in (3.4) has similar forms to \( h(l) \), i.e.,

\[
 h(l) = \phi(l) + (r - \gamma)(l - 1),
\]

and \( h(1) = \phi(1) = -\delta < 0 \), so by the analogous analysis with Corollary 3.1, we know that at least there exists a number \( l^* > 1 \) such that \( h(l^*) = 0 \).

At last, by smooth fit principle and using the continuity of \( f(x) \) at \( x^* \), we solve the following equation,

\[
 C_1(x^*)^r = (x^* - q),
\]

\[
 C_1l^r(x^*)^{r-1} = 1.
\]

Then \( x^* = \frac{l^r - 1}{r - 1} q \), \( C_1 = \frac{(l^r - 1)^{r-1}}{(r - 1)^r - q^{1-r}} \). So

\[
 f(x) = \begin{cases} 
 \frac{(l^r - 1)^{r-1}}{(r - 1)^r - q^{1-r}} x^r, & \text{if } 0 \leq x < x^*, \\
 x - q, & \text{if } x \geq x^*. 
\end{cases} \tag{3.10}
\]

It’s easy to prove that the \( f(x) \) satisfies all the conditions in Theorem 3.3, therefore \( V(x) = f(x) \) and the optimal stopping time is \( \tau^* = \inf \{ t : \tilde{S}_t > x^* \} \). It remains to examine the uniqueness of \( l^* \). By simple calculation, we have \( h''(l) = \sigma^2 + \int_1^{\infty} (\ln(1 + z))^2(1 + z)^{\nu}(dz) > 0 \) which means that \( h(l) \) is convex. So by \( h(1) < 0 \), the \( l^* \) is unique, thus we end the proof. \( \square \)

**Remark 3.5.** If there is no jump term in (2.1), it’s the classical price problem for Black-Scholes model. \( h(l) = 0 \) becomes a quadratic equation of \( l \),

\[
 \frac{\sigma^2}{2} l^2 + \left( \bar{r} - \delta - \frac{\sigma^2}{2} \right) l - \bar{r} = 0.
\]

This equation has two solutions

\[
 l_1' = \frac{1}{\sigma} \left( \sqrt{\left( \frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma} \right)^2 + 2\delta + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}} \right) > 1,
\]

\[
 l_2' = \frac{1}{\sigma} \left( - \sqrt{\left( \frac{\sigma}{2} - \frac{\gamma - r + \delta}{\sigma} \right)^2 + 2\delta + \frac{\sigma}{2} + \frac{\gamma - r + \delta}{\sigma}} \right) \leq 1.
\]

This is exactly the result in Xia and Zhou [23](2007).
Remark 3.6. Using the Markov property of $S_t$, by (2.3) and (2.6), we can obtain the expression of the value process

$$V_t = \underset{\tau \in T_t}{\text{esssup}} \mathbb{E} \left[ e^{-r(t-\tau)} (S_\tau - q e^{\gamma \tau}) \bigg| F_t \right]$$

$$= e^{\gamma t} \underset{\tau \in T_t}{\text{esssup}} \mathbb{E} \left[ e^{-r(t-\tau)} (e^{-\gamma \tau} S_t \exp \left( \left[ r - \delta - \frac{1}{2} \sigma^2 + \int_{-1}^{+\infty} \ln(1 + z) \nu(dz) \right] (\tau - t) + \sigma (W_\tau - W_t) + \int_{t}^{\tau} \int_{-1}^{+\infty} \ln(1 + z) \tilde{N}(dz, ds) - q e^{\gamma (t-\tau)} \right) \bigg| F_t \right]$$

$$= e^{\gamma t} \sup_{\tau \in T_0} \mathbb{E} \left[ e^{-r(t-\tau)} \left( x \exp \left( \left[ r - \delta - \frac{1}{2} \sigma^2 + \int_{-1}^{+\infty} \ln(1 + z) \nu(dz) \right] \tau + \sigma W_\tau + \int_{0}^{\tau} \int_{-1}^{+\infty} \ln(1 + z) \tilde{N}(dz, ds) - q e^{\gamma \tau} \right) \bigg|_{x = e^{-\gamma \tau} S_t} \right]$$

$$= e^{\gamma t} V(e^{-\gamma \tau} S_t).$$

(3.11)

Remark 3.7. The reason why we can’t handle the case $\delta = 0$ as Xia and Zhou [23] did is that $\lim_{n \to +\infty} \mathbb{E} \left[ e^{-rn} f(\tilde{S}_n 1_{(\tau \geq n)}) \right]$ is hard to prove to be 0 for Lévy model, the reader maybe can find a approach to fix it.

4. Ranges of fair values of parameters

In Section 2, since there is no arbitrage, parameters $q$, $c$, $\gamma$ shall satisfy (2.7). In the view of Theorem 3.4, there are two cases can be dealt with as follows.

If $x^* \leq x = S_0$, by (3.10), $f(x) = x - q$. Comparing with (2.7), we know that the fee $c$ shall be 0. And by the definition of $\tau^*$, we know that the optimal time of the stock loan is 0, which means that the optimal choice is not having this loan, the bank does not charge a service fee for its service since the stock price is large, so the bank and the client do not have enough incentive to do business.

If $x^* > x = S_0$, by (3.10), $f(x) = \frac{(l^* - 1)^{l^*-1}}{(l^*)^{l^*}} q^{1-l^*} x^{l^*}$, so the parameters $q$, $c$, $\gamma$ shall satisfy

$$\frac{(l^* - 1)^{l^*-1}}{(l^*)^{l^*}} q^{1-l^*} x^{l^*} = x - q + c. \quad (4.12)$$
Rewriting the equality by dividing with \(x\),
\[
\frac{(l^* - 1)^{-1}}{(l^*)^r} \left( \frac{q}{x} \right)^{1-r} = 1 - \frac{q - c}{x}.
\] (4.13)
Letting \(\frac{q}{x} \rightarrow +\infty\) we have
\[
\lim_{x \rightarrow +\infty} \frac{q - c}{x} = 1,
\] (4.14)
where \(c\) is a function of \(\frac{q}{x}\) such that the limit is well-defined. Recalling that \(q - c\) presents the amount the borrower get from the loan while \(x\) is the initial stock price, (4.14) can be explained in the following way: for any given \(q\), the less \(x\) is, the more impossible the borrower to redeem the stock is, that is, the more likely to directly sell the stock. Moreover, the limit of the probability for redeeming can be calculated rigorously as the following:
\[
\lim_{x \rightarrow 0} P(\tau^*(x) < +\infty) \leq \lim_{x \rightarrow 0} P\left(\tau^*_\tau < +\infty\right) \leq \lim_{x \rightarrow 0} Ce^{-\xi x} = 0.
\] (4.15)
Since the idea of such analysis is similar for other the distribution of dividends, here we omit all the ranges of parameters for all other distributions.

5. **Reinvested dividends returned to the borrower on redemption**

Firstly, if all dividends are reinvested, then the share held at time \(t\), denoted it by \(A(t)\), will be \(e^{\delta t}\) due to \(dA(s) = A(s)\delta ds\) for \(\forall s \in [0, t]\), which means the dividends reinvested at time \(s\) cause an increment of the share with size \(\delta\).

Secondly, the price of all the stock held at time \(t\) is exactly \(A(t)S(t)\), where \(S(t)\) is the stock price. So the value function is
\[
V(x) = \sup_{\tau \in \mathcal{T}_0} E\left[e^{-r\tau}(e^{\delta \tau}S_\tau - qe^{r\tau})_+\right].
\]
It’s a special case of basic stock loan with \(\delta = 0\) which we can’t deal with, so we consider only reinvest part, say \(0 < \alpha < 1\), of the dividends. Applying the same analysis above, the value function is
\[
V_1(x) = \sup_{\tau \in \mathcal{T}_0} E\left[e^{-r\tau}(e^{\alpha \delta \tau}S_\tau - qe^{r\tau})_+\right],
\]
which is still a basic stock loan replacing original \(\delta\) by \((1 - \alpha)\delta\), the results remain.
6. Dividends always delivered to the borrower partly

Noticing that the value of an infinite maturity stock loan lies in the dividends gained forever, \( S_0 = \mathbb{E}\left( \int_0^{\infty} e^{-rt} \delta S_t dt \right) \), so it’s meaningless to set a stock loan delivering all the dividends to the borrower, in which the borrower can gain all the value of the stock loan simply by never redeeming. Nevertheless, delivering part, namely \( 0 < \beta < 1 \), of the dividends to the borrower is acceptable, in which the value function follows

\[
V_2(x) = \sup_{\tau \in T_0} \mathbb{E}\left[ e^{-r\tau} (S_\tau - qe^{r\tau})_+ + \int_0^\tau \beta e^{-ru} \delta S_u du \right].
\] (6.16)

We claim that

**Theorem 6.1.** If \( \delta > 0 \), then the value function is

\[
V_2(x) = \begin{cases} 
(1 - \beta) l^*(l^* - 1) l^* - 1 & \text{if } 0 \leq x < x^*; \\
(1 - \beta) l^* - 1 & \text{if } x \geq x^*,
\end{cases}
\]

where \( l^* > 1 \) is the solution of \( h(l) = 0 \).

Because the proof of Theorem 6.1 shares similar idea to Theorem 3.1, we only give its sketch here.

**Theorem 6.2.** The function \( f(x) \) is larger than value function, i.e., \( f(x) \geq V_2(x) \), if \( f(x) \) satisfies the following conditions:

1) \( f(0) = 0, f(x) \leq x \);
2) there is an \( x^* > 0 \) such that \( f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^*\}) \);
3) \( \max \{ Lf(x) + \beta \delta x, (x - q)_+ - f(x) \} = 0 \).

**Proof.** Since the Eq.(3.2) also holds here, by adding \( \mathbb{E}\left[ \int_0^\tau \beta e^{-ru} \delta S_u du \right] \) to the both sides of the Eq.(3.2) we have the following

\[
\mathbb{E}\left[ e^{-r\tau} f(\bar{S}_\tau) + \int_0^\tau \beta e^{-ru} \delta S_u du \right] - f(x) = \mathbb{E}\left[ \int_0^\tau e^{-ru} \left( Lf(\bar{S}_u) + \beta \delta S_u \right) du \right].
\] (6.17)

By conditions 3)

\[
\mathbb{E}\left[ e^{-r\tau} f(\bar{S}_\tau) + \int_0^\tau \beta e^{-ru} \delta S_u du \right] \leq f(x).
\]

By Remark 3.3, the last inequality holds for any bounded stopping time \( \tau \wedge n \), then by the Fatou’s lemma, it also holds for all stopping time, i.e.,

\[
\mathbb{E}\left[ e^{-r\tau} f(\bar{S}_\tau) + \int_0^\tau \beta e^{-ru} \delta S_u du \right] \leq f(x).
\]
Moreover, by using condition 3) again, \((x - q)_+ \leq f(x)\), so

\[
\mathbb{E} \left[ e^{-\beta t} (\tilde{S}_t - q)_+ + \int_0^t \beta e^{-rt} \delta S_t dt \right] \leq \mathbb{E} \left[ e^{-\beta t} f(\tilde{S}_t) + \int_0^t \beta e^{-rt} \delta S_t dt \right] \leq f(x).
\]

Taking \(\sup_{t \in T_0} V_2(x) \leq f(x)\), which ends the proof. \(\square\)

**Theorem 6.3.** Assume that \(\delta > 0\). If an increasing function \(f(x)\) satisfies the following conditions:

1) \(f(0) = 0, \ f(x) \leq x\);
2) there is an \(x^{**} > q\) such that \(f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^{**}\})\);
3) \(L f(x) = -\beta \delta x, \ (x - q)_+ < f(x), \ \forall x < x^{**}\);
4) \(L f(x) = -\beta \delta x, \ x - q = f(x), \ \forall x \leq x^{**}\),
then \(f(x)\) must be equal to the value function \(V_2(x)\). Moreover, the optimal stopping time is \(\tau^{**} \equiv \inf \{ t : \tilde{S}_t > x^{**} \}\).

**Proof.** Proceeding similar procedure as in the proof of Theorem 3.3, it would suffice to prove that \(f(x) = \mathbb{E} \left[ e^{-\beta t} (\tilde{S}_t - q)_+ + \int_0^t \beta e^{-rt} \delta S_t dt \right]\).

By (6.17) we have

\[
\mathbb{E} \left[ e^{-\beta t} f(\tilde{S}_t - q)_+ + \int_0^t \beta e^{-rt} \delta S_t dt \right] = \mathbb{E} \left[ e^{-\beta t} f(\tilde{S}_t - q)_+ + \int_0^t \beta e^{-rt} \delta S_t dt \right] - f(x).
\]

\[
= \mathbb{E} \int_0^{\tau^{**} \wedge n} e^{-\beta u} \left( L f(\tilde{S}_u - q) + \beta \delta S_u \right) du
\]

\[
= \mathbb{E} \int_0^{\tau^{**} \wedge n} e^{-\beta u} \left( \tilde{r} q - \delta S_u - \beta \delta \tilde{S}_u \right) 1_{\tilde{S}_u > x^{**}} du
\]

\[
= \mathbb{E} \int_0^{\tau^{**} \wedge n} e^{-\beta u} \left( \tilde{r} q - \delta S_u - \beta \delta \tilde{S}_u \right) 1_{u \geq \tau^{**}} du
\]

\[
= 0.
\]

So, since \(e^{-\beta t} f(\tilde{S}_t - q)_+\) is dominated by \(\sup_{0 \leq t \leq +\infty} e^{Y_t}\) which is integrable and \(\int_0^{\tau^{**} \wedge n} \beta e^{-rt} \delta S_t dt\) is an increasing function with respect to \(n\), by the Dominated convergence theorem and the monotone convergence theorem.
we have

\[ f(x) = \mathbb{E} \left[ e^{-r^* \gamma} f(\mathcal{S}_{r^* \gamma}) + \int_0^{r^* \gamma} \beta e^{-r \delta S_t} dt \right] \]

\[ = \lim_{n \to +\infty} \mathbb{E} \left[ e^{-r^* \gamma} f(\mathcal{S}_{r^* \gamma}) + \int_0^{r^* \gamma} \beta e^{-r \delta S_t} dt \right] \]

\[ = \mathbb{E} \left[ \lim_{n \to +\infty} e^{-r^* \gamma} f(\mathcal{S}_{r^* \gamma}) + \int_0^{r^* \gamma} \beta e^{-r \delta S_t} dt \right] \]

\[ = \mathbb{E} \left[ e^{-r^* \gamma} f(\mathcal{S}_{r^* \gamma}) + \int_0^{r^* \gamma} \beta e^{-r \delta S_t} dt \right]. \]  

(6.19)

By definition of \( \tau^* \), \( \mathcal{S}_{r^* \gamma} > x^* \), so \( f(\mathcal{S}_{r^* \gamma}) = (\mathcal{S}_{r^* \gamma} - q)_+ \), the proof is therefore complete. \( \square \)

**Proposition 6.1.** If \( f(x) \) in Theorem 6.3 exists, then \( f'(0) = \beta \).

**Proof.** By Theorem 6.3 and the definition of \( V_2(x) \),

\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} \]

\[ = \lim_{x \to 0} \mathbb{E} \left[ \frac{e^{-r^*r(x)}(S_{r^*r(x)} - qe^{r^*r(x)})}{x} \right] + \lim_{x \to 0} \mathbb{E} \left[ \frac{\int_0^{r^*r(x)} \beta e^{-ru \delta S_u} du}{x} \right]. \]  

(6.20)

The first term on RHS of the last equation of the Eq.(6.20) can be proved to be exactly 0 via the same approach as in Proposition 3.1. Since, for fixed \( \omega \), \( \tau^*(x) \) is an increasing function with respect to \( x \) and \( e^{-ru \delta S_u} \) is independent of \( x \), by the monotone convergence theorem we have

\[ \lim_{x \to 0} \mathbb{E} \left[ \frac{\int_0^{r^*r(x)} \beta e^{-ru \delta S_u} du}{x} \right] = \mathbb{E} \left[ \lim_{x \to 0} \int_0^{r^*r(x)} \beta e^{-ru \delta S_u} du \right]. \]

Moreover, as did in (4.15), we know that \( \lim_{x \to 0} \tau^*(x) = +\infty \) holds almost surely, thus by using \( S_0 = \mathbb{E} \left( \int_{0}^{+\infty} e^{-rt} \delta S_t dt \right) \),

\[ \mathbb{E} \left[ \lim_{x \to 0} \int_0^{r^*r(x)} \beta \delta \frac{e^{-ru \delta S_u} du}{x} \right] = \mathbb{E} \left[ \int_0^{+\infty} \beta \delta \frac{e^{-ru \delta S_u} du}{x} \right] = \beta. \]

\( \square \)

Finally, instead of checking the functions in Theorem 6.1 satisfy all the conditions in Theorem 6.3 in the case when \( \delta > 0 \), we show the idea why this kind of the functions is chosen.
Theorem 6.4. If \( \delta > 0 \), then the following function \( V_2(x) \) satisfies all the conditions in Theorem 6.3,

\[
V_2(x) = \begin{cases} 
  \frac{(1-\beta)(l^*-1)(l^*)^{-1}}{l^*} q^{1-l^*} x^{l^*} + \beta x, & \text{if } 0 \leq x < x^{**}, \\
  x - q, & \text{if } x \geq x^{**}, 
\end{cases}
\]

(6.21)

where \( l^* > 1 \) is the solution of \( h(l) = 0 \) and \( x^{**} = \frac{r}{(l^*-1)(l^*)^{-1}} q \). Moreover, the optimal stopping time is \( \tau^{**} \equiv \inf \{ t : \tilde{S}_t - x > x^{**} \} \).

Proof. We guess from Proposition 6.1 and Theorem 3.4 that \( f(x) = C_2 x^l + \beta x \) if \( x < x^{**} \), where \( C_2 > 0, \ l > 0 \). Using the same notation \( L \) as in Theorem 3.4 we have

\[
L f(x) = C_2 x^l h(l) - \beta \delta x.
\]

So \( L f(x) + \beta \delta x = 0 \), \( \forall 0 < x \leq x^{**} \), is equal to \( h(l) = 0 \) and we can prove \( h(l) = 0 \) has a solution, \( l^* \), larger than 1, as did in Theorem 3.4. Using the smooth fit principle, \( x^{**} \) is a continuous point of \( f(x) \) and \( f'(x) \), we know from which that

\[
C_2 (x^{**})^{l^*} + \beta x^{**} = (x^{**} - q);
\]

\[
C_2 l^* (x^{**})^{l^*-1} + \beta = 1.
\]

Solving the two equations we get \( x^{**} = \frac{r q}{(l^*-1)(l^*)^{-1}} \) and \( C_2 = \frac{(1-\beta)(l^*-1)(l^*)^{-1}}{l^*} q^{1-l^*} \). Therefore

\[
f(x) = \begin{cases} 
  \frac{(1-\beta)(l^*-1)(l^*)^{-1}}{l^*} q^{1-l^*} x^{l^*} + \beta x, & \text{if } 0 \leq x < x^{**}, \\
  x - q, & \text{if } x \geq x^{**}.
\end{cases}
\]

(6.22)

It’s easy to check that this \( f(x) \) satisfies all the conditions in Theorem 6.3, thus by Theorem 6.3, \( V_2(x) = f(x) \) and the optimal stopping time is determined by \( \tau^{**} \equiv \inf \{ t : \tilde{S}_t - x > x^{**} \} \).

Remark 6.1. We see that \( x^{**} > x^* \), so it implies that the borrower prefers not to redeem the stock when the part of dividends can be delivered. It closes to the reality due to the opportunity cost for redemption increases. If \( \beta \to 1 \), then \( x^{**} \to +\infty \) and \( C_2 \to 0 \), the value function turns to taking \( x \) as limit. So we claim that the optimal strategy for all dividends delivered to the borrower is never redeeming the stock.

It’s a natural idea to combine these two different dividends distribution, and it turns out to be quite feasible. As long as \( \alpha + \beta < 1 \), the value function
has the following form

$$V_3(x) = \sup_{\tau \in T_0} \mathbb{E}[e^{-r\tau}(e^{\alpha \delta \tau}S_\tau - q e^{\gamma \tau})_+ + \int_0^\tau \beta e^{-ru} \delta S_u du]. \quad (6.23)$$

Then the value function $V_3(x)$ has the following expression, and we omit its proof here.

**Theorem 6.5.** If $\delta > 0$, then the value function is

$$V_3(x) = \begin{cases} (1 - \beta)l^{l^{**} - 1}l^{l^{**}}q + \beta x, & \text{if } 0 \leq x < x^{***}, \\ x - q, & \text{if } x \geq x^{***}, \end{cases} \quad (6.24)$$

where $l^{**} > 1$ is the solution of the following equation

$$-\tilde{r} + (\tilde{r} - (1 - \alpha)\delta) l + \frac{1}{2} \sigma^2 l(l - 1) + \int_{-1}^{+\infty} ((1 + z)^l - 1 - lz) \nu(dz) = 0$$

and $x^{***} = \frac{\frac{\rho q}{(l^{**} - 1)(1 - \beta)}}{l^{**} + 1}.$

**Remark 6.2.** There is one kind of dividends distribution in Dai and Xu [9] not mentioned here. This is a difficult optimal stopping problem and we have no way to solve it up to now.

### 7. Example

In this section, combining our main results with the Kou and Wang’s model studied in Kou and Wang [13], we present an example to explain stock loan with jumps. Firstly we give a brief introduction of Kou and Wang’s model in which the stock price follows the following SDEs,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} V_i\right), \quad (7.25)$$

where $W_t$ is a standard Brownian motion, $N_t$ is a Poisson process with intensity $\lambda$, and $\{V_i\}_{i \geq 1}$ is a sequence of independent identically distributed (i.i.d.) random variables, which are larger than $-1$ such that $Y = log(V + 1)$ has an asymmetric double exponential distribution with the density

$$f_Y(y) = p \cdot \theta_1 e^{-\theta_1 y} 1_{\{y \geq 0\}} + q \cdot \theta_2 e^{\theta_2 y} 1_{\{y < 0\}}, \quad \theta_1 > 1, \theta_2 > 0,$$

\footnote{We are informed that this section has already been reported in [6] after our paper has been completed. However, we are not able to contact the authors, so we keep this section unchanged and wait for information from them.}

\footnote{We replace $V_i - 1$ in Kou and Wang [13] by $V_i$ to accord with our notations in this paper.}
where \( p, q \geq 0, \ p + q = 1 \), represent the probabilities of upward and downward jumps. Clearly, this is a Lévy model, therefore has all the properties mentioned before. In addition, analytical expressions for expectations involving first passage times can be obtained in this model due to the exponential distribution has no memory.

Since the solution \( l^* \) of \( h(l) = 0 \) plays a key role here, we calculate \( h(l) \) as follows,

\[
h(l) = -\tilde{r} + \tilde{\mu} l + \frac{1}{2} \sigma^2 (l - 1) + \lambda \left( p \frac{l(l - 1)}{\theta_1 - l} + q \frac{l(l - 1)}{\theta_2 + l} \right),
\]

if \(-\theta_2 < l < \theta_1\). This is a quadratic equation with respect to \( l \), the explicit expression for the solutions is available but here we are satisfied with numerical results.

Let \( r = 0.05, \sigma = 0.15, \delta = 0.01, \lambda = 0.1, \ p = 0.1, \theta_1 = 3.0, \theta_2 = 2.0 \) and let loan interest rate \( \gamma = 0.07 \) and loan size \( q = 100 \). By (7.26), we have

\( l^* = 1.8131 \),

\( x^* = 222.99 \).

Then the figure 1 of the value function on the basic stock loan \( V(x) \) is

![Figure 1. Basic value function](image-url)
Taking account of reinvestment of dividends, we let the reinvested proportion of dividends to be $\alpha_1 = 0.2$ and $\alpha_2 = 0.5$, respectively. Since the stock loan with $\alpha > 0$ is exactly the basic stock loan with $\delta' = (1 - \alpha)\delta$, by (7.26), we have

\[
\begin{align*}
I_{\alpha_1}^* &= 1.6995, \quad x_{\alpha_1}^* = 242.95 \\
I_{\alpha_2}^* &= 1.5063, \quad x_{\alpha_2}^* = 297.52.
\end{align*}
\]

Then we have the following figure 2

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Value function with different reinvested proportion}
\end{figure}

From this figure 2, it’s easy to see that value function is an increasing function with respect to $\alpha$ which can be illustrated in the following way:

As mentioned in Section 5, this stock loan is the basic stock loan with $\delta' = (1 - \alpha)\delta$ instead of $\delta$, which means that interest rate gap $\gamma - r - (1 - \alpha)\delta$ is increasing with respect to $\alpha$. Since the gap is the key point to evaluate the relative value of stock loan, the less $\gamma - r - (1 - \alpha)\delta$ is, the more valuable the loan is, exactly what the figure 2 shows.

Next, dividends delivered to the borrower is considered with $\beta_1 = 0.1$ and $\beta_2 = 0.2$. From (7.26), we know $I^*$ is the same as basic stock loan,
meanwhile, by (6.22) we know
\[ x_{\beta_1}^* = 247.77, \]
\[ x_{\beta_2}^* = 278.74. \]

We give the following figure 3

**Figure 3.** Value function with different delivered proportion

Regarding (6.22) as a function according to \( \beta \) with \( x \) fixed, then the derivation is
\[
\frac{df(\beta)}{d\beta} = -l^r (1 - \beta)^{r-1} (l^r - 1)^{r-1} q^{1-r} x^{r-1} + x
\]
\[
= \left( 1 - \frac{(1 - \beta)^{r-1} (l^r - 1)^{r-1} q^{1-r} x^{r-1}}{(l^r q)^{r-1}} \right) x
\]
\[
= \left( 1 - \left( \frac{(l^r - 1)x}{l^r q} \right)^{r-1} \right) x > 0,
\]
(7.27)

where the last equality holds for the reason that \( x \leq x^{**} = \frac{b q}{(r-1)x(1-\beta)} \). The figure 3 depicts why this happens owns to the increase of opportunity cost, which has been discussed in Remark 6.1.

Finally, we combine reinvestment and dividends. Let \( \alpha_3 = 0.2, \beta_3 = 0.2, \)
then

\[ l^*_{\alpha, \beta} = \alpha_1 = 1.6995, \]
\[ x^*_{\alpha, \beta} = x^*_{\alpha_1} = \frac{x^*_{\alpha_1}}{1 - \beta_3} = 303.69. \]

We have the following figure

\[ 0 \quad 50 \quad 100 \quad 150 \quad 200 \quad 250 \quad 300 \]
\[ 300 \quad 250 \quad 200 \quad 150 \quad 100 \quad 50 \quad 0 \]

Initial stock price \( x \)

Stock loan value \( V(x) \)

Basic value function \( \alpha = 0.2, \beta = 0 \)

\( \alpha = 0, \beta = 0.2 \)

\( \alpha = 0.2, \beta = 0.2 \)

\( (x-q)_+ \)

\[ \text{Figure 4. Value function with both reinvestment and dividends} \]

The figure 4 portrays that both reinvestment and dividends increase the value of the stock loan.

8. Appendix

**Proposition 8.1.** If a family of functions \( f(x, t) \) is convex with respect to variable \( x \) for any given parameter \( t \), then \( F(x) = \sup_{t \in T} f(x, t) \) is also a convex.

**Proof.** Assume that \( F(x) \) is not convex, then there exist \( x_1, x_2 \) and \( \lambda \in (0, 1) \) such that

\[ \lambda F(x_1) + (1 - \lambda) F(x_2) < F(\lambda x_1 + (1 - \lambda x_2)). \]

By the definition of \( F(x) \), there is a \( t^* \) such that

\[ f(\lambda x_1 + (1 - \lambda x_2), t^*) > \lambda f(x_1) + (1 - \lambda) f(x_2) \]
\[ \geq \lambda f(x_1, t^*) + (1 - \lambda) f(x_2, t^*). \]
This is a contradiction with \( f(x, t^*) \) is a convex function with respect to \( x \), so it ends the proof.

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References