Minimization of absolute ruin probability under negative correlation assumption

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HIGHLIGHTS

• Study minimization of absolute ruin with constrained law and negative correlation.
• Insurer’s liabilities and capital gains in financial market are negatively correlate.
• Derive closed-form minimal absolute ruin probability and optimal control strategy.
• The optimal strategies fail to be monotonic or continuous.
• Give economic analysis of main results.

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ABSTRACT

In this paper we consider the problem of minimizing the absolute ruin probability of an insurance company. The managers of the company control investment amount and risk exposure to minimize the absolute ruin probability. A negative correlation between insurer’s liabilities and capital gains in financial market is introduced. Under this negative correlation assumption, the explicit forms of the solutions and optimal strategies to this problem for all different parameters are derived. We find that the solutions of this problem are S-shaped and the optimal strategies fail to be monotonic or continuous.

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1. Introduction

Absolute ruin probability is an important risk measure and has been frequently considered in recent years. Studying absolute ruin probabilities helps us to investigate behaviors of a company when it is in deficit, which we will not discuss in the study of regular ruin probabilities.

In classical compound Poisson risk model, Dassios and Embrechts (1989) studied the case of exponentially distributed individual claim amounts and derived the explicit expression of absolute ruin probability by a martingale approach. Cai (2009) investigated Gerber–Shiu function under absolute ruin, and developed a system of integro-differential equations for this function.
Later, Takkar and Markussen (2003) obtained asymptotic results for a more practical case with a higher borrowing rate. In these models, when the surplus level drops below a critical level, absolute ruin occurs. But if there is a diffusion term in the surplus process, typically, investment in a Black–Scholes risky asset is introduced, or the surplus process itself is modeled by a Brownian motion, the surplus has a positive probability to bounce back from any negative level. We refer interested readers to Gerber and Yang (2007), where absolute ruin is defined as an event that \( \liminf \) of the surplus process is negative infinity. Luo and Taksar (2011) considered a diffusion approximation model including both investment and reinsurance, and obtained explicit expression of absolute ruin probability.

However, all the works mentioned above do not consider the correlation between liabilities and capital gains in financial market, which can have a huge influence. For example, American International Group, once the largest insurance company in the United States, collapsed within a few months in 2008. One of the several major mistakes made by AIG, which together contributed to its sudden collapse, is ignorance of the negative correlation between its liabilities and capital gains in financial market, according to Stein (2012, Chapter 6). Recently, Zou and Cadenillas (2014), and Bi and Zhang (2015) considered the negative correlation in their works. Therefore, we introduce this correlation in our model.

In this paper, we consider a diffusion approximation model of the insurance company’s surplus, i.e., the surplus is a drifted Brownian motion. The managers of this insurance company can control allocations of the surplus into a risky asset, a risk free asset and reinsurance purchases. And when it is in deficit, the company can borrow money to continue its business. Here the risky asset is governed by Black–Scholes dynamics, the risk free asset has a constant rate. As suggested in Stein (2012, Chapter 6), we assume that there exists a negative correlation between insurer’s risk and capital gains in financial market. Our object is to find the optimal investment and reinsurance strategies to minimize the absolute ruin probability of the company, and to derive the explicit expression of this probability. By Bellman’s stochastic principle of optimality, we get the associated HJB equation and verification theorem to show that a decreasing \( C^2(\mathbb{H}) \) solution of the HJB equation coincides with the minimal absolute ruin probability function. Under the negative correlation assumption, the two controls, investment amount and risk exposure, are not independent. This fact makes things more complicated than those in current works without negative correlation. For avoiding tedious discussion on different parameter settings in the standard way to handle this problem, we here introduce a plane geometric view to solve it and are able to find the explicit solutions for all parameters (there are 16 different parameter cases).

The most different and interesting part is that the optimal strategies fail to be monotonic or continuous. This result seems unreasonable. Indeed, it comes from that the monotonicity between the mean growth rate and the volatility of the company’s reserve may fail. So, in order to bounce back, the manager has to make a choice: to maximize the mean growth rate or to maximize the risk. From our result, the manager’s preference for risk grows with the surplus decreasing. And there exist some threshold points such that when the deficit is larger than the associated threshold point, the manager prefers higher risk to higher mean growth rate, i.e., he or she will choose the strategy which maximizes its risk, even this strategy decreases its mean growth rate. Under our correlation assumption, the balance between growth and volatility becomes more subtle.

The rest of the paper is organized as follows: In Section 2, we give a rigorous formulation of our model; In Section 3, we give the associated HJB equation and prove the verification theorem; In Section 4, we provide a plane geometric way to obtain the explicit solutions for all different parameters, which we classify into 16 settings; In Section 5 we present some examples to show the effect of the negative correlation on the optimal strategies and the minimal absolute ruin probability function, and then give an economic explanation on the lack of monotonicity and continuity; In Section 6 we summarize our conclusions in this paper.

2. Formulation

In this section we formulate our risk model and stochastic optimal control problem. We start with a diffusion approximation model of the surplus process. Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) be a filtered complete probability space satisfying the usual conditions (cf. Karatzas and Shreve, 1991). The reserve of the insurance company without any control is

\[ X_t = x + at + \sigma W_t^0, \]

where \( x \) is the initial surplus, \( W_t^0 \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \), \( a > 0 \) is a constant proportional to the safety loading of the company, and \( \sigma \) is the volatility of the insurance company’s surplus.

We assume that the insurance company can invest in financial market, and in which the risky asset follows Black–Scholes model:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sigma S_t dW_t^1, \\
    dB_t &= rB_t dt,
\end{align*}
\]

where \( r \geq 0 \) is the interest rate, \( \mu > r \) is the appreciation rate, and \( \sigma > 0 \) is the volatility of the stock; \( W_t^1 \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) and satisfies

\[
W_t^0 = \rho W_t^1 + \sqrt{1 - \rho^2} W_t^2,
\]

where \( W_t^2 \) is another standard Brownian which is independent of \( W_t^1 \) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P) \) and \( \rho \in [-1, 0] \) is a constant.

We denote the amount invested in the risky asset at time \( t \) by \( a_t \). The insurance is also allowed to purchase proportional reinsurance, then the company can manage its risk exposure level \( b_t \) by buying proportional reinsurance \( (1 - b_t) \) at time \( t \). Meanwhile it devorts its premiums to the reinsurance company at rate \( \beta (1 - b_t) \), where \( \beta > \alpha \) is proportional to the safety loading of the reinsurance company. This is a non-cheap case. Then a control strategy is a process \( \pi = \{(a_t, b_t), t \geq 0\} \). A control \( \pi \) is admissible if \( \pi \) is a predictable process with respect to filtration \( \{\mathcal{F}_t\} \), and for a priori given number \( A > 0 \) and any \( \varepsilon \geq 0 \), it holds that \( 0 \leq a_t \leq A \) and \( 0 \leq b_t \leq 1 \) almost surely. We denote the set of all admissible strategies by \( \Pi \).

When an admissible strategy \( \pi \) is applied, the dynamics of the surplus becomes

\[
dX_t^{\pi} = [\alpha - \beta (1 - b_t) + (\mu - r)a_t + \sigma X_t^{\pi}] dt + \sigma b_t dW_t^1 + \sigma a_t dW_t^0.
\]

Substituting Eq. (2.1) into the above equation, we get

\[
dx_t^{\pi} = [\alpha - \beta (1 - b_t) + (\mu - r)a_t + \sigma X_t^{\pi}] dt + (b_t \sigma + \sigma a_t) dW_t^1 + b_t \sigma \sqrt{1 - \rho^2} dW_t^2.
\]

In this model we allow for unlimited borrowing. If \( X_t - a_t \) is negative, the company actually borrows. Moreover, if the surplus level \( X_t \) is negative, the company is allowed to borrow to continue its business.

For an admissible strategy \( \pi \), the utility function is defined as probability of absolute ruin which means that \( \liminf \) of the surplus is equal to negative infinity. The event of absolute ruin under strategy \( \pi \) is

\[
\Theta^{\pi} = \{ \omega \in \Omega : \liminf_{t \to \infty} X_t^{\pi} = -\infty \}.
\]
The utility function \( V_\pi(x) \) denotes the absolute ruin probability under policy \( \pi \) with initial surplus \( x \), i.e.,
\[
V_\pi(x) = P(\sigma^T \mathbf{X}^\pi_0 = x).
\]
The objective of this paper is to find the optimal utility function
\[
V(x) = \inf_{\pi \in \Pi} V_\pi(x)
\tag{2.3}
\]
and optimal strategy \( \pi^* \) such that
\[
V_\pi(x) = V(x).
\tag{2.4}
\]
Problem (2.3)–(2.4) of minimizing absolute ruin probability is indeed an infinite time horizon stochastic optimal control problem. We aim at deriving the explicit forms of the optimal utility function and the optimal strategies for all different parameters. And we will also make some economic analysis about them.

3. HJB equation and verification theorem

In this section, we state the associated HJB equation and verification theorem to problem (2.3)–(2.4). As the derivation of the HJB equation is pretty standard, we omit it. And we only give a proof of the verification theorem, which ensures that we can solve the stochastic optimal control problem (2.3)–(2.4) via solving the associated HJB equation.

For any \( C^2(\Omega) \) function \( f \), we define an operator
\[
\mathcal{L}^{a,b}(f)(x) = \frac{1}{2} \left( \sigma^2 \alpha^2 + 2\sigma_\alpha \rho \alpha b + \sigma^2 b^2 \right) f(x)
+ \left( (\mu - \hat{\beta}) \alpha + \beta b + \nu \alpha + \alpha - \beta \right) f(x).
\tag{3.1}
\]
Assume that the optimal utility function \( V \) defined by (2.3) is twice continuously differentiable. Then by Bellman's principle of optimality (cf. Yong and Zhou, 1999, Fleming and Soner, 1993 and Promislow and Young, 2005), the associated HJB equation is
\[
0 = \inf_{0 \leq a < \infty, 0 \leq b < \infty} \left\{ \mathcal{L}^{a,b}(V)(x) \right\}
\tag{3.2}
\]
with boundary conditions
\[
V(-\infty) = 1, \quad V(\infty) = 0.
\tag{3.3}
\]
Notice that in Eq. (3.2), unlike the case without negative correlation, there is one cross term, involving two control parameters, which means the two control parameters cannot be separated, i.e., they are not independent.

To prove the verification theorem, we need some auxiliary lemmas. If \( \pi \) is an admissible control, for any \( M, N \) with \(-\infty < M \leq X^\pi_0 \leq N < \infty \), we set
\[
T_{M,N}^\pi = \inf\{t > 0 : X^\pi_t \not\in (M, N) \}.
\]

**Lemma 3.1.** For any \( M, N, x \) with \(-\infty < M < x < N < \infty \) and an admissible control \( \pi \), we have
\[
P(T_{M,N}^\pi < \infty | X^\pi_0 = x) = 1.
\]

**Lemma 3.2.** For any admissible control \( \pi \), if \( \liminf_{t \to \infty} X^\pi_t = -\infty \), then \( \lim_{t \to \infty} X^\pi_t = -\infty \); If \( \limsup_{t \to \infty} X^\pi_t = \infty \), then \( \lim_{t \to \infty} X^\pi_t = \infty \), where \( X^\pi_t \) is governed by (2.2).

We omit proofs of Lemmas 3.1–3.2 because they are similar to those of Lemmas 2.3 and 2.4 in Luo and Taksar (2011).

**Theorem 3.3.** Let \( F \) be a decreasing \( C^2(\Omega) \) solution to the HJB equation (3.2) and satisfy the boundary conditions (3.3). Then the optimal utility function \( V \) defined by (2.3) coincides with \( F \). Furthermore, if \( \pi^*(x) = (a^*(x), b^*(x)) \) is such that
\[
0 = \frac{1}{2} \left( \sigma^2 \alpha^2 + 2\sigma_\alpha \rho \alpha b + \sigma^2 b^2 \right) F''(x)
+ \left( (\mu - \hat{\beta}) \alpha + \beta b + \nu \alpha + \alpha - \beta \right) F'(x)
\]
for all \( x \in (-\infty, \infty) \). Then the optimal strategy \( \pi^* \) is of form \( \pi^* = ((a^*(X^\pi_t), b^*(X^\pi_t)), t \geq 0 \), where \( X^\pi_t \) is the corresponding optimal state risk process determined by
\[
dX^\pi_t = \left[ \alpha - \beta(1-b^*(X^\pi_t)) + (\mu - r)a^*(X^\pi_t) + rX^\pi_t \right] dt
+ \left[ b^*(X^\pi_t) \sigma + \sigma \alpha a^*(X^\pi_t) \right] dW^1_t
+ b^*(X^\pi_t) \sqrt{1 - \rho^2} dW^2_t,
\tag{3.4}
\]
that is, \( F(x) = V(x) = V_\pi(x) \).

**Proof.** For a given admissible strategy \( \pi = (a(t), b(t)), t \geq 0 \), \( X^\pi_t \) is the corresponding controlled reserve. By Lemma 3.1, we have \( T_{M,N}^\pi < \infty \) almost surely for any \(-\infty < M < N < \infty \). And let \( F \) be a \( C^2(\Omega) \) solution of Eq. (3.2). Applying Itô's formula to \( F(X^\pi_t) \) we get
\[
F(X^\pi_{T_{M,N}^\pi}) - F(x) = \int_0^{T_{M,N}^\pi} \left[ \alpha - \beta b(t) + (\mu - r)a(t) + rX^\pi_t \right] F'(X^\pi_t)
+ \frac{1}{2} \left[ \sigma^2 \alpha^2 + 2\sigma_\alpha \rho \alpha b + \sigma^2 b^2 \right] F''(X^\pi_t) dt
+ \int_0^{T_{M,N}^\pi} \left[ \sigma \rho a(t) + (\alpha + \beta)b(t) \right] F'(X^\pi_t) dW^1_t
+ \int_0^{T_{M,N}^\pi} \frac{\sigma^2}{2} \left[ \alpha^2 b^2 + \beta^2 \right] F''(X^\pi_t) dW^2_t
\geq \int_0^{T_{M,N}^\pi} \left[ \sigma \rho b(t) + \alpha a(t) \right] F'(X^\pi_t) dW^1_t
+ \frac{\sigma^2}{2} \left[ \alpha^2 b^2 + \beta^2 \right] F''(X^\pi_t) dW^2_t.
\tag{3.5}
\]
The last inequality in (3.5) comes from (3.2). We claim that, by taking expectation of the both sides of the last inequality, we can get
\[
\mathbb{E}_t[F(X^\pi_{T_{M,N}^\pi})] - F(x) \geq 0.
\tag{3.6}
\]
Now we show (3.6). Let \( f_1(t) = [\sigma \rho b(t) + \alpha a(t)] F'(X^\pi_t) dW^1_t \), then the stochastic process \( M_1(t) = \int_0^t f_1(s) ds \) is a local martingale. Indeed, define the sequence of stopping times \( T_n \) by \( T_n = \inf\{t \geq 0 : (M_1(t)) > n\} \). Since
\[
\langle M_1 \rangle(t) = \int_0^t f_1^2(s) ds \leq C(T_{M,N}^\pi)^{a \wedge 1} \to \infty
\]
due to \( a(t) \) and \( b(t) \) are bounded, \( F(x) \) is bounded on interval \([M, N]\) and Lemma 3.1, we know that \( T_n \to \infty \) as \( n \to \infty \), and \( \{M(t \wedge T_n); t \geq 0\} \) is a martingale. Therefore \( \mathbb{E}_t[M_1(t \wedge T_n)] = 0 \). Using \( a(t) \) and \( b(t) \) are bounded and \( F(x) \) is bounded on interval \([M, N]\) again, by Dominated convergence theorem, letting \( n \to \infty \) and \( t \to \infty \), we have
\[
\mathbb{E}_t[F(X^\pi_{T_{M,N}^\pi})] = \mathbb{E}_t[\int_0^{T_{M,N}^\pi} (\sigma \rho b(t) + \alpha a(t)) F'(X^\pi_t) dW^1_t] = 0.
\]
Similarly, \( \mathbb{E}_t[\int_0^{T_{M,N}^\pi} \sigma \rho b(t) + \alpha a(t) \sqrt{1 - \rho^2} F''(X^\pi_t) dW^2_t] = 0 \). Hence, the inequality (3.6) holds. We write
\[
\mathbb{E}_t[F(X^\pi_{T_{M,N}^\pi})] = \mathbb{E}_t[(M(t \wedge T_n) \wedge N) \leq N] = N.
\]
Combining Lemmas 3.1 and 3.2, we see that the process diverges to either positive infinity or negative infinity with probability 1 so we have ergodicity of the controlled process. Then letting \( N \to \infty \) we have
\[
F(x) \leq F(M \leq M) P_a(t_{M,N}^\pi < \infty, X^\pi_{T_{M,N}^\pi} = M)
= F(M) P_a(t_{M,N}^\pi < \infty, X^\pi_{T_{M,N}^\pi} = M) = M).
\tag{3.7}
\]
Since boundary conditions (3.3) and
\[ V_\pi(x) = P_\pi(\liminf_{t \to -\infty} X_t^+ = -\infty) \]
\[ = \lim_{M \to -\infty} P_\pi(\tau_{M,\infty}^\pi < \infty) \]  
(3.8)
letting \( M \to -\infty \) in (3.7) we get
\[ F(x) \leq V_\pi(x). \]
If we replace \( \pi \) by \( \pi_\pi \), then the first integrand on the left hand side of (3.5) vanishes and the inequality becomes an equality; likewise inequality (3.7) becomes an equality as well. Letting \( M \to -\infty \), we obtain
\[ F(x) = V_{\pi_\pi}(x). \]
The proof is completed. \( \square \)

4. Solution to the HJB equation

We solve Eq. (3.2) in this section. As there is a cross term involving two control parameters in (3.2), the discussion about parameters is tedious and lengthy. Here we simplify Eq. (3.2) via solving an elementary plane geometry problem. Then we can get much more convenient forms of the HJB equation in different regions and this simplified forms become much easier to solve.

Before we simplify the HJB equation, we give some discussion about the utility function \( V \). We notice that if \( r > 0 \) and \( x > -\frac{\sigma_1^2}{\sigma_1^2} \), the absolute ruin probability is zero, since we can take the strategy \( b_t \equiv 0 \) and \( a_t \equiv 0 \) and consequently the surplus process has a positive drift and a zero diffusion term. If \( r = 0 \) or the initial value of the surplus is less than \( -\frac{\sigma_1^2}{\sigma_1^2} \), then the absolute ruin probability is a strictly decreasing function of the initial value, i.e., \( V(x) \) is strictly decreasing on \((-\infty, -\frac{\sigma_1^2}{\sigma_1^2}) \) if \( r > 0 \) or on \( \mathbb{N} \) if \( r = 0 \).

For notational simplicity, we define \(-\frac{\sigma_1^2}{\sigma_1^2} = +\infty \) if \( r = 0 \) and \((-\infty, \infty) = 0 \). The set \( \{ x \in \mathbb{N} : V(x) = 0 \} \) splits into two sets: \( \{ x \in \mathbb{N} : V(x) = 0, V'(x) < 0 \} \) and \( \{ -\frac{\sigma_1^2}{\sigma_1^2}, \infty \} \). It is obvious that on the first set, \( V(x) \) satisfies Eq. (3.2) which has the form of \((x + \alpha + (\mu - r)A)V'(x) = 0 \). Then we get that the first set is a singleton \( \{-\frac{\sigma_1^2}{\sigma_1^2}, \frac{\sigma_1^2}{\sigma_1^2} \} \).

For readers, we write Eq. (3.2) here again:
\[ 0 = \inf_{0 \leq a \leq A, 0 \leq b \leq 1} \{L^a,b(f)(x)\}, \]
where
\[ L^a,b(f)(x) = \frac{1}{2}(\sigma_1^2 a^2 + 2a \sigma_1 \rho a b + \sigma_1^2 b^2 f''(x)) + (\mu - r)a b + \alpha - b \beta f'(x). \]

For a given \( x \) and a \( C^2(\mathbb{R}) \)-continuous decreasing function \( f \), if \( f''(x) \neq 0 \), \( L^a,b(f)(x) \) is a binary quadratic function of \( a \) and \( b \). Then the problem to find the minimizer and minimum of \( L^a,b(f)(x) \) under the given \( f \) and \( x \) becomes an extreme value problem of a binary quadratic function on a rectangle \([0, A] \times [0, 1] \). Define a function on \( \mathbb{R} \times \mathbb{N} \times \mathbb{R} \) by
\[ G(a, b, t) = \frac{1}{2}(\sigma_1^2 a^2 + \sigma_1 \rho a b + \frac{1}{2} \sigma_1^2 b^2) \]
\[ + (\mu - r)a b + \alpha - b \beta t. \]

It is obvious that the extreme value problem of \( L^a,b(f)(x) \) coincides with the one of the functions \( G(a, b, t) \) if \( f''(x) \neq 0 \). For a fixed \( t \), the contour lines of \( G(a, b, t) \) are non-intersecting ellipses on \( \mathbb{N} \)-plane, i.e., for an appropriate number \( M \), \( \{(a, b) : G(a, b) = M \} \) is an ellipse, and the elliptic function is
\[ M = \frac{1}{2}(\sigma_1^2 a - k_t)^2 + \sigma_1 \rho a (a - k_t)(b - k_t) + \frac{1}{2} \sigma_1^2 (b - k_t)^2 \]
\[ - \frac{1}{2} \sigma_1^2 \beta^2 - 2 \sigma_1 \rho \beta (\mu - r) + \sigma_1^2 (\mu - r)^2 \frac{t}{1 - \rho^2} \sigma_1^2 \sigma_1^2 \]
\[ \Rightarrow (r_x + \alpha - \beta)t, \]
(4.1)
where
\[ k_1 = \frac{\sigma_1 \rho - \rho (\mu - r)}{\sigma_1^2 (1 - \rho^2)}, \]
\[ k_2 = -\frac{\sigma_1 \rho - \rho (\mu - r)}{\sigma_1^2 (1 - \rho^2)}. \]
(4.2)
Both of them are negative. After some calculations, we can find that, the major-axes of all these ellipses are parallel to a first quadrant vector, which is
\[ \hat{\theta} = \left(1, \frac{(\sigma_1^2 - \sigma_1^2) - \sqrt{(\sigma_1^2 - \sigma_1^2)^2 + 4 \rho^2 \sigma_1^2 \sigma_1^2}}{2 \sigma_1 \rho} \right) \]
(4.4)
and \( \hat{\theta} \) has no relation with \( t \). From Eq. (4.1), it is easy to see that these ellipses have a common eccentricity. Then the number \( M \) in (4.1) can be considered as an indicator to characterize the size of ellipse.

Therefore, the minimum(maximum) problem of \( G(a, b, t) \) on \([0, A] \times [0, \frac{b_1}{k_2}] \), for a given \( t \), can be turned into the problem of finding the smallest(largest) ellipse, which is centered at \((k_t, k_t, \hat{t})\), has a fixed eccentricity and major-axis angle, and contacts the rectangle \([0, A] \times [0, \frac{b_1}{k_2}] \). Moreover, the size indicator \( M \) of this ellipse is the extreme value, the point of contact is the extreme point. So, the simplification of HJB equation (3.2) changes into a pure plane analytic geometric problem.

Define a function \( H(t) \) on \( \mathbb{R} \setminus \{0\} \) as the minimum of \( G(a, b, t) \) when \( t < 0 \) and as the maximum of \( G(a, b, t) \) when \( t > 0 \). It is obvious that, on the set \( \{ x \in \mathbb{N} : f''(x) \neq 0 \} \), \( \inf_{a,b \in \mathbb{R} \setminus \{0,1\}} \{L^a,b(f)(x)\} = 0 \) and \( H(\frac{b_t}{k_1}) = 0 \) are equivalent.

**Lemma 4.1.** If \( \frac{b_t}{k_1} \geq \frac{k_2}{k_1} \) and here \( k_1 \) and \( k_2 \) are defined in (4.2)–(4.3) respectively, then the function \( H(t) \) has the following form on \((-\infty, 0)\):
\[ H(t) = H_1(t) = \begin{cases} G(k_t, k_t, t), & \frac{1}{k_2} \leq t < 0, \\ G(\hat{a}, 1, t), & \frac{-\sigma_1^2 \hat{a} + \rho \sigma_1 \rho}{\mu - r} \leq t < \frac{1}{k_2}, \\ G(A, 1, t), & \frac{-\sigma_1^2 + \sigma_1 \rho A}{\mu - r} \leq t < \frac{A}{k_1}. \end{cases} \]
(4.5)
where \( \hat{a} = -\frac{1}{\sigma_1^2} (\sigma_1 \rho + (\mu - r)t) \). If \( A < \frac{k_1}{k_2} \), the function \( H(t) \) has the following form on \((-\infty, 0)\):
\[ H(t) = H_2(t) = \begin{cases} G(k_t, k_t, t), & \frac{A}{k_1} \leq t < 0, \\ G(A, \hat{b}, t), & \frac{-\sigma_1^2 + \sigma_1 \rho A}{\mu - r} \leq t < \frac{A}{k_1}, \\ G(A, 1, t), & \frac{-\sigma_1^2 + \sigma_1 \rho A}{\beta} \leq t < \frac{A}{k_1}. \end{cases} \]
(4.6)
where \( \hat{b} = -\frac{1}{\sigma_1^2} (\sigma_1 \rho A + \beta t) \). Both \( \hat{a} \) and \( \hat{b} \) are linear function of \( t \), we here omit \( t \) for simplicity.

**Proof.** Here we only solve the problem where \( A \geq \frac{k_1}{k_2} \), as the one where \( A < \frac{k_1}{k_2} \) can be solved similarly.

The contour lines of \( G(a, b, t) \) are ellipses centered at \((k_t, k_t, \hat{t})\). Then the locus of this center when \( t \) changes, is a line with slope \( \frac{k_2}{k_1} \) on \( \mathbb{N} \)-plane, see Fig. 1.
If \( t \in [\frac{1}{\sqrt{2}}, 0) \), the center \((k_1 t, k_2 t)\) is on the segment \( \mathbb{R} \), which is in the rectangle \([0, A] \times [0, 1]\). Then the minimizer of \( G(a, b, t) \) is \((k_1 t, k_2 t)\) and the minimum is \( G(k_1 t, k_2 t, t) \).

When \( t < \frac{1}{\sqrt{2}} \), the center is located outside the rectangle. Then if the minimum of \( G(a, b, t) \) is \( M \), the ellipse with equation \( G(a, b, t) = M \) on ab-plane must be tangent to line \( b = 1 \) or line \( a = A \), or just contact the rectangle at point \( (A, 1) \), and this contact point is the minimizer of \( G(a, b, t) \). As the major-axes of the contour lines are parallel to a first quadrant vector and eccentricities of them are the same, we have that, if the tangent point of line \( b = 1 \) lies on segment \([0, A] \times [1, 1]\), the tangent point of line \( a = A \) must be on line \([A] \times (1, \infty)\).

After some elementary calculation, we get that the tangent point of line \( b = 1 \) is \((\tilde{a}, \tilde{b})\), where

\[
\tilde{a} = -\frac{1}{\sigma_1^2} (\sigma_1 \rho + (\mu - r) t),
\]

which is always positive. We also have that when \(-\frac{\sigma_1^2 + \sigma_2 \sigma_1 \rho}{\mu - r} \leq t < \frac{1}{\sqrt{2}}\), the center is on the segment \( \mathbb{R} \) and \( \tilde{a} \leq A \), as in Fig. 1. Then if \(-\frac{\sigma_1^2 + \sigma_2 \sigma_1 \rho}{\mu - r} \leq t < \frac{1}{\sqrt{2}}\), the minimizer of \( G(a, b, t) \) is \((\tilde{a}, 1)\) and the minimum is \( G(\tilde{a}, 1, t) \).

If \( t < -\frac{\sigma_1^2 + \sigma_2 \sigma_1 \rho}{\mu - r}\), the center is located on the segment \( \mathbb{R} \). Notice that the second coordinate of the tangent point of the ellipse and line \( a = A \), \( \tilde{b} = -\frac{1}{\sigma_1^2} (\sigma_1 \rho A + \beta t) \), is a linear decreasing function with \( t \), then in this situation, \( \tilde{a} > A \) and \( \tilde{b} > 1 \). So the minimizer of \( G(a, b, t) \) is \((A, 1)\) and the minimum of \( G(a, b, t) \) is \( G(A, 1, t) \). \( \square \)

**Lemma 4.2.** The function \( H(t) \) has the following forms in \((0, \infty)\) as stated in Table 1 under different parameter settings:

\[
A_0 = \frac{\sigma}{\sigma_1} \left( \frac{\sigma_1 (\mu - r) - 2 \rho \sigma_1 \beta}{\sigma_1 (\beta - 2 \sigma_1 (\mu - r) \rho)} \right)
\]

\[
h_0(t) = G(A, 1, t),
\]

\[
h_1(t) = \begin{cases} G(A, 0, t), & \text{if } 0 < t \leq a_1, \\ G(A, 1, t), & \text{if } t > a_1, \end{cases}
\]

\[
h_2(t) = \begin{cases} G(0, 1, t), & \text{if } 0 < t \leq a_2, \\ G(A, 1, t), & \text{if } t > a_2, \end{cases}
\]

\[
h_3(t) = \begin{cases} G(0, 1, t), & \text{if } 0 < t \leq a_3, \\ G(A, 0, t), & \text{if } a_3 < t \leq a_1, \\ G(A, 1, t), & \text{if } t > a_1, \end{cases}
\]

\[
h_4(t) = \begin{cases} G(A, 0, t), & \text{if } 0 < t \leq a_2, \\ G(A, 1, t), & \text{if } t > a_2, \end{cases}
\]

\[
\rho < \frac{1}{2} \left( \frac{\sigma_1^2 A^2 + \sigma_2 \rho A}{\mu - r A} + \frac{\sigma_2^2 A^2}{\mu - r A} \right)
\]

\[
\text{and here } a_1 = -\frac{\sigma_2^2 + \sigma_1 \rho A}{\mu - r A} \text{ and } a_2 = -\frac{\sigma_1^2 A^2 + \sigma_2 \rho A}{\mu - r A} \text{ and } a_3 = -\frac{\sigma_1^2 A^2 - \sigma_2^2 A^2}{\mu - r A}.
\]

**Table 1**

<table>
<thead>
<tr>
<th>( H(t) )</th>
<th>( [0, \frac{2 \sigma_1^2}{\sigma_1^2} - \frac{\sigma_1 \rho A}{\mu - r A}] )</th>
<th>( [\frac{2 \sigma_1^2}{\sigma_1^2} - \frac{\sigma_1 \rho A}{\mu - r A}, \infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -1 &lt; \rho &lt; 0 )</td>
<td>( A )</td>
<td>( h_2(t) )</td>
</tr>
<tr>
<td>( -\frac{1}{2} \leq \rho &lt; 0 )</td>
<td>( [0, A] )</td>
<td>( h_2(t) )</td>
</tr>
</tbody>
</table>

Proof. For any given \( t > 0 \), the contour lines of \( G(a, b, t) \) on ab-plane are ellipses centered at \((k_1 t, k_2 t)\), which is located in the third quadrant, and whose major-axes are parallel to a first quadrant vector. Then the maximum of \( G(a, b, t) \) only may happen at points: \((0, 1), (A, 0)\) and \((A, 1)\).

So far the simplified HJB equation (3.2) is done. Under a given parameter setting, we can find the corresponding \( H(t) \) from the above lemmas and rewrite HJB equation (3.2) in a much more convenient form as \( H^\prime (\frac{\rho}{1 + \rho}) = 0 \) comparing the case where \( f^\prime (x) = 0 \). Then by the standard way we can get the solution to Eq. (3.2) and the associated feedback function.

To describe our results more clearly, before we state them, we define some functions as follows:

\[
F_1(x) = \frac{2(\pi + \beta - \rho)(1 + \rho^2) \sigma_1^4}{\sigma_1^2 \beta^2 - 2 \sigma_1 \beta \sigma_2 (\mu - r) + \sigma_2^2 (\mu - r)^2}
\]

\[
F_2(x) = \frac{\sigma_1}{\beta^2} \left( \sigma_1 (\pi + \beta - \rho) \sigma_2 (\mu - r) \left[ (\sigma_1 (\pi + \beta - \rho)^2 (\mu - r)^2 \sigma_1^2 \right] \right.
\]

\[
F_3(x) = \frac{\sigma_1^2 A^2 + \sigma_2 \rho A + \frac{\sigma_2^2 A^2}{\mu - r A} + \frac{\sigma_2^2 A^2}{\pi + \beta - \rho} \sigma_1^2 A^2}{\mu - r A}
\]

\[
F_4(x) = \frac{-\frac{1}{2} \sigma_1^2 A^2 + \sigma_2 \rho A + \frac{\sigma_2^2 A^2}{\pi + \beta - \rho} - \frac{\sigma_2^2 A^2}{\mu - r A}}{\pi + \beta - \rho + (\mu - r) A}
\]

\[
F_5(x) = \frac{-\frac{\sigma_1^2 A^2 - \sigma_2^2 A^2}{\mu - r A}}{\pi + \beta - \rho + (\mu - r) A}
\]

These functions are the forms of \( f^\prime (x) \) on each region. And we define some functions to describe the constants in our results:

\[
P_1(x, y) = \int_x^y e^{\frac{\beta}{\mu}} \frac{1}{(\mu - r) A} \text{d}x, \quad Q_1(x, y) = e^{\beta (\frac{1}{\mu} - \frac{1}{\mu - r}) A}.
\]

In particular, we define \( P_2(x, y) = \int_x^y e^{\frac{\beta}{\mu}} \frac{1}{(\mu - r) A} \text{d}x \) and \( Q_2(x, y) = e^{\beta (\frac{1}{\mu} - \frac{1}{\mu - r}) A} \).
Theorem 4.3. If the parameters satisfy that \(-\frac{1}{2} \leq \rho < 0, \ A > -\frac{\sigma}{2\sigma_1\rho} \) and \( \frac{\sigma_1\beta}{\sigma_1(\mu-r)} > -\frac{\sigma^2}{2\rho^2} \). The minimal absolute ruin probability function \( V(x) \) is

\[
V(x) = \begin{cases} 
1 - C_4 \int_{-\infty}^{x} e^{\beta t} \frac{1}{\tau(t)} dt, & \text{if } x \in (-\infty, x_4), \\
C_1 - C_3 \int_{x_4}^{x} e^{\beta t} \frac{1}{\tau(t)} dt, & \text{if } x \in (x_4, x_3), \\
C_2 - C_3 \int_{x_3}^{x} e^{\beta t} \frac{1}{\tau(t)} dt, & \text{if } x \in (x_3, x_2), \\
C_3 - C_2 \int_{x_2}^{x} e^{\beta t} \frac{1}{\tau(t)} dt, & \text{if } x \in (x_2, x_1), \\
0, & \text{if } x \in (x_1, \infty),
\end{cases}
\]

where

\[
\begin{align*}
x_1 &= -\alpha - \beta - \frac{\sqrt{\sigma_1^2 \rho^2 - 2\sigma_1 \rho \beta (\mu - r) + \sigma^2 (\mu - r)^2}}{r}, \\
x_2 &= -\alpha - \beta - \frac{1}{r} + \frac{\sigma_1^2 \rho^2 - 2\sigma_1 \rho \beta (\mu - r) + \sigma^2 (\mu - r)^2}{r^2}, \\
x_3 &= \frac{1}{r} \left( \frac{1}{1 + \sqrt{(1 - \rho^2) A^2 + 4 \rho \sigma^2}} - (\mu + r) A - \alpha \right), \\
x_4 &= \frac{1}{r} \left( \frac{1}{1 + \sqrt{(1 - \rho^2) A^2 + 4 \rho \sigma^2}} - (\mu + r) A - \alpha \right),
\end{align*}
\]

and

\[
\begin{align*}
C_1 &= \left[ Q_2(x_3, x_4)Q_2(x_4, x_3)Q_3(x_3, x_2)(-\infty, x_4) + Q_2(x_3, x_4)Q_3(x_4, x_3)P_2(x_4, x_3) \\
&\quad + Q_1(x_3, x_2)P_2(x_2, x_1) + P_1(x_2, x_1) \right]^{-1}, \\
C_2 &= Q_1(x_1, x_2)C_1, \\
C_3 &= Q_2(x_1, x_2)C_2, \\
C_4 &= Q_2(x_1, x_2)C_3, \\
E_1 &= 1 - C_4 P_4(-\infty, x_4), \\
E_2 &= E_1 - C_3 P_3(x_4, x_3), \\
E_3 &= E_2 - C_2 P_2(x_3, x_2)
\end{align*}
\]

The optimal investment feedback function is

\[
a^*(x) = \begin{cases} 
A, & \text{if } x \in (-\infty, x_3), \\
-\frac{1}{2} \sigma_1 \rho (\mu + r) F_2(x), & \text{if } x \in (x_3, x_2), \\
k_1 F_1(x), & \text{if } x \in (x_2, x_1), \\
0, & \text{if } x \in (x_1, \infty),
\end{cases}
\]

and the optimal reinsurance feedback function is

\[
b^*(x) = \begin{cases} 
0, & \text{if } x \in (-\infty, x_4), \\
1, & \text{if } x \in (x_4, x_2), \\
k_2 F_1(x), & \text{if } x \in (x_2, x_1), \\
0, & \text{if } x \in (x_1, \infty).
\end{cases}
\]

Proof. It is easy to see that for any given decreasing \( C(\mathbb{R}) \) function \( f \), on the set \( \{ x : f'(x) \neq 0 \} \), the equations

\[
\inf_{(a,b) \in (0, A) \times (0, 1)} \left\{ \lambda e^{-\beta f(x)} \right\} = 0
\]

and \( H'(\tau(x)) = 0 \) are equivalent. We can get the following form of \( H(t) \) under the parameter setting from Lemmas 4.1–4.2:

\[
H(t) = \begin{cases} 
H_1(t), & \text{if } t < 0, \\
H_2(t), & \text{if } t > 0.
\end{cases}
\]

Notice that \( H(t) \) is polynomial about \( t \) on each interval. Then we can get easily the explicit forms of \( f'(\tau(x)) \) from equation \( H'(\tau(x)) = 0 \) on each regions.
Theorem 4.5. The minimal absolute ruin probability function $V(\cdot)$ is a decreasing $C^2(\mathbb{R})$ function and is given by the following Tables 2 and 3 under different parameters:

Before stating closed-forms of those functions in the above tables, we define some numbers: $x_1, x_2, x_3, x_4$ are defined as in Theorem 4.3.

$x_5 = -\frac{\alpha}{2} + \frac{\sigma^2 (\mu - r) A}{r (\sigma^2 A + 2 \sigma_1 \rho A)}$

$x_6 = -\frac{\alpha}{2} + \frac{\sigma^2 (\mu - r) A - \beta}{r (\sigma^2 A - 2 \sigma^2)}$

$y_1 = x_1$

$y_2 = -\frac{\beta}{2} + \frac{1}{r} \frac{\beta^2 (\mu - r) A + \sigma^2 (\mu - r)}{r (\sigma^2 A - 2 \sigma^2)}$

$y_3 = \frac{1}{r} \left( \frac{\beta^2 (\mu - r) A + 2 \sigma_1 \rho A + \frac{\beta}{2} \sigma^2}{\sigma^2 A + \sigma_1 \rho A} - (\mu - r) A - \alpha \right)$

$y_4 = x_4$

$y_5 = x_5$

$y_6 = x_6$

$q_1 = Q_1(x_1, x_2)$

$q_2 = Q_2(x_3, x_4)$

$q_3 = q_2 Q_3(x_3, x_4)$

$q_4 = Q_4(x_5, x_6)$

$p_i = P_i(x_{i+1}, x_i), \quad i = 1, 2, \ldots, 5$

$p_1 = P_1(-\infty, x)$

$p_2 = P_2(x_1, x_2)$

$p_3 = p_4 = p_5 = 0$

$w_2 = w_1 Q_1(x,y), \quad w_3 = w_2 Q_2(x,y), \quad w_4 = w_3 Q_3(x,y), \quad w_5 = w_4 Q_4(x,y)$

$v_1 = P_1(y_1, y_2)$

$v_2 = P_2(y_2, y_3)$

$v_3 = P_3(y_3, y_4)$

$v_4 = P_4(y_4, y_5)$

$v_5 = P_5(-\infty, y)$

The functions in the Tables 2–3 and their associated optimal feedback functions are defined as follows:

$$V_{1,0} \quad \pi_{1,0}$$

$$1 - C_1^{t,0} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,0}}{\pi_{1,0}} \, ds$$

$$0 \quad 0$$

where $C_1^{t,0} = (q_2 p_3 + q_1 p_2 + p_1)^{-1}$, $C_2^{t,0} = q_1 c_1^{t,0}$, $C_3^{t,0} = q_2 c_2^{t,0}$, $E_1^{t,0} = 1 - C_1^{t,0} p_3$, $E_2^{t,0} = E_1^{t,0} - C_2^{t,0} p_2$.

$$V_{1,1} \quad \pi_{1,1}$$

$1 - C_1^{t,1} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,1}}{\pi_{1,1}} \, ds$ (A. 0)

$E_1^{t,1} - C_1^{t,1} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,1}}{\pi_{1,1}} \, ds$ (A. 0)

$E_2^{t,1} - C_2^{t,1} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,1}}{\pi_{1,1}} \, ds$ (A. 0)

$0 \quad 0$ (A. 0)

where $C_1^{t,1} = (q_2 p_4 + q_3 p_3 + q_1 p_2 + p_1)^{-1}$, $C_2^{t,1} = q_2 c_1^{t,1}$, $C_3^{t,1} = q_1 c_2^{t,1}$, $E_1^{t,1} = 1 - C_1^{t,1} p_3$, $E_2^{t,1} = E_1^{t,1} - C_2^{t,1} p_2$.

$$V_{1,2} \quad \pi_{1,2}$$

$1 - C_1^{t,2} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,2}}{\pi_{1,2}} \, ds$ (A. 0)

$E_1^{t,2} - C_1^{t,2} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,2}}{\pi_{1,2}} \, ds$ (A. 0)

$E_2^{t,2} - C_2^{t,2} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,2}}{\pi_{1,2}} \, ds$ (A. 0)

$0 \quad 0$ (A. 0)

where $C_1^{t,2} = (q_4 p_6 + q_5 p_4 + q_1 p_1 + p_1)^{-1}$, $C_2^{t,2} = q_2 c_1^{t,2}$, $C_3^{t,2} = q_1 c_2^{t,2}$, $E_1^{t,2} = 1 - C_1^{t,2} p_3$, $E_2^{t,2} = E_1^{t,2} - C_2^{t,2} p_2$. 

$$V_{1,3} \quad \pi_{1,3}$$

$1 - C_1^{t,3} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,3}}{\pi_{1,3}} \, ds$ (A. 0)

$E_1^{t,3} - C_1^{t,3} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,3}}{\pi_{1,3}} \, ds$ (A. 0)

$E_2^{t,3} - C_2^{t,3} \int_{-\infty}^t e^{\rho s} \frac{\pi_{1,3}}{\pi_{1,3}} \, ds$ (A. 0)

$0 \quad 0$ (A. 0)
Table 2

<table>
<thead>
<tr>
<th>( V(x) ) if ( -\frac{1}{2} \leq \rho &lt; 0 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{n_1}{n_0} \sigma \sigma_{n_1} &gt; -\frac{1-2\rho}{\rho} )</td>
</tr>
<tr>
<td>( V(x) )</td>
</tr>
<tr>
<td>( A )</td>
</tr>
<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
</tr>
</tbody>
</table>

Table 3

<table>
<thead>
<tr>
<th>( V(x) ) if ( 1 &lt; \rho &lt; \frac{1}{2} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{n_1}{n_0} \sigma \sigma_{n_1} \leq 1 )</td>
</tr>
<tr>
<td>( V(x) )</td>
</tr>
<tr>
<td>( A )</td>
</tr>
<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<td>( \frac{1}{2}, \frac{1}{2} )</td>
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<tr>
<td>( \frac{1}{2}, \frac{1}{2} )</td>
</tr>
</tbody>
</table>

| where \( C_1^{2,3} = (w_1 v_{16} + w_2 v_8 + w_2 v_1 + w_1 v_2 + v_4) \) |
| \( C_2^{2,3} = \frac{w_4 C_2^{2,3} + C_2^{1,3} + w_2 C_3^{2,3} + C_3^{1,3} + w_1 C_3^{2,3} + C_3^{1,3} + 1 - C_2^{2,3} v_{16}}{C_1^{2,3}} \) |
| \( E_2^{1,3} = E_1^{2,3} - E_1^{2,3} v_4, E_3^{1,3} = E_2^{2,3} - C_3^{2,3} v_3, E_4 = E_3^{2,3} - C_2^{2,3} v_2. \) |

5. Economic analysis

In this section, we give some discussion and economic analysis about our results. The explicit solutions give us insights into how a company behaves when it is in deficit, which will not be discussed under regular ruin minimization. Generally speaking, the optimal control strategy becomes more aggressive as the company's surplus decreases and becomes more conservative as it grows. Very different to the cases where all Brownian motions are independent, the optimal feedback functions fail to be monotonic, or even continuous functions of surplus level. This comes from that the mean growth rate of the insurance company's reserve is no longer monotonic about the company's volatility level.

To illustrate our results, we present several examples with plots for different parameter settings and give a slight discussion about them. Firstly, we will see two examples in which the optimal feedback functions lack continuity and monotonicity. Secondly, we will see the influence of \( \rho \) under a given parameter setting. At last, we will see the differences of strategies under different stock volatilities.

In the following two examples, we will see the lack of continuity and monotonicity.

Example 5.1. See Fig. 2. In this example, we consider a case with a high coefficient and a high investment limit. The parameters are given as follows: \( \sigma = 0.5, \sigma_1 = 0.5, \rho = -0.8, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 1.3. \) The minimal absolute ruin probability has the form of \( V_{14}(x) \). There are five threshold points: \( x_5 = -1.8478, x_4 = -0.6667, x_3 = -0.5345, x_2 = -0.1957, x_1 = 0.1500. \)

Example 5.2. See Fig. 3. In this example, we consider a case with a high coefficient and a low investment limit. The parameters are given as follows: \( \sigma = 5, \sigma_1 = 1, \rho = -0.7, r = 2, \mu = 2.5, \alpha = 1, \beta = 1.3, A = 4.7. \) The minimal absolute ruin probability has the form of \( V_{23}(x) \). There are five threshold points: \( x_5 = -5.0103, x_4 = -2.8425, x_3 = -1.1356, x_2 = -0.7107, x_1 = 0.1500. \)

To explain the reason of the discontinuity and non-monotonicity, we first review the dynamic of the insurance company's surplus here:

\[
dX_i^r = (\alpha - \beta(1 - b_i)) + (\mu - r)\sigma + \sigma X_t^r \ dt + (b_i \sigma + \sigma_1 a_i) dW_1 + b_i \sigma \sqrt{1 - \rho^2} dW_2.
\]

Under strategy \( \pi_t^1 \equiv (A, 0), \pi_t^2 \equiv (0, 1) \) or \( \pi_t^{1,2} \equiv (A, 1), \) the corresponding volatility of the insurance company at time \( t \) is \( \sigma_t^1 = \sigma_t^A, \sigma_t^2 = \sigma_t^r, \sigma_t^{1,2} = \sigma_t^{A + r} + \sigma_t^r \) and the mean growth rate is \( \gamma_t = \alpha + \beta \gamma_t^A + r \gamma_t^r, \gamma_t^{1,2} = \alpha + \beta \gamma_t^A + r \gamma_t^r. \)

If the parameter setting is defined as in Example 5.1, for any given surplus level \( x \), we have

\[
\hat{\sigma}_t^2 > \hat{\sigma}_t^1 > \hat{\sigma}_t^2
\]

(5.1)

and

\[
\hat{r}_t(x) < \hat{r}_3(x) < \hat{r}_3(x).
\]

(5.2)

where \( \hat{r}_t(x) = \alpha - \beta + (\mu - r)A + r, \hat{r}_3(x) = \alpha + \nu, \hat{r}_3(x) = \alpha + (\mu - r)A + r. \) Seeing Fig. 2, we can find that the feedback function \( \pi_t^{1,2} \) changes from \( (A, 1) \) through \( (0, 1) \) into \( (A, 0) \) while the surplus level decreases from \( x_3 \) to \( -\infty \). This means the manager's preference for risk grows with the deficit growing. Moreover, from (5.2) we can see that when the surplus is deeply low, the manager prefers higher volatility to higher mean growth rate, i.e., he or she chooses a strategy to maximize the company's risk, even though it decreases the reserve's mean growth rate.

In Example 5.1, at the discontinuity points of the feedback function, the ratio of mean growth rate and volatility are continuous, i.e.,

\[
\hat{r}_1(x) = \hat{r}_2(x) = \hat{r}_3(x) \quad \text{at } x_3 \]  

and

\[
\hat{r}_1(x) = \hat{r}_2(x) = \hat{r}_3(x) \quad \text{at } x_4.
\]

And from the explicit form of \( V_{14} \) in Theorem 4.5, we find that this ratio is just the function \( \frac{\nu}{\nu} \), So, the discontinuity of the feedback function has no influence on the utility function. Meanwhile, we have that when the surplus hits \( x_3 \) at time \( t \), the manager can adopt either the strategy \( \pi_t = (A, 0) \) or \( \pi_t = (0, 1) \) immediately, as both of them are optimal and equivalent. As the same, when the surplus hits \( x_4 \) at time \( t \), the strategy \( \pi_t = (0, 1) \) and \( \pi_t = (A, 1) \) are equivalent.

The discussion about the lack of continuity and monotonicity in Example 5.2 is very similar to Example 5.1. Notice that under the assumption in Example 5.2, we have

\[
\hat{\sigma}_t^2 > \hat{\sigma}_t^1 > \hat{\sigma}_t^2, \quad \hat{r}_2 < \hat{r}_1 < \hat{r}_3.
\]

Then the optimal feedback function changes from \( (A, 1) \) through \( (0, 1) \) into \( (0, 1) \) while the surplus level decreases from \( x_4 \) to \( -\infty \).

Comparing Examples 5.1 and 5.2, we can see that under different parameter settings, the feedback functions are very different. The balance between volatility and mean growth rate in our model is more delicate than the one in the model where all Brownian motions are independent.

Example 5.3. See Fig. 4. In this figure, we plot two examples under same parameter setting except coefficient \( \rho \) with high investment limit. The parameters are given as follows: \( \sigma = 0.5, \sigma_1 = 0.5, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 1.9. \) And the coefficient \( \rho = -0.1 \) in the first case and \( \rho = -0.9 \) in the second case. The minimal absolute ruin probability has the form of \( V_{10}(x) \) in
Fig. 2. The case with high coefficient $\rho$ and high investment limit $A$.

Fig. 3. The case with high coefficient $\rho$ and low investment limit $A$.

Fig. 4. The effect of coefficient $\rho$ with high investment limit $A$. 
case 1. There are three threshold points: \( x_3 = -0.5363, x_2 = -0.1794, x_1 = 0.1500 \). The minimal absolute ruin probability has the form of \( V_{11}(x) \) in case 2. There are four threshold points: \( y_4 = -0.9146, y_3 = -0.5653, y_2 = -0.1978, y_1 = 0.1500 \).

**Example 5.4.** See Fig. 5. In this figure, we plot two examples under same parameter setting except coefficient \( \rho \) with low investment limit. The parameters are given as follows: \( \sigma = 0.5, \sigma_1 = 0.5, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 0.07 \). And the coefficient \( \rho = -0.0001 \) in the first case and \( \rho = -0.95 \) in the second case. The minimal absolute ruin probability has the form of \( V_{20}(x) \) in case 1. There are three threshold points: \( x_3 = -0.1769, x_2 = -0.1471, x_1 = 0.1500 \). The minimal absolute ruin probability has the form of \( V_{22}(x) \) in case 2. There are four threshold points: \( y_4 = -0.5273, y_3 = -0.1999, y_2 = 0.1245, y_1 = 0.1500 \).

Obviously, when the insurance company is in large deficit, the manager needs to maximize company’s volatility to bounce back. If the correlation between the liabilities and capital gains in financial market becomes stronger, the maximal volatility which the company can get becomes smaller. Then the absolute ruin probability becomes high at low surplus level. But, when the surplus level is high, the manager is risk-averse and the strategy is conservative. If the correlation is strong, the company’s risk becomes low. For example, if \( \rho = -1 \), the two Brownian motion in diffusion term degenerate to one. Then the absolute ruin probability becomes low at high surplus level.

We can see this influence caused by \( \rho \) in both Figs. 4 and 5. The two curves in each figures intersect at point \( x_0 = \frac{\sigma_1 + (\mu - r)A}{\sqrt{\rho - 1}} \). On the left of \( x_0 \), the curve with high \( \rho \) is more concave and on the other half, it is more convex than the one with low \( \rho \), i.e., the shapes of utility functions with high coefficient \( \rho \) are more S-shaped in both cases, no matter investment limit is high or low.

The following three examples will show us the effect of the stock volatility.

**Example 5.5.** See Fig. 6. In this example, we consider a case with high coefficient and low investment limit. The parameters are given as follows: \( \sigma = 0.5, \sigma_1 = 0.01, \rho = -0.8, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 1.3 \). The minimal absolute ruin probability has the form of \( V_{22}(x) \). There are four threshold points: \( x_4 = -2.0883, x_3 = -0.2467, x_2 = 0.1101, x_1 = 0.1500 \).

**Example 5.6.** See Fig. 7. In this example, we consider a case with high coefficient and low investment limit. The parameters are
Fig. 7. The case with stock volatility $\sigma_1 = 0.5$.

Fig. 8. The case with stock volatility $\sigma_1 = 1$.

given as follows: $\sigma = 0.5, \sigma_1 = 0.5, \rho = -0.8, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 1.3$. The minimal absolute ruin probability has the form of $V_{14}(x)$. There are five threshold points: $x_5 = -1.3478, x_4 = -0.6667, x_3 = -0.5345, x_2 = -0.1957, x_1 = 0.1500$.

Example 5.7. See Fig. 8. In this example, we consider a case with high coefficient and low investment limit. The parameters are given as follows: $\sigma = 0.5, \sigma_1 = 1, \rho = -0.8, r = 2, \mu = 2.1, \alpha = 1, \beta = 1.3, A = 1.3$. The minimal absolute ruin probability has the form of $V_{14}(x)$. There are four threshold points: $x_4 = -1.3055, x_3 = -0.5400, x_2 = -0.1852, x_1 = 0.1500$.

Comparing these three pictures, when $\sigma_1$ grows, the manager’s reference for stocks becomes higher at low surplus level, and becomes lower at high surplus level. This presents another evidence: the optimal strategy becomes conservative with surplus growing and aggressive with the surplus decreasing.

6. Conclusion

This paper investigates minimizing the absolute ruin probability for an insurance company. The company chooses the optimal investment and reinsurance strategies to minimize absolute ruin probability when insurer’s liabilities and capital gains in financial market are negatively correlated. As a result, the two control parameters are not independent. We simplify the corresponding HJB equation via solving an elementary plane analytic geometry problem. Then we get much more convenient forms of the HJB equation in different regions and the simplified one becomes much easier to solve. The explicit solutions and the associated feedback functions are obtained under all 16 different parameter settings. Very different to the cases where all Brownian motions are independent, the optimal feedback functions fail to be monotonic, or even continuous functions of the surplus level. This comes from the fact that the mean growth rate of the insurance company’s reserve is no longer monotonic about the company’s volatility level. Generally speaking, the optimal control strategy becomes more aggressive as the surplus decreases and it becomes more conservative as the surplus grows.

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