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Zongxia Liang, Min Song

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TIME-CONSISTENT REINSURANCE AND INVESTMENT STRATEGIES
FOR MEAN-VARIANCE INSURER UNDER PARTIAL INFORMATION

Zongxia Liang and Min Song∗
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

ABSTRACT. In this paper, based on equilibrium control law proposed by Björk and Murgoci[4], we study an optimal investment and reinsurance problem under partial information for insurer with mean-variance utility, where insurer’s risk aversion varies over time. Instead of treating this time-inconsistent problem as pre-committed, we aim to find time-consistent equilibrium strategy within a game theoretic framework. In particular, proportional reinsurance, acquiring new business, investing in financial market are available in the market. The surplus process of insurer is depicted by classical Lundberg model, and the financial market consists of one risk free asset and one risky asset with unobservable Markov-modulated regime switching drift process. By using reduction technique and solving a generalized extended HJB equation, we derive closed-form time-consistent investment-reinsurance strategy and corresponding value function. Moreover, we compare results under partial information with optimal investment-reinsurance strategy when Markov chain is observable. Finally, some numerical illustrations and sensitivity analysis are provided.

JEL Classifications: G11, C61, G32.
Submission Classifications: IE13, IE12, IM52, IB91, IE53, IE43.
Keywords: Equilibrium control law; Time-consistent strategy; Investment-reinsurance; Partial information; Mean-variance criterion; Regime switching.

1. Introduction

To control risk exposure, insurers can invest in a financial market, purchase reinsurance contract and acquire new business (acting as a reinsurer for other insurers). Topic about optimization problems with various objectives in insurance risk management has been extensively investigated in literature. For example, Browne [6](1995) initiates
study of explicit investment strategy for an insurance company to maximize expected utility of terminal wealth or minimize ruin probability with surplus process modeled by a drifted Brownian motion. Yang and Zhang [23](2005) study optimal investment strategies for an insurer to maximize expected exponential utility of terminal wealth or maximize survival probability with surplus process satisfied by a jump-diffusion model. Furthermore, Xu et al. [22](2008), Gu et al. [10](2010), Liang et al. [15](2011) and Guan and Liang [11](2014) study optimal investment-reinsurance policies for an insurer who maximizes expected utility of terminal wealth in different situations.

Since the introduction by Markowitz[16](1952), mean-variance portfolio selection problem has become a key research topic in finance, as a result, recently many scholars consider the optimal investment-reinsurance policies for insurers under mean-variance criterion. For example, Bäuerle [3](2005) studies optimal proportional reinsurance/new business problem and gets closed-form optimal strategy under mean-variance criterion with surplus process driven by classical Cramér Lundberg model. Bai and Zhang[1](2008) use linear quadratic method and dual method to derive explicit optimal investment-reinsurance policies for an insurer under mean-variance criterion, where the surplus of the insurer is depicted by Cramér Lundberg model and an approximated diffusion model. Zeng et al. [26](2010) assume that the surplus of an insurer is modeled by a jump-diffusion process, and use stochastic maximum principle to derive closed-form optimal investment policies under benchmark and mean-variance criteria.

However, since mean-variance criterion does not satisfy the iterated-expectation property, stochastic control problem for mean-variance criterion is time-inconsistent in the sense that Bellman optimality principle does not hold, which means a control maximizes the mean-variance utility at time zero may not be optimal for mean-variance utility at latter time. Strotz[18](1956) proposes another way to handle time-inconsistent problem: finding time-consistent strategies instead of treating it as a pre-committed problem. Recently, Ekeland and Pirvu [7](2008) first provide a precise definition of equilibrium concept in continuous time. Basak and Chabakauri[2](2010) use dynamic programming method to

On the other hand, regime-switching models have become popular among academia and practitioners since Hamilton[12](1989) pioneers the econometric applications of regime-switching models, which can reflect different market states (saying Bullish versus Bearish), see Zhou and Yin[27](2003). Mostly, regime switching is modeled by a continuous Markov chain, in practice, the Markov chain sometimes is not directly observable and has to be inferred from observable price processes of stocks or other assets, therefore hidden Markov model has been largely used in optimal investment problems concerning partial information. Since Lakner[13](1995) reclaims utility maximization with partial information, numerous work emerges. For example, Sass and Haussmann[19] (2004), Rieder and Bäuerle[17](2005) consider portfolio optimization problem with asset’s drift process modeled by an unobservable continuous time Markov chain. Xiong and Zhou[21](2007) study mean-variance portfolio selection problem under partial information and Elliott et al.[9](2010) consider a similar problem under a hidden Markovian regime-switching model. More recently, Liang and Bayraktar[14](2014) investigate optimal investment-reinsurance problem with unobservable claim size and intensity.

Unfortunately, as far as we are concerned, little literature of insurance cares about time-consistent strategy for mean-variance insurers under partial information. But as the time-consistency of the strategy is a basic requirement of rational decision making and insurers are only accessible to partial information of the market state in many situations, in this paper, we try to consider an optimal investment and reinsurance problem for mean-variance insurers under partial information and aim to find the corresponding time-consistent strategy. In particular, we
assume that the drift rate of stock is Markov-modulated and it can not be observed directly by the insurer and the claim process suffered by insurance company is modeled by classical compound Poisson process. Also, proportional reinsurance and acquiring new business is available in insurance market. Different from Zeng et al. [24] (2011) and [25] (2013) assuming that insurer’s risk aversion level is a constant and Björk et al. [5] (2014) assuming that insurer’s risk aversion is inversely proportional to wealth, we assume that insurer’s risk aversion is different for different market states and the instant risk aversion is determined by his or her estimation of the market state right now, and the estimation is made by filtering Markov chain using data of the stock’s price process. To solve this partially observed optimization problem, we first transfer it to an equivalent problem with complete observations using reduction technique, and to get time-consistent investment-reinsurance strategy for this time-inconsistent mean-variance problem, we then derive an extended HJB equation and corresponding verification theorem. Finally we compare the investment-reinsurance strategy under partial information with strategy adopted if the market state can be observed directly, effects of partial information on investment-reinsurance policy are illustrated.

The paper is organized as follows: The model of insurer’s wealth process with proportional reinsurance and optimization objective are presented in Section 2, moreover, definitions of equilibrium control law (time-consistent strategy) and value function are given. In Section 3 we reduce the partially observable problem to an equivalent completely observable problem and derive the generalized extended HJB equation. In Section 4 we solve the generalized HJB equation explicitly by making an Ansatz. In Section 5 we derive the equilibrium control law and the corresponding equilibrium value function given the market state is observable. Section 6 provides a sensitivity analysis to clarify the effects of several parameters on equilibrium control law and compares partially and completely observable cases. Section 7 is a conclusion.
2. The risk model

In this section we consider an insurer whose proportional reinsurance is available. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbf{P})\) is a filtered complete probability space and \(\mathcal{F}_t\) is the information of the market available up to time \(t\). \([0,T]\) is a fixed time horizon. All the processes introduced below are assumed to be adapted to \(\{\mathcal{F}_t, t \in [0,T]\}\). The insurer’s surplus process is modeled by classical Lundberg model

\[
dX(t) = cd t - d \{ \sum_{i=1}^{N_t} Y_i \},
\]

where \(c\) is premium rate of the insurer, \(Y_i\) is size of the \(i\)th claim, the total amount of claims up to time \(t\) is denoted by homogeneous Poisson process \(N_t\) with intensity \(\lambda > 0\), and \(N = \{N_t\}\) is independent of \(\{Y_i\}\). All the claims \(Y_i, i = 1, 2, 3, \cdots\) are assumed to be independent and identically distributed (iid) with \(\mathbb{E}[Y_i] = \nu_1\) and \(\mathbb{E}[Y_i^2] = \nu_2\). To prevent the insurer from bankruptcy immediately, \(c > \lambda \nu_1\) is required. For simplicity, we set \(c = \lambda \nu_1 (1 + \eta)\) according to expectation principle, where \(\eta > 0\) is the safety loading. In addition, reinsurance is allowed and we consider the proportional reinsurance here. Denote the reinsurance proportion by \(h(t)\), which means that \(100(1 - h(t))(h(t) \geq 0)\) of the insurance risk is divided to a reinsurer at time \(t\). When the \(i\)th claim \(Y_i\) occurs, the insurer pays only \(h(t)Y_i\) while the reinsurer pays the rest. However, based on expectation principle, the insurer has to pay a premium at the rate of \((1 + \theta)\lambda \nu_1 (1 - h(t))(\theta > 0)\) to the reinsurer due to the reinsurance business. In general, \(\theta > \eta\), otherwise, arbitrage will exist. The insurer can hedge its insurance risk by reinsurance strategy \(h(t)\). If \(h(t)\) is small, the insurer takes a little risk of insurance by himself and divides most of the risk to the reinsurer. \(h(t) > 1\) means taking new reinsurance business. Subsequently, in this case, the surplus process \(X(t)\) takes the following form:

\[
dX(t) = \lambda \nu_1 [h(t)(1 + \theta) + (\eta - \theta)]dt - h(t)d \{ \sum_{i=1}^{N_t} Y_i \}.
\]

Besides taking reinsurance strategy, the insurer can also invest in financial market. We consider the financial market with one bond and one risky asset. The bond evolves according to

\[
dS_0(t) = rS_0(t)dt,
\]
$r > 0$ is risk-free interest rate. The stock price process $S_1 = \{S_1(t)\}$ is given by
\[
dS_1(t) = S_1(t)\{\mu(t)dt + \sigma dW(t)\},
\]
where $\sigma > 0$ is a constant, $W = \{W(t)\}$ is a Brownian motion w.r.t. $\{\mathcal{F}_t, \ t \geq 0\}$, $\mu(t) = \mu(I(t))$, $I(t)$ is a stationary continuous-time Markov chain process which represents the market state. We assume that processes $W(t)$ and $I(t)$ are independent of each other, economically thinking, the evolution of any particular security does not affect the whole market, but the market state will have impact on the security through parameters. Assume that there are $d$ regimes for the market state, the state space of $I(t)$ is defined by $\{e_1, \cdots, e_d\}$, where $e_k$ is the $k$-th unit vector in $\mathbb{R}^d$ and $(I(t))_{t \geq 0}$ has a generator $Q = (q_{ij})_{d \times d}$. Then $\mu(I(t)) = \mu I(t)$ where $\mu = (\mu_1, \ldots, \mu_d)' \in \mathbb{R}^d$. For any vector or matrix $M$, we denote $M'$ as its transpose.

Let $\pi(t)$ denote the money account invested in the risky asset $S_1$ and $h(t)$ be the retention level at time $t$. An investment-reinsurance strategy is described by a pair process $u(t) = (\pi(t), h(t))$. Corresponding to a strategy $(\pi(t), h(t))$, the insurer’s wealth process $X(t)$ follows the following dynamic:
\[
\begin{cases}
dX(t) = \left[rX(t) + (\mu(t) - r)\pi(t) + \lambda \nu_1(h(t)(1 + \theta) + (\eta - \theta))\right]dt \\
X(0) = x_0.
\end{cases}
\]  
(2.1)
An investment-reinsurance strategy is said to be admissible if $\{u(t)\}$ is $\{\mathcal{G}_t\}$-progressively measurable, where $\mathcal{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ is the filtration generated by stock price process $\{S_1(t)\}$ and the claim process, $\mathbb{E}[\int_0^T \pi^2(t)dt] < \infty$ and Eq.(2.1) has a unique strong solution. The set of all admissible strategies is denoted by $\mathcal{U}$.

The objective functional we adopt here is mean-variance type utility of the insurer’s terminal wealth and is given by
\[
J(t, x, u(\cdot)) = \mathbb{E}_{t,x}[X^u(T)|\mathcal{G}_t] - \frac{\gamma(t)}{2} \mathbb{V}ar_{t,x}[X^u(T)|\mathcal{G}_t]
= \mathbb{E}_{t,x}[X^u(T)|\mathcal{G}_t] - \frac{\gamma(t)}{2} \{\mathbb{E}_{t,x}[(X^u(T))^2]|\mathcal{G}_t\}
\]  
(2.2)
where $\mathbb{E}_{t,x}[\cdot|\mathcal{G}_t]$ and $\mathbb{V}ar_{t,x}[\cdot|\mathcal{G}_t]$ are the conditional expectation and variance given filtration $\{\mathcal{G}_t\}$ with $X_t = x$, respectively, $\gamma(t) = \mathbb{E}[^{\gamma'}I(t)|\mathcal{G}_t]$, $\gamma = (\gamma_1, \cdots, \gamma_d)' \in \mathbb{R}^d$ represents the insurer’s risk aversion under different market states and we assume that $\gamma_1 \leq \cdots \leq \gamma_d$. To ensure the
existence of the equilibrium control and corresponding value function, we require the non-degeneracy condition: there exists a $\delta > 0$ such that $\gamma_i > \delta$ for $1 \leq i \leq d$.

The insurer’s main goal is to maximize the objective functional $J(t, x, u(\cdot))$ in (2.2) by adopting admissible control law, i.e., we have the following mean-variance problem:

$$\max_{u \in \mathcal{U}} \left\{ J(t, x, u(\cdot)) \right\}.$$  \hspace{1cm} (2.3)

Due to the fact that this objective functional concerns non-linear term of expectation of the terminal wealth and the insurer’s risk aversion changes over time, the optimization problem is clearly time-inconsistent, and Bellman’s stochastic principle of optimality, which says that if a control law is optimal on the full time interval $[0, T]$, then it is also optimal for any subinterval $[t, T]$, fails in this case (pointed out by Björk and Murgoci[4](2010)). To treat this time-inconsistent problem seriously, it becomes popular in the literature to view the problem within a game theoretic framework and look for Nash subgame perfect equilibrium solutions. Ekeland and Pirvu[7](2008) first provide a precise definition of the equilibrium concept in continuous time to deal with optimal investment and consumption problem under hyperbolic discounting, then Björk and Murgoci[4](2010) generalize the equilibrium concept to handle more general time-inconsistent optimal control problems. In this paper, we study the partially observed time-inconsistent optimization problem (2.2)-(2.3) based on definition of equilibrium control law given by Björk and Murgoci[4](2010), and readers interested in this economic insight are referred to their paper.

**Definition 2.1.** For any fixed $(t, x)$, consider an admissible strategy $\hat{u}$ (a candidate equilibrium law). For any fixed admissible control law $u$, and $h > 0$, one can define a new control law $u_h$ by

$$u_h(s, y) = \begin{cases} u(s, y), & \text{for } t \leq s < t + h, \\ \hat{u}(s, y), & \text{for } t + h \leq s \leq T. \end{cases}$$

If

$$\liminf_{h \to 0^+} \frac{J(t, x, \hat{u}(\cdot)) - J(t, x, u_h(\cdot))}{h} \geq 0 \quad (2.4)$$

for all $u \in U$, then $\hat{u}$ is called an equilibrium control law and the equilibrium value function is given by

$$V(t, x) = J(t, x, \hat{u}). \quad (2.5)$$

3. Problem reduction and extended HJB equation

In this section, we first formulate an auxiliary problem with complete observation which is equivalent to control problem (2.2)-(2.3), then by solving an extended HJB equation (3.6), we get equilibrium control law and the corresponding value function.

3.1. The reduction. In order to reduce this partially observable problem to an equivalent one with complete observation, we define the conditional probabilities

$$p_k(t) = P(I(t) = e_k | G_t), \quad k = 1, \ldots, d,$$

i.e., the Wonham-filter process of the Markov chain. Let $p(t) = (p_1(t), \ldots, p_d(t))'$, then the following statement holds (cf. Elliott et al. [8], Rieder and Bäuerle [17]):

**Lemma 3.1.** There exists a $\{G_t\}$-Brownian motion $\{\hat{W}(t)\}$ such that

1. **filter processes** $p_k(t)$ satisfy for $t \geq 0$

   $$dp_k(t) = \sum_j q_{jk}p_j(t)dt + \frac{1}{\sigma}(\mu_k - \hat{\mu}(t))p_k(t)d\hat{W}(t),$$

   and in the matrix form we can rewrite it as follows:

   $$dp(t) = Q'p(t)dt + \frac{1}{\sigma}M(t)p(t)d\hat{W}(t),$$

   where $M(t) = \text{Diag}(\mu - \hat{\mu}(t)I_d)$ is a $d \times d$ diagonal matrix and

   $$\hat{\mu}(t) = \sum_{k=1}^d \mu_k p_k(t) = E[\mu(t)|G_t].$$

2. $\mu(t)dt + \sigma dW(t) = \hat{\mu}(t)dt + \sigma d\hat{W}(t)$.

3. $\sigma(S_1(s), s \leq t) = \sigma(W(s), s \leq t) = \sigma(p(s), s \leq t)$.

From (2) of Lemma 3.1, we have $\hat{W}(t) = W(t) + \int_0^t \sigma^{-1}(\mu(s) - \hat{\mu}(s))ds$ and $\tilde{W} = (\hat{W}(t))_{t \in [0, T]}$ is usually called innovation process. Then the control model with complete observation can be characterized for $u =$
\[(\pi, h) \in U[0, T] \text{ by the following:} \]
\[
\begin{align*}
\frac{d \tilde{X}(t)}{dt} &= [r \tilde{X}(t) + (\mu(t) - r) \pi(t) + \lambda \nu_1(h(t)(1+\theta) + (\eta - \theta))] \, dt \\
&+ \pi(t) \sigma d\tilde{W}(t) - h(t) d\{\sum_{i=1}^{N_i} Y_i\}, \\
\tilde{X}(0) &= x_0, \\
dp(t) &= Q p(t) dt + \frac{\sigma}{2} M(t)p(t) d\tilde{W}(t), \\
q_k(0) &= P(I(0) = \epsilon_k), \quad k = 1, \ldots, d.
\end{align*}
\] (3.1)

For any given \[P(I(0) = \epsilon_k), \quad k = 1, \ldots, d, \] is initial estimated distribution of market state. Then objective functional in the reduced model is given by
\[
\tilde{J}(t, x, p, u(\cdot)) = E_{t,x,p}[\tilde{X}^u(T)] - \frac{\gamma^2}{2} Var_{t,x,p}[\tilde{X}^u(T)]
\]
\[
= E_{t,x,p}[\tilde{X}^u(T)] - \frac{\gamma^2}{2} \{E_{t,x,p}[(\tilde{X}^u(T))^2]\}
\]
\[
- \{E_{t,x,p}[\tilde{X}^u(T)]^2\},
\]
where \(E_{t,x,p}\) is the conditional expectation given \(\tilde{X}^u(t) = x, p(t) = p\).

**Lemma 3.2.** For any given \(u \in U[0, T]\), \(\tilde{J}(t, x, p, u(\cdot)) = J(t, x, u(\cdot))\).

**Proof.** To prove \(\tilde{J}(t, x, p, u(\cdot)) = J(t, x, u(\cdot))\), it is enough to show that \(E_{t,x}[X^u(T)|G_t] = E_{t,x,p}[\tilde{X}^u(T)]\) and \(E_{t,x}[(X^u(T))^2|G_t] = E_{t,x,p}[(\tilde{X}^u(T))^2]\).

By Lemma 3.1, comparing SDEs (2.1) and (3.1) satisfied by \(X^u(t)\) and \(\tilde{X}^u(t)\), respectively, we have \(X^u(T) = \tilde{X}^u(T)\). Then statement (3) in Lemma 3.1 yields \(E_{t,x}[X^u(T)|G_t] = E_{t,x,p}[\tilde{X}^u(T)]\) and \(E_{t,x}[(X^u(T))^2|G_t] = E_{t,x,p}[\tilde{X}^u(T)]^2\).

By Lemma 3.2, we can see that the objective functional depends on history of \((S_1(s))_{s \leq t}\) only through \(p(t)\). Thus problem (2.3) is equivalent to mean-variance problem
\[
\max_{u \in U^t} \{\tilde{J}(t, x, p, u(\cdot))\}. \quad (3.3)
\]

**3.2. Extended HJB equation.** We now use result of Bj"ork and Murgoci [4] (2010) to derive extended HJB equation for our completely observable optimal control problem (3.3). Firstly, we rewrite the objective functional (3.2) as follows:
\[
\tilde{J}(t, x, p, u(\cdot)) = E_{t,x,p}[F(t, p, \tilde{X}^u(T))] + G(t, p, E_{t,x,p}[\tilde{X}^u(T)]),
\] (3.4)
where \(F(t, p, y) = y - \frac{\gamma}{2} p y^2, G(t, p, y) = \frac{\gamma}{2} p y^2\). Then the corresponding equilibrium value function is
\[
\tilde{V}(t, x, p) = \tilde{J}(t, x, p, \hat{u}(\cdot)).
\] (3.5)
For any $\phi(t,x,p) \in C^{1,2,2}([0,T] \times \mathcal{R} \times \mathcal{R}^d)$, define
\[
A^u \phi(t,x,p) = \phi_t(t,x,p) + \phi_x(t,x,p) [rx + (\mu(t) - r) \pi(t) \\
+ \lambda \mu_1(h(t)(1+\theta) + (\eta - \theta))] + \frac{1}{2} \phi_{xx}(t,x,p) \sigma^2 \pi^2(t) \\
+ \phi_p^2(t,x,p) Q_p^p + \frac{1}{2} \sigma_p^2 \pi^2(t) \phi_{pp}(t,x,p) \sigma_p^2 \pi^2(t) + \pi(t) \phi_{xp}^2(t,x,p) \sigma_p^2 \pi^2(t) \\
+ \lambda E\left[\phi(t,x-h(t)Y,p) - \phi(t,x,p)\right].
\]

where $\phi_p$ is a $d \times 1$ vector with $k$-th element noted by $\phi_{pk}$, and $\phi_{pp}$ is a $d \times d$ matrix with $(\phi_{pp})_{jk} = \phi_{pjk}$. By Definition 2.1 and using the infinitesimal generator $A^u$, we can derive extended HJB equation and verification theorem as follows.

**Theorem 3.1.** (Verification Theorem) Suppose that there are functions $\hat{V}(t,x,p), g(t,x,p) \in C^{1,2,2}([0,T] \times \mathcal{R} \times \mathcal{R}^d)$ and $f(t,x,p,\tilde{p}) \in C^{1,2,2}([0,T] \times \mathcal{R} \times \mathcal{R}^d \times \mathcal{R}^d)$ satisfying the following extended HJB equation:
\[
\begin{align*}
\sup_{u \in U} \left\{ A^u \hat{V}(t,x,p) - A^u f(t,x,p,p) + A^u f_p(t,x,p) \right\} &= 0, \\
-\mathcal{A}^x (G \circ g)(t,x,p) + \mathcal{H}^u g(t,x,p) &= 0, \\
A^u f_p(t,x,p) &= 0, \\
A^u g(t,x,p) &= 0,
\end{align*}
(3.6)
\]
where the maximum of the first equation is attained at $\hat{u}(t,x,p)$ for all $(x,p) \in \mathcal{R} \times \mathcal{R}^d$, $f(t,x,p,p) = f(t,x,p,\tilde{p})$, $(G \circ g)(t,x,p) = G(t,p,g(t,x,p)) = \gamma_{\tilde{p}} g^2(t,x,p)$, and $\mathcal{H}^u g(t,x,p) = G_{y}(t,p,g(t,x,p)) A^u g(t,x,p)$. Then $\hat{u}$ is an equilibrium control law and $\hat{V}(t,x,p)$ is the corresponding equilibrium value function. Moreover, by Feynman-Kac formula, $f$ and $g$ have the following expressions:
\[
\begin{align*}
f(t,x,p,\tilde{p}) &= E_{t,x,p} \left[F(t,\tilde{p},\hat{X}^\hat{u}(T))\right], \\
g(t,x,p) &= E_{t,x,p} \left[\hat{X}^\hat{u}(T)\right].
\end{align*}
(3.7)
\]

**Proof.** The proof is similar to Theorem 4.1 of Björk and Murgoci[4](2010) and Theorem 3.1 of Wei et al.[20](2013), we omit it here. \hfill \Box

Using Eqs (3.4), (3.5) and (3.7) we have
\[
\hat{V}(t,x,p) = f(t,x,p,p) + G(t,p,g(t,x,p)).
(3.8)
\]
Since the infinitesimal generator $A^u$ is linear, the first equation in Eq.(3.6) can be simplified as
\[
\begin{align*}
A^u \hat{V}(t,x,p) - A^u f(t,x,p,p) + A^u f_p(t,x,p) - A^u (G \circ g)(t,x,p) \\
+ \mathcal{H}^u g(t,x,p) &= A^u f_p(t,x,p) + \mathcal{H}^u g(t,x,p).
\end{align*}
\]
And by using the concrete expression of infinitesimal generator $A^u$ we can rewrite Eq.(3.6) as follows:

$$
\begin{align*}
\sup_{\pi(t), \hat{h}(t)} \left\{ f_p^g(t, x, p) + \gamma' p g(t, x, p) g_x(t, x, p) + (f^p_x(t, x, p) + \gamma' p g(t, x, p))g_x(t, x, p) \right\} \\
g_x(t, x, p)[x + (\mu(t) - r)\tilde{\pi}(t) + \lambda \nu_1(\hat{h}(t)(1 + \theta) + (\eta - \theta))] \\
+ \frac{1}{2} \sigma^2 g_{xx}(t, x, p)\tilde{\pi}^2(t) + g(t, x, p)g_{xp}(t, x, p)M_p + \lambda \mathbb{E}[f^p(t, x - \alpha(t)Y, p) - f^p(t, x, p)] \\
= 0,
\end{align*}
$$

(3.9)

$$
\begin{align*}
\frac{1}{2} g_{xx}(t, x, p)\tilde{\pi}^2(t) + g(t, x, p)g_{xp}(t, x, p)M_p + \lambda \mathbb{E}[f^p(t, x - \alpha(t)Y, p) - f^p(t, x, p)] \\
= 0,
\end{align*}
$$

(3.10)

$$
\begin{align*}
\frac{1}{2} g_{xx}(t, x, p)\tilde{\pi}^2(t) + g(t, x, p)g_{xp}(t, x, p)M_p + \lambda \mathbb{E}[g(t, x - \alpha(t)Y, p) - g(t, x, p)] \\
= 0,
\end{align*}
$$

(3.11)

$$
\begin{align*}
\left\{ \begin{array}{ll}
\tilde{f}_p(T, x, p) = x - \frac{\gamma' \tilde{p}}{2} x^2, \\
\tilde{V}(T, x, p) = x, \\
g(T, x, p) = x.
\end{array} \right.
\end{align*}
$$

(3.12)

Since equilibrium control $\tilde{u}(t) = (\tilde{\pi}(t), \tilde{h}(t))$ maximizes the LHS of HJB (3.9), the equilibrium control $(\hat{\pi}(t), \hat{h}(t))$ is founded to be

$$
\hat{\pi}(t) = \frac{\langle f^p_x + \gamma' p g_{px} \rangle (\hat{u} - r) + f^p_x + \gamma' p g_{px} Y \rangle M_p}{\langle f^p_x + \gamma' p g_{px} \rangle \sigma^2},
$$

$$
\hat{h}(t) = \arg \sup_{h(t)} \{ \mathbb{E}[f^p(t, x - \hat{h}(t)Y, p) + \gamma' p g(t, x - \hat{h}(t)Y, p)] \}
$$

(3.13)

and

$$
f^p_x(t, x, p) + \gamma' p g(t, x, p)g_{xx}(t, x, p) < 0.
$$

Obviously, by substituting Eqs (3.13) into Eqs(3.10)-(3.11), one can derive explicit expressions of $f(t, x, p, \tilde{p})$ and $g(t, x, p)$ with boundary conditions (3.12), consequently one can get the equilibrium value function by Eq.(3.8).
4. Solution of the MV problem

In this section, inspired by the boundary condition (3.12), we try to solve the extended HJB equation (3.9)-(3.12) by making the following Ansatz:

\[
\begin{align*}
g(t, x, p) &= a(t, p)x + b(t, p), \\
f(t, x, p, \bar{p}) &= a(t, p)x + b(t, p) - \frac{1}{2} [A(t, p)x^2 + B(t, p)x + C(t, p)]
\end{align*}
\]

with boundary condition \(a(T, p) = 1, b(T, p) = 0, A(T, p) = 1, B(T, p) = 0\) and \(C(T, p) = 0\). We assume that \(A(t, p) > 0\) for all \((t, p)\), then the LHS of (3.9) is indeed a concave quadratic function with respect to \(\pi(t)\) and \(h(t)\), so maximum can be attained.

For simplicity, we adopt a simpler notation scheme by denoting \(a(t, p), b(t, p), A(t, p), B(t, p)\) and \(C(t, p)\) by \(a, b, A, B\) and \(C\), respectively. With the mentioned Ansatz Eqs(4.1), the equilibrium control in Eqs(3.13) can be rewritten as follows:

\[
\begin{align*}
\hat{\pi}(t) &= \frac{-a \gamma p B / 2 + \gamma p a_b b_p - r \gamma p}{\frac{1}{2} A_p^2} + \frac{(A_p - \gamma p B / 2 + \gamma p a_b b_p) / A_p^2}{\frac{1}{2} A_p^2} + \frac{(a^2 - A) (A_p - \gamma p B / 2 + \gamma p a_b b_p) / A_p^2}{\frac{1}{2} A_p^2} \\
\hat{h}(t) &= \frac{-a \gamma p B / 2 + \gamma p a_b b_p - r \gamma p}{\frac{1}{2} A_p^2} + \frac{a^2 - A}{A_p^2}
\end{align*}
\]

Since \(f(t, x, p, \bar{p}) = g(t, x, p) - \frac{1}{2} [A(t, p)x^2 + B(t, p)x + C(t, p)]\), the Eqs: \(A^u f(t, x, p) = 0\) and \(A^u g(t, x, p) = 0\) hold for all \(\bar{p}\) is equivalent to

\[
\begin{align*}
A^u [a(t, p)x + b(t, p)] &= 0, \\
A^u [A(t, p)x^2 + B(t, p)x + C(t, p)] &= 0.
\end{align*}
\]

By Eqs (4.3), (3.10), (3.11) and (4.2), we first derive the following equations satisfied by \(a(t, p)\) and \(A(t, p)\):

\[
\begin{align*}
a_t &= a[r + \frac{(a^2 - A)(\mu'p - r) + (a a_p - A_p)Mp}{A_p^2} (\mu'p - r) + \frac{\lambda a^2}{A_p^2} a^2 - A)] \\
&+ a' \frac{(a^2 - A)(\mu'p - r) + (a a_p - A_p)Mp}{A_p^2} M + Q'(p) + \frac{1}{2a^2} p' M a_{pp} M p = 0, \\
A_t &= 2A[r + \frac{(a^2 - A)(\mu'p - r) + (a a_p - A_p)Mp}{A_p^2} (\mu'p - r) + \frac{\lambda a^2}{A_p^2} a^2 - A)] \\
&+ A' \frac{2(a^2 - A)(\mu'p - r) + (a a_p - A_p)Mp}{A_p^2} M + Q'(p) + \frac{1}{2a^2} p' M a_{pp} M p \\
&+ \frac{(a^2 - A)(\mu'p - r) + (a a_p - A_p)Mp}{A_p^2} M^2 = 0
\end{align*}
\]

with boundary condition \(a(T, p) = A(T, p) = 1\). Then we can directly establish the solution of this ODE system (4.4), namely, \(A(t, p) = \exp(2r(T - t))\) and \(a(t, p) = \exp(r(T - t))\), which means that \(A(t, p) = a^2(t, p)\) and \(A_p(t, p) = a_p(t, p) = A_{pp}(t, p) = a_{pp}(t, p) = 0\). Substituting these relations into Eqs (4.2), we get a simpler form of equilibrium
control:

\[
\begin{aligned}
\hat{\pi}(t) &= \frac{a - \gamma_p pB/2 + \gamma_p \sigma^2 \sigma_p - r}{A \sigma^2} \quad B_p M_p \\
\hat{h}(t) &= \frac{a - \gamma_p pB/2 + \gamma_p \sigma^2 \sigma_p}{A \gamma_p} \quad B_p M_p \\
\end{aligned}
\]

(4.5)

It is easy to see that the equilibrium strategy is independent with the insurer’s wealth, our result is consistent with Zeng et al. [25] (2013) and Wei et al. [20] (2013). Another thing we want to point out is the phenomenon that optimal investment strategy is independent with insurer’s wealth also occurs when the insurer is of constant absolute risk aversion.

With this simpler form of equilibrium control (4.5), we can derive the ODE system satisfied by \( b(t, p) \) and \( B(t, p) \):

\[
\begin{aligned}
&b_t + b_p Q_p + \frac{1}{\sigma^2} p B_{pp} M_p + a \lambda \nu_1 (\eta - \theta) + (\mu - \gamma_p p - r) \cdot \\
&[b_p - \frac{a + \gamma_p p (ab - B/2)}{A \sigma^2} + \frac{a + \gamma_p p (ab - B/2)}{\sigma^2}] \lambda \nu_2 = 0, \\
&B_t + B_p Q_p + \frac{1}{\sigma^2} p B_{pp} M_p + 2 A \lambda \nu_1 (\eta - \theta) + \\
&2(\mu - \gamma_p p - r) \cdot [b_p - \frac{a + \gamma_p p (ab - B/2)}{\sigma^2}] \lambda \nu_2 = 0
\end{aligned}
\]

(4.6)

with boundary condition \( b(T, p) = B(T, p) = 0 \).

By Feynman-Kac formulas, we can solve Eqs(4.6) as follows:

\[
\begin{aligned}
&b(t, p) = \mathbb{E} \left\{ \int_t^T \frac{1}{\gamma} Z(s) \left[ (\mu' Z(s) - r) \right]^2 + \lambda \theta^2 \nu_2^2 \right\} ds \bigg| Z(t) = p \\
&B(t, p) = 2a(t) b(t, p)
\end{aligned}
\]

(4.7)

where stochastic process \( Z = (Z(t)) \in \mathcal{R}^d \) is a solution of the following SDE:

\[
dZ(t) = (Q' - \mu Z(t) - r) \frac{Z(t)}{\sigma^2} dt + \frac{1}{\sigma} M(t) dW(t)
\]

and \( \tilde{M} = \text{Diag}(\mu - \mu' Z(t) I_d) \) is a \( d \times d \) matrix.

Finally, we solve equation satisfied by \( C(t, p) \):

\[
\begin{aligned}
C_t + C_p Q_p + \frac{1}{2 \sigma^2} p M C_{pp} M_p + \frac{1}{\gamma^2} \left[ (\mu Z(s) - r) \right]^2 + \lambda \theta^2 \nu_2^2 \\
+ 2b_p \left[ \frac{a + \gamma_p p (ab - B/2)}{\sigma^2} - b_p' M_p \right] \lambda \nu_2 - \frac{(b_p' M_p)^2}{\sigma^2} = 0
\end{aligned}
\]

(4.8)

Let \( N(t, p) = C(t, p) - b^2(t, p) \), by Eqs(4.6) and (4.8), we have

\[
N_t + N_p Q_p + \frac{1}{2 \sigma^2} p M N_{pp} M_p + \frac{1}{\gamma^2} \left[ (\mu Z(s) - r) \right]^2 + \lambda \theta^2 \nu_2^2 = 0
\]

(4.9)

with boundary: \( N(T, p) = 0 \). By Feynman-Kac formula, we have

\[
N(t, p) = \mathbb{E} \left\{ \int_t^T \frac{1}{\gamma^2 Z(s)} \left[ (\mu Z(s) - r) \right]^2 + \lambda \theta^2 \nu_2^2 \right\} ds \bigg| Z(t) = p
\]

(4.10)
Theorem 4.1. The equilibrium control for the MV problem (2.3) and (3.3) is given by
\[
\begin{align*}
\hat{x}(t) &= \frac{1}{\sigma^2} \left( \frac{\mu'p - r}{\gamma'p} - b'_p Mp \right) e^{r(t-T)}, \\
\hat{h}(t) &= \frac{\theta
u_1}{\gamma'p\nu_2} e^{r(t-T)}
\end{align*}
\]
and the corresponding equilibrium value function is given by
\[
V(t, x) = \tilde{V}(t, x, p) = e^{(T-t)x + b(t, p) - \frac{\gamma'p}{2}N(t, p)},
\]
where \(b(t, p)\) and \(N(t, p)\) are given by (4.7) and (4.10).

Proof. Combining (4.5) and (4.7), it is easy to get (4.11). By (3.8) and (4.1), we have
\[
\tilde{V}(t, x, p) = a(t, p)x + b(t, p) - \frac{\gamma'p}{2} [A(t, p)x^2 + B(t, p)x + C(t, p)] + \frac{\gamma'p}{2} [a(t, p)x + b(t, p)]^2
\]
\[
= e^{(T-t)x + b(t, p) - \frac{\gamma'p}{2}N(t, p)},
\]
where the second equation is from \(A(t, p) = a^2(t, p) = e^{2(T-t)}, B(t, p) = 2a(t, p)b(t, p)\) and \(N(t, p) = C(t, p) - b^2(t, p)\). By Lemma 3.2 and Eq.(3.5), statement \(V(t, x) = \tilde{V}(t, x, p)\) follows. \(\Box\)

Remark 4.1. Viewing Lemma 3.1 and (4.11), the equilibrium reinsurance strategy \(\hat{h}(t)\) depends on \(p(t)\), thus depends on \((S_1(t))\), this coincides with econometric intuition, as different paths of stock price will change the insurer’s risk aversion, hence change the optimal reinsurance strategy. By Lemma 3.2 and (3.7), given the equilibrium control law and the information filtration \((\mathcal{G}_s)_{0 \leq s \leq T}\), both the conditional expectation and conditional variance of the terminal wealth are given by
\[
\begin{align*}
\mathbb{E}_{t,x}[X^u(T)|\mathcal{G}_t] &= \mathbb{E}_{t,x,p}[\tilde{X}^u(T)] = e^{(T-t)x + b(t, p)}, \\
\text{Var}_{t,x}[X^u(T)|\mathcal{G}_t] &= \text{Var}_{t,x,p}[\tilde{X}^u(T)] = N(t, p).
\end{align*}
\]

Remark 4.2. If \(\mu = \mu_11_d\), which means that premium rate of the stock is independent of market state, then we have \(p_k(t) = P(I(t) = e_k|\mathcal{G}_t) = P(I(t) = e_k)\) and \(M = 0_{d \times d}\), therefore equilibrium investment in stock degenerates to \(\hat{\pi}(t) = \frac{1}{\sigma^2} \frac{\mu_1 - r}{\gamma'p} e^{r(t-T)}\). The term \(b'_p Mp\) in (4.11) appears due to that stock price process contains some information about market state. Since the claim process of insurer is independent of market state, similar term does not appear in the equilibrium reinsurance strategy \(\hat{h}(t)\). Further more, if also \(\gamma = \gamma_11_d\), then our result coincides with the special case in Zeng et al.[25](2013).
5. Equilibrium results in completely observable case

In this section, we assume that the information available to the insurer is filtration \( \{\mathcal{F}_t\}_{t \in [0,T]} \). That is, in addition to information generated by stock price process and insurer’s claim process, the information about state of the market is also available. Consequently, investment-reinsurance strategy \((\pi(t), h(t))\) is no longer required to be \( \{\mathcal{G}_t\}_{t \in [0,T]} \) measurable.

The wealth process of insurer is still given by (2.1), but comparing to (2.2), the objective functional changes to

\[
J_c(t, x, i, u(\cdot)) = \mathbb{E}_{t,x} \left[ X^u(T) | \mathcal{F}_t \right] - \frac{\gamma}{2} \text{Var}_{t,x} \left[ X^u(T) | \mathcal{F}_t \right] = \mathbb{E}_{t,x,i} \left[ X^u(T) \right] - \frac{\gamma}{2} \left\{ \mathbb{E}_{t,x,i} \left[ (X^u(T))^2 \right] - \mathbb{E}_{t,x,i} \left[ (X^u(T))^2 \right] \right\},
\]

and problem (2.3) becomes

\[
\max_{u \in \mathcal{U}_c} \left\{ J_c(t, x, i, u(\cdot)) \right\},
\]

where \( \mathcal{U}_c \) represents set of all admissible control law. For any \( \varphi(t, x, i) \in C^{1,2}(0, T] \times \mathcal{R} \), let

\[
\mathcal{A}^u_c \varphi(t, x, i) = \varphi_t(t, x, i) + \varphi_x(t, x, i)[r x + (\mu(t) - r) \pi(t) + \lambda v(t)(1 + \theta) + (\eta - \theta)]) + \frac{1}{2} \varphi_{xx}(t, x, i) \sigma^2 h(t) + \lambda \mathbb{E}[\varphi(t, x) - h(t) Y, p] - \varphi(t, x, p)] + \sum_{j=1}^{d} q_{ij} \varphi(t, x, j).
\]

Similar to Section 3, we have following verification theorem for completely observable case.

**Theorem 5.1.** (Verification Theorem) Suppose that there are functions \( V_c(t, x, i), g(t, x, i) \in C^{1,2}(0, T] \times \mathcal{R} \) and \( f(t, x, i, j) \in C^{1,2}(0, T] \times \mathcal{R} \) satisfying the following extended HJB equation: \( \forall(t, x), \)

\[
\begin{aligned}
\sup_{u \in \mathcal{U}_c} \left\{ \mathcal{A}^u_c V_c(t, x, i) - \mathcal{A}^u_c f_c(t, x, i, i) + \mathcal{A}^u_c f_c(t, x, i) \right\} = 0,
- \mathcal{A}^u_c (G_c \circ g_c)(t, x, i) + \mathcal{H}^u_c g_c(t, x, i) = 0,
\mathcal{A}^u_c f_c(t, x, i) = 0,
\mathcal{A}^u_c g_c(t, x, i) = 0,
f_c^{ij}(T, x, p) = \frac{\gamma j}{2} x^2,
V_c(T, x, p) = x, g_c(T, x, p) = x,
\end{aligned}
\]

where the maximum of the first equation is attained at \( \hat{u}_c(t, x, i) \) for all \( (x, i) \in \mathcal{R} \times \mathcal{N} \), \( f_c^{ij}(t, x, i) = f_c(t, x, i, j) \), \( (G_c \circ g_c)(t, x, i) = G_c(t, x, i, g_c(t, x, i)) \)

\[
= \frac{\gamma j}{2} g_c^2(t, x, i) \quad \text{and} \quad \mathcal{H}^u_c g_c(t, x, i) = \frac{\partial (G_c \circ g_c)(t, x, i)}{\partial y} A^u_c g_c(t, x, i).
\]

Then \( \hat{u}_c \) is an equilibrium control law, and \( V_c(t, x, i) \) is the corresponding equilibrium value function. Moreover, by Feynman-Kac formula, \( f \)
and \( g \) have the following expressions:
\[
\begin{align*}
    f_c(t, x, i, j) &= \mathbb{E}_{t,x,i}[X_{\tilde{u}_c}(T) - \frac{\gamma_j}{2}(X_{\tilde{u}_c}(T))^2], \\
    g_c(t, x, i) &= \mathbb{E}_{t,x,i}[X_{\tilde{u}_c}(T)].
\end{align*}
\]

By similar procedure done in Section 4 and tedious calculation, we can solve the extended HJB equation (5.3), and the corresponding equilibrium control law and value function are as follows:

**Theorem 5.2.** The equilibrium control for the MV problem (5.2) is given by
\[
\begin{align*}
    \hat{\pi}_c(t, i) &= \frac{\mu_i - r}{\gamma_i \sigma^2} e^{r(t-T)}, \\
    \hat{h}_c(t, i) &= \frac{\theta \nu_1}{\gamma_i \nu_2} e^{r(t-T)},
\end{align*}
\]

and the corresponding equilibrium value function is given by
\[
V_c(t, x, i) = e^{(T-t)}x + b_c(t, i) - \frac{\gamma_i}{2} N_c(t, i),
\]

where \( b_c(t, i) \) and \( N_c(t, i) \) satisfy following ODE systems:
\[
\begin{align*}
    \frac{db_c(t, i)}{dt} + \sum_{j=1}^d q_{ij} b_c(t, j) + \frac{1}{\nu_1} \left( \frac{(\mu_i - r)^2}{\sigma^2} + \lambda \theta \nu_2 \right) e^{r(T-t)} + \lambda \nu_1 (\eta - \theta) e^{r(T-t)} &= 0, \\
    b_c(T, i) &= 0,
\end{align*}
\]

and
\[
\begin{align*}
    \frac{dN_c(t, i)}{dt} + \sum_{j=1}^d q_{ij} N_c(t, j) + \frac{1}{\nu_1} \left( \frac{(\mu_i - r)^2}{\sigma^2} + \lambda \theta \nu_2 \right) e^{r(T-t)} + \lambda \nu_1 (\eta - \theta) e^{r(T-t)} &= 0, \\
    N_c(T, i) &= 0.
\end{align*}
\]

Our completely observable case has some connection with the mean-variance asset-liability problem studied in Wei et al.[20](2013), the main difference is that in our model, liability follows a compound poisson jump process, no longer a geometric Brownian motion process, and the insurer can partially hedge risk of liability by reinsurance.

**Remark 5.1.** Comparing to the equilibrium strategy (4.11) for partially observable case, strategy (5.5) for completely observable case is no longer continuous and jumps at time when the market state changes. As stated in Lemma(3.2) and (4.11), the objective functional and equilibrium control law depend on the history of \((S_1(s))_{s \leq t}\) only through \(p(t)\), we find that when the market state is observable, strategy (5.5) is independent of stock’s price process.
Remark 5.2. Contrary to Remark 4.1, the conditional expectation and conditional variance of the terminal wealth for completely observable case are given by

\[
\begin{align*}
E_{t,x,i}[X^u(T)] &= e^{(T-t)x + b_c(t,i)}, \\
\text{Var}_{t,x,i}[X^u(T)] &= N_c(t,i),
\end{align*}
\]

where \( b_c(t,i) \) and \( N_c(t,i) \) are given by (5.7) and (5.8).

Remark 5.3. Given \( p(0) = e_k \) and \( I(0) = e_k \), which is equivalent to say that insurer knows the market state for \( t = 0 \). For time \( 0 \leq t \leq T \), we have \( E[h(t)] \leq E[h_c(t)] \). Actually it is equivalent to prove \( E[\frac{1}{\gamma p(t)}] \leq E[\sum_{i=1}^d \frac{p_i(t)}{\gamma_i}] \) follows, which means in the partially observable case, the insurance company will transfer more liability risk to reinsurer and hold less for itself. This result coincides with economic insight, since partial information means more risk. By similar procedure, we have \( E[\mu r - \sum_{i=1}^d \frac{p_i(t)}{\gamma_i}] \) if \( \frac{\mu - r}{\gamma_d} \leq ... \leq \frac{\mu - r}{\gamma_1} \), and \( E[\mu r - \sum_{i=1}^d \frac{p_i(t)}{\gamma_i}] \) if \( \frac{\mu - r}{\gamma_1} \geq ... \geq \frac{\mu - r}{\gamma_d} \). Since there is a term \( b' \gamma p \) in (4.11), it is difficult to compare \( E[\hat{\pi}(t)] \) and \( E[\hat{\pi}_c(t)] \), we will give some numeric results in next section.

But for \( Q = 0_{d \times d} \), our problem degenerates to Bayesian case, some results for CRRA utility are given in Baurle and Rieder [17](2005).

6. Sensitivity analysis

In this section we give some numerical examples to show the effect of parameters on equilibrium investment-reinsurance policy and corresponding value function under partial information and compare results with equilibrium strategy in completely observable case. Since neither the optimal investment amounts nor the optimal proportions are deterministic, we only study the exact amounts of allocations or proportions in expectation sense.
Without loss of generality, we assume that \( d = 2 \), i.e., the market state is either "bullish" or "bearish" correspondingly to Regime 1 and Regime 2. Throughout this section, basic parameters influenced by market state are shown in Table 1. Note that \(-q_{11} = q_{12} = q_1\) and \(-q_{21} = -q_{22} = q_2\). Other parameters are given by \( r = 0.04, \sigma = 0.2, \nu_1 = 0.6, \nu_2 = 1, \theta = 1.5, \eta = 1, T = 10, p_1(0) = 0.3, p_2(0) = 0.7, x_0 = 2. \)

<table>
<thead>
<tr>
<th>Regime</th>
<th>( \mu_i )</th>
<th>( \gamma_i )</th>
<th>( q_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>0.2</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>0.05</td>
<td>0.9</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 1. values of the parameters in our model

To calculate \( b(t, p) \) and \( N(t, p) \), one can use Monte Carlo simulation method to calculate (4.7) and (4.10), but the amount of calculation is rather big, therefore we try to solve \( b(t, p) \) and \( N(t, p) \) directly by (4.6), (4.9) and PDE numerical method. Noting that \( p_1(t) + p_2(t) \equiv 1 \), let \( k(t, p_1) = b(t, p_1, 1 - p_1) \) and \( l(t, p_1) = N(t, p_1, 1 - p_1) \), then \( k_{p_1} = b_{p_1} - b_{p_2}, k_{p_1 p_1} = b_{p_1 p_1} - 2b_{p_1 p_2} + b_{p_2 p_2} \), for \( l(t, p_1) \), partial derivatives respect to \( p_1 \) just similar. Then we can get PDEs satisfied by \( k(t, p_1) \) and \( l(t, p_1) \):

\[
\begin{aligned}
&k_t + (q_e - \frac{\mu_e - r}{\sigma^2} m_e)k_{p_1} + \frac{m_e^2}{2\sigma^2} k_{p_1 p_1} + \frac{1}{\gamma_e} (\frac{\mu_e - r}{\sigma^2})^2 \\
&+ \lambda \frac{\hat{q}^2 \nu_1^2}{\nu_2^2} + \lambda \nu_1 (p - \theta) e^{(T-t)} - \frac{\nu_{e}^2}{\gamma_e} = 0, \\
l_t + q_e l_{p_1} + \frac{m_e^2}{2\sigma^2} l_{p_1 p_1} + \frac{1}{\gamma_e} (\frac{\mu_e - r}{\sigma^2})^2 + \lambda \frac{\hat{q}^2 \nu_1^2}{\nu_2^2} = 0, \\
k(T, p_1) = 0, l(T, p_1) = 0,
\end{aligned}
\]

(6.1)

where \( q_e = q_2 - (q_1 + q_2) p_1, \mu_e = \mu_2 + (\mu_1 - \mu_2) p_1, m_e = (\mu_1 - \mu_2) p_1 (1 - p_1) \) and \( \gamma_e = \gamma_2 + (\gamma_1 - \gamma_2) p_1 \). Then equilibrium strategy in (4.11) becomes

\[
\hat{\pi}(t) = \frac{1}{\sigma^2} (\frac{\mu_e - r}{\gamma_e} - k_{p_1} m_e) e^{(t-T)}, \quad \hat{h}(t) = \frac{\theta \nu_1}{\gamma_e \nu_2} e^{(t-T)}.
\]

6.1. Sensitivity analysis of the equilibrium strategy and value function. Some numerical illustrations and sensitivity analysis for the equilibrium investment-reinsurance strategy and the corresponding value function are presented in this subsection.

Figures 1-3 show the impacts of parameters: generator of transition probability \( Q \), premium rate of stock \( \mu \), insurer’s risk aversion \( \gamma \) on the equilibrium investment strategy. Comparing equilibrium strategies
under \((q_1, q_2) = (0.3, 0.6)\) and \((q_1, q_2) = (0.6, 0.6)\) in Figure 1, when 
t is near time \(T = 10\), smaller \(q_1\) means larger amount invested in s-
tock, however when \(t\) is near time 0, this relation may changes. Since 
\(q_2\) does not change, smaller \(q_1\) means longer average time the Markov 
chain stays in Regime 1 and stationary distribution of the Markov 
chain has larger probability for Regime 1, also Regime 1(bullish market) 
means large premium and small risk aversion. When \(t = T\), we have 
\(k_{p_1}(T, p_1) = 0\), so \(E[\hat{\pi}(T)]\) mainly depends on the stationary distribution 
of the Markov chain, by analysis above, smaller \(q_1\) means larger 
investment in stock, this also explains strategies for \((q_1, q_2) = (0.6, 0.6)\) 
and \((q_1, q_2) = (0.3, 0.3)\) converge at time \(T\). However, when \(t = 0\), the
distribution of Markov chain is given, the main effect is $k_{p_1}(0, p_1)$, as shown in Figure 1, smaller $q_1$ leads to bigger $k_{p_1}(0, p_1)$. Figures 2 shows that investment in stock increases with bigger stock premium, this coincides with economic intuition. Figure 3 shows that when $t = T$, insurer with low risk aversion invests more in stock, since the stationary distribution does not change and $k_{p_1}(T, p_1) = 0$, however, similar to Figure 1, when $t$ goes to 0, this relation inverses also due to term $k_{p_1} m_e$, which means that insurer with lower risk aversion for bullish market prefers to invest relatively less when the probability for bullish market is small, and invest relatively more when the probability becomes large. Figure 4 shows how $k_{p_1}(t, p_1)$ develops with time $t$, for parameters given in our model, $k_{p_1}(t, p_1)$ is nonnegative, increases as $p_1$ becomes larger and decreases as time goes by with final term $k_{p_1}(T, p_1) = 0$.

Figures 5-6 show the impacts of parameters: generator of transition probability $Q$, insurer’s risk aversion $\gamma$ on the equilibrium reinsurance strategy. Since there is no similar term $b_{p_i} M p$ in $\hat{h}(t)$ as pointed in Remark 4.2, many monotonous properties preserves. Figure 5 shows that given $q_2$ determined, smaller $q_1$ leads to higher insurer’s retention level, and given $q_1$ determined, smaller $q_2$ means lower insurer’s retention level. This is mainly due to different $(q_1, q_2)$ lead to different stationary distribution as pointed above, actually the difference between reinsurance strategies for $(q_1, q_2) = (0.3, 0.3)$ and $(q_1, q_2) = (0.6, 0.6)$ is relatively small as shown in Figure 5. Figure 6 shows that the insurer’s retention...
level increases as insurer’s risk aversion decreases, this is because when insurer’s risk aversion is low, insurer is more risk seeking and transfer less liability risk to reinsurance company to reduce cost for reinsurance, with lower risk aversion, the insurer may even acquire new business ($\hat{h}(t) > 1$).

Figures 7-9 show the influence of parameters: generator of transition probability $Q$, premium rate $\mu$, insurer’s risk aversion $\gamma$ on the equilibrium value function, these parameters influence equilibrium investment-reinsurance strategy and hence change $b(t, p)$ and $N(t, p)$ representing expectation and variance of wealth correspondingly. Always more investment in stock and higher retention level mean larger expectation and variance simultaneously, then synthetical effect is complex. Comparing value functions in Figure 7 for $(q_1, q_2) = (0.3, 0.6)$ and
\((q_1, q_2) = (0.3, 0.3)\), although insurer significantly invests more in stock and keeps higher retention level for the first case (see Figure 1 and Figure 5), value functions are almost the same. Then comparing value functions for \((\mu_1, \mu_2) = (0.2, 0.05)\) and \((\mu_1, \mu_2) = (0.1, 0.5)\) in Figure 8, we find that bigger \(\mu_1\) may lead to smaller \(V(t, x_0)\) for time near 0. Figure 9 depicts value functions for different \(\gamma\), comparing \((\gamma_1, \gamma_2) = (0.5, 0.9)\) and \((\gamma_1, \gamma_2) = (0.3, 0.9)\), we find that smaller \(\gamma_1\) leads to smaller \(V(t, x_0)\), this result contradicts to Y.Zeng et al.(2013) who investigate equilibrium value function with no regime switching and all processes is observable.

6.2. **Comparison between partially and completely observable cases.** In this subsection, we mainly compare different equilibrium investment reinsurance strategies and value functions for partially observable and completely observable cases. We only consider \(p_1(0) = 0\) and \(I(0) = e_2\) and \(p_1(0) = 1\) and \(I(0) = e_1\), which is equivalent to require that insurer knows the market state for \(t = 0\) as pointed in Remark 5.3. In Figures 10 - 12, “par” and “com” represent partial information and complete information case correspondingly.

In Figures 10 and 11, we can find that investment amount and retention rate in completely observable case is bigger than in partially observable case. Since \(k_{p_1} \geq 0\) shown in Figure 4 and \(\frac{\mu_1 - r}{\gamma_1} \geq \frac{\mu_2 - r}{\gamma_2}\) is satisfied, this result coincides with Remark 5.3. We also find that, as time goes by, expected investment and retention level converge for different initial market states in both partially and completely observable cases.

In Figure 12, we find that equilibrium value for completely observable case is smaller than for partially observable case, and for case that initial market is bearish, difference is significant. The underlying reason in shown in Figure 13, more investment in stock and higher retention level for completely observable case shown in Figures 10 and 11 results larger expectation of wealth, but results in even much larger variance.
7. Conclusion

In this paper, we consider an optimal investment and reinsurance problem under partial information for insurer with mean-variance utility within a game theoretic framework in light of equilibrium control law proposed by Björk and Murgoci\cite{4}. Specially, we assume that the surplus process of insurer follows classical Lundberg model and the drift process of risky asset is depicted by an unobservable Markov-modulated regime switching model. Different from Zeng et al.\cite{24, 25}(2011, 2013) and Björk et al.\cite{5}(2014), risk aversion of the insurer in our model changes over time due to his or her estimation of instant market state given information before generated by stock price process. Inspired by
Björk and Murgoci[4](2010), we provide a verification theorem without proof and derive explicitly the time-consistent investment-reinsurance strategy and corresponding value function. In addition, we give another verification theorem given that the market state modeled by a continuous-time Markov chain is observable, time-consistent strategies and corresponding value function are also derived. Some comparison results between partially observable case and completely observable case are illustrated. Further more, we present some numeric illustrations and sensitivity analysis to demonstrate results we have derived.

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References


Highlights for review

- Study optimal investment and reinsurance problem under partial information
- The performance of the control problem is mean-variance utility
- Find time-consistent equilibrium strategy in a game theoretic framework
- Derive closed-form optimal investment-reinsurance strategy and value function
- Compare results under partial information with complete information