Optimal dynamic asset allocation of pension fund in mortality and salary risks framework

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ABSTRACT

In this paper, we consider the optimal dynamic asset allocation of pension fund with mortality risk and salary risk. The managers of the pension fund try to find the optimal investment policy (optimal asset allocation) to maximize the expected utility of terminal wealth. The market is a combination of financial market and insurance market. The financial market consists of three assets: cashes with stochastic interest rate, stocks and rolling bonds, while the insurance market consists of mortality risk and salary risk. These two non-hedging risks cause incompleteness of the market. By martingale method and dynamic programming principle we first derive the approximate optimal investment policy to overcome the difficulty, then investigate the efficiency of the approximation. Finally, we solve an optimal assets liabilities management (ALM) problem with mortality risk and salary risk under CRRA utility, and reveal the influence of these two risks on the optimal investment policy by numerical illustration.

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1. Introduction

As a major factor of social security system, the pension funds now are definitely confronted with many serious risks, such as...
finance risk and mortality risk as well as salary risk during the diversification process suiting the demands in the market. The assets liabilities management (ALM) for a company has become a vital important act to control these different threats. The ALM theory is pioneered by Anglo-Saxon financial institutions during the 1970s to study the mismatches between the assets and liabilities. There are two fundamental methods to solve ALM problem, i.e., the so-called stochastic dynamic programming and martingale methods. The stochastic dynamic programming method is first proposed by Merton (1971) and relies on the stochastic control and theory of Hamilton–Jacobi–Bellman (HJB) equations. Recently, Menoncin and Scaillet (2006) applied this method to life annuity. By using the theory of Lagrange multipliers, the martingale method was beautifully developed by Cox and Huang (1989) in the setting of complete market. Some successful applications in ALM problem without mortality risk can be referred to Boulier et al. (2001) and Deelstra et al. (2003, 2004).

It is well-known that one can get the same solution for one optimization problem by these two methods in the setting of complete market but it is not sure in the setting of incomplete market. Hainaut and Devolder (2006) showed that these two methods also get the same solution on the same optimization problem in the setting of incomplete market for a pure endowments insurance case. Whereafter, Hainaut and Devolder (2007) successfully solved an optimized ALM problem with mortality risk and dividends in incomplete market by stochastic dynamic programming. The solution obtained in incomplete market is indeed an approximation of the target process by a projection from the expanded space, thus it is not the exact optimal solution in self-financed space and so one has great doubt about efficiency of the approximation.

However, to the best of our knowledge in incomplete markets, no literature studies the exact optimal solution in self-financed space and how to get it, so the approximation is optimal in some sense as soon as its efficiency is confirmed. To demonstrate the efficiency of approximation, we first consider a degenerated ALM problem whose exact optimal solution can be easily obtained and compare exact solution with approximation by numerical illustration. Then, we solve the ultimate ALM problem with two non-hedging risks to be more close to real market by efficient approximation.

The first non-hedging risk considered in this paper is the mortality intensity risk. It is influential in a long time insurance contract and makes the mortality risk become a systematic risk. The mortality risk is first described by a doubly stochastic counting process in Brémaud (1981) and the related models were also established by the comparison of the mortality risk with credit risk in financial market. In particular, the total number of death is a Poisson process with stochastic intensity. For describing stochastic mortality intensity better, the affine models are widely expanded, including the Ornstein–Uhlenbeck (OU) process and CIR models, which have been both minutely studied by Luciano et al. (2012). In this paper we will use the OU model to describe mortality intensity because it has an explicit expression. But when the actuarial one turns to be stochastic, the interplay of stochastic intensity between assets and liabilities causes the difficulty of calculating further liabilities. We calculate the explicit expressions of interactional terms and use the property of multidimensional Gaussian processes to get over this difficult.

The second non-hedging risk investigated in this paper is the salary risk because some advanced defined benefit (DB) pension plans are related to employees’ salaries of retired time. These kinds of pension plans develop rapidly because they can promote employees’ initiative in working and guarantee the same living standards after retiring. In this paper, we also introduce a risk factor to describe the working atmosphere in a department, which is indispensable in salary besides financial market. Following this motivation, Blake et al. (2001) first studied the pension plans related to the salary risk. Cairns et al. (2006) studied the optimal dynamic asset allocation for defined contribution pension plans. The reader also refers to Hainaut and Deelstra (2011), Guan and Liang (2014), He and Liang (2013a,b) for the recent works about defined contribution pension plans. With the same idea, the salary risk is described by an exponential Brownian motion in this paper, while the non-hedging Brownian motion describing the initiative of employees.

The rest of the paper is organized as follows. Sections 2 and 4 present the mathematical models of assets and actuarial liabilities, respectively. Section 5 establishes two non-hedging risk models related to the ALM problems. Section 5 formulates the optimization problem and derives its general solution by Lagrange multipliers. Section 6 compares the optimal strategies in self-financed space with the approximation obtained by dynamic programming, and demonstrates the efficiency of this approximation. In Section 7, we first get the closed-form of the optimal solution for the optimization problem with the two non-hedging risks under CRRA utility, which is more qualified to be a utility function than CARA. Then we give a numerical illustration to show how the economic behaviors of mortality and salary risks impact on the optimal strategies. Finally, we point out the essence of salary risk and mortality risk. The last section is a conclusion.

2. Assets

We consider a complete financial market composed of three assets: cashes with stochastic interest rate, stocks and rolling bonds. The financial probability space is denoted by \((\Omega^1, \mathcal{F}^1, \mathbb{P}^1, P^1, \mathbb{F})\), where the Brownian motions \((W^1_t)\) and \((W^0_t)\) on \((\Omega^1, \mathcal{F}^1, \mathbb{P}^1)\) are independent, and \(\mathcal{F}^1 = \sigma(\bigcup_{t\geq0} \mathcal{F}^1_t)\).

The existence of a unique equivalent measure \(Q^1\) is guaranteed by the completeness of the financial market. Under these assumptions we characterize the dynamics of the three assets as follows:

The stochastic interest rate is modeled by the following Vasicek’s model:

\[
\begin{align*}
\frac{dr_t}{\sigma} &= r_t (\bar{r} - r_t) dt + \sigma_t dW^r_t \\
&= r_t (\bar{r} - r_t - \frac{\sigma_t}{\theta_t}) dt + \sigma_t (dW^r_t + \theta_t dt),
\end{align*}
\]

where the \((W^r_t)\) is a Brownian motion under the \(Q^1\). The parameters \(r, \bar{r}\) and \(\sigma_t\) are positive constants but the \(\theta_t\) is a negative constant.

The rolling bond \([K^f_t]\) with maturity \(K\) is defined by the following SDE:

\[
\frac{dK^f_t}{K^f_t} = r_t dt - \sigma_t n(K)(dW^r_t + \theta_t dt)
\]

\[
= r_t dt - \sigma_t n(K) d\tilde{W}^r_t,
\]

where the \((\tilde{W}^r_t)\) is a Brownian motion under the \(Q^1\). The parameters \(r, \bar{r}\) and \(\sigma_t\) are positive constants but the \(\theta_t\) is a negative constant.

The rolling bond \([K^f_t]\) with maturity \(K\) is defined by the following SDE:

\[
\frac{dK^f_t}{K^f_t} = r_t dt - \sigma_t n(K)(dW^r_t + \theta_t dt)
\]

\[
= r_t dt - \sigma_t n(K) d\tilde{W}^r_t,
\]

where the \(n(K)\) is determined by the maturity of the rolling bond: \(n(K) = \frac{1}{\theta_t}(1 - e^{-\theta_t K})\).

The stock \((S_t)\) is a geometric Brownian motion satisfying the following SDE:

\[
\frac{dS_t}{S_t} = r_t dt + \sigma_{st} (dW^s_t + \theta_t dt) + \sigma_s (dW^r_t + \theta_t dt)
\]

\[
= r_t dt + \sigma_{st} d\tilde{W}^s_t + \sigma_s d\tilde{W}^r_t,
\]

where the \(\sigma_{st}, \sigma_s\) and \(\theta_t\) are positive constants and the stock’s risk premium is \(\nu_s = \sigma_s \theta_t + \sigma_{st} \theta_t\).
The neutral measure $Q^f$ can be formulated by
$$\left. \frac{dQ^f}{dp}\right|_{\mathcal{F}_t} = \exp\left\{ -\frac{1}{2} \int_0^t (\theta^2 + \theta_t^2) du - \int_0^t (\theta_t dW^a_u + \theta^a_t dW^b_u) \right\}.$$ 

Moreover, if we assume that the fund is self-financed, the wealth process can be uniquely determined by the investment strategies $\{\pi^1_t, \pi^2_t\}$, where the $\pi^1_t$ is the fraction of the total asset invested in stocks at time $t$, and the $\pi^2_t$ is the fraction in rolling bonds. Thus, the SDE of wealth $\{X_t\}$ is as follows.

$$dX_t = \left(1 - \pi^1_t - \pi^2_t\right) r_t dt + \pi^1_t dR^F_t + \pi^2_t dS_t,$$

$$= \left(\tau_t + \pi^1_t \nu_t + \pi^2_t \nu_t\right) dt + \pi^1_t \sigma dW^s_t + (\pi^1_t \sigma_{\nu} - \pi^2_t \sigma_{n(K)}) dW^b_t.$$ 

The whole self-financed processes with different investment strategies form a self-financed space, denoted by $\Lambda^F_t(x)$.

$$\Lambda^F_t(x) = \left\{ X_t : \exists \text{ an } \{\mathcal{F}_t\}-\text{adapted process } \{(\pi^1_t, \pi^2_t)\} \right\}$$

$$e^{-\int_0^t \tau_s dX_s} = x + \int_t^T \pi^1_s X_s \frac{d}{s} e^{-\int_0^t \tau_s dX_s} dS_s$$

$$+ \int_t^T \pi^2_s X_s \frac{d}{s} e^{-\int_0^t \tau_s dX_s} dR_s. \quad (2.4)$$

3. Non-hedging risks

The financial market is complete but the insurance market is usually incomplete in reality. There are two non-hedging risks in the insurance market, which result in the incompleteness indeed and influence the further liabilities directly. This section discusses these two non-hedging risks.

3.1. Mortality risk

The stochastic mortality is modeled by OU process

$$d\lambda_t = \alpha (\lambda_t - \mu) dt + \sigma dW^a_t,$$ \hspace{1cm} (3.1)

where $\alpha > 0$, $\sigma \geq 0$, and $\{W^a_t\}$ is a Brownian motion under probability measure $P^a$. It is well-known that the OU process has the following expression:

$$\lambda_t = \lambda_0 e^{\alpha t} + \int_0^t e^{\alpha (s-t)} dW^a_s.$$ \hspace{1cm} (3.2)

As in Møller (1998), the longevities of $n_t$ employees are i.i.d. exponential random variables, denoted by $\tau_1, \tau_2, \ldots, \tau_{n_t}$, and are defined on a probability space $(\Omega^f, \mathcal{F}_{\infty}^f, P^f)$, where the filtration $\mathcal{F}^f$ is generated by $\{W^f_t\}_{t \geq 0}$ and $\{\xi_i : i = 1, \ldots, n_t\}$. To be accurate, the longevity can be considered as a stopping time related to stochastic mortality by

$$\tau_i = \inf\left\{ s : \int_0^s \lambda_u du > \xi_i \right\}, \quad i = 1, \ldots, n_t,$$

where $\xi_i$ is a normal exponential distributed random variable and is independent of the above Brownian motions, i.e., it has probability density function $f(x) = \frac{1}{\xi_i} e^{-\xi_i x}$. Then the survival probability of this longevity can be written as

$$S(t, s) = P(t \geq s | \tau > t)$$

$$= \mathbb{E}\left\{ P(\zeta > \int_t^s \lambda_u du) | \mathcal{F}_t \right\}$$

$$= \mathbb{E}\left\{ \exp\left( -\frac{1}{2} \int_0^s \theta^2 + \theta_u du - \int_0^s \theta_u dW^a_u + \theta^a_u dW^b_u \right) | \mathcal{F}_t \right\}$$

$$\triangleq \exp\left( -\int_0^s f(t, t + u) du \right), \quad (3.3)$$

where $f(t, s)$ is the forward death intensity of the stochastic mortality (cf. Luciano et al. (2012)). When stochastic mortality intensity is modeled by the stochastic process (3.2), the forward death intensity is

$$f(t, s) = \lambda_t e^{\alpha (s-t)} - \frac{\sigma^2}{2\alpha^2} (e^{\alpha (s-t)} - 1)^2.$$ \hspace{1cm} (3.4)

Assume that $\{N_t\}_{t \geq 0}$ is the total number of deaths at instant $t$, which is defined by a counting process

$$N_t = \sum_{i=1}^{n_t} l_{\omega(\tau_i \leq t)}(\omega),$$

$I_A(\cdot)$ is an indicator function of $A$ and the filtration $\mathcal{F}^a$ mentioned above can be exactly chosen as one generated by $\{N_t\}$. By a simple calculation we know that the compensated process $\{N_t\}$ of the counting process $\{N_t\}$ is a martingale under $P^f$ and has the following expression:

$$\tilde{N}_t = N_t - \int_0^t (n_k - n_{k-}) \lambda_s du. \quad (3.5)$$

3.2. Salary risk

The terminal benefit provided by a salary-related pension fund is a proportion of member’s final salary. Because the employees are both from the same department with numerous members, we only investigate the pension fund counting initially $n_t$ members with the same age $t$ and salary $\{M_t\}$. The stochastic salary correlated to the financial market is

$$dM_t = \left(\tau_t + v_m\right) dt + \sigma_m dW^m_t + \sigma_m dW^s_t + \sigma_m dW^b_t,$$ \hspace{1cm} (3.6)

where the constant parameters $\sigma_m, \sigma_m$ and $\sigma_m$ denote the correlation between the salaries and the interest rates, the correlation between the salaries and the stocks, and the embedded volatility of salaries, respectively. The parameter $\tau_t + v_m$ is the stochastic growth rate of the salary and $\sigma_m$ is determined by $v_m, \theta$, $\sigma_m$, $\theta$, $\sigma_m$, and $\sigma_m$ in Eq. (3.6). The process $\{W^m_t\}$ is a Brownian motion defined on a probability space $(\Omega^m, \mathcal{F}_{\infty}^m, P^m)$, where the $\mathcal{F}^m$ is the filtration generated by the $\{W^m_t\}$. The process $\{W^m_t\}$ is independent of $\{W^s_t, W^b_t\}$ and represents the intrinsic randomness of the salary, and this randomness also reflects the incentive in working. As this salary risk cannot be traded, the process $\{W^m_t\}$ is a source of incompleteness. The $Q^m$ is an actuarial neutral measure (under which the process $\{W^m_t\}$ is a Brownian motion) defined by

$$\left. \frac{dQ^m}{dp}\right|_{\mathcal{F}_t} = \exp\left( -\frac{1}{2} \int_0^t \theta^2 + \theta_u du - \int_0^t \theta_u dW^m_u + \theta^m_u dW^m_u \right).$$ \hspace{1cm} (3.7)
4. Liabilities

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be the filtrated complete probability space as a combination of the financial market and the insurance market:

\[
\Omega = \Omega^F \times \Omega^I \times \Omega^m, \\
\mathcal{F}_t = \mathcal{F}_t^F \otimes \mathcal{F}_t^I \otimes \mathcal{F}_t^m \wedge \mathcal{N}, \\
\mathcal{F} = \mathcal{F}_\infty, \\
\mathbb{P} = \mathbb{P}^F \times \mathbb{P}^I \times \mathbb{P}^m,
\]

where the \(\sigma\)-algebra \(\mathcal{N}\) is generated by all subsets of null sets from \(\mathcal{F}^F \otimes \mathcal{F}^I \otimes \mathcal{F}^m\). The liability of the company in the pension fund is the expectation of total capitals paid out to survivors at retirement time \(T\). For an alive employee at time \(T + s(0 < s < T - T)\), the company will pay out \(\rho M_t e^{\int_t^{T + s} \nu ds}\) as a pension, where \(M_t\) is the salary at retirement time \(T\) and \(\rho \leq 1\) is a constant proportion. Thus, for an alive employee at retirement time \(T\), the liability of company at time \(T\) is

\[
L(T) = \mathbb{E}\left[\int_T^{T + s} e^{-\int_t^{T + s} \tau ds} \rho M_t e^{\int_t^{T + s} \nu ds} e^{-\int_t^T \lambda u du} ds \middle| \mathcal{F}_T\right] = \rho M_t \int_T^{T + s} \mathbb{E}\left[\exp\left(-\int_t^T \lambda u du\right) \middle| \mathcal{F}_T\right] ds, \\
= \rho M_t \int_T^{T + s} \exp\left(-\int_0^\infty f(T, t + u) du\right) ds, \\
= \rho M_t \int_T^{T + s} \exp\left[-\int_0^\infty \left[-\lambda_T e^{au} + \frac{\sigma_u^2}{2a^2}(e^{au} - 1)^2\right] du\right] ds, \\
= M_t \rho \int_T^{T + s} \exp\left[-\frac{\lambda_T}{2a^2}\left(e^{a(T-t)} - 1\right) - \frac{\sigma_u^2}{2a^2}\left(e^{a(T-t)} - 1\right)\right] ds \leq M_t D(\lambda_T),
\]

where we have just used the expressions (3.3) and (3.4). The \(D(\cdot)\) is a function of \(1_T\) and parameters \(\sigma_u, \rho\) represents a discount factor about death rate. So, the total liabilities at time \(T\) for the company is \((n_T - N_T) L(T)\) and the expectation of the total liabilities at instantaneous time \(t\) is

\[
\mathbb{E}\left[(n_T - N_T) L(T) \mid \mathcal{F}_T\right] = \mathbb{E}\left[\sum_{i=1}^{N_T} I(t_i > T) L(T) \mid \mathcal{F}_T\right] = \sum_{\tau > T} \mathbb{E}\left[I(t_i > T) L(T) \mid \mathcal{F}_T\right], \\
= (n_T - N_T) \mathbb{E}\left[I(t_i > T) M_T D(\lambda_T) \mid \mathcal{F}_T, \tau > T\right], \\
= (n_T - N_T) \mathbb{E}\left[M_T \mid \mathcal{F}_T^a \wedge \mathcal{F}_T^m\right], \\
\times \mathbb{E}\left[I(t_i > T) D(\lambda_T) \mid \mathcal{F}_T, \tau > T\right].
\]

5. Optimization problem and general solutions

This section formulates the optimization problem of the assets and liabilities management (ALM) and derives the closed-forms of the optimal investment strategies in expanded space under a given utility functions by theory of Lagrange multipliers.

5.1. Optimization problem

The ALM is to reduce the mismatches between the assets and liabilities by controlling the investment policy. In particular, for any given wealth \(x\), death intensity \(\lambda\), salary \(m\), interest rate \(r\) and \(n\) observed deceases, the value function \(V(t, x, n, \lambda, m, r)\) at time \(t\) is

\[
V(t, x, n, \lambda, m, r) = \sup_{\mathcal{X}_T} \left\{ \mathbb{E}[U(\mathcal{X}_T - (n_T - N_T) L(T)) \mid \mathcal{F}_T, \mathcal{F}_T, \mathcal{F}_T] \right\},
\]

where \(U(\cdot)\) is a utility function that is concave and strictly increasing. The wealth processes are chosen from admissible set \(\mathcal{A}_t\) defined by (5.2), which is an expansion of the self-financed space \(\mathcal{A}_T^\gamma\). In economics, the expanded space \(\mathcal{A}_t\) is delimited by a budget constrain: the expectation of the deflated terminal wealth is not bigger than the current wealth, i.e.,

\[
\mathcal{A}_t(x) = \left\{ \mathcal{X}_T : \mathbb{E}[H(t, T) \mathcal{X}_T \mid \mathcal{F}_T] \leq x \right\}.
\]

where \(H(t, T)\) is the deflator of the combined market and is defined as a product of the financial deflator and the salary actuarial change of measure. It has the following expression:

\[
H(t, T) = e^{-\int_l^T \tau d\theta_u^T} \left( \frac{\sigma_u^2}{2a^2} \right)_T \left( \frac{\theta_u^T}{\theta_u^T} \right)_T, \\
\exp\left(-\int_t^T \tau d\theta_u^T - 1 + T \int_t^T (\theta_m^T + \theta_r^T + \theta_l^T) du - \int_t^T (\theta_m^T dW_u^o + \theta_r^T dW_u + \theta_l^T dW_u^o)\right).
\]

Since the real admissible set is a self-financed space \(\mathcal{A}_T^\gamma\) but not the expanded space \(\mathcal{A}_t\), the above optimization problem (5.1) is not the optimal ALM problem in real markets. So the optimization problem concerned in this paper should be

\[
V(t, x, n, \lambda, m, r) = \sup_{\mathcal{X}_T \in \mathcal{A}_T^\gamma} \left\{ \mathbb{E}[U(\mathcal{X}_T - (n_T - N_T) L(T)) \mid \mathcal{F}_T] \right\}.
\]

However, the optimal problem formulated in self-financed space is really tough solved due to the incompleteness of the market and the ill definition of utility CRRA to negative terminal surplus, which is a significant utility mentioned in Hainaut and Devolder (2007). But the optimal investment strategies in self-financed space can be approximated by optimization in expanded space via the martingale method or dynamic programming (cf. Hainaut and Devolder (2006)), so in the following subsection we first solve the optimization problem in expanded space.

5.2. Optimization problem in expanded space

The method to solve the optimization problem in expanded space is theory of Lagrange multipliers developed by Cox and Huang (1989) in complete market, the reader can refer to Karatzas and Shreve (1998) for the details. The Lagrange multiplier to the budget constrain (5.2) is a positive real number and we denote it by \(y_t\). Thus, the Lagrangian function is

\[
\mathcal{L}(t, x, n, \lambda, m, r, \mathcal{X}_T, y_t) = \mathbb{E}[U(\mathcal{X}_T - (n_T - N_T) L(T)) \mid \mathcal{F}_T] + y_t \left[x - \mathbb{E}(H(t, T) \mathcal{X}_T) \mid \mathcal{F}_T\right].
\]
The sufficient condition to obtain the optimal wealth $\tilde{X}^*_T$ is that the wealth $\tilde{X}^*_T$ is a saddle point of the Lagrangian function (5.4) corresponding to the optimal Lagrange multiplier $y^*_T > 0$. So the value function can be formulated by the Lagrangian function

$$V(t, x, n, \lambda, m, r) = \inf_{Y_t \in \mathbb{R}^+} \left( \sup_{X_t} \mathcal{L}(t, x, n, \lambda, m, r, X_t, y_t) \right).$$

There exists a continuous function $I(\cdot)$ satisfying $U(I(x)) = x$ under the assumptions that utility $U(\cdot)$ is a strictly concave and increasing $C^1$ function. Differentiating (5.4) with respect to $\tilde{X}_t$ yields that the optimal terminal wealth is a function of optimal Lagrange multiplier $y^*_T$ and has the following expression:

$$\tilde{X}^*_T = I(y^*_T H(t, T)) + (n_k - N_T)L(T), \quad (5.5)$$

and the optimal Lagrange multiplier $y^*_T$ is determined by the budget constraint:

$$x = E[H(t, T)I(y^*_T H(t, T)) + (n_k - N_T)L(T)| \mathcal{F}_t]. \quad (5.6)$$

The value function can also be calculated (if $y^*_T$ is known) by

$$V(t, x, n, \lambda, m, r) = E[U(I(y^*_T H(t, T)))| \mathcal{F}_t]. \quad (5.7)$$

It is obvious that the optimal terminal wealth $\tilde{X}^*_T$ in (5.5) depends on observed deceases $N_T$ at time $T$. When death occurs, $\tilde{X}^*_T$ must be a jump process, so it is not in the self-financed space. Thus the next step is to obtain the approximate optimal strategy in self-financed space by projection of optimal terminal wealth from expanded space. The projection methods can be implemented by martingale method and dynamic programming method and these two methods are proved to have the same performance (cf. Hainaut and De Volder (2006)). Hence, in the next section we will solve a single risk management problem by these two methods directly.

6. Comparison of optimal strategy and approximation

This section compares the exact optimal investment strategy with the approximate strategy obtained by projection to demonstrate the efficiency of this approximation method. In some original DB pension fund, benefit is really determined, thus the terminal surplus can be guaranteed positive in a conservative strategy as long as the initial solvency guaranteed. But in our salary-related benefit, terminal surplus can be negative in any strategies because the terminal salary always has possibilities of explosion. Therefore, to get an exact optimal investment, exponential (CARA) utility functions are used in this section for the well-defined negative surplus and some convenience of calculation. Besides, the interest rate is constant and the death intensity grows with a constant rate for more simplicity.

Let $r_t = r$, $\sigma_t = 0$ in (2.1), $\sigma_t = 0$ in (2.3), $\sigma_t = 0$ in (3.1), and $\sigma_{mr} = 0$ in (3.6). Then the rolling bond defined by (2.2) becomes a risk-less asset which provides a constant growth rate $r$, so we only have one risky asset in markets and a non-hedging salary risk in this simplified management problem. It is worth mentioning the reason we hold a non-hedging risk is that the approximation by projection equals the optimal investment strategy in self-financed space once the market becomes complete.

6.1. Martingale method

First, we use the martingale method to obtain the approximation. The optimal Lagrange multiplier defined by (5.6) can be calculated by

$$-\frac{1}{\alpha} \ln y^*_n = \frac{1}{E[H(t, T)| \mathcal{F}_t]} \left[ x - E[(n_k - N_T)H(t, T)L(T)| \mathcal{F}_t] \right]$$

$$+ \frac{1}{\alpha} E[H(t, T) \ln H(t, T)| \mathcal{F}_t]$$

$$= \frac{1}{e^{-r(T-t)}} \left[ x - (n_k - N_T)T^{-1}P_t M(t) D \right.$$}

$$+ \frac{1}{e^{-r(T-t)}} \left( -\tilde{r} - \frac{\sigma_t^2}{2} + \frac{\sigma_{mr}^2}{2} \right) (T-t) \right].$$

Substituting $y^*_n$ into Eq. (5.5), the expression of optimal terminal wealth is

$$\tilde{X}^*_T = -\frac{1}{\alpha} \ln y^*_T H(t, T)) + (n_k - N_T)L(T)$$

$$= e^{-(T-t)x} + \int_0^T \frac{\theta_t}{\alpha} (dW^u + \theta_t du) + \int_0^T \frac{\theta_m}{\alpha} (dW^m + \theta_m du)$$

$$+ (n_k - N_T)M(t) - (n_k - N_T)\tau^{-1}P_t M(t) D$$

$$= e^{-(T-t)x} + \frac{1}{\alpha} \left( \int_0^T \theta_t d\tilde{W}^u + \theta_m d\tilde{W}^m \right)$$

$$+ (n_k - N_T)M(t) \tilde{D} == E^Q \left[ (n_k - N_T)M(t) D| \mathcal{F}_t \right].$$

where $D = D(\lambda_T)$ is a constant since the mortality process now is not stochastic. The deflated wealth can be decomposed into several stochastic integrals w.r.t. Brownian motions and compensated process (see (3.5)):

$$E[\tilde{X}^*_T H(t, T)| \mathcal{F}_t] = E^Q \left[ \tilde{X}^*_T e^{-r(T-t)}| \mathcal{F}_t \right]$$

$$= E^Q \left[ x + \frac{e^{-r(T-t)}}{\alpha} \left( \int_0^T \theta_t d\tilde{W}^u + \theta_m d\tilde{W}^m \right) \right.$$

$$\left. + (n_k - N_T)\tau^{-1}P_t M(t) D (n_k - N_T)\tau^{-1}P_t M(t) D| \mathcal{F}_t \right]$$

$$= x + \frac{e^{-r(T-t)}}{\alpha} \left( \int_0^T \theta_t d\tilde{W}^u + \theta_m d\tilde{W}^m \right)$$

$$+ \int_0^T (n_k - N_T)\tau^{-1}P_t e^{-r(u-1)} M(t) D (\sigma_m d\tilde{W}_u^m + \sigma_m d\tilde{W}_u^m)$$

$$- \int_T^T \tau^{-1} P_t e^{-r(u-1)} M(t) D \left( \int_0^T \frac{dN_u - (n_k - N_T)\lambda_u du}{\tilde{n}} \right), \quad (6.1)$$

where $\tilde{W}^u$ and $\tilde{W}^m$ are Brownian motions under the neutral measure $Q$ defined by a product of the neutral measures $Q^u$ and $Q^m$.

Comparing (6.1) with the martingale decomposition of the self-financed process $X_t$ in (2.4) rewritten by

$$e^{-r(t-s)} X_s = x + \int_s^T e^{-r(u-s)} \pi_u X_s \sigma d\tilde{W}_u^1, \quad (6.2)$$

and letting the coefficients of $d\tilde{W}_u$ be equal in (6.1) and (6.2), we see that the approximate optimal strategy in self-financed space is

$$\tilde{\pi}^*_u \approx \frac{\theta_t + \alpha D(n_k - N_T)\pi_m e^{-r(T-u)}}{\alpha X_u \sigma} e^{r(T-u)}.$$

6.2. Dynamic programming

The second method to obtain the optimal investment policy is called stochastic dynamic programming. By expression (5.7), the
optimal value function is
\[
V(t, x, n, m) = -\frac{1}{\alpha} \mathbb{E}[H(t, T)|\mathcal{F}_t] \\
= -\frac{1}{\alpha} \exp\left(-\alpha e^{(T-t)}(x - (n_k - n_f)_{t\rightarrow T} P_t m)D\right) \\
\times \exp\left(-\left(\frac{\theta_1^2}{2} + \frac{\theta_2^2}{2}\right)(T-t)\right).
\]

Using Itô’s formula for jump processes, the expectation of the value function at \(t+\triangle t\) is
\[
\mathbb{E}\left[V(t+\triangle t, x_{t+\triangle t}, N_{t+\triangle t}, M_{t+\triangle t})|\mathcal{F}_t\right] = V(t, x, n, m) + \mathbb{E}\left[\int_t^{t+\triangle t} G^*(s, X_s, N_s, M_s)ds|\mathcal{F}_t\right] \\
+ \mathbb{E}\left[\int_t^{t+\triangle t} (V(s, X_s, N_s, M_s) - V(s, X_s, N_{s-}, M_s))dN_s|\mathcal{F}_t\right].
\]

The function \(G^*(s, X_s, N_s, M_s)\) has the following expression:
\[
G^*(s, X_s, N_s, M_s) = V_t + (r + \pi_t) X_s V_t \\
+ \left(\sigma\sigma_t + \sigma_t^2\right) M_s V_t + \frac{1}{2}\sigma^2\sigma_t^2 X_t^2 V_t \\
+ \frac{1}{2}\left(\sigma^2_0 + \sigma^2_t\right) M_t V_t + \pi_t \sigma_m M_t X_t X_m,
\]
where \(V_t, X_t, V_{XX}, V_{XM}\) and \(V_{XXM}\) are partial derivatives of the value function w.r.t. time, wealth and salary respectively.

By maximizing the continuous term \(G^*\) in Eq. (6.4), it is easy to see that the approximate optimal strategy \(\tilde{\pi}_t^u\) is
\[
\tilde{\pi}_t^u = \frac{-V_t X_t + \alpha m M_t V_t}{\sigma^2 X_t X_m}.
\]

Using Eq. (6.3), the partial derivatives \(V_t, V_{XX}\) and \(V_{XM}\) are
\[
\frac{\partial V(t, x, n)}{\partial x} = -ae^{(T-t)}V(t, x, n), \\
\frac{\partial^2 V(t, x, n)}{\partial x^2} = (ae^{(T-t)})V(t, x, n), \\
\frac{\partial^3 V(t, x, n)}{\partial x^2 \partial m} = (n_k - n_f)_{t\rightarrow T} P_t D(\alpha e^{(T-t)})V(t, x, n),
\]
respectively. Therefore, the approximation obtained by dynamic programming is
\[
\tilde{\pi}_t^u = \frac{\theta_t + \alpha D(n_k - n_f) m \sigma_m e^{-\alpha u}}{\alpha X_t \sigma_t e^{(T-u)}},
\]
and it is equal to the approximation obtained by martingale method. The identity is not a coincidence, the reader can refer to Hainaut and Devolder (2007).

6.3. Exact solution

To demonstrate the efficiency of the approximation, the exact optimal strategy should be given now. This subsection presents the exact optimal investment strategy in self-financed space. The exact optimal can be gained thanks to the exponential form of CARA and the property of exponential martingale.

**Theorem 6.1.** Let \(r_t = \bar{r}, \sigma_t = 0\ in (2.1), \sigma_r = 0\ in (2.3), \sigma_x = 0\ in (3.1), \sigma_m = 0\ in (3.6), and the utility function be \(U(x) = -\frac{1}{\alpha} e^{-\alpha x}\). Considering the optimization problem (5.1) in self-financed space \(A^t_x(x)\), the optimal investment strategy \(\pi^*_t\) is determined by
\[
\pi^*_t = \frac{\theta_t + \alpha D(n_k - n_f) m \sigma_m e^{-\alpha u}}{\alpha e^{(T-u)}X_t \sigma_t}.
\]

**Proof.** Seeing Appendix A, the terminal surplus at time \(T\) can be decomposed into the following integrals w.r.t. Brownian motions and the compensated process.
\[
X_T = (n_k - n_f) L(T) = e^{(T-t)}C_0 + \int_t^T C_1 d\tilde{W}_u + \int_t^T C_2 d\tilde{N}_u \\
+ \int_t^T C_3 du + \int_t^T C_4 d\tilde{W}_u^m,
\]
where the functions \(C_0, C_1, C_2, C_3\) and \(C_4\) are respectively
\[
C_0 = X_t - D(n_k - n_f) M_t, \\
C_1 = e^{\alpha(T-u)}(\alpha X_t \sigma_t - D(n_k - n_f) M_t \sigma_m), \\
C_2 = -D \alpha e^{\alpha(T-u)} M_t, \\
C_3 = D \alpha e^{\alpha(T-u)} M_t (n_k - n_f)_{u\rightarrow T} P_t \lambda_u, \\
C_4 = -D(n_k - n_f) e^{\alpha(T-u)} M_t \sigma_m.
\]

By (5.3) and the property of exponential martingale, the value function can be expressed by
\[
V(t, x, n, m) = \sup_{X_t \in A^t_x(x)} \mathbb{E}\left[-\frac{1}{\alpha} \exp\left(-\alpha e^{(T-t)} C_0 + \int_t^T C_1 d\tilde{W}_u \\
+ \int_t^T C_2 d\tilde{N}_u + \int_t^T C_3 du + \int_t^T C_4 d\tilde{W}_u^m\right)\right]|\mathcal{F}_t
\]
\[
= \sup_{X_t \in A^t_x(x)} \mathbb{E}\left[-\frac{1}{\alpha} \exp\left(-\alpha e^{(T-t)} C_0 + \int_t^T \left(\frac{1}{2} \alpha^2 C_1^2 - \alpha C_1 \theta_t\right) du\right) \\
+ \int_t^T -\alpha C_4 dN_u \\
- \int_t^T (e^{-\alpha C_2} - 1)(n_k - n_f) \lambda_u du \\
+ \int_t^T (e^{-\alpha C_2} - 1 + \alpha C_2)(n_k - n_f) \lambda_u du\right]|\mathcal{F}_t
\]
\[
= \sup_{X_t \in A^t_x(x)} \mathbb{E}\left[-\frac{1}{\alpha} \exp\left(-\alpha e^{(T-t)} C_0 + \int_t^T \left(\frac{1}{2} \alpha^2 C_1^2 - \alpha C_1 \theta_t\right) du\right) \\
+ \int_t^T -\alpha C_4 dN_u \\
+ \int_t^T (e^{-\alpha C_2} - 1 + \alpha C_2)(n_k - n_f) \lambda_u du\right].
\]

Since \(\{\pi_u\}\) is an \(\{\mathcal{F}_u\}\)-adopted process, maximizing the expression in conditional expectation in any time \(u(t \leq u < T)\) can maximize the value function. Thus it is sufficient to maximize the following expression:
\[
-\frac{1}{\alpha} \exp\left(-\alpha e^{(T-t)} C_0 + \int_t^T \left(\frac{1}{2} \alpha^2 C_1^2 - \alpha C_1 \theta_t\right) du\right) \\
+ \int_t^T (e^{-\alpha C_2} - 1 + \alpha C_2)(n_k - n_f) \lambda_u du \\
+ \int_t^T -\alpha C_4 du + \int_t^T \left(\frac{1}{2} \alpha^2 C_2^2 - \alpha C_4 \theta_m\right) du.
\]
Because the functions $C_0, C_2, C_3$ and $C_4$ are independent of $\pi$, we only need to maximize $\frac{1}{2}\alpha^2 C_1^2 - \alpha c C_1(\theta_t)$, it is easy to see that $\frac{\partial}{\partial \theta_t}$ maximizes $\frac{1}{2}\alpha^2 C_1^2 - \alpha c C_1(\theta_t)$ due to the convexity of quadratic function. So the optimal investment strategy $\pi^*$ in self-financed space is
\[
\pi^*_u = \frac{\theta_t + \alpha D(n_u - N_u)M_u\sigma M_n e^{(T-u)}}{\sigma e^{(T-u)}X_u\sigma}.
\]

6.4. Numerical illustration

This subsection simulates the strategy processes and the wealth process under the optimal investment strategy and the approximation, then we compare them intuitively. The optimal investment strategy $\pi^*_u$ and the approximate strategy $\hat{\pi}^*_u$ are respectively
\[
\pi^*_u = \frac{\theta_t + \alpha D(n_u - N_u)M_u\sigma M_n e^{(T-u)}}{\sigma e^{(T-u)}X_u\sigma},
\]
\[
\hat{\pi}^*_u = \frac{\theta_t + \alpha D(n_u - N_u)M_u\sigma M_n e^{(T-u)}}{\sigma e^{(T-u)}X_u\sigma}.
\]

Because $t = u < T$ is an increasing function of $u(t < u < T)$ and equals 1 at $u = T$, $\hat{\pi}^*_u$ equals $\hat{\pi}^*_T$ at terminal time $T$. This reveals the consistency between the exact optimal solution and the approximation. The consistency is commonsense for there is no deceases and the market is complete at the terminal instant. Moreover, the approximate strategy shows more risk aversion than the exact optimal solution because $\hat{\pi}^*_u$ is not bigger than $\pi^*_u$ at all. After the analysis by expression of two strategies, more comparisons are made by numerical illustration with following parameters: $t = 50, T = 65, T^* = 110, \alpha = 0.1, \sigma_a = 0, \lambda_t = 0.01, r = 0.02, \sigma_t = 0.15, \theta_t = 0.35, \sigma_m = 5/100, \theta_m = 0.1, \sigma_m^s = 0.02, M_t = 2500, \rho = 0.3, n_u = 100, \alpha = 0.05$ and $X_u = L(t)$.

The consistency at terminal time and risk aversions are verified in Fig. 2. Also, $\sup_{t < u < T} |\pi^*_u - \hat{\pi}^*_u| = 4.01\%$ shows that absolute error in investment strategies is just 5% and can be allowed in real investment. Fig. 1 shows that the trajectory of $X^*$ coincides with $X^*$ which means that the wealth process controlled by approximate optimal strategies are close to the exact optimal wealth processes. By some simple calculations in Matlab, the fractional error in wealth processes is obtained by $\sup_{t < u < T} |X^*_u - X^*_u| = 3.32\%$, which means that the behavior of the approximate processes is good enough. Thus the efficiency of approximation by martingale method and dynamic programming method can be confirmed in both aspects of theoretical analysis and numerical simulations.

7. Solution of double risks

This section solves the optimal ALM problem with mortality risk and salary risk by dynamic programming, and reveals the influence of these two risks on the optimal strategy. The CRRA utility will be used here because it is more qualified than CARA for the optimal ALM problem.

7.1. Expression of approximation

**Theorem 7.1.** Let $\sigma > 0$ in (2.1), $\sigma_x > 0$ in (2.3), $\sigma_a > 0$ in (3.1) and the utility function be $U(x) = -\frac{1}{\gamma}x^\gamma$ $(0 < \gamma < 1)$. Considering the optimization problem (5.1), the value function over expanded space $A_t(x)$ is
\[
V(t, x, n, \lambda, m, r) = \frac{1}{\gamma} B(r_t)^{1-\gamma} \left( x - (n_u - n) m \hat{D}(\lambda) \right)^\gamma,
\]
and the optimal investment strategy $\hat{\pi}^*_u$ obtained by dynamic programming is
\[
\hat{\pi}^*_u = \left( \frac{\sigma^2 + \sigma_n^2}{\sigma_n^2 n(K) X_u} \right) + \left( \frac{\lambda_t}{n(K)} \right) + \left( \frac{\sigma_m M_n(u - N_u)}{\sigma_n^2 n(K) X_u} \right) + \left( \frac{\gamma n_t(T-u)}{n(K) X_u (1-\gamma)} \right),
\]
where the equity($u$) = $X_u - (n_u - N_u) M_u \hat{D}(\lambda_u)$, and the function $D(\lambda_u)$ is determined by the following equations:
\[
\hat{D}(\lambda_u) = \int_{\lambda_u}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} D(y) f(x, y|\lambda_u) e^{-x} dx dy dz,
\]
\[
f(x, y|\lambda_u) = \frac{1}{(2\pi)^{3/2} \det(\Sigma)^{1/2}} \exp \left( -\frac{1}{2} (x - y)^T - \mu)^T \Sigma^{-1} (x - y)^T - \mu \right),
\]
\[
D(\lambda_u) = \rho \int_0^{T} \int_{-\infty}^{\lambda_u} (s - T) ds - \lambda_T n_s (s - T) + \frac{\sigma_n^2}{2 \sigma^2} \left( s - T - \frac{1}{2} n_s (2(s - T)) - 2 n_s (s - T) \right) ds,
\]
\[
\mu(\lambda_u) = \left( \frac{n_s (T - s) \lambda_u}{e^{(T-s) \lambda_u}} \right), \quad \Sigma = \left( \begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array} \right).
\]
Proof. Using (4.2), (5.6) and (5.7), the optimal Lagrange multiplier \( y^* \) is

\[
y^*_t = \frac{1}{\beta} \left[ \frac{x - xE[H(t, T)(n_t - N)]L(t)}{E[H(t, T)^{\gamma - 1}] | F_t} \right] = 1 \left[ \frac{x - (n_t - N)(H(t, T)M | F_t^n \vee F_t^m) \left\{ \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) T \hat{D}(\lambda(t)) \right] \right\}^{\gamma}}{B(t)} \right],
\]

and the value function \( V(\cdot) \) over the expanded space \( A_t(x) \) is

\[
V(t, x, n, \lambda, m, r) = \frac{1}{\beta} \left[ \frac{x - xE[H(t, T)^{\gamma - 1}] | F_t]}{E[H(t, T)^{\gamma - 1}] | F_t} \right] \left\{ \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) m \hat{D}(\lambda(t)) \right] \right\}^{\gamma},
\]

where \( B(t) \) is defined by (7.2), and the calculation of \( \hat{D}(\lambda(t)) \) can be found in Appendix B.

By Itô’s formula for jump processes, the expectation of the value function \( V(\cdot) \) at \( t + \Delta t \) is

\[
\mathbb{E} \left[ V(t + \Delta t, X_{t + \Delta t}, N_{t + \Delta t}, \lambda_{t + \Delta t}, M_{t + \Delta t}, n_{t + \Delta t}) | F_t \right] = V(t, x, n, \lambda, m, r) + \mathbb{E} \left[ \int_t^{t + \Delta t} \mathbb{G}(s, X_s, N_s, \lambda_s, M_s, r_s) | F_t \right] + \mathbb{E} \left[ \int_t^{t + \Delta t} V(s, X_s, N_s, \lambda_s, M_s, r_s) | F_t \right] - \mathbb{E} \left[ V(s, X_s, N_s, \lambda_s, M_s, r_s) dN_s | F_t \right].
\]

The value function \( \mathbb{G}(s, X_s, N_s, \lambda_s, M_s, r_s) \) is calculable as

\[
\mathbb{G}(s, X_s, N_s, \lambda_s, M_s, r_s) = V_s + \alpha_s (r - r_s) V_X + (r_s + \pi^*_s v_s + \pi^*_s v_r) X s V_X
\]

\[+ (r_s + v_m) M_s V_M + a \lambda_s V_A + \frac{1}{2} a^2 V_{RR} + \frac{1}{2} \left( (\pi^*_s)^2 + (\pi^*_s)^2 - \pi^*_s \sigma (K)_s \right) X^2 V_{XX} + \frac{1}{2} \left( (\sigma^2_m + \sigma^2_m + \sigma^2_m + \sigma^2_m) M^2_M V_{MM} + \frac{1}{2} a^2 V_{LL} + \sigma_r (\pi^*_s \sigma - \pi^*_s \sigma (n)_s X s V_X + \sigma_r \sigma_m M_s V_{RM} + \sigma_m (\sigma^2_s \sigma_r - \sigma^2_s \sigma_r(n)_s) M s X V_M, \]

where \( V_s, V_X, V_M, M_s, V_{MM}, V_{ML}, V_{XX}, V_{RM}, V_{MM} \) and \( V_{XM} \) are first and second order partial derivatives of \( V(\cdot) \) with respect to interest rate, wealth, salary, and mortality, respectively. By maximizing \( \mathbb{G} \), the approximation optimal strategy \( \hat{\pi}_u^* \) is obtained.

\[
\hat{\pi}_u^* = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right)^{\gamma} \right] \hat{\pi}_u,
\]

\[
\pi^*_u = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right)^{\gamma} \right] \pi^*_u
\]

By Eq. (7.4), the derivatives \( V_s, V_X, V_{MM}, V_{LM}, V_{XX} \) and \( V_{RM} \) are determined by

\[
\frac{\partial V(t, x, n, \lambda, m, r)}{\partial x} = (\text{equity})^{\gamma - 1} B(t)^{1 - \gamma},
\]

\[
\frac{\partial^2 V(t, x, n, \lambda, m, r)}{\partial x^2} = (\gamma - 1) (\text{equity})^{\gamma - 2} B(t)^{1 - \gamma},
\]

\[
\frac{\partial^2 V(t, x, n, \lambda, m, r)}{\partial x \partial \sigma} = \gamma n_t (T - t) (\text{equity})^{\gamma - 1} B(t)^{1 - \gamma},
\]

\[
\frac{\partial^2 V(t, x, n, \lambda, m, r)}{\partial x \partial \sigma} = - (\gamma - 1) (n_t - n) \hat{D}(\lambda(t)) (\text{equity})^{\gamma - 2} B(t)^{1 - \gamma}.
\]

Substituting these partial derivatives into (7.5), the approximate optimal investment strategies are

\[
\hat{\pi}^*_u = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right) \right] \frac{\gamma n_t (T - u) \text{equity}(u)}{n_t \hat{D}(\lambda(t)) (1 - \gamma) B(t)},
\]

\[
\hat{\pi}^*_u = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right) \right] \frac{\gamma n_t (T - u) \text{equity}(u)}{n_t \hat{D}(\lambda(t)) (1 - \gamma) B(t)},
\]

7.2 Numerical illustration

This subsection gives the numerical illustration of the optimal strategy of the optimal ALM problem with salary risk and mortality risk, and compares this optimal strategy with the other optimal strategy of the optimal ALM problem having no salary risk and mortality risk. Firstly, we provide the investment strategies with determined actuarial salary and mortality intensity, denoted them by \( \hat{\pi}^*_u \) and \( \hat{\pi}^*_u \):

\[
\hat{\pi}^*_u = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right) \right] \frac{\gamma n_t (T - u) \text{equity}(u)}{n_t \hat{D}(\lambda(t)) (1 - \gamma) B(t)},
\]

\[
\hat{\pi}^*_u = \left[ \frac{\gamma}{\gamma - 1} \left( \frac{\gamma}{\gamma - 1} \left[ x - (n_t - N) \hat{D}(\lambda(t)) \right] \right) \right] \frac{\gamma n_t (T - u) \text{equity}(u)}{n_t \hat{D}(\lambda(t)) (1 - \gamma) B(t)},
\]
Fig. 3. Wealth, equity and impact factors: salary and $\tilde{D}$ (death discount).

Fig. 4. Optimal mean proportions of wealth.

where $\text{equity}(u) = X_u - M_t e^{\int_0^t \sigma r dB_s} \left( n - N_u \right) \rho \left( \int_t^{T_u} s - T P_t ds \right) T - u P_u$ is the actuarial equity which is a function of non-random actuarial $M_t$ and mortality $T - u P_u$.

Comparing (7.7) with (7.6), the proportions of investment are increasing when equity increases or liability decreases, which means that the company takes prudent investment strategies when equity decreases, i.e., the salary and mortality influence investment strategies.

The simulation considers a department with $n_x = 100$ employees at same age $t = 50$. The default retirement age is $T = 65$ and maturity of pension is $T^* = 110$ with proportion $\rho = 0.3$. The parameter of utility function is $\gamma = 0.05$ and we suppose that the initial wealth is 105% times the total actuarial liability at initial time: $X_0 = 105\% M_t n_x \int_0^T s - T P_t ds$. The other parameters are listed in Table 1.

Figs. 3–4 depict the influence of salary and mortality perfectly. Fig. 3 shows trajectories of wealth, equity, salary and $\tilde{D}$. When the mortality intensity become stochastic, the further death intensity becomes less than non-stochastic actuarial one according to (3.4), thus the expected longevity becomes longer and expected liability becomes higher. So the investment strategies turn to be prudent, this fact indeed has been reflected by equity in (7.6) and (7.7). But as time goes on, randomness of mortality intensity at time $T$ decreases, so the expected longevity decrease and the expected equity increase with respect to non-stochastic one as in Fig. 3–$\tilde{D}$ in stochastic case is much bigger than non-random actuarial case, thus the equity in stochastic case is lower than non-stochastic one at the beginning.

Table 1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value1</th>
<th>Value2</th>
<th>Value3</th>
<th>Value4</th>
<th>Value5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>$a_t$</td>
<td>12.72%</td>
<td>$\bar{r}$</td>
<td>3.88%</td>
<td>$\sigma_r$</td>
</tr>
<tr>
<td>Rolling bond</td>
<td>$K$</td>
<td>15</td>
<td>$n(K)$</td>
<td>6.70</td>
<td>$\delta_k$</td>
</tr>
<tr>
<td>Stock</td>
<td>$\sigma_{st}$</td>
<td>0.1%</td>
<td>$\sigma_s$</td>
<td>15.24%</td>
<td>$\delta_s$</td>
</tr>
<tr>
<td>Mortality</td>
<td>$\sigma$</td>
<td>10.94%</td>
<td>$\sigma_m$</td>
<td>0.016%</td>
<td>$\lambda_m$</td>
</tr>
<tr>
<td>Salary</td>
<td>$\theta_m$</td>
<td>-35%</td>
<td>$\nu_m$</td>
<td>-1.52%</td>
<td>$M_t$</td>
</tr>
</tbody>
</table>

Then we study the influence of salary risk. Because the further liability is a proportion of expected salary, the liability increases and equity decreases when salary increases. In Fig. 3, when the stochastic salary is less than the actuarial salary, the equity in stochastic case is much higher than one in non-random actuarial case.

Generally speaking, the influence of salary risk and mortality risk are both exerted by equity, so we focus on equity and investment strategies. Comparing the equity in Fig. 3 with one in Fig. 4, the trend of the investment strategies and the equity are similar, which coincides with the expression of strategies in (7.6) and (7.7).

Also, comparing the $y$-axis in Fig. 4, we find that the proportion of stock is a little bigger than that of rolling bond at the same time. To verify this observation, the optimal mean proportions of wealth in stocks and rolling bonds are illustrated in Fig. 5. When equity keeps positive, this is reasonable since the risk premium of stock is higher than that of rolling bond in Table 1.
8. Conclusion

The main contribution of this paper is to demonstrate the efficiency of approximation by martingale method and dynamic programming in the setting of incomplete market, and reveal the influence of salary risk and mortality risk on the optimal investment strategy by solving an optimal AML problem. In particular, we consider the optimal dynamic asset allocation of pension fund with mortality risk and salary risk, in which the pension fund manager maximizes the expected utility of terminal surplus under these two non-hedging risks.

The ALM problem consists of the asset phase and the liability phase. Cashes, stocks and rolling bonds are three investment options in the asset phase, which have two independent risk factors described by Brownian motions. There are two non-hedging risks in the liability phase, called salary risk and mortality risk. The further liability is complicated owe to the interplay of stochastic intensity between assets and liabilities, but it also can be expressed thanks to the independence of filtration and property of Gaussian processes. Moreover, the main task of the company is to maximize the terminal surplus under the CRRA utility, which makes the strategies relate to equity of a company.

The incompleteness of market causes a difficulty we have to deal with, but fortunately we know an approximation method by projecting from an expanded space. This method is beautifully used by Hainaut and Devolder directly without considering the efficiency. In order to study the efficiency of the approximation, we get an exact solution in a simple case with CARA utility and compare the exact solution with the approximation solution in an acceptable error. In fact, the wealth process controlled by approximation optimal strategies is the unique process in self-financed space, which minimizes the second moment of difference between the approximation process and the optimal wealth process in an expanded space.

Using this efficient approximation, an optimal assets liabilities management (ALM) problem with mortality risk and salary risk under CRRA utility is solved after some complicated calculations in Appendix B. The influences of salary and mortality risk are revealed as follows: The randomness of mortality intensity makes the further liability bigger, thus the investment becomes prudent in the whole time. The randomness of salary makes the liability change with the rise or fall of salary in real time, thus the investment strategy is negative correlated to salary.

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Appendix A. Calculation of terminal surplus $X_T$

**Proof.** The terminal surplus $X_T$ can be decomposed into the following stochastic integrals w.r.t. Brownian motions and the compensated process. By Itô formula we have

$$X_T - (n_k - N_k)L(T) = e^{	ilde{X}_{T-}} a e^{(T-t)\tilde{X}_T} - [(n_k - N_k)T - (n_k - N_k)_{T-}P_t + (n_k - N_k)_{T-}P_t]$$

$$P^m_{\text{martingale}}$$

$$\times \int_{T}^{\infty} e^{\tilde{X}_{T-t}} e^{-(T-t)\tilde{M}_t} \rho \int_{T}^{\infty} \exp \left[ -\lambda(T-t) \frac{1}{a} (e^{\frac{\alpha(t)}{a}} - 1) \right] ds$$

**Fig. 5.** Optimal mean proportions of wealth.

Appendix B. Calculation of $\tilde{D}$

**Proof.** By the definition of $\tilde{D}(\lambda_s)$ in (7.3), we know that

$$\tilde{D}(\lambda_s) = \mathbb{E}[\int_{\tau}^{T} D(\lambda_s) | \mathcal{F}_\tau, \tau > s]$$

$$= \mathbb{E}\left[ \int_{\tau}^{T} \lambda(u) du < \zeta \right] D(\lambda_s) | \mathcal{F}_\tau.$$ (B.1)
where the $\zeta$ is a normal exponential distributed and is independent of the filtration $\{F_t^\omega\}$, the $D(\cdot)$ is an integral containing parameter $\lambda_T$, which does not have a primitive function, and which is defined by (4.1). The joint density function of $(\int_s^T \lambda_T d\omega, \lambda_T, \zeta)$, denoted as $f(x, y, z|\lambda_T)$, can be obtained by a simple calculation and using the independence between $\zeta$ and $\{F^\omega\}$, the variable $z$ can be separated from $f(x, y, z|\lambda_T)$, so we have the expression of $\hat{f}(x, y, z|\lambda_T)$:

$$\hat{f}(x, y, z|\lambda_T) = \mathbb{P}\left(\int_s^T \lambda_T d\omega \in dx, \lambda_T \in dy, \zeta \in dz|\lambda_T\right).$$

By a simple calculation and using the independence between $\zeta$ and $\{F^\omega\}$, the variable $z$ can be separated from $f(x, y, z|\lambda_T)$, so we have the expression of $f(x, y, z|\lambda_T)$:

$$\tilde{f}(x, y, z|\lambda_T) = \mathbb{P}
\left(\int_s^T \lambda_T d\omega \in dx, \lambda_T \in dy \right) \mathbb{P}(\zeta \in dz|\lambda_T).$$

Thus

$$f(x, y|\lambda_T) = \exp\left\{-\frac{1}{2}((x, y)^T - \mu(\lambda_T))^T \Sigma^{-1}((x, y)^T - \mu(\lambda_T))\right\}. \quad (B.3)$$

Substituting (B.3) into (B.1) and (B.2), we have

$$D(\lambda_T) = \mathbb{E}\left[I \left(\int_s^T \lambda_T d\omega < \zeta\right) D(\lambda_T|\lambda_T)\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x < z) D(y|f(x, y, z|\lambda_T)) dxdydz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y|f(x, y, z|\lambda_T)) e^{-2\alpha dxdydz}. \quad \square$$

References


