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OPTIMAL ASSETS ALLOCATION AND BENEFIT OUTGO POLICIES OF
DC PENSION PLAN WITH COMPULSORY CONVERSION CLAIMS

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Abstract. In this paper, we study optimal asset allocation and benefit outgo policies of DC(defined contribution) pension plan. We extend He and Liang model \cite{5, 6}(2013a,2013b) to describe dynamics of individual fund scale during distribution period. The fund scale is affected by investment return, benefit outgo and mortality credit. The management of the pension plan controls the asset allocation and benefit outgo policies to achieve the objective of pension members. The goal of the management is to minimize accumulated deviations between the actual benefit outgo and a preset target during the whole distribution period. The performance function(criterion) is the weighted average of the square and linear deviations to express more penalty on negative deviation than positive deviation. Using HJB(Hamilton-Jacobi-Bellman) equations and variational inequality methods, the closed-forms of the optimal policies are derived. The counterintuitive effect of the optimal proportion allocated in the risky asset with respect to the fund scale is also derived, and the optimal benefit outgo has the form of the spread method. Moreover, we use Monte Carlo Methods(MCM) to analyze economic behaviors of the optimal asset allocation and benefit outgo policies.

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1. Introduction

In this paper, we study optimal asset allocation and benefit outgo policies of DC(defined contribution) pension plan with compulsory conversion claims. Before conversion time, the pension fund is allowed

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to be invested in a risk-free asset and a risky asset, and the management of the fund chooses the asset allocation policies to achieve the objectives of pension members. The optimal control theory has been extensively applied in the asset allocation problems of the pension plan. Cairns[3](2000) and Josa-Fombellida and Rincón-Zapatero[10](2004) assume that the risky asset follows geometric Brownian motions and study the optimal asset allocation policies. The results show that the optimal asset allocation policy is counterintuitive, i.e., the proportion allocated in the risky asset increases with respect to the decrease of the fund scale, and vice versa.

In addition to the asset allocation policy, the benefit outgo policy is the other important control variable for the management. In recent years, one seldom studies the optimal control problem of the DC pension plan from the individual member’s perspective. Most literatures, based on the total members’ perspective, study the optimal asset allocation and contribution policies of DB(defined benefit) pension plan to minimize solvency risk and contribution risk. Josa-Fombellida and Rincón-Zapatero[9](2001) and Ngwira and Gerrard[15](2007) study the optimal asset allocation and contribution policies of the DB pension plan, and the optimal contribution policy explains the spread effects, i.e., the optimal contribution rate is the pre-set target multiplied by the spread.

In order to deeply investigate the optimal control problem of the DC pension plan from the individual member’s perspective, we extend He and Liang model [5][6](2013a, 2013b) to describe the dynamics of the fund scale before the conversion time. This is the first time to establish the continuous time stochastic differential equation that the fund scale dynamics satisfy during the distribution period by approximating the discrete time model in Blake and et al.[2](2003). In this model, the dynamics of the fund scale are mainly affected by the investment return,
benefit outgo and the mortality credit. For simplicity, we choose the De Moivre Model (cf. [11]) to characterize the force of mortality function.

In order to secure the old-care utility at the higher age, most DC pension plans have compulsory conversion claims, i.e., the fund is fully allocated in the risk-free asset and is converted into annuities at the conversion time. Horneff and et al. [7] (2008) study the optimal conversion time problem in the discrete time framework. In the model, the authors simulate the benefit outgo performances and compare the old-care utilities between the pension plans with different conversion times. The results state that the longer conversion time will enhance the utility of the pension member and reduce the security. But we find that it is difficult to treat conversion time as a control variable by the limitation of the stochastic control methods. In [14] (2007), Milevsky and Young control the conversion time to achieve the CRRA (constant relative risk aversion) utility of the pension member. This problem is well solved by the multi-plicate separable characteristics of the CRRA utility. In this paper, the conversion time is a predetermined exogenous variable and we choose the optimal one by numerical analysis.

For the performance criterion aspects, the pension member and the management form the principle-agent relationship. The management of the pension plan should choose appropriate control policies to maximize the old-care utility of the pension member. The objectives of the management are divided into two categories: maximization of the CRRA and CARA (constant absolute risk aversion) utilities of the benefit outgo, and minimization of the deviations between the actual benefit outgo and a pre-set target. The literatures on the former objectives include Battocchio and Menoncin [11] (2004) and Milevsky and Young [14] (2007). They choose power utility and exponential utility of the benefit outgo as the performance criterions, respectively. The idea of the latter objective originates from the work of Cairns [14] (2000). The
widely recognized performance criterion in the optimal pension management problem is to minimize the fluctuation risk of the benefit outgo. The common approach to measuring the risk is to consider the deviation between the actual benefit outgo and a pre-set target. In order to penalize more on the negative deviation than the positive deviation, the square and linear deviations are both contained in the performance function, we refer the interested readers to Chang et al.[4] (2003).

Using HJB equations and variational inequality methods and similar procedures in Ngwira and Gerrard[15](2007), we solve the optimal stochastic control problem and derive the closed-forms of the optimal asset allocation and benefit outgo policies. The results of Lions and Sznitman[13](1984) guarantee the dynamics of the fund scale are uniquely determined by the stochastic differential equation with respect to the optimal feedback functions. Furthermore, we use MCM to investigate the impacts of the conversion time and the penalty degree for the negative deviation on the optimal asset allocation and benefit outgo policies.

The rest of this paper will be organized as follows. In Section 2, we extend He and Liang model[5, 6](2013a,2013b) to describe the dynamics of the fund scale before the conversion time. The management chooses the proportion allocated in the risky asset and the benefit outgo as control variables. By modeling the investment return, benefit outgo and the mortality credit, we establish the stochastic differential equation that the dynamics of the fund scale satisfy. We choose minimizing weighted average of the square and linear deviations between the actual benefit outgo and the pre-set target as a criterion. In Section 3, using HJB equations and variational inequality methods, we get the closed-forms of optimal asset allocation and benefit outgo policies. We also investigate the impacts of the conversion time and the penalty degree
for the negative deviation on the optimal policies by numerical analysis in Section 4. The conclusions of this paper are given in Section 5.

2. The stochastic optimal control problem

In this paper, we study the optimal asset allocation and benefit outgo policies of the DC pension plan with compulsory conversion claims. In order to secure the old-care utility at the higher age, most pension plans have compulsory conversion claims. Before the conversion time, the fund is allowed to be invested in a risk-free asset and a risky asset. The management of the fund dynamically chooses the proportion allocated in the risky asset and the benefit outgo to achieve the objectives. After the conversion time, the fund is compulsorily converted into annuities and the fund is fully invested in the risk-free asset.

We extend He and Liang model [5, 6] (2013a, 2013b) to describe the dynamics of the fund scale during the distribution period, which is the first time to establish the stochastic differential equation that the dynamics of the fund scale satisfy based on the individual pension member’s perspective.

From the individual member’s perspective, we model the dynamics of the individual member’s fund scale precisely by actuarial principles. The change of the fund scale is generated by the investment return, benefit outgo and the mortality credit. In the DC pension plan, the benefit outgo is not predetermined and it is affected by the fund scale of the individual account. We seldom study the DC pension management problem from the total members’ perspective. Furthermore, it is easy to estimate the expected benefit outgo of the individual member. From the total members’ perspective, which is often used in the DB (Defined Benefit) pension management, their funds are accumulated in a single account and are invested and distributed as a whole. As the scale and the structure of the members at different ages are dynamical year to
year, it needs strong assumptions and approximations to estimate the expected benefit outgo and the expected fund scale of the total members.

Denote
\[
\Delta \delta_t^\frac{1}{n} = \pi \frac{S_t^1 - S_t^1}{S_t^1} + (1 - \pi) \frac{S_t^0 - S_t^0}{S_t^0},
\]
where \(S_t^0\) and \(S_t^1\) are the values of the risk-free asset and the risky asset at time \(t\), respectively. \(n\) is a big natural number, i.e., \([t, t + \frac{1}{n})\) is a small enough time interval. \(\pi\) is the proportion allocated in the risky asset, and it is an important control variable in the model. The admissible range of \(\pi\) is \((-\infty, +\infty)\), i.e., the short sale of the risk-free asset and the risky asset are both permitted. This may be inconsistent with the pension management regulations. But, this is a time-inconsistent optimal control problem, the optimal control policies are the functions of the fund scale \(y(t)\) and the time \(t\). If we consider the constraints on the control policies, it is hard to get the closed-form solutions. So, we enlarge the actual admissible range.

During the distribution period, the individual member’s fund scale is affected by the following three factors: investment return, benefit outgo and the mortality credit. In the model, the benefit outgo is the other important control variable. In addition, we use the similar methods in He and Liang\([5, 6]\)(2013a, 2013b) to model the mortality credit.

The discrete time form of the fund scale dynamics is as follows:
\[
Y(t + \frac{1}{n}) = (Y(t)(1 + \Delta \delta_t^\frac{1}{n}) - p(t)\frac{1}{n} \frac{1}{1 - \frac{1}{n} d_{xo} + t},
\]
where \(Y(t)\) is the fund scale at time \(t\). \(p(t)\) is the benefit outgo, and it is also an important control variable in the model. The admissible range of \(p(t)\) is \((-\infty, +\infty)\). It may be different from the pension management practice. Obviously, \(p(t)\) should be greater than 0 and smaller than the fund scale \(y(t)\) to maintain the sustainability of the pension fund. For similar reasons, if we consider the constraints on \(p(t)\), the optimization may beyond the boundary. As it is a time-inconsistent optimal control
problem, the boundary will be the two-dimensional function of time $t$ and fund scale $y(t)$. It is quite difficult to solve the kind of stochastic optimal control problem with constraints on the control policies. In order to make the conclusions more simply and precise, we enlarge the admissible range of $p(t)$. Fortunately, according to empirical evidences from the pension management practice, the optimal control policies seldom beyond the actual admissible ranges and the optimal control policies make sense.

$x_0$ is pension member’s age at the beginning of the distribution period, i.e., the age of retirement. $\frac{1}{n} q_{x_0+t}$ is an actuarial symbol which is the conditional probability that a person is alive at the age $x_0 + t$ and will die within the next $\frac{1}{n}$ time interval. The last item in equation (2.1) reflects the mortality credit effects, i.e., the total fund scale of all the members at time $t$ will be distributed equally among the alive members at time $t + \frac{1}{n}$.

In order to establish the continuous-time risk model, we make the following approximations:

$$\frac{1}{n} q_{x_0+t} = \frac{1 - e^{-1/n \mu(x_0+t+s) ds}}{1 - e^{-1/n \mu(x_0+t+s) ds}} = e^{1/n \mu(x_0+t+s) ds} - 1 \approx \mu(x_0 + t) \frac{1}{n} = O\left(\frac{1}{n}\right),$$

where $\mu(x_0 + t)$ is the force of mortality function at the age of $x_0 + t$.

Then, the dynamics of the individual member’s fund scale satisfy the following equations:

$$Y(t + \frac{1}{n}) = Y(t)(1 + \Delta\delta_{x_0}^\frac{1}{n}) - p(t) \frac{1}{n} + Y(t)\mu(x_0 + t) \frac{1}{n} + O\left(\frac{1}{n}\right).$$

For simplicity, we introduce the De Moivre Model(see [11]) to characterize the force of mortality function as follows:

$$\mu(x_0 + t) = \frac{1}{\omega - x_0 - t},$$
where \( \omega \) is the maximum age of the alive persons.

Besides, the return of the risky asset is supposed to follow the diffusion process. The value of the risky asset and the risk-free asset are governed by the following stochastic differential equations on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), respectively:

\[
\begin{align*}
    dS^1(t) &= S^1(t)(c dt + \sigma dB(t)), \\
    dS^0(t) &= rS^0(t)dt,
\end{align*}
\]

where \( c \) and \( \sigma \) are separately the expected return and the volatility of the risky asset. \( r \) is the risk-free interest rate. \( \{B(t), t \geq 0\} \) is a standard Brownian motion on a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual conditions (cf. [12]). \( \mathcal{F}_t \) represents the information available at time \( t \) and any decision made up to time \( t \) is based on this information.

By the above approximations, we transform the discrete time model into the continuous time model, i.e., the dynamics of the fund scale can be written by the following stochastic differential equation:

\[
\begin{align*}
    dY(t) &= \left\{ [\pi(c - r) + r]Y(t) + \frac{Y(t)}{\omega - x_0 - t} - p(t) \right\} dt + \pi \sigma Y(t) dB(t), \\
    Y(0) &= y_0,
\end{align*}
\]

where \( y_0 \) is the initial fund scale, i.e., the accumulated fund scale at the time of retirement.

In order to secure the old-care utility at the higher age, most pension plans have compulsory conversion claims. Suppose that the conversion time is \( T \), which is an exogenous variable in the model. After the conversion time, the fund is converted into annuities and is fully invested in the risk-free asset, i.e.,

\[
\pi(t) = 0, \quad p(t) = \frac{Y(T)}{a_{x_0+T}}, \quad \forall T \leq t \leq \omega - x_0,
\]

where \( a_{x_0+T} \) is the initial value of one unit annuity at the age of \( x_0 + T \), and it is expressed by

\[
\tilde{a}_{x_0+T} = \int_0^\infty e^{-rs} s p_{x_0+T} ds = \int_0^\infty e^{-rs} e^{-\int_0^r \mu(x_0 + T + u) du} ds = \frac{\omega - x_0 - T}{2},
\]
where \( sP_{x_0+T} \) is the conditional probability that a person is alive at the age of \( x_0 + T \), and will be still alive at the age of \( x_0 + T + s \).

For the performance criterion aspects, as the pension member and the management form the principle-agent relationship, the management should choose optimal policies to achieve the goals of the pension members.

According the theoretical economic studies, the goal of the pension member is the maximization of the old-care utility after retirement. So, the goal of the management is to provide stable and sustainable benefit outgo for the pension members. The widely recognized performance criterion in the optimal pension management problem is to minimize the fluctuation risk of the benefit outgo. There is a pre-set target of the benefit outgo and it is calculated by considering the economic growth, inflation and salary increase, etc. The pre-set target is an exogenous variable and it reflects the required benefit to maintain a high standard retirement life. The actual benefit outgo should be close to the target and both the positive and negative deviations are penalized in the objective functions.

Considering the economic background of the problem, we do not only focus on the accumulated scale of the benefit outgo, or the downside risk of the benefit outgo. It is a multi-period asset allocation and benefit distribution problem, and the objective is that the fund is distributed stably and sustainable during the whole retirement periods. So, we focus on the deviations from the target. According to the above reasons, we seldom choose CARA, CRRA, VAR and ES utilities as the performance criterions. Furthermore, it is very difficult to get the closed-form of solutions under these performance criterions.

The common approach to measuring the risk is to consider the deviations between the actual benefit outgo and a pre-set target. We use minimization of the quadratic deviations of the actual benefit outgo
and a pre-set target as the objective functions, as in Ngwira and Gerrard (2007). In this circumstance, it equally penalizes positive and negative deviations. In fact, the pension member likes to obtain the benefit larger than expected, and dislikes to obtain the benefit smaller than expected. So we add the linear deviation in the objective functions. The positive weight coefficient of the linear deviations reflects more penalty on negative deviations than positive deviations.

We define the value function $V(y; t)$ as follows:

$$V(y; t) = \min_{(\pi, p) \in \Pi_t} \{ J(\pi, p, y; t) \}$$

$$\equiv \min_{(\pi, p) \in \Pi_t} E(\pi, p, y; t) \{ \int_t^T e^{-rs} p_{x_0} [(p(s) - NP)^2 - \beta (p(s) - NP)] ds \} + \alpha e^{-rT} p_{x_0} [(Y(T) - \bar{a}_{x_0} T \cdot NP)^2 - \beta (Y(T) - \bar{a}_{x_0} T \cdot NP)] \}

(2.3)

where $E_{(\pi, p, y; t)}[\cdot]$ means conditional expectation given the initial value $Y(t) = y$, $p(t) = p$, $\pi(t) = \pi$ at time $t$. The performance function is $J(\pi, p, y; t)$. $\Pi_t$ is the set of all admissible policies whose initial values are replaced by $p(t) = p$, $\pi(t) = \pi$. $NP$ is the pre-set target of expected benefit outgo. It reflects the required benefit to maintain a high standard retirement life. It is an exogenous variable and $NP > 0$. $\beta$ is a weight variable and it measures the penalty degree on the negative deviations. Traditionally, we use minimization of the quadratic deviations of the actual benefit outgo and a pre-set target as the objective functions, as in Ngwira and Gerrard (2007). In this circumstance, $\beta = 0$ and it equally penalizes positive and negative deviations. In fact, the pension member likes to obtain the benefit larger than expected, and dislikes to obtain the benefit smaller than expected. So we add the linear deviation in the objective functions. $\beta$ should be positive to penalize negative deviations more than positive deviations. The penalty degree $\beta$ depends on the pension member’s risk averse levels. $\alpha$ is another weight variable and it measures the importance of the utility obtained after the conversion. So $\alpha \geq 0$ and depends on the individual member’s preference. The asset allocation policy $\pi^*$ and the benefit outgo policy
are called optimal control process if \((\pi^*, p^*) \in \Pi_t\) and satisfy

\[ V(y; t) = J(\pi^*, p^*, y; t). \]

Thus, the continuous-time stochastic optimal control problem (2.2)-(2.3) for the DC pension plan has been established. We aim at deriving closed-forms of the optimal policies \((\pi^*, p^*)\) to solve the optimal control problem (2.2)-(2.3).

3. Solutions of the Stochastic Optimal Control Problem

In this section, we use Itô stochastic calculus and variational methods to solve the optimal control problem (2.2)-(2.3) and give expressions of the optimal policies \((\pi^*, p^*) \in \Pi_t\). First, we establish the HJB equations that value function and optimal feedback function of the stochastic control problem satisfy. Second, we derive the optimal feedback functions from the HJB equation. Third, the results in Lions and Sznitman [13] (1984) guarantee that the solution of the stochastic differential equation for the fund scale dynamics is uniquely determined by the optimal feedback functions. The unique solution of the SDE is the so-called the optimal state/wealth process (cf. [17] (1999)). Finally, we get closed-forms of the optimal asset allocation and benefit outgo policies by compositions of the optimal feedback functions and the optimal state process.

First, we derive the associated HJB equations with the stochastic control problem (2.2)-(2.3). Denote its solution by \( \varphi(\pi, p, y; t) \). Using variational methods and Itô formula, we know that \( \varphi(\pi, p, y; t) \) satisfies the following HJB equations:

\[
0 = \min_{\pi, p} \left\{ \frac{\partial \varphi}{\partial y} \left[ y(\pi (c-r)+r + \frac{1}{\omega-x_0-t})-p] + \frac{1}{2} \frac{\partial^2 \varphi}{\partial y^2} \pi^2 \sigma^2 y^2 \right. \right. \\
+ \left. \left. \frac{\partial \varphi}{\partial t} e^{-rt} e^{-\int_0^t \mu(x_0+s)ds} [(p - NP)^2 - \beta(p - NP)] \right\} \right\} \quad (3.1)
\]
with boundary condition
\[
\varphi(\pi, p, Y(T); T) = \alpha e^{-rT}e^{-\int_0^T \mu(x_0+s)ds}[(Y(T) - \ddot{x}_0 + T \cdot NP)^2 - \beta(Y(T) - \ddot{x}_0 + T \cdot NP)].
\] (3.2)

Differentiating equation (3.1) with respect to \(t\) and then letting these derivatives w.r.t. \(t\) and \(p\) be zero, respectively, we get the optimal feedback functions \(\pi^*_t(y, t)\) and \(p^*_t(y, t)\) as follows:

\[
\begin{align*}
\pi^*_t(y, t) &= \frac{\varphi_y(r-c)}{\varphi_y+y^2}, \\
p^*_t(y, t) &= \frac{\varphi_e^{-r}e^{-\int_0^t \mu(x_0+s)ds}}{2} + NP + \frac{\beta}{2}.
\end{align*}
\]

By the boundary condition (3.2) we guess \(\varphi(\pi^*_t, p^*_t, y; t)\) has the following forms:

\[
\varphi(\pi^*_t, p^*_t, y; t) = e^{-rT}e^{-\int_0^T \mu(x_0+s)ds}Q(t)(y^2 - 2S(t)y + R(t)),
\] (3.3)

where \(Q(t)\), \(S(t)\) and \(R(t)\) are undetermined functions of \(t\). The boundary conditions are:

\[
\begin{align*}
Q(T) &= \alpha, \\
S(T) &= \frac{(\omega - x_0 - T)NP}{2} + \frac{\beta}{2}, \\
R(T) &= \frac{(\omega - x_0 - T)^2}{4}NP^2 + \frac{(\omega - x_0 - T)\beta}{2}NP.
\end{align*}
\] (3.4)

So the optimal feedback functions can be rewritten as follows:

\[
\begin{align*}
\pi^*_t(y, t) &= \frac{(y-S(t)(r-c))}{y^2}, \\
p^*_t(y, t) &= Q(t)(y - S(t)) + NP + \frac{\beta}{2}.
\end{align*}
\] (3.5)

Thus we only need to prove that (3.3) holds for all \(0 < t \leq T\).

It is easy to see from (3.5) that

\[
\begin{align*}
\varphi_y &= 2e^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q(t)(y - S(t)), \\
\varphi_{yy} &= 2e^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q(t), \\
\varphi_t &= -re^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q(t)(y^2 - 2S(t)y + R(t)) \\
&+ \mu(x_0 + t)e^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q(t)(y^2 - 2S(t)y + R(t)) \\
&+ e^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q'(t)(y^2 - 2S(t)y + R(t)) \\
&+ e^{-rT}e^{-\int_0^t \mu(x_0+s)ds}Q(t)(-2S'(t)y + R'(t)).
\end{align*}
\] (3.6)
Substituting (3.5) and (3.6) into (3.1), the associated HJB equation can be rewritten as follows again:

\[
0 = rQ(t)[y^2 - 2S(t)y + R(t)] + \mu(x_0 + t)Q(t)[y^2 - 2S(t)y + R(t)] \\
- Q(t)[y^2 - 2S(t)y + R(t)] + Q(t)[2S'(t)y - R'(t)] \\
+ 2Q(t)[y - S(t)][\frac{(c-r)^2(y-S(t))}{\sigma^2}] - ry - \mu(x_0 + t)y \\
+ Q(t)(y - S(t)) + NP + \frac{\beta}{2} - Q(t)\frac{(c-r)^2(y-S(t))^2}{\sigma^2} \\
- [Q(t)(y - S(t)) + \frac{\beta}{2} + \frac{\beta}{2}Q(t)(y - S(t)) + \frac{\beta}{2}].
\]

(3.7)

Equating the coefficient for the quadratic factor to zero in (3.7), we get the following Riccati ordinary differential equation:

\[
Q' + \left(\frac{1}{\omega - x_0 - t} + r - \frac{(c-r)^2}{\sigma^2}\right)Q - Q^2 = 0.
\]  

(3.8)

Let \(Q(t) \equiv u^{-1}(t)\). Then

\[
u' - \left(\frac{1}{\omega - x_0 - t} + r - \frac{(c-r)^2}{\sigma^2}\right)u + 1 = 0
\]

(3.9)

with boundary condition

\[u(T) = \frac{1}{\alpha}.
\]

The solution of (3.9) is as follows:

\[u(t) = \frac{1}{\alpha}(\omega - x_0 - T) + \frac{1}{M}(\omega - x_0 - t)e^{M(T-t)} - \frac{1}{M}e^{M(T-t)} + \frac{1}{M^2}\
(\omega - x_0 - t)e^{M(T-t)}
\]

with

\[M = r - \frac{(c-r)^2}{\sigma^2}.
\]

Thus, the solution of (3.8) is

\[Q(t) = \frac{1}{\alpha}(\omega - x_0 - t)e^{M(T-t)} + \frac{1}{M}(\omega - x_0 - t)e^{M(T-t)} - \frac{1}{M}e^{M(T-t)} + \frac{1}{M^2}.
\]  

(3.10)

Equating the coefficient for the linear factor to zero in (3.7), we have

\[S' - (\frac{1}{\omega - x_0 - t} + r)S + NP + \frac{\beta}{2} = 0.
\]  

(3.11)

The solution of (3.11) is

\[
S(t) = \begin{cases}
\frac{\omega - x_0 - T}{r}e^{r(t-T)}\left(\frac{\omega - x_0 - T}{r}NP + \frac{\beta}{2}\right) \\
+ \frac{\omega - x_0 - T}{r}e^{r(t-T)}\left(NP + \frac{\beta}{2}\right) \\
\times \left[ e^{r(t-T)} \frac{1}{r} \frac{\omega - x_0 - T}{r^2(\omega - x_0 - T)} + \frac{1}{r^2(\omega - x_0 - T)} \right].
\end{cases}
\]  

(3.12)
Similarly, equating the independent factor to zero in (3.7), we can get ordinary differential equation that \( R(t) \) satisfies:

\[
R' - \frac{1}{\omega - x_0 - t} r - \frac{Q'}{Q} R - \left[ \frac{(c - r)^2}{\sigma^2} S^2 + Q S^2 - 2 N P \cdot S - \beta S + \frac{\beta^2}{4} \right] = 0.
\]

(3.13)

As \( Q(t) \) are \( S(t) \) have been solved in (3.7) and (3.10), it is easy to solve the differential equation (3.13) with boundary conditions in (3.4). Since \( R(t) \) is not related to the optimal feedback functions of (3.5), we omit the calculations here.

Therefore, we get the closed-forms of the optimal feedback functions \( \pi^*(y, t) \) and \( p^*(y, t) \) by (3.5), (3.10) and (3.12). It is easy to see that \( \pi^* \) and \( p^* \) are functions of time \( t \) and the fund scale \( y \) at time \( t \).

Now we turn to establishing the optimal asset allocation policy \( \pi^*(t) \) and the optimal benefit outgo policy \( p^*(t) \). Let \( \pi^*(\cdot, \cdot) \) and \( p^*(\cdot, \cdot) \) be defined by (3.5). Then the following SDE

\[
\begin{cases}
    dY^*_t = \left\{ [\pi^*(Y^*_t(t), t)(c - r) + r] Y^*_t(t) + \frac{Y^*_t(t)}{\omega - x_0 - t} p^*(Y^*_t(t), t)(t) \right\} dt \\
    + \pi^*(Y^*_t(t), t) \sigma Y^*_t(t) dB(t), \\
    Y^*_0 = y_0
\end{cases}
\]

(3.14)

has a unique solution \( Y^*_t(t) \). We use the similar approach in Øksendal and Sulem(2007) to prove the verification theorem on the stochastic optimal control problem (2.2)-(2.3). The functions \( \pi^*(y, t) \) and \( p^*(y, t) \) defined by (3.5) and the function \( \varphi(\pi^*(y, t), p^*(y, t), y; t) \) defined by (3.3) are the optimal feedback functions and the value function, respectively.

The unique solution \( \{Y^*_t(t), t \geq 0\} \) of (3.14) is the optimal state process. The optimal asset allocation and benefit outgo policies \( (\pi^*(t), p^*(t)) \) are compositions of the optimal feedback functions \( (\pi^*(\cdot, \cdot), p^*(\cdot, \cdot)) \) and the optimal state process \( Y^*_t(t) \), i.e., \( \pi^*(t) = \pi^*(Y^*_t(t), t), p^*(t) = p^*(Y^*_t(t), t) \).

We also have \( V(y, t) = \varphi(\pi^*(y, t), p^*(y, t), y; t) = J(\pi^*, p^*, y; t) \).

We summarize the main results of this section as follows.
Theorem 3.1. Let $Q(\cdot)$ and $S(\cdot)$ be defined by (3.10) and (3.12), respectively. Then

1. The $\pi_*(\cdot,t)$ and $p_*(\cdot,t)$ defined by (3.5) are the optimal feedback functions of the stochastic optimal control problem (2.2)-(2.3);

2. The stochastic process $Y_*(t)$ uniquely determined by SDE (3.14) is the optimal state process of the stochastic optimal control problem (2.2)-(2.3);

3. The compositions of the optimal feedback functions $(\pi_*(\cdot,t), p_*(\cdot,t))$ and the optimal state process $Y_*(t)$, i.e., the stochastic processes $\pi_*(t) = \pi_*(Y_*(t), t)$ and $p_*(t) = p_*(Y_*(t), t)$ are the optimal asset allocation policy and benefit outgo policy of the problem (2.2)-(2.3), respectively. The $(Y_*, (\pi_*, p_*))$ is the optimal pair.

4. If the $R(\cdot)$ is known, the value function $V(y, t)$ of the stochastic optimal control problem (2.2)-(2.3) is $\varphi(\pi_*(y, t), p_*(y, t), y; t)$ defined by (3.3).

4. Analysis of the Optimal Control Policies

In this section, we investigate economic behaviors of the optimal asset allocation policy $\pi^*$ and the optimal benefit outgo policy $p^*$. First, we will theoretically prove that the counterintuitive effects of the optimal feedback asset allocation function with respect to the fund scale are also valid in this model, i.e., $\frac{\partial \pi^*}{\partial y} < 0$ for $\forall \: 0 < t \leq T$. It is easy to see from (3.5) and (3.12) that

$$\frac{\partial \pi^*}{\partial y} = \frac{c-r}{\sigma^2 y^2} S - \frac{c-r}{\sigma^2 y^2} \frac{\omega-x_0-T}{\omega-x_0-t} e^{r(t-T)} \left\{ \frac{(\omega-x_0-T) NP + \beta}{2} + \frac{(NP+\beta)}{2} \right\}.$$ 

Let

$$k(t) = e^{-r(t-T)} \frac{1}{r} \frac{\omega-x_0-t}{\omega-x_0-T} - \frac{1}{r} - e^{-r(t-T)} \frac{1}{r^2(\omega-x_0-T)} - \frac{1}{r^2(\omega-x_0-T)}.$$ 

Differentiating $k(t)$ with respect to $t$, we get

$$\frac{\partial k}{\partial t} = -e^{-r(t-T)} \frac{\omega-x_0-t}{\omega-x_0-T} < 0.$$
So $k(t)$ is a decreasing function of $t$. As $t \leq T < \omega - x_0$ and $k(T) = 0$, we have $k(t) > 0$, $\forall \ t \leq T < \omega - x_0$, i.e., $\frac{\partial k}{\partial t} < 0 \ \forall \ t \leq T$.

Thus, the counterintuitive effects of the optimal feedback asset allocation function with respect to the fund scale are also valid. The increasing fund scale will decrease the proportion allocated in the risky asset. In this circumstance, the management of the pension fund could achieve higher old-care utility for the members by the current fund scale, and he (or she) should reduce the investment in the risky asset to avoid the risk of value fluctuations, and vice versa.

Using the similar procedures, we can prove that $\frac{\partial p^*}{\partial (NP)} > 0$. In fact, since $k(t) > 0, \forall t \leq T$, we know that

$$\frac{\partial \pi^*}{\partial (NP)}(y, t) = \frac{e^{-r\frac{\omega - x_0 - T}{\omega - x_0}} e^{r(t-T)} \left[\frac{\omega - x_0 - T}{2} + \frac{1}{r} \frac{\omega - x_0 - T}{r} e^{-r(t-T)} \right] \left[1 + \frac{1}{r^2(e^{-r(t-T)} r^2)} \right]}{y \sigma^2} > 0.$$ 

$NP$ is the pre-set target of the expected benefit outgo and it is an exogenous variable, which is determined according to the economic growth, inflation and salary increase, etc. The theoretical results show that the increasing expected benefit outgo will increase the proportion allocated in the risky asset. In this circumstance, the pre-set target increases when the economic growth rate and the inflation are both higher than usual, and the benefit outgo should be raised to maintain the purchasing power of the members. The management of the pension plan should invest more in the risky asset to increase the fund scale and stand the value fluctuations.

Second, we investigate implications of the optimal feedback benefit outgo function $p^*$. The results show the form of the spread method. It is trivial to get that $p^*$ is the sum of the pre-set target $NP$ and the spread in (3.5). In this model, the spread is divided into two items, the fixed item and the variable item. The fixed item indicates that
the deviation between the optimal and the expected benefit outgo has positive correlation with respect to the deviation between the fund scale \( y \) and \( S(t) \).

We theoretically prove that the optimal feedback benefit outgo function has positive correlation with respect to the fund scale, i.e., \( \frac{\partial p^*}{\partial y} > 0 \). It is easy to see

\[
\frac{\partial p^*}{\partial y} = Q = \frac{1}{\alpha} (\omega - x_0 - t) e^{M(T-t)} + \frac{1}{M} (\omega - x_0 - t) e^{M(T-t)} - \frac{1}{M^2} e^{M(T-t)} + \frac{1}{M^2}.
\]

Denote

\[
l(t) = \frac{1}{M} (\omega - x_0 - t) e^{M(T-t)} - \frac{1}{M} (\omega - x_0 - T) e^{M(T-t)} + \frac{1}{M^2} e^{M(T-t)} + \frac{1}{M^2}.
\]

Differentiating \( l(t) \) with respect to \( t \), we have

\[
\frac{\partial l(t)}{\partial t} = - \frac{1}{M} e^{M(T-t)} - (\omega - x_0 - t) e^{M(T-t)} + \frac{1}{M} e^{M(T-t)} < 0.
\]

So \( l(t) \) is a decreasing function of \( t \). As \( t \leq T < \omega - x_0 \) and \( l(T) = 0 \), we also have

\[
l(t) > 0, \quad \forall t \leq T < \omega - x_0.
\]

Thus \( \frac{\partial p^*}{\partial y} > 0 \), for \( \forall t \leq T \). So the positive correlation between the optimal feedback benefit outgo function and the fund scale has been proved. The increasing fund scale will increase the benefit outgo to the pension members. In this circumstance, the fund scale is larger and the management could pay more benefit to the members to increase their old-care utilities. Under this performance criterion, as it penalizes more on the negative deviation than the positive deviation, the increasing fund scale will increase the optimal benefit outgo even more.

It is obvious to see that increasing \( NP \) will increase the optimal benefit outgo at the same time. In this part, we investigate the impacts of \( NP \) on the spread \( p^* - NP \), i.e., the deviation between the optimal feedback benefit outgo function and the pre-set target.
In fact, we have
\[
\frac{\partial (p^\ast \Box NP)}{\partial (NP)} = -Q \frac{\partial S}{\partial (NP)} = (\omega - x_0 - T)e^{-\frac{(\omega - x_0)^2}{2\sigma^2}}(T - t) \times \left\{ \frac{-\omega - x_0 - t}{2} + e^{-r(t-T)} \left( \frac{-\omega - x_0 - t}{r} \right) + e^{-r(t-T)} \left( \frac{1}{r^2} \right) - e^{-r(t-T)} \left( \frac{1}{r^2} \right) \right\} < 0
\]
holds for \( \forall t \leq T \) due to \( l(t) > 0 \) for \( \forall t \leq T < \omega - x_0 \).

The results show that increasing expected benefit outgo will reduce the deviation between the optimal and expected benefit outgo. In this circumstance, the pre-set target \( NP \) is higher, so it is difficult for the management to pay the benefit high enough to match the expectations.

Third, we use numerical methods to investigate the impacts of conversion time on the optimal control policies. In order to secure the old-care utility at the higher age, most pension funds have compulsory conversion claims. Before the conversion time, the fund is allowed to be invested in the risky asset to increase the fund scale as well as the old-care utility of the member. After the conversion time, the fund is all converted into annuities and is fully invested in the risk-free asset according to the security concerns. Literatures on the subjects include Yaari[16](1965), Blake et al.[2](2003) and Horneff et al.[7](2006), etc. They investigate the optimal asset allocation policies before the conversion and compare the differences of the utilities achieved under different conversion times. In this part, we investigate impacts of the conversion time on the optimal asset allocation and the benefit outgo policies by numerical analysis.

In the numerical analysis procedures, we use MCM to randomly generate 10000 paths of the fund scale processes. At each time step, the fund scale is re-calculated by the \( \pi^\ast \) and \( p^\ast \) of the last step and a randomly generated Brownian motion, as in (3.14). Then, \( \pi^\ast \) and \( p^\ast \) of the step are re-calculated by the current fund scale according to (3.5). The \( \pi^\ast \) and \( p^\ast \) at each time step are the averages of the \( \pi^\ast s \) and \( p^\ast s \) of the 10000 simulation tracks. The procedures are periodically repeated
10000 times and the average of the optimal control policies is used for the investigation.

According to the empirical evidences from the capital market and the pension management practice, we make the following assumptions for the variables. In the life insurance research, the maximum age of the alive persons is $\omega = 110$. The age at the beginning of the distribution period is $x_0 = 60$, i.e., the retirement age is 60. According to the data of the capital market, we choose the risk-free interest rate as $r = 0.02$ and it is the yield of the one year U.S. bonds. The expected return and the volatility of the risky asset are $c = 0.08$ and $\sigma = 0.33$, respectively. These coefficients are estimated by the investment return of the U.S. stock index. Furthermore, we make some assumptions on the individual member’s account. The initial fund scale is $y_0 = 60$ (ten thousand dollars), and the pre-set target for the expected benefit outgo is $NP = 3$ (ten thousand dollars). Since the DC pension plan provides the majority of the total old-care security, we believe the above assumptions are reasonable. Furthermore, $\alpha$ measures the importance of the old-care utility provided after the conversion time, and it depends on the preference of the individual member. $\beta$ measures penalty degree on the negative deviations, and it depends on the risk averse level of the individual member. We choose $\alpha = 0.5$ and $\beta = 1$ arbitrarily as an example. We have tried several combinations of the two weight variables and the conclusions are similar with the conclusions in the paper.

In order to investigate the impacts of the conversion time on the optimal control policies, $T$ varies as $T = 5, 10, 15, 20$ in this part.

Figure 1 shows the impacts of the conversion time $T$ on the optimal proportion allocated in the risky asset $\pi^*$. The results show that increasing conversion time decreases the proportion allocated in the risky asset. Furthermore, the optimal asset allocation policies have convergent effects with respect to time. As the conversion time is longer, the fund is allowed to be invested in the risky asset for a longer period. This
will increase the fund scale and promote the performance of the pension fund. So, the management of the pension fund should reduce the investment in the risky asset to avoid the risk, and vice versa. As time passes by, the return obtained from the initial investment increases the fund scale, and this results in lower demands for the risky asset later.

Figure 2 shows the impacts of the conversion time $T$ on the optimal benefit outgo $p^*$. The results show that increasing conversion time increases the benefit outgo. The fund with longer conversion time has larger investment return due to the longer time investment in the risky asset. The management could pay more benefit outgo in this circumstance. Otherwise, the management should reduce the benefit outgo, especially the benefit outgo before the conversion time to maintain the payment ability after the conversion.

The last, we use similar methods to investigate impacts of the penalty degree for the negative deviation on the optimal control policies. In this part, we suppose the conversion time is $T = 20$ and the penalty degree $\beta$ varies as $\beta = 0, 0.5, 1, 1.5$. When $\beta = 0$, it is the square deviation
Figure 2. The impacts of the conversion time $T$ on the optimal benefit outgo $p^*$. 

$\omega = 110, x_0 = 60, r = 0.02, c = 0.08, \sigma = 0.33, y_0 = 60, NP = 3, \alpha = 0.5, \beta = 1.$

minimization problem and it equally penalizes the negative and positive deviations.

Figure 3. The impacts of penalty degree $\beta$ on the optimal proportion allocated in the risky asset $\pi^*$. 

$\omega = 110, x_0 = 60, r = 0.02, c = 0.08, \sigma = 0.33, y_0 = 60, NP = 3, T = 20, \alpha = 0.5.$
Figure 3 shows impacts of the penalty degree $\beta$ on the optimal proportion allocated in the risky asset $\pi^*$. The results show that the increasing $\beta$ increases the proportion allocated in the risky asset. Larger $\beta$ means higher penalty degree for the negative deviation between the actual and expected benefit outgo. The management of the pension fund should invest more in the risky asset to increase the fund scale and pay more benefit to reduce the probability of negative deviations.

Figure 4 shows the impacts of the penalty degree $\beta$ on the optimal benefit outgo $p^*$. The results show that increasing $\beta$ increases the optimal benefit outgo. The management of the pension plan has to invest more in the risky asset to increase the fund scale and pay more benefit to achieve higher old-care utility.

5. Conclusions

In this paper, we study the optimal asset allocation and benefit outgo policies of the DC pension plan. Extending He and Liang model [5, 6] (2013a, 2013b), we model the dynamics of the individual fund scale
during the distribution period. The management of the pension fund controls the asset allocation and benefit outgo policies to achieve the objectives of pension members. Most pension plans are compulsorily converted into annuities at some higher age to secure the old-care utility. The main goal of the management is to minimize the weighted average of the square and linear deviations between the actual and expected benefit outgo to penalize more on the negative deviation. Using HJB variational methods, the closed-forms of the optimal policies are derived. The counterintuitive effect of the optimal asset allocation policy with respect to the fund scale is also established. The optimal benefit outgo is the expected benefit outgo plus the spread which consists of the fixed and the variable items. Furthermore, we use Monte Carlo Methods to show that increasing conversion time will decrease the proportion allocated in the risky asset and increase the benefit outgo both before and after the conversion time. Longer time of investment in the risky asset will increase the fund scale and promote the performance of the pension fund. Increasing penalty degree for the negative deviation will increase the proportion allocated in the risky asset and increase the benefit outgo. The management should increase the investment in the risky asset to increase the fund scale and the benefit outgo as well as reduce the probability of negative deviations.

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**REFERENCES**


Highlights of IME-D-14-00341

- Study optimal asset allocation and benefit outgo policies of DC pension plan
- Derive closed-forms of the optimal policies under a new criterion
- The continuous-time risk model associated with the problem is initially introduced
- Economic behaviors of the optimal policies are analyzed by MCM