Optimal dividend and investing control of an insurance company with higher solvency constraints

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**Article Info**

Article history:
Received July 2009
Received in revised form August 2011
Accepted 24 August 2011

**MSC:**
primary 91B30
91B28
93E20
secondary 60H10
60H30
60H05

**Keywords:**
Optimal dividend policy
Optimal return function
Solvency
Stochastic regular-singular control
Proportional reinsurance
Probability of bankruptcy
Stochastic differential equations

**Abstract**

This paper considers the optimal control problem of a large insurance company under a fixed insolvency probability. The company controls proportional reinsurance rate, dividend pay-outs and investing process to maximize the expected present value of the dividend pay-outs until the time of bankruptcy. This paper aims at describing the optimal return function as well as the optimal policy. As a by-product, the paper theoretically sets a risk-based capital standard to ensure the capital requirement that can cover the total risk.

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**1. Introduction**

In this paper we consider the optimal control problem of a large insurance company in which the dividend pay-outs, investing process and the risk exposure are controlled by management. The investing process in a financial market may contain an element of risk, so it will impact security and solvency of the company (see Theorem 4.1 below). Moreover, the company has a minimal reserve as its guarantee fund to protect the insureds and attract a sufficient number of policy holders. We assume that the company can only reduce its risk exposure by proportional reinsurance policy for simplicity. The objective of the company is to find a policy, consisting of risk control and a dividend payment scheme, which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy. This is a mixed regular-singular control problem on a diffusion model which has received renewed interest recently, e.g. He and Liang (2008) and references therein, Højgaard and Taksar (2001, 1999, 1998), Harrison and Taksar (1983), Paulsen and Gjessing (1997) and Radner and Sheph (1996). Optimizing dividend pay-outs is a classical problem in actuarial mathematics, for which earlier work is given in e.g. Borch (1969, 1967) and Gerber (1972). We notice that some of these papers seem not to take security and solvency into consideration and so the results therein may not be commonly used in practice because the insurance business is a business affected with a public interest, and insureds and policy-holders should be protected against insurer insolvencies (see Williams and Heins, 1985, Riegel and Miller, 1963, and Welson and Taylor, 1959). The policy, making the company go bankrupt before termination of the contract between the insurer and policy holders or the policy of low solvency (see Bowers et al., 1997), is not the best way and should be prohibited even though it can win the highest profit. Therefore, one of our motivations is to consider the optimal control problem of a large insurance company under higher solvency and security, and to find the best equilibrium policy between making profit and improving security.

Unfortunately, there are very few results concerning the optimal control problem of a large insurance company based on higher solvency and security. Paulsen (2003) studied this kind of
optimal control for a diffusion model via properties of a return function. Some of our results are somewhat like that of Paulsen (2003), but both approaches used are very different. He et al. (2008) investigated the optimal control problem for a linear Brownian model. However, we find that the case treated in He et al. (2008) is a trivial case, that is, the company in the model in He et al. (2008) will never go to bankruptcy, it is an ideal model in concept, and it indeed does not exist in reality (see Theorem 4.2 below). Because the probability for bankruptcy for the model treated in the present paper is very large (see Theorem 4.1 below), our results cannot be directly deduced from He et al. (2008). Therefore, to solve these problems we need to use the idea invented in He et al. (2008), the stochastic analysis and PDE method to establish a complete setup for further discussing the optimal control problem of a large insurance company under higher solvency and security in which the dividend pay-outs, investing process and the risk exposure are controlled by management. This is another one of our motivations. This paper is the first systematic presentation of the topic, and the approach here is rather general, so we anticipate that it can deal with other models. We aim at deriving the optimal return function, the optimal retention rate and dividend payout level. The main result of this paper will be presented in Section 3 below. As a by-product, the paper theoretically sets a risk-based capital standard to ensure the capital requirement that can cover the total given risk. Moreover, we also discuss how the risk and minimum reserve requirement affect the optimal reactions of the insurance company by the implicit types of solutions and how the optimal retention ratio and dividend payout level are affected by the changes in the minimum reserve requirement and risk faced by the insurance company.

The paper is organized as follows: In Section 2 we establish a stochastic control model of a large insurance company. In Section 3 we present the main result of this paper and its economic and financial interpretations, and discuss how the risk and minimum reserve requirement affect the optimal retention ratio and dividend payout level of the insurance company. In Section 4 we give an analysis on the risk of the stochastic control model treated in the present paper and study relationships among investment risk, underwriting risk and the insolvency probability. In Section 5 we give some numerical samples to portray how the risk and minimum reserve requirement affect the dividend payout level of the insurance company. The proofs of the theorems and lemmas which study the properties of the probability of bankruptcy and optimal return function will be given in the Appendix.

2. Mathematical model

To give a mathematical formulation of the optimization problem treated in this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) denote a filtered probability space. For the intuition of our diffusion model we start from the classical Cramér–Lundberg model of a reserve (risk) process. In this model claims arrive according to a Poisson process \(N_t\) with intensity \(\nu\) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The size of each claim is \(U_i\). Random variables \(U_i\) are i.i.d. and are independent of the Poisson process \(N_t\) with finite first and second moments given by \(\mu\) and \(\sigma^2\) respectively. If there is no reinsurance, dividend pay-outs or investments, the reserve (risk) process of the insurance company is described by

\[
r_t = n_0 + pt - \sum_{i=1}^{N_t} U_i,
\]

where \(p\) is the premium rate. If \(\eta > 0\) denotes the safety loading, the \(p\) can be calculated via the expected value principle as

\[
p = (1 + \eta)\nu\mu.
\]

In a case where the insurance company shares risk with the reinsurance, the size of the claims held by the insurer become \(U^{(a)}_t\), where \(a\) is a (fixed) retention level. For proportional reinsurance, \(a\) denotes the fraction of the claim covered by cedent. Consider the case of cheap reinsurance for which the reinsurance company uses the same safety loading as the cedent, the reserve process of the cedent is given by

\[
r_t^{(a, \eta)} = u + p^{(a, \eta)} t - \sum_{i=1}^{N_t} U^{(a)}_i,
\]

where \(p^{(a, \eta)} = (1 + \eta)\nu\mathbb{E}[U^{(a)}_t]\). Then as \(\eta \to 0\)

\[
\{\eta p^{(a, \eta)} t \}_{t \geq 0} \overset{D}{\to} BM(\mu(a)t, \sigma^2(a)t) \tag{2.1}
\]

and \(BM(\mu, \sigma^2)\), stands for Brownian motion with the drift coefficient \(\mu\) and diffusion coefficient \(\sigma\) on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The passage to the limit works well in the presence of a big portfolio. We refer the reader for this fact and for the specifics of the diffusion approximations to Emanuel et al. (1975), Grandell (1977, 1978, 1990), Harrison (1985), Iglehart (1969), and Schmidli (1994).

Throughout this paper we consider the retention level to be the control parameter for the selected alternative at each time by the insurance company. We denote this value by \(a(t)\). If there is no dividend pay-outs or investments, in view of (2.1), we can assume that in our model the reserve process \(\{R_t\}\) of the insurance company is given by

\[
dR_t = a(t)\mu dt + a(t)\nu dW^1_t,
\]

where \(U^{(a)}_t = aU_t\), \(\mu(a) = a\mathbb{E}[U_t]\) and \(\sigma^2(a) = a^2\sigma^2\). And the reserve invested in a financial asset is the price process \(\{P_t\}\) governed by

\[
dP_t = \rho P_t dt + \sigma P_t dW^2_t,
\]

where \(\tau > 0, \sigma \geq 0, \mathbb{E}[W^1_t]_{t \geq 0}\) and \(\mathbb{E}[W^2_t]_{t \geq 0}\) are two independent standard Brownian motions on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). The case of \(\sigma = 0\) corresponds to the situation where only risk-free assets, such as bonds or bank accounts, are used for investments. A policy \(\pi\) is a pair of non-negative càdlàg \(\mathcal{F}_t\)-adapted processes \(\{a_\pi(t), L^\pi_t\}\), where \(a_\pi(t)\) corresponds to the risk exposure at time \(t\) and \(L^\pi_t\) corresponds to the cumulative amount of dividend pay-outs distributed up to time \(t\). A policy \(\pi = \{a_\pi(t), L^\pi_t\}\) is called admissible if \(0 \leq a_\pi(t) \leq 1\) and \(L^\pi_t\) is a non-negative, non-decreasing, right-continuous function. When \(\pi\) is applied, the resulting reserve process is denoted by \(\{R^\pi_t\}\). We assume that the initial reserve \(R^\pi_0\) is a deterministic value \(x\). In view of the independence of \(W^1\) and \(W^2\), the dynamics for \(R^\pi_t\) is given by

\[
dR^\pi_t = (a_\pi(t)\mu + \rho R^\pi_t) dt + \sqrt{a_\pi^2(t)\sigma^2 + \sigma^2} \cdot (R^\pi_t)^2 dW_t - dL^\pi_t, \tag{2.2}
\]

where \(\{W_t\}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Moreover, we suppose that the insurance company has a minimal reserve \(m\) as its guarantee fund to protect the insureds and attract a sufficient number of policy holders, that is, the company needs to keep its reserve above \(m\). The company is considered bankrupt as soon as the reserve falls below \(m\). We define the time of bankruptcy by \(\tau^* = \inf\{t \geq 0 : R^\pi_t \leq m\}\). Obviously, \(\tau^*\) is an \(\mathcal{F}_t\)-stopping time.

We denote by \(\Pi\) the set of all admissible policies. For any \(b \geq 0\), let \(\Pi_b = \{\pi \in \Pi : \int_0^\infty \mathbb{E}[1_{\{R^\pi_t > b\}}] dL^\pi_t = 0\}\). Then it is easy to see
that $Π = Π_0$ and $b_1 > b_2 \Rightarrow Π_{b_1} \subset Π_{b_2}$. For a given admissible policy $π$ we define the optimal return function $V(x)$ by

$$f(x, π) = \mathbb{E} \left\{ r^T e^{-ct} dl^*_t \right\},$$

$$V(x, b) = \sup_{π \in Π_{b_1}} \left\{ f(x, π) \right\},$$

$$V(x) = \sup_{b \in B} \{ V(x, b) \}$$

and the optimal policy $π^*$ by

$$J(x, π^*) = V(x),$$

where

$$Δ := \{ b : P[r^T \leq T] \leq π, f(x, π) = V(x, b) \text{ and } π_0 \in Π_{b_0} \},$$

$c > 0$ is a discount rate, $r^T$ is the time of bankruptcy $r^T$ when the initial reserve $x = b$ and the control policy is $π_0$. $1 = \varepsilon$ is the standard of security and less than solvency for given $ε > 0$.

The main purpose of this paper is to find the optimal return function $V(x)$ and the optimal policy $π^*$. Throughout this paper we assume that $r < c$ in view of $V(x) = ∞$ for $r > c$ (see Heijgaard and Taksar, 2001).

### 3. Main result

In this section we first introduce an auxiliary Hamilton–Jacobi–Bellman (HJB) equation, then we present the main result of this paper, finally we give economic and financial interpretations of the main result.

**Lemma 3.1.** Let $h \in C^2(m, ∞)$ satisfy the following HJB equation

$$\max_{a \in [0,1]} \left\{ \frac{1}{2} [σ^2 a^2 + σ_π^2 a^2] h''(x) + [μa + rχ]h'(x) - ch(x) \right\} = 0, \quad x ≥ m$$

with boundary condition $h(m) = 0$. Then

(i) $h'(x) > 0, \forall x ≥ m$

(ii) There exists a unique $b_0 > x_0$ such that $h'(b_0) = 0$ and $(x - b_0) ≥ m$ except $b_0$, where $x_0 = \frac{σ^2 (1-a^2)}{μ}$, $a^*$ is a constant in $(0, 1)$.

**Proof.** The proof of this lemma is standard and can be proved by the same way as in the proof of He and Liang (2009), Shreve et al. (1984) and Paulsen and Gjessing (1997). So we omit it here. □

Assume that $h(x)$ is a solution of (3.1). Define functions $F_b(x)$ and $a^*(x)$ by

$$F_b(x) = \begin{cases} 0, & 0 ≤ x < m, \\ h'(b), & m ≤ x ≤ b, \\ x - b + F_b(b), & x ≥ b \end{cases}$$

and

$$a^*(x) = \begin{cases} λx, & 0 ≤ x ≤ x_0, \\ 1, & x ≥ x_0. \end{cases}$$

respectively, where $λ = \frac{μ}{σ^2 (1-a^2)}$. It easily follows that $F_b \in C^2((m, ∞))(b)$. Now we can present the main result of this paper as follows. We will give rigorous proof of the main result in the Appendix.

**Theorem 3.1.** Let level of risk $ε ∈ (0, 1)$ and time horizon $T$ be given.

(i) If $P[r^T ≤ T] ≤ ε$ then the optimal return function $V(x)$ is $F_{b_0}(x)$ defined by (3.2), and $V(x) = F_{b_0}(x) = V(x, 0) = f(x, π^*)$. The optimal policy $π^*_{b_0}$ is $[a^*(R^T_{b_0}), L^*_{b_0}]$, where $[R^T_{b_0}, L^*_{b_0}]$ is uniquely determined by the following stochastic differential equation

$$\left\{ \begin{array}{l} dl^*_{b_0} = (a^*(R^T_{b_0})μ + rR^T_{b_0})dt \\
+ \sqrt{(a^*(R^T_{b_0}))^2 σ^2 + σ^2 \cdot (R^T_{b_0})^2} dW_t - dl^*_t \\
m ≤ R^T_{b_0} ≤ b_0, \\
\int_0^∞ I(t,h^*_{b_0}) (t) dt = 0. \end{array} \right.$$ (3.4)

The solvency of the company is bigger than $1 - ε$.

(ii) If $P[r^T ≤ T] > ε$ then there is a unique optimal dividend $b^*(≥ b_0)$ satisfying $P[r^T ≤ T] = ε$. The optimal return function $V(x)$ is $F_{b^*}(x)$ defined by (3.2), that is,

$$V(x) = F_{b^*}(x) = \sup_{b \in B} \{ V(x, b) \},$$

where

$$b^* = \min\{b : P[r^T ≤ T] = ε\} = \min\{b : b \in Δ\} \in Δ$$

and

$$Δ := \{ b : P[r^T ≤ T] ≤ ε, f(x, π) = V(x, b) \text{ and } π_0 \in Π_{b_0} \}.$$

Moreover,

$$V(x) = V(x, b^*) = J(x, π_{b^*})$$

and the optimal policy $π^*_{b^*}$ is $[a^*(R^T_{b^*}), L^*_{b^*}]$, where $[R^T_{b^*}, L^*_{b^*}]$ is uniquely determined by the following stochastic differential equation

$$\left\{ \begin{array}{l} dl^*_{b^*} = (a^*(R^T_{b^*})μ + rR^T_{b^*})dt \\
+ \sqrt{(a^*(R^T_{b^*}))^2 σ^2 + σ^2 \cdot (R^T_{b^*})^2} dW_t - dl^*_t \\
m ≤ R^T_{b^*} ≤ b^*, \\
\int_0^∞ I(t,h^*_{b^*}) (t) dt = 0. \end{array} \right.$$ (3.8)

The solvency of the company is $1 - ε$.

(iii) For any $x ≤ b_0$,

$$\frac{F_{b^*}(x)}{F_{b_0}(x)} = \frac{h'(b^*)}{h'(b_0)} < 1.$$ (3.9)

**Economic and financial explanation of Theorem 3.1 is as follows:**

(1) For a given level of risk and time horizon, if probability of bankruptcy is less than the level of risk, the optimal control problem of (2.4) and (2.5) is the traditional one, then the company has higher solvency, so it will have a good reputation. The solvency constraints here do not work. This is a trivial case. In view of Theorem 4.2 below, the model treated in He et al. (2008) can be reduced to this trivial case.

(2) If probability of bankruptcy is larger than the level of risk, the traditional optimal policy will not meet the standard of security and solvency, the company needs to find a sub-optimal policy $π^*_{b^*}$ to improve its solvency. The sub-optimal reserve process $R^T_{b^*}$ is a diffusion process reflected at $b^*$, and the process $L^*_{b^*}$ is the process which ensures the reflection. The sub-optimal action is to pay out everything in excess of $b^*$ as dividend and pay no dividend when the reserve is below $b^*$, and $a^*(x)$ is the sub-optimal feedback control function.
(3) On the one hand, the inequality (3.9) states that \( \pi_0^\kappa \) will reduce the company’s profit, while on the other hand, in view of (3.6) and \( \mathbb{P}(\tau_b^{\kappa} = T) = \varepsilon \) as well as Lemma 6.7 below, the cost of improving solvency is minimal. Therefore the policy \( \pi_0^\kappa \) is the best equilibrium action between making profit and improving solvency.

**Effect of the risk level \( \varepsilon \) and minimum reserve requirement \( m \) on the optimal reaction and dividend payout level of the insurance company** is given as follows:

(4) We see from the Fig. 4 (based on PDE (6.2) satisfied by solvency probability) that the dividend payout level \( b^* \) is an increasing function of minimum reserve requirement \( m \). Using comparison theorem for one-dimensional Itô process we know that the reserve process \( R_t^{\kappa} \) of the insurance company is also an increasing function of \( b^* \). Therefore, since the sub-optimal feedback control function \( a^\varepsilon(x) \) is increasing with respect to \( x \), by Theorem 3.1 we conclude that the optimal retention ratio \( a^\varepsilon(R_t^{\kappa}) \) increases with \( m \), that is, increasing minimum reserve requirement will improve the optimal retention ratio. However, this increasing action must result in lower profit because the optimal return function \( V(x, b^*) \) is a decreasing of \( b^* \) (see Lemma 6.7). So the process \( L_t^{\kappa} \) is a decreasing function of \( m \).

(5) We see from the Fig. 3 that the dividend payout level \( b^* \) is a decreasing function of the risk \( \varepsilon \). So, by the same argument as in (4) above, the optimal retention ratio \( a^\varepsilon(R_t^{\kappa}) \) decreases with \( \varepsilon \), and the process \( L_t^{\kappa} \) increases with \( \varepsilon \).

(6) We also see from the Fig. 6 that, given the risk \( \varepsilon \), the dividend payout level \( b \) is an increasing function of underwriting risk \( \sigma^2 \), so it decreases the company’s profit.

**Remark 3.1.** Because He et al. (2008) had no continuity of probability of bankruptcy and actual \( b^* \), the authors of He et al. (2008) did not obtain the best equilibrium policy \( \pi_0^\kappa \).

**Remark 3.2.** By (6.2) one knows that the equation \( \psi(T, m, b^*) = 1 - \phi(T, m, b^*) = \varepsilon \) can set a risk-based capital standard \( (m, b^*) \) to ensure the capital requirement that can cover the total given risk \( \varepsilon \), then establish the optimal return function, the optimal retention rate and dividend payout level via Theorem 3.1.

**Remark 3.3.** By using the same approach as in Haøjgaard and Taksar (2001) we can show that the \( b^* \) is an increasing function of \( \sigma^2 \), so the company has the possibility of making a larger gain from the reinvestments. We omit the analysis here. We focus on the effect of investment risk on the probability of bankruptcy as the topic of this paper in the next section.

4. **Analysis on risk of a large insurance company**

The first result of this section is the following, which states that the company has to find the optimal policy to improve its solvency.

**Theorem 4.1.** For \( b \geq m > 0 \), let \( \{R_t^{\kappa}, L_t^{\kappa}\} \) be defined by the following SDE (see Lions and Sznitman, 1984)

\[
\frac{dR_t^{\kappa}}{(a^\varepsilon(R_t^{\kappa}))\mu + R_t^{\kappa})dt} + \sqrt{\left(\frac{a^\varepsilon(R_t^{\kappa})}{\mu + R_t^{\kappa}}\right)^2 \sigma^2 + \sigma^2_p \cdot (R_t^{\kappa})^2} \cdot dW_t + - \frac{1}{\mu + R_t^{\kappa}} \cdot dL_t^{\kappa},
\]

\[
m \leq R_t^{\kappa} \leq b, \quad - \int_0^\infty \mathbb{E}[R_t^{\kappa} - b] \cdot dL_t^{\kappa} = 0, \quad R_0^{\kappa} = b.
\]

Then

\[
\mathbb{P}(\tau_b^{\kappa} \leq T) \geq \varepsilon(b, T) \equiv \frac{4 \left(1 - \Phi \left(\frac{b-m}{\sqrt{\varepsilon}}\right)\right)^2}{\exp \left(\frac{\lambda \mu + \sigma^2 \cdot \varepsilon}{\sigma^2 \cdot \varepsilon}\right)} > 0,
\]

where \( \tau_b^{\kappa} = \inf\{t : R_t^{\kappa} \leq m\}, k = (\lambda^2 \sigma^2 + \sigma^2_p)^2, \lambda = \frac{\mu}{\sigma_2(1 - \sigma_1^2)} \).

**Proof.** Since \( a^\varepsilon(x) \) is a bounded Lipschitz continuous function, the following SDE

\[
dR_t^{\kappa} = (a^\varepsilon(R_t^{\kappa})) \mu + R_t^{\kappa} \cdot \sigma dW_t + \sqrt{\left((a^\varepsilon(R_t^{\kappa})) \mu + R_t^{\kappa} \cdot \sigma^2 \right) \sigma^2} \cdot dW_t + R_t^{\kappa} = b
\]

has a unique solution \( R_t^{\kappa} \) using the comparison theorem for a one-dimensional Itô process, we have

\[
\mathbb{P}(R_t^{\kappa} \geq R_t^{\kappa}) = 1.
\]

Let \( Q \) be a measure on \( \mathcal{F}_t \) defined by

\[
\mathbb{E}[Q(\omega)] = M_t(\omega) dP(\omega),
\]

where

\[
M_t = \exp \left\{ - \int_0^t \frac{a^\varepsilon(R_s^{\kappa}) \mu + R_s^{\kappa} \cdot \sigma^2}{\sqrt{\left(a^\varepsilon(R_s^{\kappa}) \mu + R_s^{\kappa} \cdot \sigma^2 \right) \sigma^2}} dW_s - \frac{1}{2} \int_0^t \frac{a^\varepsilon(R_s^{\kappa}) \mu + R_s^{\kappa} \cdot \sigma^2}{\sqrt{\left(a^\varepsilon(R_s^{\kappa}) \mu + R_s^{\kappa} \cdot \sigma^2 \right) \sigma^2}} ds \right\}.
\]

Since \( \{M_t\} \) is a martingale w.r.t. \( \mathcal{F}_t \), we have \( \mathbb{E}[M_T] = 1 \). Using the Girsanov theorem, we know that \( Q(\omega) \) is a probability measure on \( \mathcal{F}_t \) and the process \( \{R_t^{\kappa}\} \) satisfies the following SDE

\[
dR_t^{\kappa} = \sqrt{\left((a^\varepsilon(R_t^{\kappa})) \sigma^2 + \sigma^2_p R_t^{\kappa} \right) \sigma^2} \cdot dW_t + R_t^{\kappa} = b,
\]

where \( \hat{W}_t \) is a Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q) \).

In view of (4.3), \( R_t^{\kappa} \geq R_t^{\kappa} \geq m \) for any \( t \geq 0 \), so we can define \( \rho(t) \) by

\[
\hat{\rho}(t) = \frac{1}{a^\varepsilon(R_t^{\kappa}) \sigma^2 + \sigma^2_p R_t^{\kappa} \sigma^2} \cdot dW_t + R_t^{\kappa} = b + \hat{W}_t,
\]

where \( \hat{W}_t \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q) \).

Moreover, for \( t \geq 0 \)

\[
\hat{\rho}(t) = \frac{1}{a^\varepsilon(R_t^{\kappa}) \sigma^2 + \sigma^2_p R_t^{\kappa} \sigma^2} \cdot dW_t + R_t^{\kappa} = b + \hat{W}_t,
\]

and define \( R_t^{\kappa} \) by \( R_t^{\kappa} \). Then \( \rho(t) \) is a strictly increasing function and

\[
\hat{R}_t^{\kappa} = b + \hat{W}_t,
\]

where \( \hat{W}_t \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q) \).

So \( \rho(t) \leq \frac{1}{\kappa} t \) and \( \rho^{-1}(t) \geq \kappa t \). As a result

\[
\mathbb{Q}[\inf\{t : R_t^{\kappa} \leq m\} \leq T] = \mathbb{Q}[\inf\{t : \hat{R}_{\rho^{-1}(t)}^{\kappa} \leq m\} \leq T]
\]

\[
= \mathbb{Q}[\inf\{t : \hat{W}_t \leq m - b\} \leq \rho^{-1}(T)]
\]
\[ p(t) \leq \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \]

the standard normal distribution function. By virtue of (4.4),
\[ Q[p(t) \leq m] \leq T] = \int_{-\infty}^{\infty} Q[p(t) \leq m] dQ(\omega) \]

\[ = \int_{-\infty}^{\infty} M_t dQ(\omega) \]

\[ \leq \mathbb{E}^2[M_t^2] \mathbb{P}\left[ \inf[t : R_i^{11} \leq m] \leq T \right]^\frac{1}{2}. \]

Substituting (4.5) and
\[ \mathbb{E}^2[M_t^2] \leq \exp \left\{ \frac{(\lambda \mu + r)^2}{t} \right\} \]

into (4.6), we get
\[ \mathbb{P}\left[ \inf[t : R_i^{11} \leq m] \leq T \right] \geq \frac{4 \left[ 1 - \phi \left( \frac{b-m}{\sqrt{\kappa T}} \right) \right]^2}{\exp \left\{ \frac{(\lambda \mu + r)^2}{t} \right\}} > 0. \]

Thus by (4.3)
\[ \mathbb{P}[\tau_b^{11} \leq T] \geq \mathbb{P}\left[ \inf[t : R_i^{11} \leq m] \leq T \right] \]

\[ \geq \varepsilon(b, \sigma^2, \sigma_p^2, T) \equiv \frac{4 \left[ 1 - \phi \left( \frac{b-m}{\sqrt{\kappa T}} \right) \right]^2}{\exp \left\{ \frac{(\lambda \mu + r)^2}{t} \right\}} > 0. \] (7.4)

The economic interpretation of Theorem 4.1 is the following.

1. The lower boundary \( \varepsilon(b, \sigma^2, \sigma_p^2, T) \) of the bankrupt probability for the company is an increasing function of \( \sigma_p^2 \); thus the reinvestments will make the company less likely to go bankrupt.

2. The lower boundary \( \varepsilon(b, \sigma^2, \sigma_p^2, T) \) of the bankrupt probability for the company is an increasing function of \( m \), so the minimum reserve requirement \( m \) will increase the risk of the company going into bankruptcy.

3. The lower boundary \( \varepsilon(b, \sigma^2, \sigma_p^2, T) \) of the bankrupt probability for the company is a decreasing function of \( b \), so the optimal dividend payout barrier should stay reasonably high so that the company gets good solvency.

4. The company does have larger risk before the contract between insurer and policy holders goes into effect (i.e., \( 0 < T < T \)), but the lower boundary \( \varepsilon(b, \sigma^2, \sigma_p^2, T) \) is positive for any \( T > 0 \), the company has to find an optimal policy to improve the ability of the insurer to fulfill its obligation to policy holders.

Now we prove the second result of this section.

**Theorem 4.2.** Let \( m = 0 \) in Theorem 4.1. Then for any \( T \) and \( b \)
\[ \mathbb{P}[\tau_b^{11} \leq T] = 0. \]

**Proof.** Let \( \tau_b^{11} = \inf[t : R_i^{11} = 0, R_i^{11} = b], \tau_n = \inf[t : R_i^{11} = 2^{-2n}x_0], A = \{ \tau_b^{11} \leq T \} \) and \( B_n = \{ \tau_n \leq T \} \). Then for any \( n > 0 \) \( A \subseteq B_n \). As a result,
\[ \mathbb{P}[A] = \mathbb{P}[A \cap B_n] \leq \mathbb{P}[A | B_n]. \]

Noting that \( (R_i^{11}) \) is a Markov process, we have
\[ \mathbb{P}[A | B_n] = \mathbb{P}\left[ \inf[t : R_i^{11} \leq 0] \leq T \right] \]
\[ \leq \mathbb{P}^{2^{-2n}x_0}\left[ \inf[t : R_i^{11} \leq 0] \right] \]
\[ \leq \mathbb{P}^{2^{-2n}x_0}\left[ \inf[t : R_i^{11} \leq 2^{-2n}x_0] \text{ or } \sup_{0 \leq t \leq T} R_i^{11} \geq 2^{-n}x_0 \right] \]
\[ = 1 - \mathbb{P}^{2^{-2n}x_0}\left[ \inf[t : R_i^{11} \leq 2^{-2n}x_0] \text{ and } \sup_{0 \leq t \leq T} R_i^{11} \leq 2^{-n}x_0 \right] \]
\[ \equiv 1 - \mathbb{P}(D). \]

Using the definition of \( a^*(x) \), on the set \( D \)
\[ R_i^{11} = 2^{-2n}x_0 \exp\left[ \left( \lambda\mu + r - \frac{1}{2}(\lambda^2\sigma^2 + \sigma_p^2) \right) t \right] \]
\[ \geq \sqrt{\lambda^2\sigma^2 + \sigma_p^2}W_1 \]
\[ \leq 2^{-2n}x_0 \exp[X_1], \]

where \( X_1 \) is a Brownian motion with drift. So
\[ f(n) := \mathbb{P}^{2^{-2n}x_0}\left[ \inf[t : R_i^{11} \leq 2^{-2n}x_0] \text{ and } \sup_{0 \leq t \leq T} R_i^{11} \leq 2^{-n}x_0 \right] \]
\[ = \mathbb{P}\left[ \inf[X_t \leq -n \ln 2 \text{ and } X_t \leq n \ln 2] \quad 0 \leq t \right] \rightarrow 1 \]
as \( n \to \infty \). Thus \( \mathbb{P}[\tau_b^{11} \leq T] = 0 \) follows from \( \mathbb{P}[\tau_b^{11} \leq T] \leq 1 - f(n) \). □

The interpretation of Theorem 4.2 is that when \( m = 0 \) the company of the model will never go to bankruptcy. Indeed, this is an ideal model and does not exist in reality. Thus the assumption \( m > 0 \) in this paper is reasonable and more closer to the real world.

5. Numerical examples

In this section we consider some numerical samples to demonstrate the bankrupt probability is a decreasing function of dividend payout level \( b \) or initial reserve \( x \) based on the PDE (6.2) below. The dividend payout level \( b(x, m, T) \) decreases with \( m \), increases with \( m, \sigma^2, T \) and the equation \( \psi(T, b, m, x) = \varepsilon \) (see (6.2)).

**Example 5.1.** Let \( \sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1 \) in PDE (6.2) below. Figs. 1 and 2 of the bankrupt probability \( 1 - \phi(T, x) \) state that solvency will improve with dividend payout level \( b \) or initial reserve \( x \), but the company’s profit will reduce (see Lemma 6.7 below).

**Example 5.2.** Let \( \sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1 \) and solve \( b(\varepsilon) \) by \( 1 - \phi(T, b) = \varepsilon \), we get Fig. 3. It shows that the risk \( \varepsilon \) greatly impacts on dividend payout level \( b \). The dividend payout level \( b \) decreases with the risk \( \varepsilon \), so the risk \( \varepsilon \) increases the company’s profit.

**Example 5.3.** Let \( \sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1 \) and solve \( b(\varepsilon) \) by \( 1 - \phi(T, b) = \varepsilon \), we get Fig. 4 below. The two curves in this figure show that the minimum reserve requirement \( m \) increases dividend payout level \( b \), but decreases the company’s profit.
Example 5.4. Let $\sigma^2 = \mu = 1$, $\sigma_p^2 = 2$, $m = 1$ and solve $b(\epsilon)$ by $1 - \phi(T, b) = \epsilon$, we get Fig. 5. It portrays that the dividend payout level $b$ is an increasing function of time horizon $T$, so it decreases the company’s profit.

Example 5.5. Let $\mu = 1$, $\sigma_p^2 = 2$, $m = 1$ and solve $b(\epsilon)$ by $1 - \phi(T, b) = \epsilon$, we get Fig. 6. It portrays that the dividend payout level $b$ is an increasing function of underwriting risk $\sigma_p^2$, so it decreases the company’s profit.

6. Properties on bankrupt probability and $V(x, b)$

In this section, to prove Theorem 3.1, we list some lemmas on the properties of bankrupt probability and $V(x, b)$ which will be used later. The rigorous proofs of these lemmas will be given in the Appendix below.

Lemma 6.1. The probability of bankruptcy $P[\tau^b_T \leq T]$ is a decreasing function of $b$, where $\tau^b_T := \tau^b_T$. 
Lemma 6.2.

\[ \lim_{b \to \infty} \mathbb{P}[\tau_b^x \leq T] = 0. \]  

(6.1)

Lemma 6.3. Let \( \phi(t, x) \in C^1(0, \infty) \cap C^2(m, b) \) and satisfy the following partial differential equation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\phi_t(t, x) = \frac{1}{2}[a^2(x)\sigma^2 + \sigma_x^2] \phi_x(t, x) \\
\quad \quad + [a^2(x) \mu + \rho \sigma_x] \phi_x(t, x), \\
\phi(0, x) = 1, \quad \text{for } m \leq x \leq b, \\
\phi(t, m) = 0, \quad \phi(t, b) = 0, \quad \text{for } t > 0.
\end{array} \right.
\]

(6.2)

Then \( \phi(T, x) = 1 - \psi_b^x(T, x) \), i.e., \( \psi_b^x(T, x) \) is the probability that the company will survive on time interval \([0, T]\), the function \( \psi_b^x(t, x) \) is defined by

\[ \psi_b^x(t, x) := \mathbb{P}[\tau_b^x \leq t]. \]

where \( \tau_b^x := \tau_b^{x^*} \), i.e., probability of bankruptcy for the process \( \{R_t^x\}_{t \geq 0} \) with the initial asset \( x \) and a dividend barrier \( b \) is employed before time \( t \) where \( a^* (\cdot) \) is defined by (3.3).

Let \( \sigma(x) := \frac{1}{2}[a^2(x)\sigma^2 + \sigma_x^2] \) and \( \mu(x) := a^* (x)\mu + \rho \sigma_x \). Then Eq. (6.2) becomes

\[ \phi(t, x) = a^2(x)\phi_x(t, x) + \mu(x) \phi_x(t, x). \]

By the properties of \( a^* (\cdot) \), it is easy to show that \( \sigma(x) \) and \( \mu(x) \) are continuous in \([m, b]\). So there exists a unique solution \( (6.2) \) and the solution is in \( C^1(0, \infty) \cap C^2(m, b) \). Moreover, \( a^* (\cdot) \) and \( \mu(x) \) are bounded on \([m, x_0]\) and \((x_0, b)\) respectively.

Lemma 6.4. Let \( \psi_b^x(t, x) \) be a solution of Eq. (6.2). Then the \( \phi_b^x(T, b) \) is a continuous function of \( b \) on \([b_0, \infty]\).

Lemma 6.5. Let \( F_b(x) \) be defined by (3.2) and \( b_0 \) be given by part (ii) of Lemma 3.1. Then

\[ \mathcal{L} F_b(x) \leq 0, \quad \text{for all } x \geq 0, \]

(6.4)

where

\[ \mathcal{L} = \frac{1}{2}(a^2 \sigma^2 + \sigma_x^2) \frac{d^2}{dx^2} + (a \mu + \rho \sigma) \frac{d}{dx} - c. \]

Lemma 6.6. (i) For any \( b \leq b_0 \) we have \( V(x, b) = V(x, b_0) = V(x) = F_{b_0}(x) = J(x, \pi_{b_0}^x) \). Moreover, the optimal policy is \( \pi_{b_0}^x = \{a^*(R_t^{x}), \{i_t^{b_0}\}_{t \geq 0}\} \), where \( \{R_t^{x}, \{i_t^{b_0}\}_{t \geq 0}\} \) is uniquely determined by the SDE (3.4).

(ii) For any \( b \geq b_0 \) we have \( V(x, b) = F_b(x) = J(x, \pi_b^x) \). The optimal policy \( \pi_b^x = \{a^*(R_t^{x}), \{i_t^{b_0}\}_{t \geq 0}\} \), where \( \{R_t^{x}, \{i_t^{b_0}\}_{t \geq 0}\} \) is uniquely determined by the SDE (3.8).

The Lemma 6.6 mainly deals with relationships among \( F_b(x) \), \( V(x, b) \) and \( V(x, b) \) defined by (2.3).

Lemma 6.7. For any \( b \geq b_0 \) and \( x \geq m \),

\[ \frac{d}{db} V(x, b) < 0. \]

Moreover, if \( b_1, b_2 \geq b_0 \) and \( x \leq \min\{b_1, b_2\} \), then

\[ \frac{V(x, b_1)}{V(x, b_2)} = \frac{h'(b_2)}{h'(b_1)}. \]

(6.5)

(6.6)

Acknowledgments

This work is supported by Projects 11071136 and 10771114 of NSFC, Project 20060003001 of SRFDP, the SRF for ROCS, SEM and the Korea Foundation for Advanced Studies. We would like to thank the institutions for the generous financial support. We are very grateful to the referees for the careful reading of the manuscript, correction of errors, and valuable suggestions which improved the main results of this paper very much. Special thanks also go to the participants of the seminar on stochastic analysis, finance and insurance at Tsinghua University for their feedback and useful conversations. The authors also thank Jicheng Yao for very valuable discussions on Lemma 6.4.

Appendix

In this section we will give the proofs of theorem and lemmas we were concerned with throughout this paper.

Proof of Theorem 3.1. If \( \mathbb{P}[\tau_{b_0}^x \leq T] \leq \varepsilon \), then the conclusion is obvious because it is just the optimal control problem without constraints.

Assume that \( \mathbb{P}[\tau_{b_0}^x \leq T] > \varepsilon \). By Lemmas 6.1 and 6.2, there exists a unique \( b^* (\geq b_0) \) such that

\[ b^* = \min\{b: \mathbb{P}[\tau_b^x \leq T] = \varepsilon\} = \min\{b: b \in \mathbb{R}\}, \]

(6.1)

\[ \mathbb{P}[\tau_{b_0}^x \leq T] > \varepsilon, \quad \forall b \leq b^*, \]

\[ \mathbb{P}[\tau_{b_0}^x \leq T] \leq \varepsilon, \quad \forall b \geq b^* \]

By Lemma 6.7, we know that \( V(x, b) \) is decreasing w.r.t. \( b \), so \( b^* \) satisfies (3.5). Using Lemma 6.4, we get \( b^* \in \mathbb{R} \) and \( \mathbb{P}[\tau_{b_0}^{x^*} \leq T] = \varepsilon \). Moreover, by Lemma 6.6 and (A.1), we have

\[ F_{b^*}(x) = V(x, b^*) = J(x, \pi_{b^*}^x) = V(x). \]

So the optimal policy associated with the optimal return function \( V(x) \) is \( \{a^*(R_t^{x^*}), \{i_t^{b^*}\}_{t \geq 0}\} \), where \( \{R_t^{x^*}, \{i_t^{b^*}\}_{t \geq 0}\} \) is determined uniquely by (3.8). The inequality (3.9) is a direct consequence of (6.6).

Proof of Lemma 6.1. The proof of this lemma is the same as that of Theorem 3.1 in He et al. (2008); we omit it here. □
Proof of Lemma 6.2. Using the same argument as in the proof of Theorem 3.1 in He et al. (2008), we have for some $n > 3$ and large $b \geq \max\{1, m^n\}$

$$P \left[ Q_{b,T}^{(2)} \leq T \right] \geq P (\tau_b^{(2)} \leq T).$$

(8.2)

Let $R^{(2)}_t$ be the unique solution of the following SDE

$$dR^{(2)}_t = \left( \mu R^{(2)}_t + rR^{(2)}_t + \frac{\sigma^2}{\sigma_p^2} \right) R^{(2)}_t \, dt + \frac{\sigma}{\sigma_p} R^{(2)}_t \, dW_t,$$

$$R^{(2)}_0 = 0.$$ 

(8.3)

Then by the comparison theorem on SDE (see Ikeda and Watanabe, 1981)

$$P \left[ Q_{b,T}^{(2)} \leq T \right] \geq P (\tau^{(2)}_t \leq T) = \mathbb{E} \left[ \int_0^T I_{\{\tau^{(2)}_t \leq T\}} \, dt \right],$$

(8.4)

as a result.

Firstly, we estimate $P \left[ Q_{b,T}^{(2)} \leq T \right]$. Using the H"older inequality and $\alpha'(x) \leq 1$, it follows from SDE (8.3) that

$$\sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \leq b^2 + 6 \mu^2 T^2 + 6 \sigma^2 T \mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right] \leq \left( 3 \sqrt{b} \right)^2 + 6 \mu^2 T^2 + 6 \sigma^2 T \mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right].$$

(8.5)

Taking mathematical expectation at both sides of (8.5) and using the B-D-G inequality, we derive

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right] \leq \left( 3 \sqrt{b} \right)^2 + 6 \mu^2 T^2 + 12 \sigma^2 T \mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right] + 6 \sigma^2 T \mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right].$$

(8.6)

Combining Markov inequality and the inequality (8.7), we conclude that

$$P \left[ \sup_{0 \leq t \leq T} R^{(2)}_t \geq b \right] \leq \frac{\mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right]}{b^2} \leq \frac{\left( 3 \sqrt{b} \right)^2 + 6 \mu^2 T^2 + 12 \sigma^2 T \mathbb{E} \left[ \sup_{0 \leq t \leq T} (R^{(2)}_t)^2 \right]}{b^2}.$$ 

(8.8)

Remark 1.1. The proof of Theorem 3.2 in He et al. (2008) seems wrong, so we can use the way of proving Lemma 6.2 to correct it. Theorem 3.2 in He et al. (2008) is indeed a direct consequence of Lemma 6.2.

Proof of Lemma 6.3. Let $(R^{(2)}_t, L_s(t))$ denote $(R^{(2)}_s, L_s(t))$ defined by SDE (4.1). Since $(R^{(2)}_s, L_s(t))$ is a continuous process, by the generalized Itô formula, we have

$$\phi(T - (t \wedge \tau^{(2)}_s), R^{(2)}_s) = \phi(T, x) + \int_0^{t \wedge \tau^{(2)}_s} \left[ \frac{1}{2} \sigma^2 + \sigma^2_p \cdot (R^{(2)}_s)^2 \right] \phi_x(T - s, R^{(2)}_s) \, ds$$

+ $\int_0^{t \wedge \tau^{(2)}_s} \left[ \sigma^2 (R^{(2)}_s)^2 \sigma^2_p + \sigma^2_p \cdot (R^{(2)}_s)^2 \right] \phi_x(T - s, R^{(2)}_s) \, ds$

+ $\int_0^{t \wedge \tau^{(2)}_s} \left[ \sigma^2 (R^{(2)}_s)^2 \sigma^2_p + \sigma^2_p \cdot (R^{(2)}_s)^2 \right] \phi_x(T - s, R^{(2)}_s) \, ds$

+ $\int_0^{t \wedge \tau^{(2)}_s} \left[ \sigma^2 (R^{(2)}_s)^2 \sigma^2_p + \sigma^2_p \cdot (R^{(2)}_s)^2 \right] \phi_x(T - s, R^{(2)}_s) \, ds$.

(8.9)

By B-D-G inequalities, we get

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t(t)|^2 \right] \leq 4 \left( \sigma^2 + \sigma^2_p \right) T < +\infty,$$

which implies that

$$P \left[ \sup_{0 \leq t \leq T} |M_t(t)| = +\infty \right] = 0.$$ 

Thus by (8.9),

$$\lim_{b \to +\infty} P \left\{ \inf_{0 \leq t \leq T} \{R^{(2)}_t\} \leq m \right\} = 0.$$ 

(8.10)

So the inequalities (A.2), (A.4), (A.8) and (A.10) yield that

$$\lim_{b \to +\infty} P \left\{ \tau_b^{(2)} \leq T \right\} = 0. \quad \square$$

Remark 1.1. The proof of Theorem 3.2 in He et al. (2008) seems wrong, so we can use the way of proving Lemma 6.2 to correct it. Theorem 3.2 in He et al. (2008) is indeed a direct consequence of Lemma 6.2.
Let \( \tau = T \) and taking mathematical expectation at both sides of (A.11) yields that
\[
\phi(T, x) = \mathbb{E}[\phi(T - (T \wedge \tau_b^1), R_{T \wedge \tau_b^1}^y)]
= \mathbb{E}[\phi(0, R_0^y) 1_{\tau_b^1 > T}] + \mathbb{E}[\phi(T - \tau_b^1, m) 1_{\tau_b^1 < T}]
= \mathbb{E}[1_{\tau_b^1 > T}] = 1 - \psi(T, x). \quad \Box
\]

Now we use the PDE method to prove Lemma 6.4.

**Proof of Lemma 6.4.** Let \( x = by, 0 \leq y \leq 1 \) and \( \theta^b(t, y) = \phi^b(t, (b - m)y + m) \). Then Eq. (6.2) becomes
\[
\begin{cases} 
\theta^b_0(t, y) = \sigma^b(y)w_y(t, y) + \mu^b(y) = \mu[(b - m)y + m] \quad (b > b_0), \\
\theta^b_0(0, y) = 1, \quad \text{for} \ 0 \leq y \leq 1, \\
\theta^b_0(t, 0) = \theta^b_0(t, +) = 0, \quad \text{for} \ t > 0.
\end{cases}
\]

In view of (A.12), the proof of Lemma 6.4 reduces to proving \( \lim_{b_2 \to b_1} \theta^b_2(t, 1) = \theta^b_1(t, 1) \) for fixed \( b_1 > b_0 \). Let \( w(t, y) = \theta^b_2(t, y) - \theta^b_1(t, y) \). Since \( \theta^b(t, y) \) is continuous at \( y = 1 \) for any \( b > b_0 \), we only need to show that
\[
\int_0^t \int_0^1 w^2(s, y) \, dy \, ds \to 0, \quad \text{as} \ b_2 \to b_1.
\]

Let \( \sigma^b(y) = \sigma[(b - m)y + m]/(b - m)^2, \mu^b(y) = \mu[(b - m)y + m]/(b - m) \). Then (A.12) translates into
\[
\begin{cases} 
\mu^b_1(t, y) = \sigma^b_1(y)w_y(t, y) + \mu^b_2(y)w_y(t, y) \\
\sigma_2^b(y) - \sigma^b_1(y)\theta^b_1(t, y) \\
\theta^b_2(0, y) = 0, \quad \text{for} \ 0 < y < 1, \\
\theta^b_2(t, y) = 0, \quad \text{for} \ t > 0.
\end{cases}
\]

Multiplying both sides of the first equation in (A.14) by \( w(t, z) \), and then integrating both sides of the resulting equation on \([0, 1] \times [0, 1] \), we get
\[
\int_0^t \int_0^1 w(s, y) \, w_y(t, y) \, dy \, ds = \int_0^t \int_0^1 [\sigma^b_2(y) - \sigma^b_1(y)] \int_0^1 w(s, y) \, dy \, ds \, dw_y.
\]

Now we look at terms on both sides of (A.15).

Firstly, we have
\[
\int_0^t \int_0^1 w(s, y) \, w_y(t, y) \, dy \, ds = \int_0^1 \frac{1}{2} w^2(t, y) \, dy.
\]

Secondly, we deal with terms \( E_i, i = 1, \ldots, 4 \) as follows.

It is easy to see from the expression of \( a^2(\cdot) \) that there exist positive constants \( D_1, D_2 \) and \( D_3 \) such that \( \mu[(b_2y)/b_2^2] \leq D_1 \) and \( \sigma^2_2(y) \geq D_2 > 0 \) for \( y \geq 0 \), and \( \sigma^2_2(y) \leq D_3 \) for \( y \in (0, b_0/m - b - m] \cup \{\infty\} \). As a result, for any \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \)
\[
E_1 = \int_0^t \int_0^1 [\sigma^b_2(y) - \sigma^b_1(y)] \int_0^1 w(s, y) \, dy \, ds \, dw_y.
\]

In order to estimate \( E_3 \), we decompose \( E_3 \) as follows:
\[
E_3 = \int_0^t \int_0^1 [\sigma^b_2(y) - \sigma^b_1(y)] \int_0^1 \lambda_2 w^2(s, y) \, dy \, ds \, dw_y.
\]
Therefore we conclude that
\begin{equation}
\int_0^t \int_0^1 \theta_2^b(s, y) w(s, y) \, dy \, ds.
\end{equation}

Let
\begin{equation}
B_2^b(b_2) = \frac{L(2 - b_1) - 1}{4\lambda_7} \int_0^t \int_0^1 \theta_2^b(s, y)^2 \, dy \, ds.
\end{equation}

Then
\lim_{b_2 \to b_1} B_2^b(b_2) = 0.

which, together with (A.24), implies that
\begin{equation}
E_4 \leq B_2^b(b_2) + \lambda_7 \int_0^t \int_0^1 w_2^b(s, y) \, dy \, ds.
\end{equation}

Choosing \(\lambda_1, \lambda_2, \lambda_3\) small enough such that \(\lambda_1 + \lambda_2 + \lambda_3 < D_2\), we can conclude from (A.15), (A.17), (A.18), (A.23) and (A.25) that there exist constants \(C_1, C_2\) such that
\begin{equation}
\int_0^1 w_2^b(t, y) \, dy \leq C_1 \int_0^t \int_0^1 w_2^b(s, y) \, dy \, ds + C_2[B_2^b(b_2) + B_2^b(b_2)].
\end{equation}

Using the Gronwall inequality, we get
\begin{equation}
\int_0^t \int_0^1 w_2^b(s, y) \, dy \, ds \leq C_2[B_2^b(b_2) + B_2^b(b_2)] \exp[C_1 t].
\end{equation}

So
\begin{equation}
\lim_{b_2 \to b_1} \int_0^t \int_0^1 \left( \theta_2^b(s, y) - \theta_2^b(s, y) \right)^2 \, dy \, ds = 0.
\end{equation}

Thus we complete the proof. \(\Box\)

\textbf{Proof of Lemma 6.5.} If \(x < m\) then by (3.2), \(F_b(x) = 0\). It suffices to prove (6.4) for \(m \leq x \leq b\), then
\begin{equation}
\mathcal{L}F_b(x) = \frac{\mathcal{L}h(b)}{h(b)},
\end{equation}

where \(h(b)\) is a solution of (3.1), so \(\mathcal{L}F_b(x) \equiv 0\) follows from Lemma 3.1. If \(x > b\) then by using \(F'_b(b) \geq 0\) for \(b \geq b_0\)
\begin{equation}
\mathcal{L}F_b(x) = -\frac{1}{2} \left( \mu^2 + \mu \sigma^2 + \sigma^2 \right) \sigma^2 F'_b(x) + (\mu + \sigma^2) F'_b(x) - cF_b(x)
\leq \left( \mu + \sigma^2 \right) F''_b(b) \leq \left( \mu + \sigma^2 \right) F''_b(b) \leq \left( \mu + rb \right) - cF_b(b)
\leq \mathcal{L}F_b(b) - \frac{1}{2} \left( \sigma^2 + \mu^2 \sigma^2 \right) F'_b(b) \leq 0.
\end{equation}

Thus the proof follows. \(\Box\)

\textbf{Proof of Lemma 6.6.} The proof basically follows the same arguments as in the proof of Theorem 5.2 in He and Liang (2008) and so we omit it. \(\Box\)

\textbf{Proof of Lemma 6.7.} The lemma is a direct consequence of Lemmas 6.5 and 6.6. \(\Box\)

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