WHAT IS THE OPTIMAL CHOICE? ELA OR ELID PENSION PLANS

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Abstract. In the ELA(Equity Linked Annuity) pension plan, the survival member receives the mortality credit, and leaves no bequest at the time of death, while the member receives no mortality credit and receives the fund scale as bequest in the ELID(Equity Linked Income Drawdown) pension plan. The pension member controls the asset allocation and benefit outgo policies to achieve the objectives. We explore the square deviations between the actual benefit outgo and the pre-set target as the negative old-care utility provided by benefit outgo, and the square and negative linear deviations between the actual bequest and the pre-set target are also included in the performance criterion as the negative utility provided by bequest. Using HJB (Hamilton-Jacobi-Bellman) equations and variational inequality methods, the closed-form optimal policies of the ELA and ELID pension plans are derived. It’s the first time to investigate the impacts of personal health status and bequest motive on the optimal choice between the ELA and ELID pension plans under the integrated objective functions. The worse health status and higher bequest motive result in higher old-care utility of the ELID pension plan, and vice versa. The worse personal heath status increases the proportion allocated in the risky asset and increases the benefit outgo in both pension plans. The bequest motive has positive impacts on the proportion in the risky asset and negative impacts on the benefit outgo in the ELID pension plan.

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1. Introduction

In this paper, we study the optimal asset allocation and benefit outgo policies during the distribution phases of ELA and ELID pension plans with personal health status and bequest motive.

At the early stage of the DC(Defined Contribution) pension development, the fund scale is converted into annuities at the time of retirement in order to avoid the risk of exhausted. As the fund scales are not adequate to provide high standard old-care utilities in most of the DC plans, the investment in the risky asset is allowed during some period of the distribution phase. In the ELA and ELID pension plans, the pension member usually chooses the proportion allocated in the risky asset as control variable to achieve the objectives. It is well known that the stochastic optimal control theory has been extensively applied in the asset allocation problems of the pension plan under the framework of Merton[15](1971). Cairns et al.[7](2000) and Josa-Fombellida and Rincón-Zapatero[12](2004) assume that the risky asset follows geometric Brownian motions and study the optimal asset allocation policies. Therefore, these give the possibility of studying the optimal asset allocation and benefit outgo policies in ELA and ELID pension plans.

In the ELA plan, the fund scale of the dead member is equally distributed by the survival members as mortality credit, and the survival member leaves no bequest at the time of death. While the member receives no morality credit and receives the fund scale as the bequest at the time of death in the ELID plan. In this paper we extend He and Liang[10](2013) model to describe the dynamics of the fund scales of the two pension plans in continuous-time models. In the practice of ELA pension management, as in Milevsky and Robinson[16](2000), and Albrecht and Maurer[1](2001), the benefit outgo is distributed as annuities by recalculating the fund scale every time intervals. Since the pension members have more flexibility to choose the optimal benefit outgo processes in current trends, we choose optimal benefit outgo as control variables in our model. In order to avoid exhaustion of the fund scale, we also suppose existence of compulsory conversion time to annuities as in Milevsky and Young[17](2002).
For the performance criterion aspects, the pension member usually chooses appropriate control policies to maximize the old-care utility after retirement. The objectives of the pension member are divided into two categories: maximization of the CRRA (constant relative risk aversion) and CARA (constant absolute risk aversion) utilities of the benefit outgo, and minimization of the deviations between the actual benefit outgo and a pre-set target. The literatures on the former objectives include Battocchio and Menoncin[2](2004), and Milevsky and Young[18](2007). They choose power utility and exponential utility of the benefit outgo as the performance criterions, respectively. The idea of the latter objective originates from the work of Cairns[7](2000). The widely recognized performance criterion in the optimal pension management problem is to minimize the fluctuation risk of the benefit outgo. The common approach to measuring the risk is to consider the deviations between the actual benefit outgo and a pre-set target. The latter objective is used in the model of ELA plan. In the ELID plan, in order to integrate the (negative) utility provided by bequest in the model, in this paper we include the square and negative linear deviations between the bequest and the pre-set target in the objective function which represent the preference of bequest, i.e., the bequest motive. The integrated performance function is aspired by Chang et al.[8](2003) for integrating the negative linear deviation as the preference of positive deviation. We aim at studying the dynamical asset allocation problem under the integrated performance criterion with the (negative) utilities provided both by the benefit outgo and the bequest.

In fact, the most important problem the pension members concern is the optimal choice between the ELA and ELID pension plans. We believe that the personal health status and the bequest motive have effects on the optimal choice of ELA and ELID, as well as the optimal control policies of the two pension plans. Empirical studies and statistical results imply the following reasons to choose the ELA plan or the ELID plan. Bernheim[3](1991) finds that the ELID plan is the optimal choice of members with strong bequest motives. The conclusions in
Brown[6](2001), and Finkelstain and Poterba[9](2002) confirm the theoretical model in Brugiavini[5](1993) that the members with worse health status prefer the ELID plan to ELA plan. In this paper, we suppose the pension member has personal health status and bequest motive, as in Blake, et al.[4](2003), and find the evidence of the impacts of personal health status and bequest motive on the optimal choice between ELA and ELID plans.

It is the first time we study the stochastic optimal control problem of the ELA and ELID pension plans under the integrated objective function with personal health status and bequest motive. Using HJB equations and variational inequality methods, as similar procedures in Ngwira and Gerrard[19](2007), we solve the optimal stochastic control problem and derive the closed-forms of the optimal asset allocation and benefit outgo policies. The results of Lions and Sznitman[14](1984) guarantee the dynamics of the fund scale is uniquely determined by the stochastic differential equation with respect to the optimal feedback functions. Furthermore, we use MCM(Monte Carlo Methods) to investigate the impacts of the personal health status and the bequest motive on the optimal control policies with respect to time.

The rest of this paper will be organized as follows. In Section 2, we establish stochastic differential equations to model the dynamics of the fund scales in the ELA and ELID plans, respectively. The minimization of the integrated negative old-care utilities provided by the benefit outgo and bequest is the objective function. In Section 3, using HJB equations and variational inequality methods, we establish the closed-form optimal asset allocation and benefit outgo policies of the ELA and ELID plans. In Section 4, we use MCM to investigate the impacts of the personal health status and bequest motive on the optimal control policies with respect to time. The conclusions of this paper are given in Section 5.

2. The stochastic optimal control problem

In this paper, we study the optimal asset allocation and benefit outgo policies of the ELA and ELID pension plans during the distribution
phase with personal health status and bequest motive. The pension
member controls the asset allocation and the benefit outgo policies to
achieve the optimal performance criterion. In order to protect the fund
scale from exhausted, the pension plans have compulsory conversion
claims to annuities.

As providing stable and sustainable benefit outgo is the objective
of the pension member, minimizing the square deviations between the
actual benefit outgo and the pre-set target is the performance criterion
in the ELA plan. In order to describe the bequest motive effects, the
square and negative linear deviations between the bequest and the pre-
set target are integrated in the performance criterion. The weight of
the linear deviation represents the degree of bequest motive.

In this paper, we suppose the pension member has personal health
status. Indeed, the insurance business is operated under the law of
large members, and the mortality credit is calculated by the statistical
mortality results of the total members in the plan. However, in the
individual utility perspective, pension members have different health
statuses and different mortality probabilities. These result in different
performance function, as well as optimal control policies of the pension
members, and the optimal choice between the ELA and ELID pension
plans. Now we extend He and Liang[10](2013) model to describe the
dynamics of the fund scales of the two pension plans by continuous-time
models.

2.1. Stochastic model of the ELA pension plan.
In order to describe the investment return dynamics, denote
\[
\Delta \eta^e_t = \pi_1 \frac{S^1_{t+\frac{1}{n}} - S^1_t}{S^1_t} + (1 - \pi_1) \frac{S^0_{t+\frac{1}{n}} - S^0_t}{S^0_t},
\]
where \( S^0_t \) and \( S^1_t \) are the value of the risk-free asset and the risky
asset at time \( t \), respectively. \( n \) is a big enough natural number. \( \pi_1 \) is
the proportion allocated in the risky asset, and it is an important con-
trol variable in the model. And the admissible domain is \(( -\infty, +\infty ))
, i.e., \( \pi_1 \in ( -\infty, +\infty ) \) or short sales of the risk-free asset and the risky
asset are both permitted. These hypotheses are used to avoid the time-inconsistent problem, and it is the compromise between the closed-form solution and the coordination with the pension management practice.

During the distribution phase of the ELA pension plan, the individual fund scale is affected by investment return, benefit outgo and mortality credit. In traditional ELA pension plan, the benefit outgo is determined by the current fund scale according to actuarial principles to protect the fund from exhausted, i.e., the fund scale is recalculated every time interval and the benefit is distributed as annuities. In the current pension management practice, the members are given more flexibilities to choose the appropriate benefit outgo to achieve individual optimal performance criterion. So, the benefit outgo is also an important control variable in the model.

We use similar methods in He and Liang[10, 11] (2013, 2015) to model the mortality credit. The discrete time-form of the fund scale dynamics is as follows:

$$Y_1(t + \frac{1}{n}) = \{Y_1(t)(1 + \Delta \eta^t_n) - p_1(t)\frac{1}{n}\} \frac{1}{1 - \frac{1}{n}q^{O}_{x_0+t}},$$  \hspace{1cm} (2.2)

where $Y_1(t)$ is the fund scale at time $t$, $p_1(t)$ is the benefit outgo, and it is an important control variable in the model. And we suppose that $p_1(t) \in (-\infty, +\infty)$ due to avoiding the time inconsistent problem considerations. Fortunately, according to empirical evidence from the pension management practice, the optimal control policies seldom beyond the actual admissible domains and the optimal control policies make sense. $x_0$ is pension member’s age at the beginning of the distribution phase, i.e., the age of retirement. $\frac{1}{n}q^{O}_{x_0+t}$ is an actuarial symbol which is the conditional probability that a person is alive at the age $x_0 + t$ and will die within the next $\frac{1}{n}$ time interval based on all the members’ statistical results. The mortality credit is calculated by the statistical mortality results of all the members of the pension plan. The last item in Eq.(2.2) represents the mortality credit, i.e., the fund scales of all the members at time $t$ will be distributed equally among the survival members at time $t + \frac{1}{n}$. 
In order to establish continuous-time risk model, we make the following approximations:

\[
\frac{\frac{1}{n} q_{x_0+t}}{1 - \frac{1}{n} q_{x_0+t}} = 1 - e^{-\int_0^t \mu^O(x_0+t+s)ds} - 1 = e^{\int_0^t \mu^O(x_0+t+s)ds} - 1 \\
\approx \mu^O(x_0 + t) \frac{1}{n} = O\left(\frac{1}{n}\right), \quad (2.3)
\]

where \(\mu^O(x_0 + t)\) is the force of mortality function at the age of \(x_0 + t\). It is the force of mortality of all the members, and it’s different from the personal force of mortality. For simplicity, we use the constant force of mortality model in the paper, i.e., \(\mu^O(x_0 + t) \equiv \mu^O\). Then, by (2.1)-(2.3), the dynamics of the individual member’s fund scale satisfies the following equations:

\[
Y_1(t + \frac{1}{n}) = Y_1(t)(1 + \Delta n^t) - p_1(t) \frac{1}{n} + Y_1(t)\mu^O \frac{1}{n} + o\left(\frac{1}{n}\right).
\]

Besides, the following stochastic differential equations are used to describe the dynamics of the risk-free and the risky asset returns. The values of the risky asset and the risk-free asset are governed by the following stochastic differential equations on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), respectively:

\[
\begin{align*}
dS^1(t) & = S^1(t)\left\{cdt + \sigma dB(t)\right\}, \\
dS^0(t) & = rS^0(t)dt,
\end{align*}
\]

where \(c\) and \(\sigma\) are the expected return and the volatility of the risky asset, respectively. \(r\) is the risk-free interest rate. \(\{B(t), t \geq 0\}\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual conditions(cf.[13]). \(\mathcal{F}_t\) represents the information available at time \(t\) and any decision made up to time \(t\) is based on this information.

By the above approximations, we transform the discrete-time model into the continuous-time model, i.e., the dynamics of the fund scale can be written by the following stochastic differential equation:

\[
\begin{align*}
dY_1(t) & = \left\{[\pi_1(c - r) + r]Y_1(t) + \mu^O Y_1(t) - p_1(t)\right\}dt \\
& \quad + \pi_1 \sigma Y_1(t)dB(t), \\
Y_1(0) & = y_0,
\end{align*}
\]

(2.4)
where \( y_0 \) is the initial fund scale, i.e., the accumulated fund scale at the time of retirement.

As the benefit outgo is a control variable in the model, the methods to protect the fund scale from exhausted should be implemented. We suppose that the conversion time is \( T \). At that time, the fund scale is all invested in the risk-free asset and distributed as annuities, i.e.,

\[
\pi_1(t) = 0, \quad p_1(t) = \frac{Y_1(T)}{\ddot{a}_{x_0+T}}, \quad \forall \quad T \leq t \leq \omega - x_0,
\]

where \( \omega \) is the maximum age of the life table. \( \ddot{a}_{x_0+T} \) is the initial value of one unit annuity at the age of \( x_0 + T \) from all the members’ perspective, and it is expressed by

\[
\ddot{a}_{x_0+T} = \int_0^{\omega-x_0-T} e^{-rs} s p_{x_0+T} ds
\]

\[
= \int_0^{\omega-x_0-T} e^{-rs} e^{-\int_0^s \mu^O du} ds
\]

\[
= \frac{1-e^{-(r+\mu^O)(\omega-x_0-T)}}{r+\mu^O},
\]

where \( s p_{x_0+T} \) is the conditional probability that a person is alive at the age of \( x_0 + T \), will be still alive at the age of \( x_0 + T + s \). As the actuarial principles of the annuities are also based on the statistical survival results of all the members, we use \( \mu^O \) as the force of mortality in the calculations.

According the theoretical economic studies, the objectives of the pension members are the maximization of the old-care utility after retirement. The widely recognized performance criterion in the optimal pension management problem is to minimize the fluctuation risk of the benefit outgo. There is a pre-set target of the benefit outgo. The actual benefit outgo should be close to the pre-set target, and both the positive and negative deviations be penalized.

The common approach to measuring the risk is to consider the deviations between the actual benefit outgo and a pre-set target. We use minimization of the square deviations between the actual benefit outgo and the pre-set target as the objective functions, as in Ngwira and Gerrard [19](2007). Therefore we define the value function \( V_1(y; t) \)
as follows:

\[
\begin{align*}
V_1(y; t) &= \min_{(\pi_1, p_1) \in \Pi_t} \left\{ J_1(\pi_1, p_1, y; t) \right\} \\
&= \min_{(\pi_1, p_1) \in \Pi_t} \mathbb{E}_{(\pi_1, p_1, y; t)} \left\{ \int_t^T e^{-rs} s p_{x_0}^S (p_1(s) - NP)^2 ds \right. \\
&\quad + \left. \alpha e^{-r(T-t)} P_{x_0}^S (Y_1(T) - \bar{a}Ox_0^T \cdot NP)^2 \right\}, \\
\end{align*}
\]

(2.5)

where \(\mathbb{E}_{(\pi_1, p_1, y; t)}[\cdot]\) means conditional expectation given the initial value \(Y(t) = y, \ p_1(t) = p_1, \ \pi_1(t) = \pi_1\) at time \(t\). The performance criterion is \(J_1(\pi_1, p_1, y; t)\). \(\Pi_t\) is the set of all admissible policies whose initial values are replaced by \(p_1(t) = p_1, \pi_1(t) = \pi_1\). \(NP\) is the pre-set target of expected benefit outgo. It represents the required benefit to maintain a high standard retirement life. It is an exogenous variable and \(NP > 0\).

\(s p_{x_0}^S\) is the personal conditional probability that a person is alive at the age of \(x_0\), will be still alive at the age of \(x_0 + s\). \(s p_{x_0}^S = e^{-\int_0^s \mu^S ds}\), where \(\mu^S\) is the personal force of mortality. Since the pension members have different health statuses, they have different survival/mortality probabilities. We use personal probabilities in the performance criterion of the individual pension member. \(\alpha\) is the weight variable which measures the importance of the (negative) utility provided by the annuities, and \(\alpha > 0\). The items in (2.5) are the deviations between the actual benefit outgo and the pre-set target, and the deviation between the fund scale and the expected present value of the annuities at the time of conversion. The items represent the negative old-care utilities provided by the benefit outgo during the survival period, and the annuities after the conversion, respectively. The objective function is to minimize the above deviations, i.e., maximize the old-care utilities provided by benefit outgo and annuities.

The asset allocation policy \(\pi_1^*\) and the benefit outgo policy \(p_1^*\) are called optimal(or optimal control processes) if \((\pi_1^*, p_1^*) \in \Pi_t\) and satisfies

\[
V_1(y; t) = J_1(\pi_1^*, p_1^*, y; t).
\]

Thus, the continuous time stochastic optimal control problem (2.4)-(2.5) of the ELA pension plan has been established. We aim at deriving closed-forms of the optimal policies \((\pi_1^*, p_1^*)\) to solve the optimal control problem(2.4)-(2.5).
2.2. Stochastic model of the ELID pension plan.

In the ELID pension plan, the pension members control the asset allocation and benefit outgo policies to achieve the objectives. Being different from the ELA pension plan, the member does not receive mortality credit, but receives the fund scale as bequest at the time of death in the ELID pension plan. The dynamics of the fund scale in the ELID pension plan is governed by the following stochastic differential equations:

\[
\begin{align*}
\frac{dY_2(t)}{dt} &= \left\{ \pi_2(c-r)+\pi \right\}Y_2(t) - p_2(t) \right\}dt + \pi_2\sigma Y_2(t)dB(t), \\
D_2(\tau_d) &= Y_2(\tau_d), \\
Y_2(0) &= y_0,
\end{align*}
\]

(2.6)

where \( Y_2(t) \) is the fund scale at time \( t \), \( \pi_2(t) \) is the proportion allocated in the risky asset at time \( t \), and \( p_2(t) \) is the benefit outgo at time \( t \). They are important control variables in the model with \( \pi_2(t) \in (-\infty, +\infty) \) and \( p_2(t) \in (-\infty, +\infty) \) for any \( t \geq 0 \), respectively. \( D_2(\tau_d) \) is the bequest at time \( \tau_d \), and it equals the fund scale at that time, \( \tau_d \) is the time of death and it is a random variable.

For the performance criterion aspects, besides the (negative) utility provided by the benefit outgo, the (negative) utility provided by bequest is also included in the performance criterion. The square deviation and the negative linear deviations between the actual bequest and the preset target are integrated in the objective functions. The single square deviation represents the scenario of the member with no bequest motive and the final distribution should be close to the pre-set target. The scenario is fit for the members who are at older ages or have no child to support, and they have little bequest motive. The negative linear deviation represents the scenario of the members with bequest motives. The scenario is fit for the members who are at younger ages or have family to support, and they have bequest motives. As the objective is to minimize the performance function, the negative linear deviation represents the preference for larger bequest, i.e., the bequest motive.
Hence, we define the value function $V_2(y; t)$ as follows:

$$
V_2(y; t) = \min_{(\pi_2; p_2) \in \Pi_t} \left\{ J_2(\pi_2, p_2, y; t) \right\}
= \min_{(\pi_2; p_2) \in \Pi_t} \mathbb{E}_{(\pi_2, p_2, y; t)} \left\{ \int_t^T e^{-rs} s p_{x_0}^S (p_2(s) - NP)^2 ds + \alpha e^{-rT} T p_{x_0}^S (Y_2(T) - \bar{u}^O_{x_0+T} \cdot NP)^2 \right\},
\right. \tag{2.7}
\left. Y_2(0) = y_0, \right.
$$

where $\mathbb{E}_{(\pi_2, p_2, y; t)}[\cdot]$ means conditional expectation given the initial value $Y(t) = y$, $p_2(t) = p_2$, $\pi_2(t) = \pi_2$ at time $t$. The performance function is $J_2(\pi_2, p_2, y; t)$. $\Pi_t$ is the set of all admissible policies $\pi_2(t) \in (-\infty, +\infty)$ and $p_2(t) \in (-\infty, +\infty)$, whose initial values are replaced by $p_2(t) = p_2$, $\pi_2(t) = \pi_2$. $s p_{x_0}^S$ is the personal conditional probability that a person is alive at the age of $x_0$, will be still alive at the age of $x_0 + s$. $\bar{u}^O_{x_0+T}$ is the initial value of one unit annuity at the age of $x_0 + T$ from all the members’ perspective. $\tau_d$ is the pension member’s time of death. It is a random variable, and is determined by the member’s personal health status. $\mathbb{E}_{(\tau_d)}[\cdot]$ means the expectation with respect to the random time of death. $\beta_1$ is the weight variable which measures the importance of the (negative) utility provided by the last distribution of the member with no bequest motive, and $\beta_1 > 0$. $\beta$ is another weight variable which measures the importance of the (negative) utility provided by the bequest of the member with bequest motive, and $\beta > 0$.

The value function has the following forms by some simple probability transformations:

$$
V_2(y; t) = \min_{(\pi_2; p_2) \in \Pi_t} \left\{ J_2(\pi_2, p_2, y; t) \right\}
= \min_{(\pi_2; p_2) \in \Pi_t} \mathbb{E}_{(\pi_2, p_2, y; t)} \left\{ \int_t^T e^{-rs} s p_{x_0}^S (p_2(s) - NP)^2 ds + \int_t^T e^{-rs} s p_{x_0}^S \beta_1 (Y_2(s) - NP)^2 \right\}
\tag{2.8}
\right. \left. + \alpha e^{-rT} T p_{x_0}^S (Y_2(T) - \bar{u}^O_{x_0+T} \cdot NP)^2 \right\},
\right. \left. Y_2(0) = y_0. \right.
$$

The items in equation (2.8) are the deviations between the actual benefit outgo and the pre-set target, the square and the negative linear deviations between the bequest and the target, and the deviation between the fund scale and the expected present value of the annuities at the time of conversion. The items represent the negative old-care utilities provided by the benefit outgo during the survival period, the
bequest at the time of death, and the annuities after the conversion, respectively. The objective function is to minimize the above deviations, i.e., maximize the old-care utilities provided by benefit outgo, annuities and bequest.

The asset allocation policy $\pi_2^*$ and the benefit outgo policy $p_2^*$ are called optimal control process if $(\pi_2^*, p_2^*) \in \Pi_t$ and satisfies

$$V_2(y; t) = J_2(\pi_2^*, p_2^*, y; t).$$

Thus, the continuous-time stochastic optimal control problem (2.6)-(2.8) for the ELID pension plan has been established. We aim at deriving closed-forms of the optimal policies $(\pi_2^*, p_2^*)$ to solve the optimal control problem (2.6)-(2.8).

It’s the first time to study the optimal control problem of the ELID pension plan with the performance criterion integrating the (negative) utilities of the benefit outgo and the bequest. In the next section, we will derive the closed-form solutions of the stochastic control problems of the ELA and ELID pension plans. Furthermore, we study the impacts of personal health status and the bequest motive on the optimal control policies, and the optimal choice between the ELA and ELID pension plans.

3. Solutions of the stochastic optimal control problem

In this section, we use Itô stochastic calculus and variational methods to solve the optimal control problems (2.4)-(2.5) and (2.6)-(2.8). First, we establish the HJB equation that value function and optimal feedback functions of the stochastic control problem satisfy. Second, we derive the optimal feedback functions from the HJB equation. Third, the results in Lions and Sznitman[14](1984) guarantee that the solution of the stochastic differential equation of the fund scale dynamics is uniquely determined by the optimal feedback functions. The unique solution of the SDE is the so-called the optimal state(or wealth) process(cf.[20](1999)). Finally, we get closed-forms of the optimal asset allocation and benefit outgo policies by composing of the optimal feedback functions and the optimal state process.
First, we derive the associated HJB equations with the stochastic control problems (2.4)-(2.5) and (2.6)-(2.8). We synthesize the solutions of the stochastic optimal control problems in the following HJB equations (3.1):

Denote the solutions of the equations by \( \varphi_{1,2}(\pi_{1,2}, p_{1,2}, y; t) \), where \( \varphi_1(\pi_1, p_1, y; t) \) and \( \varphi_2(\pi_2, p_2, y; t) \) are the solutions of the stochastic control problems with the optimal feedback functions of asset allocation and benefit outgo policies in the ELA and ELID pension plans, respectively.

Using variational methods and Itô formula, we know that \( \varphi_{1,2}(\pi_{1,2}, p_{1,2}, y; t) \) satisfy the following HJB equations:

\[
0 = \min_{\pi_{1,2}, p_{1,2}} \left\{ \frac{\partial \varphi_{1,2}}{\partial y} y(\pi_{1,2}(c-r)+r+\delta_{1,2}\mu^O) - p_{1,2}(t) \right\}
\]

\[
+ \frac{1}{2} \frac{\partial^2 \varphi_{1,2}}{\partial y^2} \pi_{1,2} \sigma^2 y^2 + \frac{\partial \varphi_{1,2}}{\partial t} + e^{-\delta t} e^{-\int_0^t \mu^s ds} (p_{1,2}(t) - NP)^2
\]

\[
+ (1 - \delta_{1,2}) e^{-\delta t} e^{-\int_0^t \mu^s ds} \mu^S [\beta_1(y(t)-NP)^2 - \beta(y(t)-NP)] \} \quad (3.1)
\]

with boundary condition

\[
\varphi_{1,2}(\pi_{1,2}, p_{1,2}, Y_{1,2}(T); T) = \alpha e^{-\delta T} e^{-\int_0^T \mu^s ds} (Y_{1,2}(T) - \delta_{20+T} \cdot NP)^2, \quad (3.2)
\]

when \( \delta_1 = 1 \) and \( \delta_2 = 0 \), Eq. (3.1) are the HJB equations of the stochastic optimal control problems of the ELA and ELID pension plans, respectively.

Differentiating Eq. (3.1) with respect to \( \pi_{1,2} \) and \( p_{1,2} \), and then letting these derivatives be zero, respectively, we get the following optimal feedback functions \( \pi_{1,2}^*(y; t) \) and \( p_{1,2}^*(y, t) \):

\[
\begin{align*}
\pi_{1,2}^*(y; t) &= \frac{\alpha \pi_{1,2}}{\sigma^2 y^3} e^{(r-c)} \\
p_{1,2}^*(y, t) &= \frac{\partial \varphi_{1,2}}{\partial y} e^{(r+c)} + NP
\end{align*}
\]

According to the boundary condition (3.2) we guess \( \varphi_{1,2}(\pi_{1,2}^*, p_{1,2}^*, y; t) \) has the following forms:

\[
\varphi_{1,2}(\pi_{1,2}^*, p_{1,2}^*, y; t) = e^{-\delta t} e^{-\int_0^t \mu^s ds} Q_{1,2}(t)(y^2 - 2S_{1,2}(t)y + R_{1,2}(t)), \quad (3.3)
\]

where \( Q_1(t) \), \( S_1(t) \) and \( R_1(t) \) are undetermined functions of \( t \) in the ELA pension plan, and \( Q_2(t) \), \( S_2(t) \) and \( R_2(t) \) are undetermined functions in
the ELID pension plan. The boundary conditions are

\[
\begin{align*}
Q_{1,2}(T) &= \alpha, \\
S_{1,2}(T) &= \frac{1-e^{-(r+\mu^s)(\omega-x_0-T)}}{r+\mu^s}NP, \\
R_{1,2}(T) &= \frac{1-e^{-(r+\mu^s)(\omega-x_0-T)}}{r+\mu^s}^2NP. \\
\end{align*}
\]  

(3.4)

So the optimal feedback functions can be rewritten as follows:

\[
\begin{align*}
\pi_{1,2*}(y; t) &= \frac{(y-S_{1,2}(t))}{\sigma^2}, \\
p_{1,2*}(y, t) &= Q_{1,2}(t)(y - S_{1,2}(t)) + NP. \\
\end{align*}
\]  

(3.5)

Thus we only need to prove that (3.1) holds for all \(0 < t \leq T\).

It is easy to see from (3.3) that

\[
\begin{align*}
\frac{\partial^2 \pi_{1,2}}{\partial y^2} &= 2e^{-(r+\mu^s)\delta}(t)(y - S_{1,2}(t)), \\
\frac{\partial^2 \pi_{1,2}}{\partial t^2} &= 2e^{-(r+\mu^s)\delta}(t), \\
\frac{\partial^2 \pi_{1,2}}{\partial y \partial t} &= -re^{-(r+\mu^s)\delta}(t)(y^2 - 2S_{1,2}(t)y + R_{1,2}(t)) \\
&+ \delta e^{-(r+\mu^s)\delta}(t)(y^2 - 2S_{1,2}(t)y + R_{1,2}(t)) \\
&+ e^{-(r+\mu^s)\delta}(t)(y^2 - 2S_{1,2}(t)y + R_{1,2}(t)) \\
&+ e^{-(r+\mu^s)\delta}(t)(y^2 - 2S_{1,2}(t)y + R_{1,2}(t)). \\
\end{align*}
\]  

(3.6)

Substituting (3.5) and (3.6) into (3.1), the associated HJB equation can be rewritten as follows:

\[
\begin{align*}
0 &= rQ_{1,2}(t)[y^2 - 2S_{1,2}(t)y + R_{1,2}(t)] + \mu^sQ_{1,2}(t)[y^2 - 2S_{1,2}(t)y + R_{1,2}(t)] \\
&- Q_{1,2}'(t)[y^2 - 2S_{1,2}(t)y + R_{1,2}(t)] + Q_{1,2}(t)[2S_{1,2}(t)y - R_{1,2}(t)] \\
&+ 2Q_{1,2}(t)[y - S_{1,2}(t)][(e-r)^2(y - S_{1,2}(t))] - ry - \delta_1 \mu^Oy \\
&+ Q_{1,2}(t)(y - S_{1,2}(t)) + NP - Q_{1,2}(t)[(e-r)^2(y - S_{1,2}(t))] \\
&- [Q_{1,2}(t)(y - S_{1,2}(t))]^2 - (1 - \delta_1) \mu^S[\beta_1(y - NP)^2 - \beta(y - NP)]. \\
\end{align*}
\]  

(3.7)

In the stochastic optimal control problem of the ELA pension plan, \(\delta_1 = 1\). Equating the coefficient of quadratic factor to zero in (3.7), we get the following Riccati ordinary differential equation:

\[
Q_1' + (2\mu^O - \mu^S + r - \frac{(c-r)^2}{\sigma^2})Q_1 - Q_1^2 = 0. 
\]  

(3.8)

Let \(Q(t) \equiv u^{-1}(t)\). Then

\[
u' - (2\mu^O - \mu^S + r - \frac{(c-r)^2}{\sigma^2})u + 1 = 0
\]

(3.9)

with boundary condition

\[
u(T) = \frac{1}{\alpha}.
\]

The solution of (3.9) is

\[
u(t) = \left(\frac{1}{\alpha} - \frac{1}{M}\right)e^{M(t-T)} + \frac{1}{M}
\]
with
\[ M = r + 2\mu^O - \mu^S - \frac{(c - r)^2}{\sigma^2}. \]
Thus, the solution of (3.8) is
\[ Q_1(t) = \frac{1}{\left(\frac{1}{\alpha} - \frac{1}{M}\right)e^M(t-T) + \frac{1}{M}}. \tag{3.10} \]
Equating the coefficient of linear factor to zero in (3.7), we have
\[ S'_1 - (\mu^O + r)S_1 + NP = 0. \tag{3.11} \]
The solution of (3.11) is
\[ S_1(t) = \left[1 - e^{-\frac{(\mu^O + r)(\omega - x_0 - T)}{r + \mu^O}}\frac{NP}{\mu^O + r}\right]e^{(\mu^O + r)(t-T)} + \frac{NP}{\mu^O + r}. \tag{3.12} \]
Let the constant term be zero in (3.7), we can get ordinary differential equation that \( R_1(t) \) satisfies, and the expression of \( R_1(t) \) is
\[ R'_1 - (\mu^O + r)\frac{Q_1'}{Q_1}R_1 - \frac{(c - r)^2}{\sigma^2}\frac{Q_1'}{Q_1}S_1^2 + Q_1^2 - 2NP \cdot S_1 = 0. \tag{3.13} \]
As \( Q_1(t) \) are \( S_1(t) \) have been solved in (3.10) and (3.12), it’s easy to solve the differential equation (3.13) with boundary conditions in (3.4). Since \( R_1(t) \) is not related to the optimal feedback functions of (3.5), we omit the calculations here.

Similarly, in the stochastic optimal control problem of the ELID pension plan with \( \delta_2 = 0 \), equating the coefficient for the quadratic factor to zero in (3.7), we get the following ordinary differential equation:
\[ Q'_2 + (-\mu^S + r - \frac{(c - r)^2}{\sigma^2})Q_2 - Q_2^2 + \mu^S \cdot \beta_1 = 0. \tag{3.14} \]
Making the following transformations:
\[ Q'_2 = Q_2^2 - K \cdot Q_2 - \mu^S \cdot \beta_1 = (Q_2 - w_1)(Q_2 - w_2), \]
where
\[ K = -\mu^S + r - \frac{(c - r)^2}{\sigma^2} \]
and
\[ w_{1,2} = \frac{K \pm \sqrt{K^2 + 4\mu^S \cdot \beta_1}}{2}, \]
we get
\[ \frac{dQ_2}{(Q_2 - w_1)(Q_2 - w_2)} = dt. \tag{3.15} \]
Integrating both sides of Eq. (3.15), we have
\[
\frac{1}{w_1 - w_2} \int_t^T \left( \frac{1}{Q_2 - w_1} - \frac{1}{Q_2 - w_2} \right) dQ_2 = \int_t^T ds.
\]
The solution of (3.14) is then as follows:
\[
Q_2(t) = \frac{w_2(\alpha - w_1) - w_1(\alpha - w_2)e^{(w_1 - w_2)(T - t)}}{\alpha - w_1 - (\alpha - w_2)e^{(w_1 - w_2)(T - t)}}.
\] (3.16)

Equating the coefficient of linear factor to zero in (3.7), we have
\[
S'_2 - (r + \frac{\mu^S \cdot \beta_1}{Q})S_2 + NP + \frac{\mu^S(NP \cdot \beta_1 + \frac{\beta}{\tau})}{Q_2} = 0.
\] (3.17)
The solution of the ordinary differential equation (3.17) is
\[
S_2(t) = \frac{1}{w_2(\alpha - w_1) - w_1(\alpha - w_2)e^{(w_1 - w_2)(T - t)}} \cdot \left\{ \left( \frac{\alpha(w_2 - w_1)1 - e^{-(r+\mu^O)(\omega-x_0-T)}}{r + \mu^O} \right)NP \cdot e^{(r+\frac{\mu^S \cdot \beta_1}{Q_2})(t-T)} \right. \right.
\]
\[
- \left. \frac{[NP \cdot w_2 + \mu^S(NP \cdot \beta_1 + \frac{\beta}{\tau})](\alpha - w_1)}{r + \mu^S \cdot \beta_1} \right. \left. \cdot e^{(r+\frac{\mu^S \cdot \beta_1}{Q_2})(t-T) - 1} \right) \right.
\]
\[
\left. \times \left( e^{(w_1 - w_2)(T - t)} - e^{(r+\frac{\mu^S \cdot \beta_1}{Q_2})(t-T)} \right) \right\}. \] (3.18)

Let the constant term be zero in (3.7), we can get ordinary differential equation that $R_2(t)$ satisfies:
\[
R'_2 - (\mu^S + r - \frac{Q'_2}{Q_2})R_2 - \left[ \frac{(c - r)^2}{\sigma^2} - S_2^2 + Q_2S_2^2 \right] - 2NP \cdot S_2 - \frac{\mu^S \cdot \beta_1 \cdot NP^2}{Q_2} - \frac{\mu^S \cdot \beta \cdot NP}{Q_2} = 0.
\] (3.19)

Similarly, as $Q_2(t)$ and $S_2(t)$ have been solved in (3.16) and (3.18), it’s easy to solve the differential equation (3.19) with boundary conditions in (3.4). However, since $R_2(t)$ is not related to the optimal feedback functions of (3.5), we don’t care the expression of $R_2(t)$, and omit its calculations here.

Therefore, we get the closed-forms of the optimal feedback functions $\pi_{1,2*}(y, t)$ and $p_{1,2*}(y, t)$ by (3.5), (3.10) and (3.12), as well as (3.5), (3.16) and (3.18), respectively. It is easy to see that $\pi_{1,2*}$ and $p_{1,2*}$ are functions of time $t$ and the fund scale $y$ at time $t$. 
Now, we turn to establishing the optimal asset allocation policy \( \pi_{1,2}^*(t) \) and the optimal benefit outgo policy \( p_{1,2}^*(t) \). Let \( \pi_{1,2}^*(\cdot, \cdot) \) and \( p_{1,2}^*(\cdot, \cdot) \) be defined by (3.5). Then the following two SDEs

\[
\begin{align*}
\frac{dY_{1,2}^*}{dt} &= \left\{ \left[ \pi_{1,2}^*(Y_{1,2}^*(t), t)(c-r)+r \right] Y_{1,2}^*(t) + \delta_{1,2} \mu \sigma Y_{1,2}^*(t) \right. \\
& \quad \left. - p_{1,2}^*(Y_{1,2}^*(t), t)(t) \right\} dt + \pi_{1,2}^*(Y_{1,2}^*(t), t) \sigma Y_{1,2}^*(t) dB(t), \quad (3.20)
\end{align*}
\]

have unique solutions \( Y_{1,2}^*(t) \) and \( Y_{2,2}^*(t) \), respectively. We use the similar approach in Øksendal and Sulem [21] (2007) to prove the verification theorem on the stochastic optimal control problems (2.4)-(2.5) and (2.6)-(2.8). The functions \( \pi_{1,2}^*(y; t) \) and \( p_{1,2}^*(y; t) \) defined by (3.5) and the function \( \varphi_{1,2}(\pi_{1,2}^*(y; t), p_{1,2}^*(y; t), y; t) \) defined by (3.3) are the optimal feedback functions and the value function, respectively. The unique solution \( \{ Y_{1,2}^*(t), t \geq 0 \} \) of (3.20) is the optimal state process. The optimal asset allocation and benefit outgo policies \( (\pi_{1,2}^*(t), p_{1,2}^*(t)) \) are the compositions of the optimal feedback functions \( (\pi_{1,2}^*(\cdot, t), p_{1,2}^*(\cdot, t)) \) and the optimal state process \( Y_{1,2}^*(t) \), i.e., \( \pi_{1,2}^*(t) = \pi_{1,2}^*(Y_{1,2}^*(t), t) \), \( p_{1,2}^*(t) = p_{1,2}^*(Y_{1,2}^*(t), t) \). We also have \( V_{1,2}(y, t) = \varphi_{1,2}(\pi_{1,2}^*(y; t), p_{1,2}^*(y; t), y; t) = J_{1,2}(\pi_{1,2}^*, p_{1,2}^*, y; t) \).

4. Analysis of the Optimal Control Policies

In this section, we use MCM to study the impacts of personal health status and bequest motive on the optimal asset allocation and benefit outgo policies of the ELA and ELID pension plans. Furthermore, we establish the optimal old-care utility boundary between the ELA and ELID pension plans in the coordinates of health status v.s. bequest motive plane.

In the numerical analysis procedures, we use MCM to randomly generate 10000 paths of the fund scale processes. At each time step, the fund scale is re-calculated by the \( \pi_{1,2}^* \) and \( p_{1,2}^* \) of the last step and a randomly generated Brownian motion, as in (3.20). Then, \( \pi_{1,2}^* \) and \( p_{1,2}^* \) of the step are re-calculated by the current fund scale according to (3.5). The \( \pi_{1,2}^* \) and \( p_{1,2}^* \) at each time step are the averages of the \( \pi_{1,2}^* \) and
$p_{1,2}^*$ of the 10000 simulation tracks. The procedures are periodically repeated 10000 times and the average of the optimal control policies is used for investigation.

According to the empirical evidence from the capital market and the pension management practice, we make the following assumptions for the variables. In the life insurance area, the maximum age of the alive persons is $\omega = 110$. The age at the beginning of the distribution phase is $x_0 = 60$, i.e., the retirement age is 60. According to the data of the capital market, we choose the risk-free interest rate as $r = 0.02$ and it’s the yield of the one year U.S. bonds. The expected return and the volatility of the risky asset are $c = 0.08$ and $\sigma = 0.33$, respectively. These coefficients are estimated by the investment return of the U.S. stock index. Furthermore, we make some assumptions on the individual member’s pension account. The initial fund scale is $y_0 = 30$ (ten thousand dollars), and the pre-set target for the expected benefit outgo is $NP = 3$ (ten thousand dollars). Since the DC pension plan provides the majority of the total old-care security, the above assumptions represent the scenario of adequate fund scales. It’s the current old-care situations in most countries. The pension member should allocate more proportion in the risky asset, and stand the inadequate benefit outgo to make the DC pension plan stable and sustainable.

$\mu^O = 0.05$ is the force of mortality of all the pension members and it’s the statistical mortality results of the large numbers. $\mu^S = 0.075$ is the force of mortality of the member with personal health status, and the slight increase represents the slightly worse health status of the member. As the increase of the force of mortality with respect to age is not considered in the model, the results may have some defects. It’s also the compromise between the closed-form solution and the coordination with the practice.

Furthermore, $T = 15$ is the compulsory conversion time. At that time, all the fund scale is converted into annuities and fully invested in the risk-free asset. $\alpha = 0.5$ measures the importance of (negative) utility provided by the annuities after the conversion time.
\( \beta_1 = 0.005 \) measures the importance of (negative) utility provided by the last distribution of the member without bequest motive. As the objective function is the minimization of deviations, and the bequest(last distribution) of the member at the younger age is several times of the pre-set target \( NP \), \( \beta_1 \) should be small to maintain the comparability. \( \beta = 0.25 \) measures the importance of (negative) utility provided by the bequest of the member with bequest motive. As some members are at younger ages, or have family to support, they have bequest motives. The negative linear deviation in the objective function represents the improvement of the old-care utility by the bequest received.

First, according to the above assumptions on the key parameters of the model, we simulate the optimal proportions allocated in the risky asset and optimal benefit outgos of the ELA and ELID pension plans with respect to time \( t \). In Figure 1. and Figure 2., the optimal proportion in the ELA plan is much lower than in the ELID plan, and the optimal benefit outgo in the ELA plan is much larger than in the ELID plan. As the larger bequest will improve the old-care utility enormously, the member in the ELID plan should increase the risky investment, and decrease the benefit outgo to enlarge the fund scale, as well as increase the old-care utility provided by bequest. In Figure 1., as time passes
Figure 2. Optimal benefit outgo $p_1^*$ and $p_2^*$ in ELA and ELID plans, and the optimal bequest process $y_2^*$ in ELID plan with respect to time $t$ by, the optimal proportions reveal convergent effects. These are due to the following two reasons: the higher proportion of risky investment increases the fund scale, and the larger fund scale results in the decrease in the risky investment due to the counterintuitive effects, as in Josa-Fombellida and Rincón-Zapatero[12](2004). As time approaches the compulsory conversion time, the old-care utility provide by bequest is less likely to be realized, the member in the ELID plan needs not to stand the value fluctuation of the risky asset to enlarge the fund scale. In Figure 2., we find the optimal bequest process is almost at the same level of the initial fund scale over 15 years, the higher return of the risky investment and the less benefit outgo result in the old-care utility improvement by the larger bequest. Besides that, there are jumps of optimal benefit outgo around the conversion time of both plans. These are due to the larger weight $\alpha$ on the (negative) utility provided by the annuities, and the member has to reduce the benefit outgo during the survival period to increase the annuities, and the old-care utility. The jump magnitude will be reduced as the weight variable $\alpha$ becomes smaller. Next, we study the impacts of the personal health status $\mu^S$ on the optimal control policies. In Figure 3. and Figure 4., the higher personal forces of mortality increase the optimal proportions allocated
Figure 3. The effects of $\mu^S$ on optimal proportion allocated in the risky asset $\pi_1^*$ in ELA plan with respect to time $t$.

Figure 4. The effects of $\mu^S$ on optimal proportion allocated in the risky asset $\pi_2^*$ in ELID plan with respect to time $t$.

in the risky asset in both of the ELA and ELID plans. The member with worse health status inclines to invest more in the risky asset to enlarge the fund scale, and distribute adequate benefit outgo to increase the old-care utility in the former time. As the survival probability of the member with worse health status is smaller at the older age, the decrease of old-care utility due to the inadequate benefit outgo in the
latter time has little impacts in the overall performance criterion. Besides that, in the ELID plan, the bequest motive of the member with worse health status increase the risky investment even more to have the opportunity of receiving larger bequest.

Figure 5. The effects of $\mu^s$ on optimal benefit outgo $p^*_1$ in ELA plan with respect to time $t$

Figure 6. The effects of $\mu^s$ on optimal benefit outgo $p^*_2$ in ELID plan with respect to time $t$
In Figure 5. and Figure 6., the higher personal forces of mortality increase the optimal benefit outgo in both of the ELA and ELID plans in the former time. As the survival probability decreases more rapidly of the member with worse health status, the deviations between the actual benefit outgo and the pre-set target in the former time are more important in the performance criterion. The member of worse health status inclines to distribute more benefit outgo to decrease the deviations and improve the old-care utility. As time passes by, the higher benefit outgo in the former time reduces the distribution potential in the latter time. The member of worse health status has to stand the lower benefit outgo in the latter time and the lower annuities after the conversion, as the deviations of these are less important in the performance criterion. Besides that, in the ELID plan, the bequest motive decreases the optimal benefit outgo of the members in all health statuses compared with the counterparts in the ELA plan.

![Figure 7](image)

**Figure 7.** The effects of $\beta$ on optimal proportion allocated in the risky asset $\pi^*_2$ in ELID plan with respect to time $t$

The third, we study the impacts of bequest motive degree $\beta$ on the optimal control policies. In the ELA pension plan, the survival pension member receives mortality credit and has no bequest at the time of death, so the bequest motive degree has no impact on the optimal old-care utility, as well as the optimal control polices. In Figure 7. and
Figure 8. The effects of $\beta$ on optimal benefit outgo $p_2^*$ in ELID plan with respect to time $t$

Figure 8., in the ELID plan, the higher bequest motive degree increases the proportion allocated in the risky asset, and decreases the benefit outgo in the former time. In the former time, the optimal policy of the member with higher bequest motive degree is to enlarge the fund scale, so the survival member could receive the larger bequest at the time of death. These result in old-care utility improvement. So, the member with higher bequest motive increases the risky investment and decreases the benefit outgo. The decrease of the old-care utility by the inadequate former benefit outgo is compromised by the old-care utility improvement generated by the larger bequest. As time passes by, the fund scale of the members with higher bequest motive is accumulated more rapidly. They could afford the higher benefit outgo and the lower risky investment in the latter time. Besides that, the fund scale will be converted into annuities and there is no bequest after the conversion. As time approaches the conversion time, the fund scale needs to be distributed more rapidly by the member with higher bequest motive.

The last, we study the impacts of health status $\mu^S$ and the bequest motive degree $\beta$ on the optimal old-care utilities in the ELA and ELID pension plans. Since we do not derive the explicit expressions of $R_1(t)$
and $R_2(t)$, we use the following methods to calculate the optimal objective functions at time 0 of the two pension plans. Using MCM, we simulated 10000 tracks of the optimal fund scale processes and recalculate the optimal proportions in risky asset and optimal benefit outgos at every time intervals. We use the average of the simulated accumulations of deviations to calculate the objective functions $V_0$ at time 0.

![Figure 9. The effects of $\mu_S$ on the objective functions $V_0$ of the ELA and ELID plans.](image)

Since the objective function is to minimize the square deviations of the benefit outgo minus the linear deviation of bequest, the smaller objective function represents higher optimal old-care utility. In Figure 9., the higher personal force of mortality increases the optimal old-care utility as the deviations of the benefit outgo in the latter time are less important in the performance criterion and we can increase the benefit outgo in the former time to improve the old-care utility. Besides, the old-care utility is improved more rapidly in the ELID plan, as the old-care utility improvement generated by the larger bequest is realized at the younger age of the member with worse health status. In Figure 10., the bequest motive has no impacts on the optimal old-care utility in the ELA plan. Furthermore, the higher bequest motive degree improves the old-care utility enormously in the ELID plan. The higher bequest motive directly increases the old-care utility provided by bequest and
it plays an important role in the integrated performance criterion. In Figure 11, we study the optimal choice between the ELA and ELID pension plans of the member with different health statuses and bequest motives. According to the hypotheses of the parameters, ELA plan is the optimal choice by the member with better health status and lower bequest motive. The old-care utility improvements generated by the
bequests of these members are less likely to be realized and they prefer to receive more benefit outgo during the survival period due to the mortality credit effects and no bequests. On the contrary, ELID plan is the optimal choice of the member with worse health status and higher bequest motive. The old-care utility improvements generated by the bequests of these members are more likely to be realized. The old-care utility improvement generated by the bequest exceeds the old-care utility improvement generated by the higher benefit outgo due to the mortality credit effects in the ELA plan.

5. Conclusions

In this paper, we study optimal asset allocation and benefit outgo policies during the distribution phase of the ELA and ELID pension plans with personal health status and bequest motive of the pension member. It’s the first time to study the stochastic optimal control problem with the integrated performance functions synthesizing negative old-care utilities provided by both the benefit outgo and the bequest. Using HJB equations and variational inequality methods, we derive the closed-forms of the optimal asset allocation and benefit outgo policies. Furthermore, we use MCM to investigate the impacts of the personal health status and the bequest motive on the optimal control policies with respect to time. The worse health status and higher bequest motive result in higher old-care utility of the ELID pension plan, while the ELA pension plan is the optimal choice of the member with better health status and lower bequest motive. In the former time during the distribution phase, the worse health status increases the proportion allocated in the risky asset and the benefit outgo in both pension plans. The bequest motive has positive impacts on the optimal proportion in the risky asset and negative impacts on the optimal benefit outgo in the ELID pension plan. As time passes by, since these optimal control policies change the fund scales, and the old-care utilities provided by bequests are less likely to be realized, the optimal policies are convergent and reverse in the latter time.
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REFERENCES


Highlights for review

- Study optimal asset allocation and benefit outgo polices of ELA and ELID plans
- Derive closed-forms of the optimal policies under new criterion
- The new criterion is negative old-care utility provided by benefit outgo and bequest
- Heath status and bequest motive impact on the optimal choice between ELA and ELID