OPTIMAL MANAGEMENT OF DC PENSION PLAN IN A STOCHASTIC INTEREST RATE AND STOCHASTIC VOLATILITY FRAMEWORK

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ABSTRACT. This paper investigates an optimal investment strategy of DC pension plan in a stochastic interest rate and stochastic volatility framework. We apply an affine model including the Cox-Ingersoll-Ross (CIR) model and the Vasicek mode to characterize the interest rate while the stock price is given by the Heston’s stochastic volatility (SV) model. The pension manager can invest in cash, bond and stock in the financial market. Thus, the wealth of the pension fund is influenced by the financial risks in the market and the stochastic contribution from fund participant. The goal of the fund manager is, coping with the contribution rate, to maximize the expectation of the constant relative risk aversion (CRRA) utility of the terminal value of the pension fund over a guarantee which serves as an annuity after retirement. We first transform the problem into a single investment problem, then derive explicit solution via stochastic programming method. Finally, numerical analysis is given to show the impact of financial parameters on the optimal strategies.

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1. Introduction

Pension fund management has recently become a popular significant subject for retirement system. There are two main kinds of pension
fund, the defined contribution pension plan (DC) and the defined benefit pension plan (DB). In the DC pension plan, the fund participant contributes part of his salary to the plan and the risk is taken by himself. In the DB pension plan, the benefit at retirement is fixed in advance and the contribution from the pension plan participation is important to ensure the plan’s balance. In recent years, DC pension plan have become popular in the pension fund management.

Contrast to the DB pension plan, the benefit of DC plan often comes up with a cash lump sum or a life annuity at retirement. The benefit relies heavily on fund portfolio. So an efficient investment strategy is very important during the accumulation phase of pension plan. A discrete DC pension plan model was firstly derived by Vigna and Haberman [26] (2001). They derived the optimal investment strategy for the DC fund member by stochastic dynamic programming method. Later Haberman and Vigna [16] formulated the model of DC pension fund in a market with $n$ risk assets and studied downside risk for the pension member at the same time. These works generally intend to maximize the utility of terminal value of the pension fund and only cared about financial risk during accumulation phase. However, the pension member has to face the annuity risk when the annuity is bought at retirement. Gerrard et al. [11] (2004)’s work introduced an income drawdown option to hedge the risk of annualization main at retirement for DC fund. Moreover, Cairns et all [3] (2006) considered the case when the goal of the pension manager was to maximize the annuity bought at retirement and used the salary as a numeraire. They derived explicit solutions of optimal investment strategies in the case when the salary could be efficiently hedged while numerical method was used for non-hedgeable salary.

However, because a pension fund often takes a long time, generally 20 to 40 years, it is crucial to take into account the risk of interest rate. Besides, the DC pension plan leaves the plan member facing the risks from the market and does not guarantee any minimum benefit. So the DC pension plans with minimum guarantee are attractable. Boulier et al. [2] (2001) modeled the interest rate by Orstein-Uhlenbeck process and obtained the optimal allocations to maximize the CRRA utility of terminal value over a guarantee which was an annuity after retirement.
Deelstra et al. [6](2004) extended the result to the case of stochastic contribution rate and general minimum guarantee. The market in their model is complete and martingale method was beautifully applied in their work. Giacinto et al. [12](2011) used stochastic programming method in DC pension plan when the wealth of the pension fund had to be higher than a solvency level all the time in the accumulation phase. More researches about guarantee in a DC pension plan can refer to Jesen and Sørensen [20](2001), Deelstra et al. [5](2003) and references therein. In addition, inflation risk is also important in the long run. Battocchio and Mennoncin [1](2004), Zhang et al. [29](2007), Han and Hung [17](2012) introduced inflation risk in the pension plan and derived optimal investment strategies to maximize the CRRA utility of real terminal value. Inflation indexed bonds were included to hedge inflation risk. Also, mean-variance efficient frontier and strategies with inflation risk in DC pension plan was firstly obtained in Yang et al. [27](2013).

Most of the researches concern only with stock price driven by a geometric Brownian motion. However, the stock price may have different features in the real world and real data in the market tends to support the stochastic volatility model for a stock. Gao [10](2009) studied the optimal investment strategy under the constant elasticity of variance (CEV) model both before and after retirement. Legendre transform was efficiently used in his work to find the explicit solution. Apart from the CEV model, the Heston’s SV model is also a good tool to characterize the stock price. Heston’s SV model has been adopted in various literatures to price derivatives in the market, see Seep [24](2008), Deelstra and Rayée [7](2013). In recent years, the Heston’s SV model was also widely used in the area of reinsurance. Li et al. [22](2012), Yi et al. [28](2013) and Zhao et al. [30](2013) all contributed to obtain the optimal reinsurance and investment strategy with stock price given by the Heston’s SV model.

As is stated above, stochastic interest rate and stochastic volatility have been concerned separately in the research of DC pension fund. However, it is more practical to concerns these two risks together. This
paper intends to search the optimal investment strategy for DC pension plan in a stochastic interest rate and stochastic volatility framework similar to Grzelak and Oosterlee [13](2011). The interest rate here follows an affine model including the Cox-Ingersoll-Ross (CIR) model and the Vasicek mode while the stock price is given by the Heston’s SV model. We aim at extending Deelstra et al.[5](2003) ’s work from a complete market to an incomplete market caused by stochastic volatility. The contribution rate is also stochastic, which may make our problem unsolvable. In order to get a solvable contribution rate, the contribution rate is closely related with the financial market and assumed to be driven by a diffusion process similar to the stock price in this paper. We require that the terminal value of the pension fund must exceed a given guarantee which is considered as an annuity from retirement time to death. The mortality correlated with the annuity is random and characterized by a deterministic force. The goal of the pension fund manager is to maximize the expectation of the CRRA utility of terminal value over the guarantee at retirement. To obtain an explicit solution, we firstly use the techniques in Boulier et al.[2](2001) to transform the fund process from a non-self-financing problem to a single investment problem. However, since the market is incomplete, martingale method does not work here. We will apply the stochastic programming method in this paper to obtain the optimal investment strategy, and then compare it with the case of a complete market.

This paper is organized as follows: In Section 2, we characterize the model with cash, bond and stock by a stochastic interest rate and stochastic volatility framework. The contribution rate and guarantee are also given. In Section 3, we transform the original problem to a self financing problem, and derive the explicit solution by stochastic programming method. Numerical analysis is given in Section 4. Section 5 concludes this paper.

2. Formulations of Model and Optimization Problem

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})\) be a filtered complete probability space and \(\mathcal{F}_t\) represents the information available before time \(t\) in the market. The pension fund starts at time 0 and the retirement time is \(T\). We
assume that all the processes introduced below are well defined and adapted to \( \{ \mathcal{F}_t \}_{t \geq 0} \).

2.1. The Financial Market. The financial market consists of a cash, a bond and a stock. We assume that the instantaneous interest rate is given by the following stochastic differential equation:

\[
    dr(t) = (a - br(t))dt - \sqrt{k_1 r(t)} + k_2 dW_r(t), \quad r(0) = r_0,
\]

where \( a, b, k_1 \) and \( k_2 \) are positive constants, \( W_r(t) \) is a standard Brownian motion on \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \). This interest rate model has already been used in Deelstra et al. [5](2003) and includes the CIR model and Vasicek model as special cases (corresponding to \( k_2 = 0 \) and \( k_1 = 0 \), respectively). In the case \( k_2 = 0 \), the condition: \( 2a > b \) is required to ensure \( r(t) > 0 \).

The risk free asset (i.e. cash) \( S_0(t) \) satisfies the following equation

\[
    \frac{dS_0(t)}{S_0(t)} = r(t)dt, \quad S(0) = S_0.
\]

Besides, zero-coupon bonds are also available in the market. Zero-coupon bonds are assets which deliver a payment of $1 at maturity. Following Deelstra et al.[5](2003), the price of \( B(t, s) \) at time \( t \) with maturity \( s \) is given by the following backward stochastic differential equation:

\[
    \begin{cases}
    d\frac{B(t,s)}{B(t,s)} = r(t)dt + h(t)(s-t)\sqrt{k_1 r(t)} + k_2(\lambda_r\sqrt{k_1 r(t)} + k_2 dt + dW_r(t)), \\
    B(s, s) = 1,
    \end{cases}
\]

where \( h(t) = \frac{2(\exp(mt) - 1)}{m-(b-k_1\lambda_r)\exp(mt)} \), \( m = \sqrt{(b-k_1\lambda_r)^2 + 2k_1} \) and \( \lambda_r \sqrt{k_1 r(t)} + k_2 \) is the market price of risk of \( W_r(t) \).

Moreover, the explicit form of \( B(t, s) \) is given by

\[
    B(t, s) = \exp(-h(s-t)r(t) + h_1(s-t)),
\]

and

\[
    h_1(t) = \frac{k_2 t - k_2 h(t) + (a + \frac{k_2 b}{k_1})}{k_1} \log\left\{ \frac{2(m-b-k_1\lambda_r)^{2m}}{m-(b-k_1\lambda_r) + e^{mt}(m + b - \lambda_r k_1)} \right\}.
\]

The maturity of the bond \( B(t, T) \) is \( T - t \) which varies continuously over time. Since there does not exist zero-coupon bonds with any maturity \( t > 0 \) in the market, it is unrealistic to invest in \( B(t, T) \). So we introduce a introduce a rolling bond with a constant maturity \( K \) similar to Boulier et al.[2](2001). We can invest in the rolling bond to hedge the risk of interest rate.
The rolling bond $B_K(t)$ follows the following stochastic differential equation:

$$
\frac{dB_K(t)}{B_K(t)} = r(t)dt + h(K)\sqrt{k_1 r(t)} + k_2(\lambda_r \sqrt{k_1 r(t)} + k_2 dt + dW_r(t)).
$$

(2.5)

In fact, since the rolling bond is only correlated with the interest rate, it can be replicated by the zero coupon bond and cash in the market. The relation between the rolling bond and zero-coupon bonds is as follows:

$$
\frac{dB(t, s)}{B(t, s)} = \left(1 - \frac{h(s - t)}{h(K)}\right) \frac{dS_0(t)}{S_0(t)} + \frac{h(s - t)}{h(K)} \frac{dB_K(t)}{B_K(t)}.
$$

(2.6)

The third asset in the market is a stock which is given by the Heston’s SV model and also relates to the risk of interest rate as follows:

$$
\begin{align*}
\frac{dS(t)}{S(t)} &= r(t)dt + \sigma_S \sqrt{k_1 r(t)} + k_2(\lambda_r \sqrt{k_1 r(t)} + k_2 dt + dW_r(t)) \\
&\quad + \nu L(t)dt + \sqrt{L(t)}dW_S(t), \\
\frac{dL(t)}{L(t)} &= \alpha(\delta - L(t))dt + \sigma_L \sqrt{L(t)}dW_L(t),
\end{align*}
$$

(2.7)

where $\alpha, \delta, \sigma_L, \nu$ are positive constants. Also, $W_S(t)$ and $W_L(t)$ are standard Brownian motions with $\text{Cov}(W_S(t), W_L(t)) = \rho_{SL} t$. We assume that $W_r(t)$ is independent of $W_S(t)$ and $W_L(t)$. We can see that in the above model the market price of risk of $W_S(t)$ is $\nu \sqrt{L(t)}$, but the market price of risk of $W_L(t)$ can be arbitrage chosen. So in the Heston’s SV model we will not get an unique risk neutral measure and the market is thus incomplete. Moreover, we need a condition: $2\alpha \delta > \sigma^2_L$ ensuring $L(t) > 0$.

2.2. Defined contribution pension fund management.

2.2.1. Contribution process. In the defined contribution pension management, the contributor contributes part of his salary to the pension fund before retirement. The contribution is very important in a DC pension fund. We assume that the contribution rate in our model is a stochastic process driven by

$$
\frac{dC(t)}{C(t)} = \mu dt + \sigma_C \sqrt{k_1 r(t)} + k_2(\sqrt{k_1 r(t)} + k_2 dt + dW_r(t)) + \sigma_C(\nu L(t)dt + \sqrt{L(t)}dW_S(t)).
$$

(2.7)

The contribution rate here is similar to the stock index in the market. We will see that this kind of contribution rate can be replicated by the assets although the market is incomplete.
2.2.2. Annual guarantee. In the DC pension plan, we require that the pension fund must exceed a minimum guarantee at retirement $T$. The minimum return guarantee was considered in Deelstra et al. [5](2003). However, in this paper we consider the annual guarantee introduced by Boulier et al.[2](2001). The death time in their model is constant and we extends the form of their guarantee to the case with a random time of death. The fund guarantees at least $g(t), t \in [T, T']$ annuity at time $t$, and $T'$ is the date of death and random. The annual guarantee in our model thus is as follows:

$$G(T) = \int_T^{T'} g(s) B(T, s) s^{-T} p_T dT, \quad (2.8)$$

where $g(s) = g(T) e^{g(s-T)}$ and $w$ is the largest survival age, $s^{-T} p_T$ is the probability that the contributor will survive to $s$ given that she/ he is still alive at $T$. $g$ can be seen as inflation which caused the increasing cost of life. $s^{-T} p_T$ can be calculated by a given mortality $\lambda(t)$ and satisfies $s^{-T} p_T = e^{-\int_T^{s} \lambda(u)du}$. To simplify the model, we consider here a mortality with deterministic force. We adopt the Abraham De Moivre model (cf. Kohler and Kohler (2000) [21]), in which $\lambda(t) = w - t$. So

$$s^{-T} p_T = e^{-\int_T^{s} \lambda(u)du} = e^{-\int_T^{s} \frac{w}{w-t}du} = \frac{w-s}{w-T}.$$ 

2.3. The optimization problem. During the accumulation phase, the pension fund manager will receive a continuously income from the contributor and can invest in the assets in the market. Assume that there are no transactions costs or taxes in the market and short buying is allowed, the wealth of the pension fund is as follows:

$$\begin{cases}
\displaystyle{dX(t) = u_0(t)X(t)\frac{dS(t)}{S(t)} + u_B(t)X(t)\frac{dB(t)}{B(t)} + u_S(t)X(t)\frac{dS(t)}{S(t)} + C(t)dt,} \\
X(0) = X_0,
\end{cases} \quad (2.9)$$

where $u_0(t), u_B(t)$ and $u_S(t)$ are the proportions of money invested in the cash, rolling bond and the stock, respectively.

Substituting (2.2), (2.5) and (2.6) into the last equation, we can get

$$\begin{cases}
\displaystyle{dX(t) = r(t)X(t)dt + \nu L(t)dW(t) + \sqrt{\nu^2 L(t) + \sigma^2(t)}dW_S(t)} \\
\quad + \{u_B(t)[\nu L(t) + \sqrt{\nu^2 L(t) + \sigma^2(t)}dW(t)] + \}u_S(t)\nu L(t)dt + \sqrt{\nu^2 L(t) + \sigma^2(t)}dW_S(t) + C(t)dt,} \\
X(0) = X_0.
\end{cases} \quad (2.10)$$
Denote \( u(t) = (u_B(t), u_S(t)) \). And \( u(t) \) is called an admissible strategy if it satisfies the following conditions:

(i) \( u(t) \) is progressively measurable w.r.t. \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\).

(ii) \( \mathbb{E}\left\{ \int_0^T X(t)^2 \left[ (u_B(t)h(K) + u_S(t)\sigma_S)^2(k_1r(t) + k_2) + u_S(t)^2L(t) \right] dt \right\} < +\infty \).

(iii) Equation (2.10) has an unique strong solution for the initial data \((t_0, r_0, L(0), X(0)) \in [0, T] \times (0, +\infty)^3\).

We denote the set of all admissible strategies of \( u(t) \) by \( \Pi \). We assume here that the pension fund guarantees at least deterministic annuity after retirement. The goal of the fund manager is to maximize the expectation of the utility of terminal value \( X(t) \) over the annual guarantee. Our optimization problem can thus be as follows:

\[
\begin{align*}
\max_{u(t) \in \Pi} E\{U(X(T) - G(T))\} \\
\text{subject to: } X(T) \geq G(T).
\end{align*}
\] (2.11)

where \( U(x) \) is the utility function which is strictly concave.

Utility function describes the preference over wealth. Many different utility functions are formulated and studied in previous literature. There are three well known main utility functions in portfolio theory: CRRA utility function, CARA utility function and the mean-variance optimization. Based on existing works, the separation of wealth and stochastic indexes in the market for the optimal value holds well for CRRA utility maximization. So the problem of CRRA utility maximization in a stochastic framework is often solvable. However, the optimal function in a CARA utility maximization usually indicates a complex relation between the wealth and interest rate. The wealth and interest rate is not separated in the optimal value. Thus, when we consider a stochastic interest rate, the explicit solution for CARA utility maximization is not easy to be derived, see Battochio and Menoncin [1] (2004). The mean-variance criterion was firstly studied by Markowitz [23] and attracted many attention. The mean-variance optimization problem can be transformed to a quadratic minimization problem by the procedure in Vigna [25](2014). This quadratic function is similar to the CRRA utility and the separation of wealth and stochastic
indexes also holds for the optimal function. Therefore, the mean-variance problem is often solvable in a stochastic environment with stochastic interest rate, which we will consider in our further works. In this paper, we consider the CRRA utility function, for which we can derive the explicit form of the solution, as follows:

\[ U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \text{ and } \gamma \neq 1, \]

where \( \gamma \) is the relative risk aversion.

3. Solution of the problem

The optimization problem (2.11) is not a single investment problem and so it is hard to solve. It involves a continuously wealth inflow caused by contribution, and also considers the minimum guarantee. In this section, we firstly introduce auxiliary processes and change the problem (2.11) into a single investment problem.

3.1. Construction of an Auxiliary Problem. Inspired by Han and Hung[17](2012), we replicate the continuously inflow \( C(t)dt \) by the following steps. Firstly, we have to price an asset \( D(t,s) \), \( s \geq t \) with final payment \( C(s) \) at maturity \( s \). Using this particular form of \( C(t) \), we see that \( D(t,s) \) is not correlated with the risk from \( W_L(t) \), and thus it can be replicated by the assets in the market. \( D(t,s) \) satisfies the following BSDE:

\[
\begin{cases}
D_t + D_C C \mu + D_r (a-br) + \frac{1}{2} D_r (k_1 r + k_2) - D_C C \sigma C_1 (k_1 r + k_2) - r D = -\lambda r D (k_1 r + k_2) + \frac{1}{2} D_C C [\sigma C_1^2 (k_1 r + k_2) + \sigma C_2^2], \\
D(s, s) = C(s).
\end{cases}
\]

(3.1)

So the explicit form of \( D(t,s) \) is the following:

\[ D(t, s) = C(t) \exp \{ f_1(s-t) + f_2(s-t)r(t) \}, \]

(3.2)

where

\[ f_2(t) = \frac{-2e^{\sqrt{\Delta_f}} + 2}{(b - \lambda r k_1 + \sigma C_1 k_1 + \sqrt{\Delta_f})e^{\sqrt{\Delta_f}} + \sqrt{\Delta_f} - (b - \lambda r k_1 + \sigma C_1 k_1)}, \]

\[ f_1(t) = \int_0^t [a f_2(s) + \frac{1}{2} k_2 f_2(s)^2 + (\lambda_r - \sigma C_1) k_2 f_2(s) + \mu] ds, \]

\[ \Delta_f = (b - \lambda r k_1 + \sigma C_1 k_1)^2 + 2k_1. \]
Moreover, \( D(t, s) \) also satisfies the following stochastic differential equation:
\[
\begin{align*}
\frac{dD(t, s)}{D(t, s)} &= r(t)dt + \sqrt{k_1 r(t)} + k_2 \lambda \sqrt{k_1 r(t)} + k_3 dt \\
&+ dW_r(t) + \sigma C_1 \nu L(t) dt + \sqrt{L(t)} dW_S(t), \quad s \geq t, \\
D(s, s) &= C(s).
\end{align*}
\] (3.3)

\( D(t, s) \) defined above represents the current value of an asset with final payment of \( C(s) \) at maturity \( s \). By integrating \( D(t, s) \) over \([t, T]\), we can get the present value of the continuously contribution \( C(s) \) from \( t \) to \( T \). Define it by \( F(t, T) = \int_t^T D(t, s) ds \). Then
\[
\begin{align*}
\frac{dF(t, T)}{F(t, T)} &= -C(t)dt + r(t)F(t, T)dt + \int_t^T D(t, s)(\sigma C_1 - f_2(s - t)) ds \\
&\times \sqrt{k_1 r(t)} + k_2 \lambda \sqrt{k_1 r(t)} + k_3 dt + dW_r(t) \\
&+ \sigma C_2 F(t, T)\nu L(t) dt + \sqrt{L(t)} dW_S(t).
\end{align*}
\] (3.4)

**Proposition 3.1.** The \( F(t, T) \) along with the continuously contribution can be replicated by the cash, bond and the stock in the market through the following equation:
\[
\frac{dF(t, T) + C(t) dt}{F(t, T)} = u_0^F(t) \frac{dS_0(t)}{S_0(t)} + u_0^F(t) \frac{dB_K(t)}{B_K(t)} + u_0^F(t) \frac{dS(t)}{S(t)},
\] (3.5)

where
\[
\begin{align*}
u_0^F(t) &= 1 - u_0^F(t), \\
u_0^F(t) &= \int_t^T D(t, s)(\sigma C_1 - f_2(s - t)) ds - \sigma C_2 F(t, T) \frac{dS(t)}{S(t)}, \\
u_0^F(t) &= \sigma C_2.
\end{align*}
\]

**Proof.** Comparing the coefficients in (3.4) and (2.5), (2.6), the proof easily follows. \( \square \)

The minimum guarantee is only related with the risk of interest rate. So, the present value of the minimum guarantee \( G(T) \) at time \( t \), can be simply obtained by conditioning on \( \mathcal{F}_t \) w.r.t the probability measure determined by the market price of risk of interest rate. Therefore, the value of \( G(T) \) at \( t \leq T \), is given by
\[
G(t) = \int_T^w g(s) B(t, s) s - T p_T ds, t \leq T. \quad (3.6)
\]
$G(t)$ satisfies the following stochastic differential equation:
\[
dG(t) = r(t)G(t)dt + \int_T^w s-T_{rg}(s)B(t, s)h(s - t)ds \sqrt{k_1 r(t) + k_2} \\
\times (\lambda_r \sqrt{k_1 r(t) + k_2} dt + dW_r(t)).
\]

The minimum guarantee $G(t)$ is only correlated with the risk of interest rate. Because there is an unique market price of risk relative to the risk of interest rate in our model, $G(t)$ can be replicated by the following cash $S_0(t)$ and bond $B_K(t)$ in the market:

\[
\begin{aligned}
  \frac{dG(t)}{G(t)} &= u^G_B(t) \frac{dS_0(t)}{S_0(t)} + u^G_B(t) \frac{dB_K(t)}{B_K(t)}, \\
  u^G_0(t) &= 1 - u^G_B(t), \\
  u^G_B(t) &= \int_{s=0}^{T} T_{rg}(s)B(t, s)h(s - t)ds.
\end{aligned}
\]

Let $Y(t) = X(t) + F(t, T) - G(t)$. We can change the former optimization problem into a single investment problem. Differentiating $Y(t)$, we get the dynamics of $Y(t)$ as follows:

\[
\begin{aligned}
dY(t) &= r(t)dt + [u^Y_B(t)h(K) + u^K_S(t)\sigma_S] \sqrt{k_1 r(t) + k_2} \lambda_r \sqrt{k_1 r(t) + k_2} dt \\
&\quad + dW_r(t) + u^Y_S(t)[\nu L(t)dt + \sqrt{L(t)}dW_S(t)],
\end{aligned}
\]

where

\[
\begin{aligned}
u^Y_B(t) &= \frac{u_B^Y(t)X(t) + u^K_B(t)F(t, T) - u^G_B(t)G(t)}{Y(t)}, \\
u^Y_S(t) &= \frac{u_S^Y(t)X(t) + u^K_S(t)F(t, T)}{Y(t)}.
\end{aligned}
\]

The condition required in the initial problem $X(T) \geq G(T)$ can be transformed into $Y(T) \geq 0$ since $Y(T) = X(T) - G(T)$ at retirement $T$. Because the auxiliary process $Y(t)$ is self-financing, $Y(0) = X(0) - F(0, T) - G(0) \geq 0$ is necessary and sufficient to ensure that admissible investment strategy exists such that $Y(T) \geq 0$. So we require that $X(0) - F(0, T) - G(0) \geq 0$ in this paper. Denote $u^Y(t) = (u^Y_B(t), u^Y_S(t))$, we call $u^Y(t)$ an admissible strategy if the initial investment strategy $u(t)$ related to $u^Y(t)$ given by (3.8) is in $\Pi$. We abuse the terminology here by saying that $u^Y(t) \in \Pi$ if it is admissible. So we can change the original problem (2.11) into a single investment problem as follows:

\[
\begin{aligned}
\max_{u^Y(t) \in \Pi} & E[U(Y(T))] \\
\text{subject to : } & Y(T) \geq 0.
\end{aligned}
\]
3.2. Solution of the auxiliary problem. Since the market is incomplete, martingale method can no longer work for the problem (3.9) here. The problem can be solved by dynamic programming method. Denote $V(t, y, r, l) = \mathbb{E}[U(Y(T)) | Y(t) = y, r(t) = r, L(t) = l]$. Then we have the following.

**Theorem 3.1.** The associated HJB equation for the above problem (3.9) is

$$
\sup_{u^Y(t) \in \Pi} \left\{ V_t + V_y y + \left( u^Y_B h(K) + u^Y_S \sigma_S \right) \lambda_r (k_1 r + k_2) + u^Y_r \nu l + V_r (a - br) \right. \\
+ \alpha V_\delta (\delta - l) + \frac{1}{2} V_{yy} y^2 (u^Y_B h(K) + u^Y_S \sigma_S)^2 (k_1 r + k_2) + \frac{1}{2} V_{yy} y^2 u^Y_S^2 l \\
+ \frac{1}{2} V_{rr} (k_1 r + k_2) + \frac{1}{2} V_{r} \sigma_L^2 l - V_{yy} y (u^Y_B h(K) + u^Y_S \sigma_S) (k_1 r + k_2) \\
+ V_{yy} \nu \sigma_L \rho_{SL} \right\} = 0.
$$

(3.10)

**Proof.** The proof is very simple, see Fleming and Soner [8](1993), Guan and Liang [18](2014), Guan and Liang [19](2014), He and Liang [18](2008) and [19](2009), so we omit it here. □

By using the first order condition of the solutions to HJB equation, we can express $u^Y_B(t)$ and $u^Y_S(t)$ in the form of $V(t, y, r, l)$ as follows:

$$
u^Y_B(t) = \frac{V_{yy} \sigma_L \rho_{SL}}{V_{yy} \sigma_L} + \frac{V_y \nu \sigma_S}{V_{yy} h(K)} + \frac{V_{gr} y}{V_{yy} h(K)} - \frac{V_r \lambda_r}{V_{yy} h(K)}.$$

(3.11)

Substituting the above two equations (3.11) into the HJB equation (3.10), we can finally get the explicit forms of $V(t, y, r, l)$ and the optimal investment strategies. They are given in the following theorem.

**Theorem 3.2.** Suppose that

$$\gamma > \max\{ \frac{2 \sigma_L \rho_{SL} \nu \alpha + \sigma_L^2 \nu^2}{\alpha^2 + 2 \sigma_L \rho_{SL} \nu \alpha + \sigma_L^2 \nu^2}, \frac{-2 \lambda_r k_1 b + k_2^2 \lambda_r^2 + 2 k_1}{b^2 - 2 \lambda_r k_1 b + k_2^2 \lambda_r^2 + 2 k_1}, 0 \}.$$
Then the optimal utility and optimal investment strategies satisfy the following equations:

\[
\begin{align*}
V(t, x, r, l) & = \frac{1}{1 - \gamma} \exp[A_1(t) + A_2(t)r + A_3(t)], \\
_uB^*(t) & = -\frac{\sigma_L \sigma_S \rho SL A_3(t)}{\gamma k h (K)} - \frac{A_2(t)}{\gamma h (K)} + \lambda_r, \\
uS^*(t) & = \frac{\sigma S^2}{\gamma} + \frac{\nu}{\gamma} \frac{A_3(t)}{s_1}, \\
\end{align*}
\]

where

\[
\begin{align*}
A_1(t) & = \int_0^T \left\{ a A_2(s) + \alpha \delta A_3(s) + \frac{1}{2} k_2 A_2^2(s) + \frac{1}{2} \nu^2 \right\} ds, \\
A_2(t) & = \begin{cases} \\
\frac{v_1 v_2}{e^{\frac{\Delta A_3(t)}{T}} - v_1 v_2}, & k_1 \neq 0, \\
\frac{1 + \frac{\lambda}{\gamma} e^{\frac{\Delta A_3(t)}{T}}}{1 - e^{\frac{\Delta A_3(t)}{T}}}, & k_1 = 0, \\
\end{cases} \\
A_3(t) & = \frac{m_1 m_2 e^{\frac{\Delta A_3(t)}{T}} - m_1 m_2}{m_1 e^{\frac{\Delta A_3(t)}{T}} - m_2} \\
\end{align*}
\]

and

\[
\begin{align*}
\Delta A_2 & = b^2 + \frac{2(1 - \gamma) \lambda_r k_1 b}{k_1} - \frac{(1 - \gamma) k_1^2}{\gamma k_1 (1 - \gamma)}, \\
\Delta A_3 & = a^2 - \frac{2(1 - \gamma)}{\gamma} \sigma_L \sigma_S \rho SL \nu \alpha - \frac{1 - \gamma}{\gamma} \nu^2 \gamma^2, \\
v_{1,2} & = \frac{\gamma b (1 - \gamma) \lambda_r k_1 \pm \sqrt{\Delta A_2}}{k_1}, \\
m_{1,2} & = \frac{\gamma a (1 - \gamma) \sigma_L \sigma_S \rho SL \nu \pm \sqrt{\Delta A_3}}{\gamma \nu \sigma_L^2 + (1 - \gamma) \sigma_S^2 \rho SL^2}.
\end{align*}
\]

Proof. See Appendix. \(\square\)

From the above formula of \(uB^*(t)\) and \(uS^*(t)\), the term of \(B_2\) and \(S_2\) are the same as the optimal investment proportions in the complete market of Deelstra et al [5](2003). These two terms aim to attain the optimal CRRA utility in a complete market. But because our market is correlated with the risk of \(W(t)\), it is not complete. Additional proportions \(B_1\) and \(S_1\) are needed to hedge this risk in our model and these two terms are closely correlated with \(L(t)\). We can also obverse from the formula that in the case \(\rho SL = 0\), that is, \(W(t)\) is independent of \(W(t)\), the proportions \(B_1\) and \(S_1\) are all equal to zero and the optimal proportions are the same as in Deelstra et al [5](2003). In this case, the parameter \(L(t)\) only causes a loss of our optimal utility. This is very natural choice: if we can not control part of risk driven by \(W(t)\) by investing in the stock, the best way for us is to adopt the same strategies as in the complete market.
3.3. Solution of the original problem. Based on the connections between \((u_Y^B(t), u_Y^S(t))\) and \((u_B(t), u_S(t))\), we can get the optimal proportions for the original problem, which is given by the following equation:

\[
\begin{align*}
    u_0^*(t) &= 1 - u_B^*(t) - u_S^*(t), \\
    u_B^*(t) &= \frac{Y^*(t)}{X^*(t)} u_B^*(t) - \frac{F(t, T)}{X^*(t)} u_B^*(t) + \frac{G(t)}{X^*(t)} u_B^*(t), \\
    u_S^*(t) &= \frac{Y^*(t)}{X^*(t)} u_S^*(t) - \frac{F(t, T)}{X^*(t)} u_S^*(t).
\end{align*}
\]  

(3.15)

In order to get the optimal utility in the case with contribution and guarantee, we need to adjust our strategies in a single investment, which is given by \(Y_B\) and \(Y_S\). The proportions of \(Y_B\) and \(Y_S\) invested in the assets maximize a single investment problem with initial wealth \(y_0\). At the same time, the contribution serves as a continuously income of the wealth. The strategy adopted here to hedge the contribution is to borrow the present value of the continuously contribution within \([0, T]\) at time 0 and regulate our investment (which is given by the term \(F_B\) and \(F_S\)) at time \(T\) to eliminate the contribution precisely. The guarantee acts as a loss of wealth at retirement. The term \(G_B\) in the formula is to put aside the initial value of the guarantee first and achieve the guarantee at retirement by investing in the bond. However, with the wealth borrowing and putting aside, a loss or benefit is added at the beginning.

4. Sensitivity Analysis

In this section, we use the Monte Carlo Methods (MCM) to give a sensitivity analysis to show the relationship between the optimal investment strategies and the parameters in our model. To compare our model with the case of a complete market, most of the parameters have been taken in Deelstra et all [5] (2003) and we also use the CIR model to characterize interest rate \((k_2 = 0)\). Unless otherwise stated, the value of the parameters are as follows: \(a = 0.018712, X_0 = 1, b = 0.2339, T = 40, r_0 = 0.05, l_0 = 0.02, C_0 = 0.15, k_1 = 0.00729316, k_2 = 0, \lambda_r = 1, K = 20, \nu = 1.5, \alpha = 0.03, \delta = 0.04, \sigma_L = 0.03, \sigma_S = 0.02, \rho_{SL} = 0.5, g = 0, w = 120, gT = 5, \mu = 0.02, \gamma = 2\).
Figure 1. optimal mean proportions of wealth

Figure 2. optimal mean proportions of wealth in the case when the risk aversion $\gamma = 4$.

Figure 1 shows the motion of optimal invest proportions from initial time to retirement. The parameters are given by the preceding paragraph. As is shown in the figure, a short position in the cash is taken firstly, which can be explained by a put option of the guarantee. However, the proportion in the cash grows fast at beginning and stably increases to about 18% at retirement. The money invested in the stock almost stays below 40% during the accumulation phase.
Proportions invested in the stock and cash increase while proportion in the bond decreases from about 110% to 40%.

Besides, the change of the mean proportions in the assets mainly take place around the initial time and the retirement. They stay steady in the middle of the time horizon. The case when \( \gamma = 4 \) is illustrated by Figure 2. \( \gamma \) is a measure of risk aversion in our model. Higher risk aversion requires us to hedge more risks in the market. The risk from the interest rate can be hedged by bond and the risk from the stock is non-hedgeable and disagreeable. So, a lower investment proportion of in the stock is adopted relative to the former case while a higher proportion is adopted in the bond. Besides, proportion in the cash is also a little higher than the preceding case. Proportion in the stock stays steady at about 20% and it is lower than the proportion in the bond all the time. The features of the movement of the proportions are almost the same as the former case.

![Figure 3](image)

**Figure 3.** Optimal mean proportions of wealth in the case when \( gT = 3 \) in the annual guarantee.

Figure 3 and Figure 4 shows the proportions of the wealth when \( gT = 3 \) and \( gT = 0 \), respectively. In the first case, proportion in the stock is around 60% during the accumulation phase. It decreases slowly first and then increases to be higher than the proportion in the bond. The proportion in the bond stays stably at about 80% before
30 years and decreases fast to only 20% at retirement. The second case depicts a DC pension plan with risks only from the market and contribution. The pension manager does not need to guarantee a least wealth at retirement. No guarantee means that the pension manager can take more risks in the market. Thus, a large proportion of 150% in the stock is adopted at beginning and proportion in the bond is only about 50%. Contrast to the first case, proportion in the stock maintains higher than proportion in the bond.

We are also cared about the influence of contribution rate on the optimal investment strategy. The management of pension fund with contribution rate $C(t) = 0.3e^{0.02t}$ is showed in Figure 5. This contribution rate is higher than the parameters adopted in Figure 1. The proportions in the bond and stock change around 60%. Since the contribution rate here is higher than the original case, it is not difficult to purchase a wealth higher than the guarantee at retirement. So, the pension manager can invest more into the stock. As is shown in this figure, proportion in the stock is always above 50% and decreases from 100% to 60%. However, we only need to invest 75% of our wealth into the bond before 30 years.

**Figure 4.** Optimal mean proportions of wealth in the case when $gT = 0$ in the annual guarantee.
Figure 5. Optimal mean proportions of wealth in the case when the contribution rate is given by $C(t) = 0.3e^{0.02t}$.

Figure 6. Optimal mean proportions of wealth when $\rho_{SL} = -1$.

However, in some situation, the risk of the stock index is negatively correlated with the risk of the volatility. Figure 6 shows the optimal investment proportions when $\rho_{SL} = -1$. In this case, the optimal proportion in the stock starts with 60% and decrease to maintain a level of 40% after about 10 years. Comparing it with the original Figure 1, we can see that the optimal proportion in the stock is negative related with the correlation coefficient $\rho_{SL}$. Inversely, the impact of $\rho_{SL}$ on
the investment of bond is not apparent. The motion of proportion in bond is almost the same as in Figure 1.

5. Conclusion

In this paper, we consider the optimal investment strategy for DC pension plans in a stochastic interest rate and stochastic volatility framework. The contribution rate in our model is also stochastic while hedgeable. The pension manager has to deal with the risk of interest, volatility and contribution. Besides, a minimum guarantee is considered to protect the pension fund at retirement. The guarantee is an annuity at retirement time and ensures a deterministic benefit from retirement to death. The interest rate follows an affine process including the CIR and Vasicek model as special cases. To hedge the risk of interest rate, a zero coupon bond is included in the market. Moreover, the market also includes cash and stock. The volatility of the stock is stochastic and follows the Heston’s SV model. The goal of the pension manager is to maximize the expectation of CRRA utility of the terminal wealth with the guarantee constraints.

However, the optimization problem is not self-financing caused by the continuously contribution and a minimum guarantee at retirement time. The technique to transform it into a self-financing problem is very traditional. Since the contribution is hedgeable, the effect of it on the wealth can be properly eliminated by adding the initial value of the continuous contribution to the initial wealth and adjusting the money invested in cash, bond and stock continuously. We do not consider the non-hedgeable contribution here since only numerical solution can be gotten in this case. The guarantee added on the wealth at retirement time can also be hedged in this way. Since the stochastic volatility can not be hedged, the market is incomplete. After transforming the problem into a self financing problem, we can get an HJB equation by stochastic programming method. The explicit solution of optimal investment strategy can be solved from the HJB equation. The relation of our problem with the case of a complete can be also observed.
6. Appendix

6.1. The proof of Theorem 3.2. Substituting (3.11) into the HJB equation (3.10), we can get

\[ 0 = V_t + V_y y + V_r (a - br) + \alpha V_t (\delta - l) + \frac{1}{2} V_{rr} (k_1 r + k_2) + \frac{1}{2} V_{yy} \sigma_l^2 \]

\[ - \frac{1}{2 V_{yy}} (V_{yy} - V_y y \lambda_r)^2 (k_1 r + k_2) - \frac{1}{V_{yy}} (V_{yl} \sigma_L \rho_{SL} + V_y) V^2. \]  

(6.1)

Similar to that of Chang and Rong [4] (2013), we may guess that the solution of \( V(t, y, r, l) \) is of the form as follows:

\[ V(t, y, r, l) = \frac{y^{1-\gamma}}{1-\gamma} g(t, r, l). \]

Differentiating \( V \), we can get

\[ V_t = \frac{g_t}{g} V, \quad V_y = \frac{(1 - \gamma)}{y} V, \quad V_{yy} = -\gamma (1 - \gamma) y^{-2} V, \]

\[ V_r = \frac{g_r}{g} V, \quad V_{rr} = \frac{g_{rr}}{g} V, \quad V_{yr} = \frac{(1 - \gamma) g_r}{g} V, \]  

\[ V_l = \frac{g_l}{g} V, \quad V_{yl} = \frac{(1 - \gamma) g_l}{g} V, \quad V_{ll} = \frac{g_{ll}}{g} V. \]  

(6.2)

Using these expressions (6.2), the equation (6.1) can be changed into the following,

\[ 0 = g_t r (1 - \gamma) g + (a - b r) g_t + \alpha (\delta - l) g_t + k_1 r + k_2 + \frac{1}{2} \sigma_l^2 \sigma_l^2 g_{ll} \]

\[ - \frac{1}{2 \gamma} g_t \frac{g_r}{g} - \lambda_r)^2 (k_1 r + k_2) - \frac{1}{2 \gamma} g (\sigma_L \rho_{SL} \frac{g_l}{g} + \nu)^2 l. \]

The above equation is a partial differential equation of \( g(t, r, l) \). We guess here \( g(t, r, l) = \exp \left[ A_1(t) + A_2(t) r + A_3(t) l \right] \). Substituting into the above equation and arranging by order of \( r \) and \( l \), we have

\[ 0 = A_1'(t) + \alpha A_1(t) + \alpha \delta A_3(t) + \frac{1}{2} k_2 A_2(t)^2 + \frac{1}{2 \gamma} [A_2(t) - \lambda_r]^2 k_2 \]

\[ + r \left[ A_2'(t) - b A_2(t) + \frac{k_1}{2} A_2(t)^2 + \frac{1}{2 \gamma} [A_2(t) - \lambda_r]^2 k_1 + (1 - \gamma) \right] \]

\[ + \left[ A_3'(t) - \alpha A_3(t) + \frac{1}{2} \sigma_l^2 A_3(t)^2 + \frac{1}{2 \gamma} \sigma_L \rho_{SL} A_3(t) + \nu \right]^2 \} \]

\[ 0 = A_1(T) = A_2(T) = A_3(T). \]  

(6.3)

The above equation is equivalent to the following three equations,

\[ 0 = A_1'(t) + \alpha A_1(t) + \alpha \delta A_3(t) + \frac{1}{2} k_2 A_2(t)^2 + \frac{1}{2 \gamma} [A_2(t) - \lambda_r]^2 k_2, \]

\[ 0 = A_1(T) = 0, \]  

(6.4)

\[ 0 = A_2'(t) - b A_2(t) + \frac{k_1}{2} A_2(t)^2 + \frac{1}{2 \gamma} [A_2(t) - \lambda_r]^2 k_1 + (1 - \gamma), \]

\[ 0 = A_2(T), \]  

(6.5)
To solve equation (6.5), rewrite it as follows:

\[ A'_2(t) = -\frac{k_1}{2\gamma}A_2(t)^2 + \left[b + \frac{1 - \gamma}{\gamma}\lambda_r k_1\right]A_2(t) - \left(1 - \gamma\right) - \frac{1 - \gamma}{2\gamma}\lambda^2 k_1. \]  

(6.7)

In the case where \( k_1 \neq 0 \), let \( \Delta_{A_2} \) be the determinant of the following equation:

\[ \frac{k_1}{2\gamma}A_2(t)^2 + \left[b + \frac{1 - \gamma}{\gamma}\lambda_r k_1\right]A_2(t) - \left(1 - \gamma\right) - \frac{1 - \gamma}{2\gamma}\lambda^2 k_1 = 0. \]  

(6.8)

By simple calculation,

\[ \Delta_{A_2} = b^2 + \frac{2(1 - \gamma)\lambda_r k_1 b}{\gamma} - \frac{(1 - \gamma)k_1^2 \lambda^2}{\gamma} - \frac{2k_1(1 - \gamma)}{\gamma}. \]

Under the condition that \( \Delta_{A_2} > 0 \), i.e., \( \gamma > \frac{-2\lambda_r k_1 b^2 + \lambda^2 k_1^2 + 2k_1}{b^2 - 2\lambda_r k_1 b \lambda^2 + 2k_1} \), we assume that \( v_{1,2} \) are the roots of equation (6.8), so

\[ v_{1,2} = \frac{\gamma b + (1 - \gamma)\lambda_r k_1 \pm \sqrt{\Delta_{A_2}}}{k_1}. \]

So the differential equation of \( A_2(t) \) can be solved by

\[ \frac{1}{v_1 - v_2} \int_t^T \left(\frac{1}{A_2(s)} - \frac{1}{A_2(s) - v_2}\right) dA_2(s) = -\frac{k_1}{2\gamma}(T - t). \]

Solving the last equation above with boundary condition \( A_2(T) = 0 \), we get

\[ A_2(t) = \frac{v_1 v_2 \exp(-\sqrt{\Delta_{A_2}(T - t)}) - v_1 v_2}{v_1 \exp(-\sqrt{\Delta_{A_2}(T - t)}) - v_2}. \]

In the case that \( k_1 = 0 \), the equation (6.6) is transformed into

\[ A'_2(t) = bA(t) - (1 - \gamma), A_2(T) = 0. \]

The above problem is easy to solve and the solution is

\[ A_2(t) = \frac{1 - \gamma}{b} \left[1 - e^{b(t-T)}\right]. \]

The solution of \( A_3(t) \) is the same as \( A_2(t) \). An observation here is that the coefficient of the term \( A_3(t)^2 \) is \( -\frac{1}{2\gamma}\sigma_L^2 - \frac{1 - \gamma}{2\sigma_L^2}r_{\rho SL} \) and it is always less than zero. Under the condition that \( \gamma > \frac{2r_{\rho SL} \Delta_{A_2} + \sigma_L^2 \nu^2}{\alpha^2 + 2r_{\rho SL} \Delta_{A_2} + \sigma_L^2 \nu^2} \),
the solution of $A_3(t)$ is

$$A_3(t) = \frac{m_1 m_2 \exp(-\sqrt{\Delta A_3(T-t)}) - m_1 m_2}{m_1 \exp(-\sqrt{\Delta A_3(T-t)}) - m_2},$$  \hspace{0.5cm} (6.9)

$$\Delta A_3 = \alpha^2 - \frac{2(1-\gamma)}{\gamma} \sigma_L \rho_{SL} \nu \alpha - \frac{1-\gamma}{\gamma} \sigma_L^2 \nu^2,$$  \hspace{0.5cm} (6.10)

$$m_{1,2} = \frac{\gamma \alpha - (1-\gamma) \sigma_L \rho_{SL} \nu \pm \gamma \sqrt{\Delta A_3}}{\gamma \sigma_L^2 + (1-\gamma) \sigma_L^2 \rho_{SL}}.$$  \hspace{0.5cm} (6.11)

After knowing $A_2(t)$ and $A_3(t)$, the solution of (6.6) is easily got by integrating and is just the form in the theorem. Substituting $V(t,y,r,l)$ into (3.11), closed forms of $u^*_Y(t)$ and $u^*_S(t)$ are derived, which is shown in this theorem. $\Box$

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References


Highlights

- Study the optimal investment strategy of DC pension plans in an incomplete market
- The stochastic interest rate follows an affine process
- The volatility is the Heston's stochastic model
- The contribution rate follows a similar stock index model with a minimum guarantee
- The closed-form solution and its sensitivity analysis are given