VALUATION OF INFINITE MATURITY STOCK LOANS WITH GEOMETRIC LÉVY MODEL

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\textbf{Abstract.} This paper deals with the pricing problem of infinite maturity stock loans where the underlying stock price follows the geometric Lévy model for incorporating more empirical features. Since the way of dividends distribution has a great influence on pricing, we aim at deriving the closed-form solutions and optimal strategy of the pricing problem subject to three specified ways of dividend distribution by variational inequality approach. The relationships among the parameters, such as loan sizes, interest rates and service fees, are also discussed. Numerical examples are included to illustrate the results.

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1. Introduction

A stock loan is a simple economy where a borrower, who owns one share of a stock, borrows a loan of amount $q$ from a lender (bank or other financial institution) with the share as collateral. The lender charges an amount $c$ ($0 \leq c \leq q$) from the borrower as the service fee. The borrower has the right to redeem the stock at any time before or on the loan maturity by repaying principal and interest to the bank, or alternatively surrender the stock. Here, for analytical tractability we mainly deal with the infinite maturity stock loan throughout this paper.

The stock loan is a currently popular financial derivative because firstly it can create liquidity by overcoming the barrier of large block sales, such
as triggering tax events or control restrictions on sales of stocks; It is also able to serve as a hedge: if the stock price goes down, the borrower may forfeit the loan at initial time and does not repay the loan; if however the stock price goes up, the client keeps all the benefits upside by repaying the principal and interest. In other words, a stock loan can help high-net-worth investors with large equity positions to achieve a variety of objectives.

Xia and Zhou [16](2007) initiated the study of above pricing problems of infinite maturity stock loan under the classical Black-Scholes framework (see [3]), i.e., the underlying stock price follows the geometric Brownian motion. They used a pure probability approach to obtain the value function as well as the optimal redeeming strategy. They also mentioned that the variational inequality approach cannot be directly applied here as in perpetual American option pricing (see [6, 7] and references therein). Liu and Xu [11](2010) also got valuation of infinite maturity capped stock loans by this approach. However, Liang, Wu and Jiang [9, 10](2010), Zhang and Zhou[17](2009) showed that the reformed variational inequality approach can achieve the same results, respectively. Zhang and Zhou [17](2009) extended it to a regime switching market, Liang, Wu and Jiang [9, 10](2010) even covered a more complicated case, i.e., with automatic termination clause, cap and margin. On the other hand, Dai and Xu [5] (2010) firstly studied the finite maturity stock loan and showed that the way how dividends is distributed really has a significant effect on pricing and optimal strategy though closed-form price formulas of this kind of stock loans are generally not available.

However, all the papers above used the classical Black-Scholes model (cf.[3]). A successful model as it is, still has weaknesses to match the reality. As stated in Cont and Tankov [4](2004), the evolution of a stock price shows evident discontinuity and scale variance, neither of which the classical Black-Scholes model has. What’s more, the empirical distribution of increments of the log-price\textsuperscript{1} results in heavier tails than the normal distribution. In addition, in real stock market, perfect hedging is not possible and options enable market participants to hedge risks that cannot be hedged by trading in the underlying only. In contrast, in the market derived by

\textsuperscript{1}The increments of the log-price is also called returns.
the classical Black-Scholes, perfect hedging is possible and options can be replicated by a self-financing strategy involving the underlying and cash. The geometric Brownian motion with regime switching is a choice of refining the Black-Scholes model, and Zhang and Zhou [17](2009) considered the pricing problem of stock loan in this refined model. Moreover, it is well-known that the geometric Lévy model is usually regarded as an improvement, not only because it remains some attractive features of Black-Scholes model but also it satisfies the properties of the real market as mentioned above.

This paper will focus on extending the analytical tractability of the pricing problem of stock loan with the classical geometric Brownian motion to alternative models with jumps. In particular, the stock loan with geometric Lévy model will be chosen as main object of the present paper. With the Lévy jump part, however, it becomes very difficult to derive analytical solutions for the pricing problem. Moreover, following Dai and Xu[5](2010), we also take the ways of dividends distribution into consideration. To the best of our knowledge, this is the first systematic presentation of the topic dealing the pricing problem of such stock loan with the geometric Lévy model. By variational inequalities(VI) approach, explicit value functions and optimal strategy of stock loan are derived here with three different ways of dividends distribution, and reasonable values of critical parameters, such as loan sizes, loan rates and service fees in terms of certain algebraic equations are also given. Our results also state that the variational inequality approach can be extended to treat the valuation problem in more general setting involving the stock loan models with jumps. The main difficult associated with the VI approach is that the corresponding VIs need to establish necessary conditions to find a certain of the functions of HJB is luckily equal to the value function. So, to overcome the difficulties, a delicate analysis will be carried out by Cramér’s estimation of ruin for Lévy process, which is proved by using the excursion theory of Lévy process (cf. [2, 8]).

The rest of the paper is organized as follows. In Section 2, the mathematical description of infinite maturity stock loan with geometric Lévy model is given. Section 3 is devoted to calculating the value function. In
Section 4, we give the relationships among the parameters. In Sections 5 and 6, we investigate the same pricing problem with different dividends distributions. Finally, numerical studies are presented in Section 7.

2. Optimal stopping problems with geometric Lévy process

In the classical Black-Scholes model, the evolution of the stock price is described by geometric Brownian motion,

$$\frac{dS_t}{S_t} = B_0^t,$$

where $B_0^t$ is a Brownian motion with drift. In the view of Lévy process as the generalization of drifted Brownian motion, the direct way to establish geometric Lévy model is replacing $B_0^t$ by a Lévy process, namely $X_t$.

We now formulate the problem in a rigorous way: the uncertainty is described by a Lévy process $\{X_t, t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration generated by $X$, $\mathcal{F}_0 = \sigma(\Omega, \emptyset)$ and $\mathcal{F} = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$. The stock price $S$ follows the following stochastic differential equation,

$$\frac{dS_t}{S_t} = dX_t = (r - \delta)dt + \sigma dW_t + \int_{-\infty}^{+\infty} z\tilde{N}(dz, dt), \quad (2.1)$$

where $r$ is risk-less interest rate; $\delta \geq 0$ is dividend yield and $\sigma > 0$ is volatility; $W_t$ and $\tilde{N}(dz, dt)$ are, respectively, the Brownian motion and compensated Poisson measure with respect to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, which will be proved to be the risk-neutral probability space of $S_t$ in the appendix below. In order to highlight the main idea of Lévy process, a restriction $\int_1^{+\infty} z\nu(dz) < +\infty$ is added here, under which condition the third term on the right hand side of Eq.(2.1) is well-defined. This restriction is reasonable since it illustrates that the effect of upward jumps is not infinity accorded with the reality.

Using Ito’s formula and following the method used in Chapter 1 of Øksendal and Sulem[12](2005), we can derive the following stochastic dynamic system,

$$S_t = x \exp\left\{r - \delta - \frac{1}{2}\sigma^2 + \int_{-\infty}^{+\infty} (\ln(1 + z) - z)\nu(dz)\right\} t + \sigma W_t$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \ln(1 + z)\tilde{N}(dz, ds), \quad (2.2)$$
where \( x = S_0 \) is the initial stock price.

Since the main idea with different distributions remains invariant, one basic distribution mechanism, namely the dividends gained by the lender before redemption, is examined in detail here while the others are sketched in Section 5 and Section 6 below. The payoff process of the basic stock loan is modeled as the following,

\[
P(t) = e^{-rt}(S_t - qe^{\gamma t})_+, \]

where \( \gamma \) is the continuously compounding stock loan interest rate larger than the risk-less rate, \( r \). According to the theory of perpetual American options (c.f.\([15]\)) , the initial value of this basic stock loan is

\[
V(x) = \sup_{\tau \in T_0} \mathbb{E}[e^{-r\tau}(S_\tau - q)e^{\gamma \tau}_+1_{(\tau < +\infty)}],
\]

where \( \tilde{r} = r - \gamma \), \( \tilde{S}_t = e^{-\gamma t}S_t \) and \( T_0 \) denotes all \( \{F_t\}_{t \geq 0} \) stopping times.

We’d like to simplify the expression above by adding a natural definition

\[
e^{-rt}(S_t - qe^{\gamma t})_+1_{(\tau = +\infty)} = \lim_{t \to +\infty} e^{-rt}(S_t - qe^{\gamma t})_+,
\]

under which

\[
\lim_{n \to +\infty} e^{-r\tau/n}(S_{\tau/n} - qe^{\gamma \tau/n})_+ = e^{-\tau}(S_t - qe^\gamma)_+, \tag{2.4}
\]

holds almost surely. Let

\[
Y_t = \ln \frac{e^{-rt}S_t}{x},
\]

then \( Y_t \) is a Lévy process with \( \mathbb{E}Y_1 = -\delta - \frac{1}{2}\sigma^2 + \int_1^{+\infty} (\ln(1+z) - z)\nu(dz) < 0 \).

By Theorem 7.2 of Kyprianou\([8]\)(2006),

\[
\lim_{t \to +\infty} Y_t = -\infty,
\]

thus

\[
\lim_{t \to +\infty} e^{-rt}S_t = \lim_{t \to +\infty} xe^{Y_t} = 0.
\]

Since \( 0 < e^{-\tau}(S_t - qe^{\gamma \tau})_+ \leq e^{-rt}S_t \), we have

\[
e^{-\tau}(S_t - qe^{\gamma \tau})_+1_{(\tau < +\infty)} = 0.
\]

Then we can rewrite (2.3) as

\[
V(x) = \sup_{\tau \in T_0} \mathbb{E}[e^{-\tau}(S_t - qe^{\gamma \tau})_+]. \tag{2.5}
\]
The value process of this stock loan is

\[ V_t(x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[ e^{-r(\tau-t)} (S_\tau - q e^{\gamma \tau})+ |\mathcal{F}_t] \right], \tag{2.6} \]

e.g.,

\[ e^{-r \tau} V_t(x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[ e^{-r \tau} (S_\tau - q) + |\mathcal{F}_t] \right], \]

where \( \mathcal{T}_t \) denotes all \( \{ \mathcal{F}_t \}_{t \geq 0} \)-stopping times \( \tau \) with \( \tau \geq t \).

To avoid arbitrage, the fair values of \( q, \gamma, c \) must satisfy

\[ V_t(x) = x - q + c. \tag{2.7} \]

To determine the range of the fair values of the parameters \( q, \gamma, c \), it suffices to calculate \( V(x) \). Because \( \tilde{r} = r - \gamma \leq 0 \), the problem we concern with is essentially to calculate the initial value of a conventional perpetual American call option with a negative interest rate. This problem poses significant challenges, largely due to the negative interest rate. We firstly make a study of some properties of value function, which helps a lot in understanding the conditions given in Section 3.

**Proposition 2.1.** \( V(x) \) is continuous and nondecreasing on \((0, +\infty)\) with \( V(0) = 0 \).

**Proof.** Rewrite (2.2) as the following,

\[ V(x) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[ g(x, \tau(\omega), \omega) \right]. \]

For fixed \( \omega \), the \( g \) as a function of \( x \) is clearly a convex function, by the linear property that expectation has, we know that \( \mathbb{E}\left[ g(x, \tau(\omega), \omega) \right] \) is also a convex function. Following the same procedure as in Proposition 8.2 given in the appendix below, we know that function \( V \) is convex, so it’s continuous. And the nondecreasing property of \( V \) follows directly from its definition. Noting that \( x = 0 \) implies \( S_t \equiv 0 \), so \( V(0) = 0 \) is directly from (2.5). \( \square \)

**Proposition 2.2.** \( (x - q)_+ \leq V(x) \leq x \) for \( x \geq 0 \).

**Proof.** If we take \( \tau = 0 \) then the inequality \( (x - q)_+ \leq V(x) \) easily follows. Since \( e^{-r \tau} (S_\tau - q e^{\gamma \tau})_+ \leq e^{-r \tau} S_\tau \leq e^{-(r-\delta) \tau} S_\tau \) and \( \{ e^{-(r-\delta) \tau} S_t, t \geq 0 \} \) is a martingale \(^3\), by applying the optional stopping theorem and the Fatou’s

\(^3\)The proof is analogous to that of Proposition 8.1 in the appendix below.
lemma we have
\[ V(x) \leq \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-(r-\delta)\tau}S\right] \]
\[ = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[\lim_{n \to +\infty} e^{-(r-\delta)\tau \wedge n}S_{\tau \wedge n}\right] \]
\[ \leq \sup_{\tau \in \mathcal{T}_0} \lim_{n \to +\infty} \inf_{x} \mathbb{E}\left[e^{-(r-\delta)\tau \wedge n}S_{\tau \wedge n}\right] \]
\[ = \sup_{\tau \in \mathcal{T}_0} \lim_{n \to +\infty} x = x. \]

\[ \square \]

**Proposition 2.3.** (Xia and Zhou[16]) Let \( k = \inf\{x > 0; x - q = V(x)\} \), where \( \inf \emptyset \equiv +\infty \). Then \( k \geq q \) and \( \{x > 0; x - q = V(x)\} = [k, +\infty) \).

**Proof.** It’s trivial for the case \( k = +\infty \). By Proposition 2.2, \( V(x) \geq (x - q) \) for \( x < q \), so \( k \geq q \). If there is a \( k_0 > k \) such that \( V(k_0) \neq k_0 - q \) then \( V(k_0) > k_0 - q \) from which we know \( \beta \equiv \frac{V(k_0) - V(k)}{k_0 - k} > 1 \).

By Proposition 2.1, \( V(x) \) is convex, so
\[ \frac{V(x) - V(k)}{x - k} \geq \frac{V(k_0) - V(k)}{k_0 - k} = \beta, \quad \forall \ x \geq k_0, \]
or
\[ V(x) \geq \beta x - k\beta + k - q, \quad \forall \ x \geq k_0, \]
which implies that \( V(x) > x \) for sufficiently large \( x \). Thus we arrive at a contradiction by Proposition 2.2. \[ \square \]

### 3. Valuations of basic stock loans by variational inequality approach

In this section we present the main result of this paper whose proof follows three steps. Firstly, a pure variational inequality approach is developed to prove that a function satisfying certain conditions including HJB equation must dominate the value function. Secondly, we try to find a certain kind of the functions in step one which is luckily equal to the value function. Finally, an explicit expression of the function mentioned in step two is derived.

Now we state the main result of this paper as follows.

**Theorem 3.1.** If \( \delta > 0 \), then the value function defined by (2.3) is
\[ V(x) = \begin{cases} 
\frac{(r-1)x^{r-1}}{(r\gamma)} - q^{1-r}x^r, & \text{if } 0 \leq x < x^*, \\
x - q, & \text{if } x \geq x^*, 
\end{cases} \]
where \(x^* = \frac{r}{r-1} q\) and \(l^* > 1\) is the solution of the following equation,
\[
h(l) = -\tilde{r} + \tilde{\mu} l + \frac{1}{2} \sigma^2 l (l - 1) + \int_{-1}^{+\infty} \left( (1 + z)^l - 1 - lz \right) \nu(dz) = 0,
\]
and \(\tilde{\mu} = r - \gamma - \delta\).

In order to state the proof of Theorem 3.1, we first introduce the following infinitesimal generator \(L\):
\[
L f(x) = -\tilde{r} f(x) + \tilde{\mu} x f'(x) + \frac{1}{2} \sigma^2 x^2 f''(x)
+ \int_{-1}^{+\infty} \left( f(x(1 + z)) - f(x) - f'(x)zx \right) \nu(dz).
\]

For more information about the infinitesimal generator \(L\), we refer the reader to Revuz and Yor[14]. Next, we try to find a function slightly larger than the value function.

**Theorem 3.2.** The function \(f(x)\) dominates the value function, i.e., \(f(x) \geq V(x)\), if \(f(x)\) satisfies the following conditions:

1) \(f(0) = 0, f(x) \leq x;\)
2) there is an \(x^* > 0\) such that \(f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^*\})\);
3) \(\max \{L f(x), (x - q)_+ - f(x)\} = 0^4\).

**Proof.** From the Itô-Tanaka formula, we know that the Meyer-Itô formula (see Protter[13]) holds for the functions \(f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^*\})\) which are not necessarily in \(C^2\). For details we refer to Chapter 6 of Revuz and Yor[14].

Using (2.1) and Itô formula we have
\[
\begin{align*}
d\tilde{S}_t &= e^{-\gamma t} S_t = e^{-\gamma t} dS_t + S_t d\tilde{S}_t \quad \text{or} \quad d\tilde{S}_t = \mu dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \quad - \gamma e^{-\gamma t} S_t dt \\
&= e^{-\gamma t} S_{t-} \left\{ (\mu - \delta) dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right\} \\
&= e^{-\gamma t} S_{t-} \left\{ (\mu - \gamma - \delta) dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right\} \\
&= \tilde{S}_{t-} \left\{ \tilde{\mu} dt + \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right\}.
\end{align*}
\]

\(^4\)This is exactly the HJB equation.
Hence by the Meyer-Itô formula
\[
df(S_t) = f'(S_t)dS_t + \frac{1}{2} f''(S_t)d[S_t]^c
\]
\[
+ d\left(\sum_{0<\mu\leq t} \left[ f(S_{\mu}) - f(S_{\mu-}) - f'(S_{\mu-})\Delta S_{\mu} \right]\right)
\]
\[
= f'(S_t)\dot{S}_t - \bar{\mu}dt + \sigma dW_t + \int_{-1}^{\infty} z\tilde{N}(dz,dt)
\]
\[
+ \int_{-1}^{\infty} \left( f(S_t(1 + z)) - f(S_{t-}) - f'(S_{t-})z\tilde{S}_{t-}\right)\tilde{N}(dz,dt).
\]
\[
+ \int_{-1}^{\infty} \left( f(S_t(1 + z)) - f(S_{t-}) - f'(S_{t-})z\tilde{S}_{t-}\right)\nu(dz)dt.
\]
So by using Itô formula again we obtain
\[
d\left(e^{-rt}f(S_t)\right)
\]
\[
= -\tilde{r}e^{-rt}f(S_t)dt + e^{-rt}df(S_t)
\]
\[
= e^{-rt}\left(-\tilde{r}f(S_t) + \bar{\mu}S_t f'(S_t) + \frac{1}{2} \sigma^2 S^2_t f''(S_t) \right)dt
\]
\[
+ e^{-rt}\left(\int_{-1}^{\infty} \left( f(S_t(1 + z)) - f(S_{t-}) - f'(S_{t-})z\tilde{S}_{t-}\right)\nu(dz)dt\right)
\]
\[
+ e^{-rt}\left(S_t f'(S_t)\sigma dW_t + \int_{-1}^{\infty} \left( f(S_t(1 + z)) - f(S_{t-})\right)\tilde{N}(dz,dt)\right)
\]
\[
= e^{-rt}\mathcal{L}f(S_t)dt
\]
\[
+ e^{-rt}\left(S_t f'(S_t)\sigma dW_t + \int_{-1}^{\infty} \left( f(S_t(1 + z)) - f(S_{t-})\right)\tilde{N}(dz,dt)\right).
\]
(3.1)

Since the second term on the right hand side (abb. RHS) of Eq.(3.1) is a martingale whose expectation is zero, taking expectations in Eq.(3.1) we have
\[
\mathbb{E}\left[e^{-rt}f(S_t)\right] - f(x) = \mathbb{E} \int_0^t e^{-ru} \left(\mathcal{L}f(S_u)\right) du.
\]
(3.2)

By the condition 3),we know \(\mathcal{L}f(x) \leq 0\), from which we deduce that
\[
\mathbb{E}\left[e^{-rt}f(S_t)\right] \leq f(x) \text{ holds for all } t \geq 0.
\]
In addition, by the same proof as in Remark 3.2 below, the inequality holds for any bounded stopping time. In fact, by truncating,
\[
\mathbb{E}\left[e^{-rt}f(S_t)\right] = \mathbb{E}\left[\lim_{n \to +\infty} e^{-rt\wedge n}f(S_{t\wedge n})\right] \leq \lim\inf_{n \to +\infty} \mathbb{E}\left[e^{-r\wedge n}f(S_{t\wedge n})\right] \leq f(x).
\]

\footnote{Since \(S(t)\) only has countable jumps, the equality holds almost everywhere with respect to Lebesgue measure on \(\mathbb{R}_+\).}
Thus the inequality holds for any stopping time $\tau$. Moreover, by condition 3), we have $(x - q)_+ \leq f(x)$, so
\[
\mathbb{E}\left[e^{-\tau(\tilde{S}_\tau - q)_+}\right] \leq \mathbb{E}\left[e^{-\tau f(\tilde{S}_\tau)}\right] \leq f(x).
\]
Taking $\sup_{\tau \in T_0}$ in the last inequality, we get $V(x) \leq f(x)$ and the proof is completed. \hfill \Box

**Remark 3.1.** Keep in mind that our main goal is to find the explicit expression of value function $V(x)$. Theorem 3.2 focuses on finding $f(x)$, which is close to $V(x)$ instead of far larger ones, so $f(x)$ is expected to share some properties with $V(x)$. As a matter of fact, Propositions 2.1, 2.2 and 2.3 provide some clues to all the conditions here.

**Remark 3.2.** To emphasize the main idea of the proof on Theorem 3.2, Eq.(3.2) is presented with deterministic time $t$ instead of bounded stopping time, such as $\tau \wedge n$. The proof for $\tau \wedge n$ follows the next steps. Firstly, notice that (3.1) is same for both cases. Secondly, by optional stopping theorem, the integral in the second term on RHS of Eq.(3.1) with upper limit $\tau \wedge n$ is still a martingale, and its expectation is zero.

In what follows, we try to find a specific function $f$ in Theorem 3.2 that is equal to the value function.

**Theorem 3.3.** Assume that $\delta > 0$ and an increasing function $f(x)$ satisfies the following conditions:
1) $f(0) = 0$, $f(x) \leq x$;
2) there is an $x^* > q$ such that $f \in C^1(R_+) \cap C^2(R_+ \setminus \{x^*\})$;
3) $L f(x) = 0$, $(x - q)_+ < f(x)$, $\forall x < x^*$;
4) $L f(x) < 0$, $x - q = f(x)$, $\forall x \geq x^*$.

Then $f(x)$ must be equal to the value function $V(x)$. Moreover, the optimal stopping time is $\tau^* \equiv \inf \left\{ t : \tilde{S}_{t^{-}} > x^* \right\}$.

In order to prove this theorem, let us first introduce Cramér’s estimation of ruin for Lévy process as a powerful tool. It is proved by using the excursion theory of Lévy process, which is explained well in Kyprianou[8] (2006) and Bertoin and Doney[2](1994), so the following lemma is stated without proof.
Lemma 3.1. Assume that $X$ is a Lévy process without monotone paths. If the conditions below hold,  
\begin{enumerate}
\item \[ \lim_{t \to +\infty} X_t = -\infty, \]
\item \[ \text{there exists a } \xi \in (0, +\infty) \text{ such that } \psi(\xi) = 0, \]
\end{enumerate}
where $\psi(\theta) = \log \mathbb{E}(\exp(\theta X_1))$ is the Laplace exponent of $X$,  
then  
\[ \lim_{x \to +\infty} e^{\xi x} P(\tau^+_x < +\infty) < +\infty, \]
where $\tau^+_x = \inf\{t : X_t > x\}$. And, as a direct consequence, there exists a constant $C$ such that the following inequality holds for any $x > 0$  
\[ e^{\xi x} P(\tau^+_x < +\infty) < C. \] (3.3)

Then, combining Lemma 3.1 and the definition of $Y_t$ from (2.4), we now prove the integrability of $\sup_{0 \leq t \leq +\infty} e^{Y_t}$. It is of great importance because $\sup_{0 \leq t \leq +\infty} e^{Y_t}$ can be used as a dominated function for $e^{-r(\tau - q)}$, here $\tau$ is a stopping time.

Corollary 3.1. If $\delta > 0$, then $\mathbb{E}\left[ \sup_{0 \leq t \leq +\infty} e^{Y_t} \right] < +\infty$.

Proof. Applying a approach being somewhat like that of Lévy -Itô decomposition in Applebaum[1] we have  
\[ \phi(\theta) = \left( -\delta - \frac{1}{2} \sigma^2 + \int_{-1}^{+\infty} (\ln(1 + z) - z) \nu(dz) \right) \theta + \frac{1}{2} \sigma^2 \theta^2 \]
\[ + \int_{-1}^{+\infty} \left( (1 + z)^\theta - 1 - \theta \ln(1 + z) \right) \nu(dz) \]  
\[ = ( -\delta - \frac{1}{2} \sigma^2) \theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{-1}^{+\infty} (1 + z)^\theta - 1 - \theta z \nu(dz). \] (3.4)

If the third term on RHS of the last equation of (3.4) is finite, we denote it by $I(\theta)$. Note that, for $z > -1, (1 + z)^\theta - 1 - \theta z$ is increasing with respect to $\theta$, thus by monotone convergence theorem $I(\theta)$ is continuous with respect to $\theta$. On the other hand, since $(1 + z)^\theta - 1 - \theta z \geq 0, \forall z > -1$ and $\theta > 1, I(\theta)$ is either finite for all $\theta > 1$ or positive infinite for some $\theta > 1$. For the latter case, using the continuity of $I(\theta)$ we know that there is a minimal number $\theta_0 > 1$ such that $\forall \theta < \theta_0, I(\theta) < +\infty$ and $I(\theta_0) = +\infty$. We’d like to study the two cases, respectively.

For the former case, $\phi(\theta)$ is a quadratic form plus a finite increasing non-negative function. Since the coefficient of $\theta^2$ is positive, $\lim_{\theta \to +\infty} \phi(\theta) = +\infty$, so using $\phi(1) = -\delta < 0$ we know that there exists a number $\xi > 1$
such that $\phi(\xi) = 0$.

For the latter case, by $\lim_{\theta \to 0} I(\theta) = +\infty$, we know $\lim_{\theta \to 0} \phi(\theta) = +\infty$, which also implies that there exists a number $\xi > 1$ such that $\phi(\xi) = 0$ due to $\phi(1) = -\delta < 0$.

So we have just proved that condition 2) of Lemma 3.1 holds and condition 1) has also been proved in Section 2 above, so by Lemma 3.1

$$e^{\xi^*} P(\tau^+_1 < +\infty) < C.$$ 

Therefore we deduce directly from the last inequality and definitions of $Y_i$ and $\tau^+_1$ that

$$P(\sup_{0 \leq t < +\infty} e^{Y_i} > Q) = P(\sup_{0 \leq t < +\infty} Y_i > \ln Q) = P(\tau^+_1 \ln Q < +\infty) < CQ^{-\xi},$$

where $Q > 0$. Furthermore, since $\xi > 1$ we have

$$\mathbb{E} \left[ \sup_{0 \leq t < +\infty} e^{Y_i} \right] = \int_0^{+\infty} P(\sup_{0 \leq t < +\infty} e^{Y_i} > Q)dQ \leq \int_0^{+\infty} CQ^{-\xi}dQ < +\infty.$$

Thus the proof is complete. \qed

We now return to the proof of Theorem 3.3.

**Proof.** It’s obvious that $f(x)$ here satisfies all the conditions in Theorem 3.2, thus $f(x) \geq V(x)$, so we only need to prove the converse inequality: $f(x) \leq V(x)$ also holds. By the definition of $V(x)$ defined by (2.3), $f(x) \leq V(x)$ can be reduced to proving existence of a stopping time $\tau^*$ such that $f(x) = \mathbb{E} \left[ e^{-\tilde{r}^* (\tilde{S}_{\tau^*} - q)} \right]$.

By (3.2) and inspecting the proof used therein

$$\mathbb{E} \left[ e^{-\tilde{r}^* \wedge \eta f(\tilde{S}_{\tau^* \wedge \eta})} \right] - f(x) = \mathbb{E} \int_0^{\tau^* \wedge \eta} e^{-\tilde{r}u} \left( Lf(\tilde{S}_{u^-}) \right) du$$

$$= \mathbb{E} \int_0^{\tau^* \wedge \eta} e^{-\tilde{r}u} \left( \tilde{r}q - \delta \tilde{S}_{u^-} \right) 1_{\tilde{S}_{u^-} > x} du$$

$$= \mathbb{E} \int_0^{\tau^* \wedge \eta} e^{-\tilde{r}u} \left( \tilde{r}q - \delta \tilde{S}_{u^-} \right) 1_{\tilde{S}_{u^-} > \tau^*} du$$

$$= 0.$$ (3.5)

Since

$$e^{-\tilde{r}^* \wedge \eta} f(\tilde{S}_{\tau^* \wedge \eta}) \leq e^{-\tilde{r}^* \wedge \eta} \tilde{S}_{\tau^* \wedge \eta} = e^{-\tilde{r}^* \wedge \eta} S_{\tau^* \wedge \eta} \leq x \sup_{0 \leq t < +\infty} e^{Y_i},$$
by the dominated convergence theorem we have
\[
f(x) = \mathbb{E}\left[e^{-r_t \wedge n} f(\bar{S}_{\tau \wedge n})\right] = \lim_{n \to +\infty} \mathbb{E}\left[e^{-r_t \wedge n} f(\bar{S}_{\tau \wedge n})\right]
\]
\[
= \mathbb{E}\left[\lim_{n \to +\infty} e^{-r_t \wedge n} f(\bar{S}_{\tau \wedge n})\right] = \mathbb{E}\left[e^{-r_\tau} f(\bar{S}_{\tau})\right].
\]
(3.6)

By definition of \(\tau^*\), \(\bar{S}_{\tau^*} \geq x^*\), so \(f(\bar{S}_{\tau^*}) = (\bar{S}_{\tau^*} - q)_+\), which, together with the last equation of (3.6), ends the proof.

**Remark 3.3.** We emphasize that \(x^*\) is predetermined, i.e., it has nothing to do with \(x\). However, \(\tau^* \equiv \inf\{t : Y_t > \ln \frac{x^*}{x}\}\), and it is indeed a function of \(x\), so we will denote it as \(\tau^*(x)\) if necessary.

Before going any further, we take a look at \(f'(0)\), which may provides some clues and ideas on deriving the explicit expression of the value function below.

**Proposition 3.1.** If \(f(x)\) in Theorem 3.3 exists, then \(f'(0) = 0\).

**Proof.** By definition,
\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \mathbb{E}\left[\frac{e^{-r\tau^*(x)}(S_{\tau^*(x)} - qe^{r\tau^*(x)})_+}{x}\right].
\]
(3.7)

Since \(\frac{e^{-r\tau^*(x)}(S_{\tau^*} - qe^{r\tau^*})_+}{x}\) is dominated by \(\sup_{0 \leq t < +\infty} \frac{e^{\gamma t}S_t}{x}\) which is independent of \(x\), by Corollary 3.1 we know that the latter is integrable. Applying the dominated convergence theorem we have
\[
\lim_{x \to 0} \mathbb{E}\left[\frac{e^{-r\tau^*(x)}(S_{\tau^*(x)} - qe^{r\tau^*(x)})_+}{x}\right] = \mathbb{E}\left[\lim_{x \to 0} \frac{e^{-r\tau^*(x)}(S_{\tau^*(x)} - qe^{r\tau^*(x)})_+}{x}\right]
\]
\[
= \mathbb{E}\left[\lim_{x \to 0} \frac{(e^{-r\tau^*}S_{\tau^*} - qe^{(\gamma - r)\tau^*})_+}{x}\right]
\]
\[
\leq \mathbb{E}\left[\lim_{x \to 0} \frac{(e^{-r\tau^*}S_{\tau^*} - q)_+}{x}\right]
\]
\[
\leq \mathbb{E}\left[\lim\sup_{x \to 0} \frac{e^{\gamma t}S_t}{x} - \frac{q}{x}\right].
\]
(3.8)

So \(f'(0) = 0\) because \(\sup_{0 \leq t < +\infty} \frac{e^{\gamma t}S_t}{x}\) is independent of \(x\) and \(\lim_{x \to 0} \frac{q}{x} = +\infty\). The proof is thus completed.

Finally, we return to the proof of Theorem 3.1. Instead of only checking that the functions in Theorem 3.1 satisfy all the conditions in Theorem 3.3,
we present the following Theorem 3.4, which is more illustrative to show the idea of finding the explicit expression of value function \( V(x) \) and the optimal stopping time.

**Theorem 3.4.** If \( \delta > 0 \), then the following function satisfies all the conditions in Theorem 3.3

\[
V(x) = \begin{cases} \frac{(l^*-1)^{l^*-1}}{(l^*)^l} q^{l^*-1} x^{l^*}, & \text{if } 0 \leq x < x^*, \\ x - q, & \text{if } x \geq x^*, \end{cases}
\]

where \( l^* > 1 \) is the unique solution of \( h(l) = 0 \) (see Theorem 3.1) and \( x^* = \frac{r}{r-\gamma} q \). Moreover, the optimal stopping time is \( \tau^* \equiv \inf \{ t : \bar{S}_t > x^* \} \).

**Proof.** Checking the procedure of calculating the value function in Liang, Wu and Jiang [10] on the classical Black-Scholes model we know that if \( x \) is smaller than a fixed number \( x^* \), then the value function is a power function. We would like to try the same idea, i.e., we guess that \( f(x) = C_1 x^l \), where \( C_1 > 0 \), \( l > 0 \) if \( x < x^* \), a number to be determined later. Then

\[
\mathcal{L}(x^l) = -\bar{r} x^l + \bar{\mu} l x^l + \frac{1}{2} \sigma^2 l(l-1) x^{l^2} + \int_{-1}^{+\infty} \left( x^l (1+z)^l - x^l - l z x^l \right) \nu(dz) \\
= x^l \left( -\bar{r} + \bar{\mu} l + \frac{1}{2} \sigma^2 l(l-1) + \int_{-1}^{+\infty} (1+z)^l - 1 - l z \nu(dz) \right) \\
= x^l h(l),
\]

so \( \mathcal{L}(x^l) = 0 \), \( \forall 0 < x < x^* \), is equal to \( h(l) = 0 \). It is easy to observe that \( h(l) \) may have solution smaller than 1. However, by Proposition 3.1, we have \( f'(0) = 0 \), therefore only those solutions being larger than 1 are needed.

Noting that \( \phi(l) \) in (3.4) has similar forms to \( h(l) \), i.e.,

\[
h(l) = \phi(l) + (r - \gamma)(l - 1),
\]

and \( h(1) = \phi(1) = -\delta < 0 \), so by the analogous analysis with Corollary 3.1, we know that at least there exists a number \( l^* > 1 \) such that \( h(l^*) = 0 \). At last, by smooth fit principle and using the continuity of \( f(x) \) at \( x^* \), we solve the following equation,

\[
C_1(x^*)^l = (x^* - q),
\]

\[
C_1 l^*(x^*)^{l^*-1} = 1.
\]
Then $x^* = \frac{-r}{\tau} q$, $C_1 = \left(\frac{(r-1)^{r-1}}{r}\right) q^{1-r}$. So
\[
f(x) = \begin{cases} 
\frac{(r-1)^{r-1}}{r} q^{1-r} x^r, & \text{if } 0 \leq x < x^*, \\
 x - q, & \text{if } x \geq x^*.
\end{cases} \tag{3.10}
\]

It is easy to prove that $f(x)$ satisfies all the conditions in Theorem 3.3, therefore $V(x) = f(x)$ and the optimal stopping time is $\tau^* = \inf \left\{ t : S_{t-} > x^* \right\}$.

Remark 3.4. Using the Markov property of $S_t$, by (2.3) and (2.6), we can obtain the expression of the value process
\[
V_t = \esssup_{r \in \mathbb{T}_t} \mathbb{E} \left[ e^{-(r - \gamma \gamma)(S_{\tau} - q e^{\gamma \tau})} | \mathcal{F}_t \right] 
= e^{\gamma t} \esssup_{r \in \mathbb{T}_t} \mathbb{E} \left[ e^{\gamma \tau \gamma}(e^{-r \tau} S_t \exp \left\{ \left[ \sigma - \frac{1}{2} \sigma^2 + \int_{-\infty}^{\infty} (\ln(1 + z) - z) \nu(dz) \right] (\tau - t) + \\
\sigma(W_t - W_t) + \int_t^\tau \int_{-\infty}^{\infty} \ln(1 + z) \tilde{N}(dz, ds) \right) - q e^{\gamma \tau}) \right] | \mathcal{F}_t \right] 
= e^{\gamma t} \sup_{r \in \mathbb{T}_0} \mathbb{E} \left[ e^{-r \gamma \gamma}(e^{\gamma \tau \gamma} x \exp \left\{ \left[ \sigma - \frac{1}{2} \sigma^2 + \int_{-\infty}^{\infty} (\ln(1 + z) - z) \nu(dz) \right] \tau + \\
\sigma W_t + \int_0^\tau \int_{-\infty}^{\infty} \ln(1 + z) \tilde{N}(dz, ds) \right) - q e^{\gamma \tau}) \right] \right|_{x = e^{\gamma \gamma} S_t} 
= e^{\gamma t} V(e^{\gamma \gamma} S_t). \tag{3.11}
\]

Remark 3.5. If there is no jump term in (2.1), the problem degenerates to the classical Black-Scholes situation. $h(l) = 0$ becomes a quadratic equation of $l$,
\[
\frac{\sigma^2}{2} l^2 + \left( \tilde{r} - \delta - \frac{\sigma^2}{2} \right) l - \tilde{r} = 0.
\]
This equation has two solutions
\[
l_1 = \frac{1}{\sigma} \left( \sqrt{\left( \frac{\sigma}{2} - \frac{\gamma - r + \tilde{\delta}}{\sigma} \right)^2 + 2\delta + \frac{\sigma}{2} \frac{\gamma - r + \tilde{\delta}}{\sigma} } \right) > 1,
\]
\[
l_2 = \frac{1}{\sigma} \left( - \sqrt{\left( \frac{\sigma}{2} - \frac{\gamma - r + \tilde{\delta}}{\sigma} \right)^2 + 2\delta + \frac{\sigma}{2} \frac{\gamma - r + \tilde{\delta}}{\sigma} } \right) \leq 1.
\]
Remark 3.6. The reason why we cannot handle the case $\delta = 0$ as in Xia and Zhou [16] is that it is hard to prove $\lim_{n \to +\infty} E\left[e^{-tn}f(\bar{S}_n)1_{(\tau^* \geq n)}\right] = 0$ in geometric Lévy model, the reader may find an approach to fix it.

4. Ranges of fair values of parameters

In Section 2, since there is no arbitrage, parameters $q$, $c$, $\gamma$ shall satisfy (2.7). In the view of Theorem 3.4, there are two cases can be dealt with.

If $x^* \leq x = S_0$, by (3.10), $f(x) = x - q$. Comparing with (2.7), we know that the fee $c$ shall be 0. And by the definition of $\tau^*$, we know that the optimal time of the stock loan is 0, which means that the optimal choice is not having initiated this loan, the bank does not charge a service fee for its service since the stock price is large, so the bank and the client do not have enough incentive to do business.

If $x^* > x = S_0$, by (3.10), $f(x) = \left(l^* - 1\right)^{l^* - 1} \frac{q - c}{x} x^{l^*}$, so the parameters $q$, $c$, $\gamma$ shall satisfy

$$\frac{(l^* - 1)^{l^* - 1}}{(l^*)^{l^*}} q^{l^* - 1} x^{l^*} = x - q + c. \quad (4.12)$$

Rewriting the equality by dividing by $x$,

$$\frac{(l^* - 1)^{l^* - 1}}{(l^*)^{l^*}} \left(\frac{q}{x}\right)^{l^* - 1} = 1 - \frac{q - c}{x}. \quad (4.13)$$

Letting $\frac{q}{x} \to +\infty$ we have

$$\lim_{\frac{q}{x} \to +\infty} \frac{q - c}{x} = 1, \quad (4.14)$$

where $c$ is a function of $\frac{q}{x}$ such that the limit is well-defined. Recalling that $q - c$ represents the amount the borrower get from the loan while $x$ is the initial stock price, (4.14) can be explained in the following way: for any given $q$, the less $x$ is, the less impossible for the borrower to redeem the stock, that is, the more likely to directly sell the stock. Moreover, the limit of the probability for redeeming can be calculated rigourously as the following:

$$\lim_{x \to 0} P(\tau^*(x) < +\infty) \leq \lim_{x \to 0} P(\tau_{\tau^*}^+ < +\infty) \leq \lim_{x \to 0} C e^{-\xi \tau^*} = 0. \quad (4.15)$$

Since the idea of such analysis is similar for other distributions of dividends, here we omit the ranges of parameters for them.
5. Reinvested dividends returned to the borrower on redemption

Firstly, if all dividends are reinvested, then the share held at time \( t \), denoted by \( A(t) \), will be \( e^{\delta t} \) since \( dA(s) = A(s)\delta ds \) for \( \forall s \in [0,t] \), which means the dividends reinvested at time \( s \) cause an increment of the share with size \( \delta \).

Secondly, the price of all the stock held at time \( t \) is exactly \( A(t)S(t) \), where \( S(t) \) is the stock price. So the value function is

\[
V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[ e^{-r\tau}(e^{\delta \tau}S_\tau - q e^{\gamma \tau})_+ \right].
\]

Since there is no way to deal with the case when \( \delta = 0 \) up to now, we consider only reinvest part, say \( 0 < \alpha < 1 \), of the dividends. Applying the same analysis above, the value function is

\[
V_1(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[ e^{-r\tau}(e^{\alpha \delta \tau}S_\tau - q e^{\gamma \tau})_+ \right],
\]

which is still a basic stock loan with original \( \delta \) replaced by \( (1 - \alpha)\delta \), the results remain true.

6. Dividends always delivered to the borrower partly

Noticing that the value of an infinite maturity stock loan lays in the dividends gained forever, \( S_0 = \mathbb{E}\left( \int_0^{+\infty} e^{-rt} \delta S_t dt \right) \), so it’s meaningless to set a stock loan delivering all the dividends to the borrower, in which the borrower can gain all the value of the stock loan simply by never redeeming. Nevertheless, delivering part, namely \( 0 < \beta < 1 \), of the dividends to the borrower is acceptable, in which the value function follows

\[
V_2(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[ e^{-r\tau}(S_\tau - q e^{\gamma \tau})_+ + \int_0^\tau \beta e^{-ru} \delta S_u du \right]. \tag{6.16}
\]

We claim that

**Theorem 6.1.** If \( \delta > 0 \), then the value function is

\[
V_2(x) = \begin{cases} 
\frac{(1-\beta)^r}{(r-1)^r} q^{1-r} x^r + \beta x, & \text{if } 0 \leq x < x^{**} \\
\frac{(1-\beta)^r}{(r-1)^r} \frac{q^{1-r} x^r}{x - q} + \beta x, & \text{if } x \geq x^{**},
\end{cases}
\]

where \( l^* > 1 \) is the solution of \( h(l) = 0 \).

Because the proof of Theorem 6.1 shares similar idea to Theorem 3.1, we only give its sketch here.
The function $f(x)$ dominates value function, i.e., $f(x) \geq V_2(x)$, if $f(x)$ satisfies the following conditions:
1) $f(0) = 0$, $f(x) \leq x$;
2) there is an $x^{**} > 0$ such that $f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^{**}\})$;
3) $\max \{L f(x) + \beta \delta x, (x - q)_+ - f(x)\} = 0$.

Proof. Since the Eq.(3.2) also holds here, by adding $E \left[ \int_0^\tau \beta e^{-rt} \delta S_r dt \right]$ to the both sides of the Eq.(3.2) we have the following
\[
E \left[ e^{-rt} f(\bar{S}_r) + \int_0^\tau \beta e^{-rt} \delta S_r dt \right] - f(x) = E \left[ \int_0^\tau e^{-rt} \left( L f(\bar{S}_u) + \beta \delta \bar{S}_u \right) du \right].
\]
By conditions 3)
\[
E \left[ e^{-rt} f(\bar{S}_r) + \int_0^\tau \beta e^{-rt} \delta S_r dt \right] \leq f(x).
\]
By Remark 3.2, the last inequality holds for any bounded stopping time $\tau \land n$, then by the Fatou’s lemma, it also holds for all stopping time, i.e.,
\[
E \left[ e^{-rt} f(\bar{S}_r) + \int_0^\tau \beta e^{-rt} \delta S_r dt \right] \leq f(x).
\]
Moreover, by using condition 3) again, $(x - q)_+ \leq f(x)$, so
\[
E \left[ e^{-rt}(\bar{S}_r - q)_+ + \int_0^\tau \beta e^{-rt} \delta S_r dt \right] \leq E \left[ e^{-rt} f(\bar{S}_r) + \int_0^\tau \beta e^{-rt} \delta S_r dt \right] \leq f(x).
\]
Taking $\sup_{t \in T_0}$, $V_2(x) \leq f(x)$, which ends the proof. \hfill \Box

Theorem 6.3. Assume that $\delta > 0$. If an increasing function $f(x)$ satisfies the following conditions:
1) $f(0) = 0$, $f(x) \leq x$;
2) there is an $x^{**} > 0$ such that $f \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+ \setminus \{x^{**}\})$;
3) $Lf(x) = -\beta \delta x$, $(x - q)_+ < f(x), \quad \forall x < x^{**}$;
4) $Lf(x) < -\beta \delta x$, $x - q = f(x), \quad \forall x \leq x^{**}$,
then $f(x)$ must be equal to the value function $V_2(x)$. Moreover, the optimal stopping time is $\tau^{**} \equiv \inf \{t : \bar{S}_t > x^{**}\}$.

Proof. Proceeding similar procedure as in the proof of Theorem 3.3, it would suffice to prove that $f(x) = E \left[ e^{-rt^{**}}(\bar{S}_{t^{**}} - q)_+ + \int_0^{t^{**}} \beta e^{-rt} \delta S_r dt \right]$. 
By (6.17) we have
\[ E \left[ e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) + \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \right] - f(x) \]
\[ = \mathbb{E} \int_0^{\tau^{*\wedge n}} e^{-ru} (\mathcal{L} f(\bar{S}_{u-}) + \beta \bar{S}_{u-}) \, du \]
\[ = \mathbb{E} \int_0^{\tau^{*\wedge n}} e^{-ru} (\bar{r} - \delta \bar{S}_{u-} + \beta \delta \bar{S}_{u-}) 1_{\bar{S}_{u-} > x} \, du \]
\[ = \mathbb{E} \int_0^{\tau^{*\wedge n}} e^{-ru} (\bar{r} - \delta \bar{S}_{u-} + \beta \delta \bar{S}_{u-}) 1_{u \geq \tau^{*\wedge n}} \, du \]
\[ = 0. \] (6.18)

So, since \( e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) \) is dominated by \( \sup_{0 \leq t \leq \infty} e^{B_t} \) which is integrable and \( \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \) is an increasing function with respect to \( n \), by the dominated convergence theorem and the monotone convergence theorem we have
\[ f(x) = \mathbb{E} \left[ e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) + \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \right] \]
\[ = \lim_{n \to +\infty} \mathbb{E} \left[ e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) + \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \right] \]
\[ = \mathbb{E} \left[ \lim_{n \to +\infty} \left( e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) + \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \right) \right] \]
\[ = \mathbb{E} \left[ e^{-\tau^{*\wedge n}} f(\bar{S}_{\tau^{*\wedge n}}) + \int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, dt \right]. \] (6.19)

By definition of \( \tau^{*\wedge n} \), \( \bar{S}_{\tau^{*\wedge n}} > x^{**} \), so \( f(\bar{S}_{\tau^{*\wedge n}}) = (\bar{S}_{\tau^{*\wedge n}} - q)_+ \), and the proof is therefore completed. \( \square \)

**Proposition 6.1.** If \( f(x) \) in Theorem 6.3 exists, then \( f'(0) = \beta \).

**Proof.** By Theorem 6.3 and the definition of \( V_2(x) \),
\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} \]
\[ = \lim_{x \to 0} \mathbb{E} \left[ \frac{e^{-\tau^{*\wedge n}} (S_{\tau^{*\wedge n}} - qe^{\tau^{*\wedge n}(x)})_+}{x} \right] \]
\[ + \lim_{x \to 0} \mathbb{E} \left[ \frac{\int_0^{\tau^{*\wedge n}} \beta e^{-rt} \delta S_r \, du}{x} \right]. \] (6.20)

The first term on RHS of the last equation of the Eq. (6.20) can be proved to be exactly 0 via the same approach as in Proposition 3.1. Since, for fixed \( \omega \), \( \tau^{*\wedge n}(x) \) is an increasing function with respect to \( x \) and \( \frac{e^{-\tau^{*\wedge n}(x)}}{x} \) is independent
of \( x \), by the monotone convergence theorem we have
\[
\lim_{x \to 0} \mathbb{E} \left[ \frac{\tau^{\ast}(x)}{x} \right] = \mathbb{E} \left[ \lim_{x \to 0} \frac{\tau^{\ast}(x)}{x} \right].
\]
Moreover, as did in (4.15), we know that \( \lim_{x \to 0} \tau^{\ast}(x) = +\infty \) holds almost surely, thus by using \( S_0 = \mathbb{E} \left( \int_0^{\infty} e^{-rt} \delta S_u dt \right) \),
\[
\mathbb{E} \left[ \lim_{x \to 0} \int_0^{\tau^{\ast}(x)} \frac{e^{-ru}S_u}{x} du \right] = \mathbb{E} \left[ \int_0^{+\infty} \frac{\beta \delta e^{-ru}S_u}{x} du \right] = \beta.
\]

\( \square \)

Finally, instead of checking that the functions in Theorem 6.1 satisfy all the conditions in Theorem 6.3 in the case when \( \delta > 0 \), we show the idea why this kind of the functions is chosen.

**Theorem 6.4.** If \( \delta > 0 \), then the following function \( V_2(x) \) satisfies all the conditions in Theorem 6.3,
\[
V_2(x) = \begin{cases} 
\frac{(1-\beta)(l^*-1)^{l-1}}{l} q^{1-l} x^l + \beta x, & \text{if } 0 \leq x < x^{**}, \\
0 & \text{if } x \geq x^{**}, 
\end{cases}
\]
where \( l^* > 1 \) is the solution of \( h(l) = 0 \) and \( x^{**} = \frac{l}{(l-1)(1-\beta)} \). Moreover, the optimal stopping time is \( \tau^{**} \equiv \inf \{ t : \tilde{S}_t < x^{**} \} \).

**Proof.** We guess from Proposition 6.1 and Theorem 3.4 that \( f(x) = C_2 x^l + \beta x \) if \( x < x^{**} \), where \( C_2 > 0, \ l > 0 \). Using the same notation \( \mathcal{L} \) as in Theorem 3.4 we have
\[
\mathcal{L} f(x) = C_2 x^l h(l) - \beta \delta x.
\]
So \( \mathcal{L} f(x) + \beta \delta x = 0, \ \forall 0 < x \leq x^{**} \), is equal to \( h(l) = 0 \) and we can prove \( h(l) = 0 \) has a solution, \( l^* \), larger than 1, as did in Theorem 3.4. Using the smooth fit principle, \( x^{**} \) is a continuous point of \( f(x) \) and \( f'(x) \), we know from which that
\[
C_2 (x^{**})^l + \beta x^{**} = (x^{**} - q);
\]
\[
C_2 l^*(x^{**})^{l-1} + \beta = 1.
\]
Solving the two equations we get \( x^{**} = \frac{l q}{(l-1)(1-\beta)} \) and \( C_2 = \frac{(1-\beta)^{(l-1)(l-1)^{-1}}}{(l)(l-1)} q^{1-l} \).

Therefore
\[
f(x) = \begin{cases} 
\frac{(1-\beta)(l^*-1)^{l-1}}{l} q^{1-l} x^l + \beta x, & \text{if } 0 < x < x^{**}, \\
x - q, & \text{if } x \geq x^{**}.
\end{cases}
\]
It is easy to check that this \( f(x) \) satisfies all the conditions in Theorem 6.3, thus by Theorem 6.3, \( V_2(x) = f(x) \) and the optimal stopping time is determined by \( \tau^{**} \equiv \inf \{ t : \tilde{S}_{t-} > x^{**} \} \).

**Remark 6.1.** We see that \( x^{**} > x^* \), so it implies that the borrower prefers not to redeem the stock when the part of dividends can be delivered. It is close to the reality because the opportunity cost for redemption increases. If \( \beta \to 1 \), then \( x^{**} \to +\infty \) and \( C_2 \to 0 \), the value function takes \( x \) as limit. So we claim that the optimal strategy for all dividends delivered to the borrower is never redeeming the stock.

It is a natural idea to combine these two different dividends distribution, and it turns out to be quite feasible. As long as \( \alpha + \beta < 1 \), the value function has the following form

\[
V_3(x) = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[ e^{-r\tau}(e^{\alpha\delta\tau}S_\tau - qe^{\gamma\tau})_+ + \int_0^\tau e^{-ru} \delta S_u du \right].
\]  

(6.23)

Then the value function \( V_3(x) \) has the following expression, and we omit its proof here.

**Theorem 6.5.** If \( \delta > 0 \), then the value function is

\[
V_3(x) = \begin{cases} 
\frac{(1-\beta)(l^{**}-1)}{(l^{**}-1)^{l^{**}-1}} q^{1-l^{**}} x^{l^{**}} + \beta x, & \text{if } 0 \leq x < x^{***}, \\
x - q, & \text{if } x \geq x^{***},
\end{cases}
\]  

(6.24)

where \( l^{**} > 1 \) is the solution of the following equation

\[-\tilde{r} + (\tilde{r} - (1 - \alpha)\delta) l + \frac{1}{2} \sigma^2 l(l - 1) + \int_{-1}^{+\infty} (1 + z)^l - 1 - l z \nu(dz) = 0\]

and \( x^{***} = \frac{r^{**} q}{(l^{**}-1)(1-\beta)} \).

**Remark 6.2.** There is one kind of dividends distribution in Dai and Xu[5] not mentioned here. This is a difficult optimal stopping problem and we have no way to solve it analytically up to now.

### 7. Example

In this section, combining our main results with the Kou and Wang’s model studied in Kou and Wang[7], we present an example to explain stock loan with jumps. First we give a brief introduction of Kou and Wang’s
model in which the stock price follows the following SDEs,\(^6\)

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t} V_i\right),
\]

where \(W_t\) is a standard Brownian motion, \(N_t\) is a Poisson process with intensity \(\lambda\), and \(\{V_i\}_{i \geq 1}\) is a sequence of independent identically distributed (i.i.d.) random variables, which are larger than \(-1\) such that \(Y = \log(V + 1)\) has an asymmetric double exponential distribution with the density

\[
f_Y(y) = p \cdot \theta_1 e^{-\theta_1 y} 1_{\{y \geq 0\}} + q \cdot \theta_2 e^{\theta_2 y} 1_{\{y < 0\}}, \quad \theta_1 > 1, \quad \theta_2 > 0,
\]

where \(p, q \geq 0, p + q = 1\), represent the probabilities of upward and downward jumps. Clearly, this is a Lévy model, therefore has all the properties mentioned before. In addition, analytical expressions for expectations involving first passage times can be obtained in this model due to the memoryless property of exponential distribution.

Since the solution \(l^*\) of \(h(l) = 0\) plays a key role here, we calculate \(h(l)\) as follows,

\[
h(l) = -\bar{r} + \bar{\mu} l + \frac{1}{2} \sigma^2 l(l-1) + \lambda \left(p \frac{l(l-1)}{(\theta_1 - l)(\theta_1 - 1)} + q \frac{l(l-1)}{(\theta_2 + l)(\theta_2 + 1)} \right)
\]

\[
= -\bar{r} + \bar{\mu} l + \frac{1}{2} \left(\sigma^2 + \frac{2\lambda p}{(\theta_1 - l)(\theta_1 - 1)} + \frac{2\lambda q}{(\theta_2 + l)(\theta_2 + 1)} \right) l(l-1),
\]

(7.26)

if \(-\theta_2 < l < \theta_1\).

This is a quadratic equation with respect to \(l\), the explicit expression for the solutions is available but here we are satisfied with numerical results. Let \(r = 0.05, \sigma = 0.15, \delta = 0.01, \lambda = 0.1, p = 0.1, \theta_1 = 3.0, \theta_2 = 2.0\) and let loan interest rate \(\gamma = 0.07\) and loan size \(q = 100\). By (7.26), we have

\[
l^* = 1.8131,
\]

\[
x^* = 222.99.
\]

Then the figure 1 of the value function on the basic stock loan \(V(x)\) is

\(^6\)We replace \(V_i - 1\) in Kou and Wang [7] by \(V_i\) to accord with our notations in this paper.
Taking reinvestment of dividends into account, we let the reinvested proportion of dividends to be $\alpha_1 = 0.2$ and $\alpha_2 = 0.5$, respectively. Since the stock loan with $\alpha > 0$ is exactly the basic stock loan with $\delta' = (1 - \alpha)\delta$, by (7.26), we have

\[
\begin{align*}
I^*_0 &= 1.6995, \quad x^*_{\alpha_1} = 242.95 \\
I^*_0 &= 1.5063, \quad x^*_{\alpha_2} = 297.52.
\end{align*}
\]

Then we have the following figure 2
From figure 2, it is easy to see that value function is an increasing function with respect to $\alpha$ which can be illustrated in the following way: As mentioned in Section 5, this stock loan is the basic stock loan with $\delta' = (1 - \alpha)\delta$ instead of $\delta$, which means that interest rate gap $\gamma - r - (1 - \alpha)\delta$ is increasing with respect to $\alpha$. Since the gap is the key point to evaluate the relative value of stock loan, the less $\gamma - r - (1 - \alpha)\delta$ is, the more valuable the loan is. This is exactly what the figure 2 shows.

Next, dividends delivered to the borrower is considered with $\beta_1 = 0.1$ and $\beta_2 = 0.2$. From (7.26), we know $l^*$ is the same as basic stock loan. Meanwhile, by (6.22) we know

\[
\begin{align*}
x^{*}_{\beta_1} &= 247.77, \\
x^{*}_{\beta_2} &= 278.74.
\end{align*}
\]

We give the following figure 3
Regarding (6.22) as a function of $\beta$ with $x$ fixed, the first derivative is

$$
\frac{df(\beta)}{d\beta} = -l^r \frac{(1 - \beta)l^{r-1}(l^r - 1)^{r-1}q^{1-r}x^r + x}{(l^r)^r} = \left(1 - \frac{(1 - \beta)l^{r-1}(l^r - 1)^{r-1}q^{1-r}x^{r-1}}{l^r q^{1-r}}\right)x > 0,
$$

(7.27)

where the last equality holds for since $x \leq x^{**} = \frac{r q}{(r-1)(1-\beta)}$. Figure 3 depicts why this happens because of the increase of opportunity cost, which has been discussed in Remark 6.1.

Finally, we combine reinvestment and dividends. Let $\alpha_3 = 0.2, \beta_3 = 0.2$, then

$$R_{\alpha_3, \beta_3} = l_{a_1} = 1.6995,$$

$$x_{\alpha_3, \beta_3}^{**} = \frac{x_{a_1}^{**}}{1 - \beta_3} = 303.69.$$

We have the following figure 4
Figure 4. Value function with both reinvestment and dividends

Figure 4 portrays that both reinvestment and dividends increase the value of the stock loan.

8. Appendix

Proposition 8.1. The probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) is risk-neutral for \(S_t\), where the dynamic of \(S_t\) follows (2.2).

Proof. By the definition of risk-neutral measure, we only need to show that the discounted process \(\{e^{-\delta t}S(t), t \geq 0\}\) is a martingale when \(\delta = 0\). By Itô formula and (2.1), we have

\[
d e^{-\delta t}S_t = S_t de^{-\delta t} + e^{-\delta t}dS_t
\]

\[
= -re^{-\delta t}S_t dt + e^{-\delta t}S_t \left[ \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right]
\]

\[
= e^{-\delta t}S_t \left[ \sigma dW_t + \int_{-1}^{+\infty} z\tilde{N}(dz, dt) \right].
\]

This ends the proof because both \(dW_t\) and \(\tilde{N}(dz, dt)\) are martingale measures. \(\square\)

Proposition 8.2. If a family of functions \(f(x, t)\) is convex with respect to variable \(x\) for any given parameter \(t\), then \(F(x) = \sup_{t \in F} f(x, t)\) is also a convex.
Proof. Assume that $F(x)$ is not convex, then there exist $x_1$, $x_2$ and $\lambda \in (0, 1)$ such that
\[ \lambda F(x_1) + (1 - \lambda)F(x_2) < F(\lambda x_1 + (1 - \lambda x_2)). \]
By the definition of $F(x)$, there is a $t^*$ such that
\[ f(\lambda x_1 + (1 - \lambda x_2), t^*) > \lambda f(x_1, t^*) + (1 - \lambda) f(x_2, t^*). \]
This is a contradiction with the convexity of $f(x, t^*)$ as a convex function with respect to $x$, and thus ends the proof. \hfill \Box

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