OPTIMAL INVESTMENT STRATEGY FOR THE DC PLAN WITH THE RETURN OF PREMIUMS CLAUSES IN A MEAN-VARIANCE FRAMEWORK

Lin He\textsuperscript{a} and Zongxia Liang\textsuperscript{b}; *

\textsuperscript{a} The School of Finance, Renmin University of China, Beijing 100872, China

\textsuperscript{b} Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Abstract. In this paper, we study the optimal investment strategy in the DC pension plan during the accumulation phase. During the accumulation phase, pension member contributes a predetermined amount of money as premiums and the management of the pension plan invests the premiums in the equities and the bonds to increase the value of the accumulation. In practice, most of the DC pension plans have return of premium clauses to protect the rights of the plan members who die during the accumulation phase. In the model, the members withdraw their premiums when they die and the difference between the premium and the accumulation (negative or positive) is distributed among the survival members. From the survival members’ point of view, when they are retired, they want to maximize the fund size and to minimize the volatility of the accumulation. We formalize the problem as a continuous-time optimal control problem in mean-variance framework. The management of the pension plan chooses the optimal investment strategy, i.e., the proportions invested in the equities and the bonds, to maximize the mean-variance utility of the pension member at the time of retirement. Using the HJB and variational inequalities methods, we establish the optimal strategy and the efficient frontier of the pension member. The impact of the risk aversion factor on the optimal investment strategy and the efficient frontier of the pension member are also given via numerical analysis.

JEL classifications: G11; C61; G23; E21; E22

MSC(2000): 91G10; 93E20; 91B70; 91B30

Keywords: DC pension plan; portfolio theory; stochastic optimal control; mean-variance framework; optimal asset allocation; return of premiums clauses.

* Corresponding author.
Email: helinmail@gmail.com (L. He), zliang@math.tsinghua.edu.cn (Z. Liang)
1. Introduction

The defined contribution (DC) pension plan plays an important role in the social security system, and it has become popular in the pension market due to the demographic evolution and the development of the equity markets. In the DC pension plan, the member contributes the predetermined amount of money as premiums during the accumulation phase. When the member is retired, the accumulation will be distributed monthly as old-age pension. The distribution is not predetermined, but depends on the mortality risk, inflation and the investment efficiency, etc. The retirement benefits of the DC pension plans depend on the returns of the fund’s portfolios, so the asset allocation decisions are crucial to DC pension management. In this paper, we study the optimal control problem on the asset allocation strategies during the accumulation phase of the DC pension plan. The accumulation phase lasts the whole working period of the pension member. The accumulation is invested in the equities and the bonds by the pension management to increase the fund size. Many literatures on the accumulation phase of the DC pension plan focus on the optimal portfolio selection problems according to some criterion. Cairns[5] and Battocchio and Menoncin[2] study the stochastic optimal control problem of the DC pension fund in continuous time. Boulier, Huang and Tailard[4] study optimal investment strategies of the DC pension plan under the stochastic interest rate. Haberman and Sung[10] study the optimal investment strategy and risk measures of the DC pension plan. Han and Hung [11] first study the optimal dynamic asset allocation for DC pension plans with stochastic interest rates and inflation in a continuous-time model.

Refer to the objective function aspect, there are two types of wildly used utility goals. One is to maximize the accumulation at the time of retirement and the other is to balance the return and the risk, i.e.,
maximizing the fund size and minimizing the volatility of the accumulation. The former goal includes three types of utility functions, the utility function exhibits constant relative risk aversion (CRRA), the utility function exhibits constant absolute risk aversion (CARA) and quadratic loss functions. The literatures concerned with the CRRA utility function are Cairns, Blake and Dowd[6], and Gao[9]. They choose the power or logarithmic utility function as the objective function. The literatures concerned with the CARA utility function are Devolder, Bosch Princep and Dominguez Fabian[7], and Battocchio and Menoncin[2]. They choose exponential utility function as the objective function. There are also some utility functions which are not widely used. Haberman and Vigna[10] minimize the quadratic loss function and Di Giacinto, Gozzi and Federico[8] use a general form of utility function to study the optimal control policies during the accumulation phase of the DC pension plan. The latter goal includes the mean-variance utility and the value at risk (VaR) utility, etc. The mean-variance utility origins from the Markowitz[15, 19], which study the single period investment allocation problem under the mean-variance utility. Richardson[19], Bajeux-Besnainou and Portait[1] and Zhou and Li[23], extend the single period mean-variance problem to the continuous time models and find the efficient frontiers. The literature on the optimal control problem of the DC pension plan is Højgaard and Vigna[12]. Vigna[20] compares the mean-variance efficiency of the CARA and CRRA utility in the DC pension plan. The problem with VaR utility function equals the problem of maximizing the terminal accumulation with the minimal guarantee. Li, Ng and Deng[14] study the mean-CaR problem. In this paper, the goal of individual pension member is to maximize the fund size and to minimize the volatility of the accumulation. We choose the mean-variance utility as our criterion. The whole problem could be formalized into a continuous time mean-variance optimal control problem. The management of the pension plan chooses the optimal investment strategies,
i.e., the proportions allocated in the equities and the bonds, to maximize the mean-variance utility of the pension member at the time of retirement.

In this paper, we study the optimal control problem of the DC pension plan with the return of premiums clauses. In order to protect the rights of the plan members who die early during the accumulation phase, most of the DC pension plans have the return of premium clauses. In this kind of actuarial clauses, the dead member can withdraw the premiums he/she contributes or the premiums accumulated by a predetermined interest rate. For simplicity, we study the former one in our model. After returning the premium, the difference (positive or negative) between the return and the accumulation will be equally distributed by the survival members. It means that the survival members will stand the mortality risk and the investment efficiency risk of the pension fund. Taking the above actuarial rules into consideration in our model is the main contributions of our paper. Blake, Cairns and Dowd[3], Milevsky and Robinson[17] transform the actuarial clauses of the DC pension plan during the distribution phase into a discrete time problem. With the help of these papers, we extend the problem with the return of premium clauses into a continuous-time stochastic control problem. Using the HJB methods, we establish the optimal proportions allocated in the equities and the bonds and find the efficient frontier of the pension member. In the paper, it’s the first time to study the optimal utility problem of the individual pension member and the accumulation of the individual member’s account is firstly described by a continuous-time stochastic process according to the actuarial rules.

The paper is organized as follows: In Section 2, we introduce the actuarial methods of the DC pension plan with the return of premiums clauses. Under the return of premiums clauses, the survival members stand the mortality risk and the investment efficiency risk as well. In
this section, we formalize the fund size process as the solution of a continuous time stochastic differential equation. In Section 3, we use the mean-variance utility as the criterion of the pension members. The survival member wants to maximize the fund size and to minimize the volatility of the accumulation. Using the HJB methods in[18, 21, 22], we establish the optimal proportions allocated in the equities and the bonds, and the efficient frontier of the pension members. In Section 4, the impacts of the risk aversion coefficient on the optimal investment strategy and the efficient frontier of the pension member are given via numerical examples. Summarizing comments are stated in the last section.

2. The continuous-time model of the DC pension plan with the return of premiums clauses

In this paper, we study the optimal investment policy of the DC pension plan with the return of premiums clauses during the accumulation phase. In the DC pension plan, the member contributes a predetermined amount of money as premiums during the accumulation phase. In the model, we suppose the premium per unit time is \( P \), which is a predetermined variable. The accumulation phase lasts the whole working period of the pension member. We suppose the accumulation period starts from the age of \( \omega_0 \) and lasts to the age of \( \omega_0 + T \), i.e., the time length of the pension fund is \( T \). During the phase, the premium is invested in the equities and the bonds by the pension management to increase the value. The proportion allocated in the equities is \( \pi \), which is a control variable. The rest \( 1 - \pi \) is allocated in the bonds. When the pension member is retired, he/she will get old-age pension from the fund and the amount is not predetermined and it is affected by the uncertainty of the mortality risk and the investment efficiency. In order to protect the rights of the plan members who die early, i.e., during the accumulation phase, most of DC pension plans have the return of premiums clauses. In this kind
of actuarial clauses, the dead member can withdraw the premiums he/she contributes or the premiums accumulated by a predetermined interest rate. For simplicity, we study the former one in our model. In the model, \( \frac{1}{n} q_{\omega_0 + t} \) is the mortality rate from time \( t \) to time \( t + \frac{1}{n} \), and \( tP \) is the accumulated premiums at time \( t \). So, the premium returned to the dead member from time \( t \) to time \( t + \frac{1}{n} \) is \( tP \frac{1}{n} q_{\omega_0 + t} \). After returning the premium, the difference (positive or negative) between the return and the accumulation will be equally distributed by the survival members.

First, we consider the differential form of the fund size \( X(t) \). The time interval is \( \frac{1}{n} \).

\[
X(t + \frac{1}{n}) = (X(t)\left(\pi \frac{S^1_{t+\frac{1}{n}}}{S^1_t} + (1-\pi) \frac{S^0_{t+\frac{1}{n}}}{S^0_t}\right) + P \frac{1}{n} - Pt \frac{1}{n} q_{\omega_0 + t}) \frac{1}{1 - \frac{1}{n} q_{\omega_0 + t}},
\]

where \( S^1_t \) and \( S^0_t \) are the prices of the equity and the bond at time \( t \), respectively. \( \pi \) is the proportion allocated in the equity. \( \frac{1}{n} q_{\omega_0 + t} \) is an actuarial symbol which stands for the probability that the person is alive at the age of \( \omega_0 + t \) will be dead in the following \( \frac{1}{n} \) time period. The last coefficient in (2.1) means that after returning the premium, the difference between the return and the accumulation will be equally distributed by the survival members. \( \pi \) is an important control variable in the model. If the management could choose the optimal \( \pi \), the goal of the pension members will be achieved.

It is obvious to see from (2.1) that the fund size is affected by the following four factors: the investment efficiency, the insurance premium, the return of premium caused by mortality and the distribution of the difference between the accumulation and the return.

Denote

\[
\Delta \delta_{\frac{1}{n}} = \pi \frac{S^1_{t+\frac{1}{n}} - S^1_t}{S^1_t} + (1-\pi) \frac{S^0_{t+\frac{1}{n}} - S^0_t}{S^0_t},
\]
then
\[ X(t + \frac{1}{n}) = (X(t)(1 + \Delta \delta_{\frac{1}{n}}) + P\frac{1}{n} - Pr\frac{1}{n}q_{\omega_0+t})(1 + \frac{1}{n}q_{\omega_0+t}), \]
where
\[ \frac{1}{n}q_{\omega_0+t} = 1 - e^{-\frac{1}{n}\mu(\omega_0+t+s)ds} \approx \mu(\omega_0 + t)\frac{1}{n} = O\left(\frac{1}{n}\right), \]
and \( \mu(t) \) is the force function of mortality.

Similarly,
\[ \frac{1}{n}q_{\omega_0+t} = 1 - e^{-\frac{1}{n}\mu(\omega_0+t+s)ds} \approx \mu(\omega_0 + t)\frac{1}{n} = O\left(\frac{1}{n}\right). \]

Hence, it is easy to see that
\[ \Delta \delta_{\frac{1}{n}} = O\left(\frac{1}{n}\right) = o\left(\frac{1}{n}\right), \]
\[ \frac{1}{n}q_{\omega_0+t} = O\left(\frac{1}{n}\right). \]

Thus
\[ X(t + \frac{1}{n}) = X(t)(1 + \Delta \delta_{\frac{1}{n}}) + X(t)\mu(\omega_0 + t)\frac{1}{n} \]
\[ + P\frac{1}{n} - Pr\mu(x_0 + t)\frac{1}{n} + o\left(\frac{1}{n}\right). \]

Next, we try to formalize the above differential form/discrete-time risk model into the continuous-time risk model. Suppose the equity price and the bond price are described by the following stochastic equations:
\[
\begin{aligned}
\frac{dS_1(t)}{dt} &= S_1(t)(c dt + \sigma dB(t)),
\frac{dS_0(t)}{dt} &= rS_0(t)dt,
\end{aligned}
\]
where \( B(t) \) is the standard Brownian Motion on a given filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). So
\[ 1 + \Delta \delta_{\frac{1}{n}} \rightarrow ((c - r)\pi + r)dt + \pi \sigma dB(t), \quad \text{as} \ n \to \infty. \]

Therefore, the fund size \( X(t) \) should be governed by the following continuous-time stochastic differential equations:
\[
\begin{aligned}
\frac{dX(t)}{dt} &= X(t)((c - r)\pi + r + \mu(\omega_0 + t))dt + Pdt - P\mu(\omega_0 + t)dt \\
&\quad + X(t)\pi \sigma dB(t),
X(0) &= x_0.
\end{aligned}
\]
To make the model more simple, we introduce the Abraham De Moivre model (cf. [13]) to characterize the force function of mortality \( \mu(t) \). In this model, the survival function \( s(t) \) and the force function \( \mu(t) \) of mortality have the following forms:

\[
\begin{align*}
  s(x + t) &= 1 - \frac{t}{x} \quad \text{for } 0 \leq t < x, \\
  \mu(x + t) &= \frac{1}{\omega - x - t} \quad \text{for } 0 \leq t < \omega - x.
\end{align*}
\]

Then the SDE (2.2) becomes the following form:

\[
\begin{align*}
  dX(t) &= X(t) \left[ (c-r)\pi + r + \frac{1}{\omega - \omega_0 - t} \right] dt + P \frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} dt + X(t) \pi dB(t), \\
  X(0) &= x_0.
\end{align*}
\]

Thus, the accumulation of the individual member’s account is described by a continuous-time stochastic process according to the actuarial rules.

Finally, we define the mean-variance optimization problem on the continuous-time risk model (2.3). From the point of view of survival members who are still alive at the age of retirement \( \omega_0 + T \), when they are retired, they want to maximize the fund size and to minimize the volatility of the accumulation. So we choose the mean-variance utility as our main criterion. The whole problem could be formalized into the continuous time optimal control problem. The management of the pension plan chooses the optimal investment strategies, i.e., the proportions allocated in the equities and the bonds to maximize the mean-variance utility of the pension member at the time of retirement. Hence the management’s optimization problem could be described as follows:

\[
\sup_{\pi \in \Pi} \{ E_{t,x} X^\pi(T), - \text{Var}_{t,x} X^\pi(T) \},
\]

where \( \Pi = \{ \pi | \pi \in [0, \infty] \} \), which means that the short sell of the bonds is permitted.

3. Solution of the optimal control problem

In this section, we work out the solution of the optimal control problem (2.4). The optimal proportions allocated in the equities and
the bonds, and the efficient frontier are established. Using the variational inequalities methods (cf. [23]), the optimal control problem (2.4) is equivalent to the following optimal control problem with value function $V(t, x)$:

$$
\begin{align*}
J(t, x, \pi) & = E_{t,x} [X^\pi(T)] - \frac{\gamma}{2} \text{Var}_{t,x} [X^\pi(T)] \\
& = E_{t,x} [X^\pi(T)] - \frac{\gamma}{2} (E_{t,x} [X^\pi(T)^2] - (E_{t,x} [X^\pi(T)])^2), \\
v(t, x) & = \sup_{\pi \in \Pi} J(t, x, \pi),
\end{align*}
$$

and optimal investment policy $\pi^*$ satisfies $v(t, x) = J(t, x, \pi^*)$. Here, $\gamma$ is the risk aversion coefficient which describes the risk averse level of individual pension member.

Denote $y^\pi(t, x) = E_{t,x} [X^\pi(T)]$, $z^\pi(t, x) = E_{t,x} [X^\pi(T)^2]$, then the value function $V(t, x)$ is

$$
v(t, x) = \sup_{\pi \in \Pi} \{ f(t, x, y^\pi(t, x), z^\pi(t, x)) \},
$$

where

$$
f(t, x, y, z) = y - \frac{\gamma}{2} (z - y^2).
$$

**Theorem 3.1.** (Verification Theorem) If there exist three real functions $F, G, H : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following generalized HJB equations:

$$
\begin{align*}
\sup_{\pi \in \Pi} \{ F_t - f_t + (F_x - f_x)[x((c - r)\pi + r + \frac{1}{\omega - \omega_0 - t}) + P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}] \\
+ \frac{1}{2} (F_{xx} - U)x^2 \pi^2 \sigma^2 \} & = 0, \\
F(T, x) & = f(T, x, x, x^2),
\end{align*}
$$

where

$$
U = f_{xx} + 2f_{xy}y_x + 2f_{xz}z_x + f_{yy}y_x^2 + 2f_{yz}y_xz_x + f_{zz}z_x^2 = y^2_x.
$$

$$
\begin{align*}
G_t + G_x [x((c - r)\pi + r + \frac{1}{\omega - \omega_0 - t}) + P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}] + \frac{1}{2} G_{xx} x^2 \pi^2 \sigma^2 & = 0, \\
G(T, x) & = x.
\end{align*}
$$

$$
\begin{align*}
H_t + H_x [x((c - r)\pi + r + \frac{1}{\omega - \omega_0 - t}) + P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}] + \frac{1}{2} H_{xx} x^2 \pi^2 \sigma^2 & = 0, \\
H(T, x) & = x^2.
\end{align*}
$$

Then $V(t, x) = F(t, x), y^\pi^* = G(t, x), z^\pi^* = H(t, x)$ for the optimal asset allocation strategy/ the optimal investment strategy $\pi^*$. 

Proof. The methods used to prove the theorem is similar to the methods in [22], we omit the details here.

In the rest of Section 3, we try to solve the HJB equations (3.2), (3.4) and (3.5), and establish the optimal investment strategy and the efficient frontier of the pension member. Obviously,

\[ f_y = 1 + \gamma y, f_{yy} = \gamma, f_z = -\frac{\gamma}{2}, \]
\[ f_t = f_x = f_{xx} = f_{xy} = f_{xz} = f_{yz} = f_{zz} = 0. \]

Differentiating (3.2) with respect to \( \pi \), we get

\[ \pi = \pi(t, x) = -\frac{F_x(c - r)}{(F_{xx} - \gamma G^2_x)x\sigma^2}. \]

Denote the \( \pi(t, x) \) by \( a^*(t, x) \). Then

\[ a^*(t, x) = -\frac{F_x(c - r)}{(F_{xx} - \gamma G^2_x)x\sigma^2}. \quad (3.6) \]

Substituting (3.6) into (3.4) and (3.5),

\[ F_t + F_x\left[x(r + \frac{1}{\omega - \omega_0 - t}) + P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}\right] - \frac{F_x^2(c - r)^2}{2(F_{xx} - \gamma G^2_x)\sigma^2} = 0 \quad (3.7) \]

and

\[ G_t + G_x\left[x(r + \frac{1}{\omega - \omega_0 - t}) + P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}\right] - \frac{F_x(c - r)^2}{(F_{xx} - \gamma G^2_x)\sigma^2} \]
\[ + \frac{G_{xx}F_x^2(c - r)^2}{2(F_{xx} - \gamma G^2_x)^2\sigma^2} = 0. \quad (3.8) \]

Suppose \( F(t, x) \) and \( G(t, x) \) have the following forms:

\[
\begin{align*}
F(t, x) &= A(t)x + B(t), \quad A(T) = 1, B(T) = 0. \\
G(t, x) &= \alpha(t)x + \beta(t), \quad \alpha(T) = 1, \beta(T) = 0.
\end{align*}
\]

Then the equations (3.7) and (3.8) become

\[
\begin{align*}
A_t x + B_t + A(t)x\left[r + \frac{1}{\omega - \omega_0 - t}\right] + A(t)P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} + \frac{A(t)^2(c - r)^2}{\gamma \alpha(t)^2\sigma^2} &= 0, \\
\alpha_t x + \beta_t + \alpha(t)x\left[r + \frac{1}{\omega - \omega_0 - t}\right] + \frac{A(t)(c - r)^2}{\gamma \alpha(t)^2\sigma^2} + \alpha(t)P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} &= 0.
\end{align*}
\]
Letting the coefficient of $x$ and the constant coefficient be equal 0 in the last two equations, we have

$$\left\{\begin{array}{l}
A_t = A(t)r + \frac{A(t)}{\omega - \omega_0 - t} = 0, \\
B_t = A(t)P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} + \frac{A(t)(c-r)^2}{2\gamma\alpha(t)\sigma^2} = 0
\end{array}\right.$$  \quad (3.9)$$

and

$$\left\{\begin{array}{l}
\alpha_t + \alpha(t)r + \frac{\alpha(t)}{\omega - \omega_0 - t} = 0, \\
\beta_t + \alpha(t)P\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} + \frac{A(t)(c-r)^2}{\gamma\alpha(t)\sigma^2} = 0.
\end{array}\right.$$  \quad (3.10)$$

The solutions of the ordinary differential equations (3.9) and (3.10) are as follows:

$$A(t) = e^{rT(\omega - \omega_0 - T)}e^{-rt(\omega - \omega_0 - t)},$$

$$\alpha(t) = e^{rT(\omega - \omega_0 - T)}e^{-rt(\omega - \omega_0 - t)},$$

$$B(t) = e^{r(T-t)}\frac{P[r(\omega - \omega_0 - 2t) - 2]}{r^2(\omega - \omega_0 - T)} - \frac{(c-r)^2t}{2\gamma\sigma^2} + \frac{(c-r)^2T}{2\gamma\sigma^2} - \frac{P[r(\omega - \omega_0 - 2T) - 2]}{r^2(\omega - \omega_0 - T)},$$

$$\beta(t) = e^{r(T-t)}\frac{P[r(\omega - \omega_0 - 2t) - 2]}{r^2(\omega - \omega_0 - T)} - \frac{(c-r)^2t}{\gamma\sigma^2} + \frac{(c-r)^2T}{\gamma\sigma^2} - \frac{P[r(\omega - \omega_0 - 2T) - 2]}{r^2(\omega - \omega_0 - T)}.$$

We get the solutions of the HJB equations (3.2) and (3.4) expressed by the following

$$F(t, x) = e^{r(T-t)}\frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}x + e^{r(T-t)}\frac{P[r(\omega - \omega_0 - 2t) - 2]}{r^2(\omega - \omega_0 - T)} - \frac{(c-r)^2t}{2\gamma\sigma^2} + \frac{(c-r)^2T}{2\gamma\sigma^2} - \frac{P[r(\omega - \omega_0 - 2T) - 2]}{r^2(\omega - \omega_0 - T)},$$

$$G(t, x) = e^{r(T-t)}\frac{\omega - \omega_0 - t}{\omega - \omega_0 - T}x + e^{r(T-t)}\frac{P[r(\omega - \omega_0 - 2t) - 2]}{r^2(\omega - \omega_0 - T)} - \frac{(c-r)^2t}{\gamma\sigma^2} + \frac{(c-r)^2T}{\gamma\sigma^2} - \frac{P[r(\omega - \omega_0 - 2T) - 2]}{r^2(\omega - \omega_0 - T)}.$$

Substituting $F(t, x)$ and $G(t, x)$ into (3.6) we have

$$a^*(t, x) = e^{-r(T-t)}\frac{(c-r)(\omega - \omega_0 - T)}{\gamma\sigma^2 x(\omega - \omega_0 - t)}.  \quad (3.11)$$

Then the optimal proportion allocated in the equities at time $t$ is

$$\pi^*(t) = a^*(t, X^{\pi^*}(t)),  \quad (3.12)$$
where \( X^\pi(t) \) is the unique solution of the following SDE:

\[
\left\{
\begin{aligned}
   dX(t) &= X(t)[(c-r)a^*(t, X(t)) + r + \frac{1}{\omega - \omega_0 - t}] dt + P^{\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t}} dB(t), \\
   X(0) &= x_0.
\end{aligned}
\right.
\]

(3.13)

Obviously, \( X^\pi(t) \) is the optimal fund size at time \( t \) and it is easy to see from the expressions of (3.11) and (3.12) that \( \pi^* \) is also the function of the risk aversion coefficient \( \gamma \). Pension members with different risk averse levels will invest different proportions in the equities and will achieve different utilities on the efficient frontier.

Then

\[
V ar_{t,x}X^\pi(T) = \mathbb{E}_{t,x}\{X^\pi(T)\}^2 - (\mathbb{E}_{t,x}X^\pi(T))^2
\]

\[
= \frac{2}{\gamma}(G(t, x) - F(t, x))
\]

\[
= \frac{(c - r)^2(T - t)}{\gamma^2 \sigma^2},
\]

(3.14)

\[
\mathbb{E}_{t,x}X^\pi(T) = G(t, x)
\]

\[
= e^{\rho(T-t) (\omega - \omega_0 - T) x + \rho(T-t) P[r(\omega - \omega_0 - 2T) - 2]} \\
\]

\[
= \frac{(c - r)^2 T}{\gamma \sigma^2} + \frac{(c - r)^2 T}{\gamma \sigma^2} P[r(\omega - \omega_0 - 2T) - 2].
\]

(3.15)

Putting (3.14) and (3.15) together, we get the efficient frontier of the following form:

\[
\mathbb{E}_{t,x}X^\pi(T) = e^{\rho(T-t) (\omega - \omega_0 - T) x + \rho(T-t) P[r(\omega - \omega_0 - 2T) - 2]} \\
- \frac{P[r(\omega - \omega_0 - 2T) - 2]}{\rho^2(\omega - \omega_0 - T)} + \frac{c - r}{\sigma} \sqrt{(T - t) V ar_{t,x}X^\pi(T)},
\]

where the \( X^\pi(t) \) is the unique solution of (3.13) corresponding to \( \pi^* \) at time \( t \) with the initial value \( x_0 \). Let \( t = 0 \), the efficient frontier at time 0 is

\[
\mathbb{E}_{0,x}X(T) = e^{\rho T} \frac{\omega - \omega_0}{\omega - \omega_0 - T} x + \rho T P[r(\omega - \omega_0) - 2] \\
\]

\[
- \frac{P[r(\omega - \omega_0 - 2T) - 2]}{\rho^2(\omega - \omega_0 - T)} + \frac{c - r}{\sigma} \sqrt{T V ar_{0,x}X(T)}
\]
with the initial optimal investment strategy

$$\pi^*(0) = e^{-rT} \frac{(c-r)(\omega - \omega_0 - T)}{\gamma \sigma^2 x_0(\omega - \omega_0)}$$

and $x_0$ is the initial fund size.

4. Sensitivity analysis

In this section, the impacts of risk aversion factor on the optimal investment strategy and the efficient frontier of the pension member are given.

First, we see from the expressions of (3.11) and (3.12) that the optimal proportion allocated in the equities at time $t$ is,

$$\pi^*(t) = a^*(t, X^\pi^*(t)) = e^{-r(T-t)} \frac{(c-r)(\omega - \omega_0 - T)}{\gamma \sigma^2 X^\pi^*(t)(\omega - \omega_0 - t)}. \quad (4.1)$$

It is obvious that $\pi^*$ decreases with respect to the risk aversion coefficient $\gamma$.

In (2.3), it is obvious that $(c-r)\pi + r + \frac{1}{\omega - \omega_0} > 0$ and $\frac{\omega - \omega_0 - 2t}{\omega - \omega_0 - t} > 0$. By the comparison theorem of the stochastic differential equations, $X^\pi^*(t)$ is an increasing function of the initial fund size $x_0$. Thus by (4.1) the proportion allocated in the equities decreases with respect to the initial fund size $x_0$.

The above conclusions show that, when the fund size is large, the management prefers to put higher proportions in the bonds to avoid the risk. While the fund size is small, the management needs to take the risk of the equity market to meet the pension member’s old-care expenses. $\gamma$ is individual pension member’s risk aversion coefficient and it describes the risk averse level of the member. The member with larger $\gamma$ has higher risk averse level and the proportion allocated in the equities should be small. See Picture 1 below. However, we can not say that $\pi^*$ increases with respect to $t$ directly because that $X^\pi^*(t)$ is the fund size at time $t$ according to the optimal strategy $\pi^*$ and it is also the function of $t$.

Next, we explore the efficient frontier of the pension member at time $t$. Similar to Markowitz model’s conclusion, the mean and the
standard deviation of the accumulation (or fund size) at time \( t \) satisfy a straight line. The straight line is the efficient frontier of the pension member at time \( t \). Different points on the line stand for different combinations of the mean and the variance of the accumulation, but they are all efficient under the mean-variance utility. See Picture 2 below. Picture 2 is based on the following:

\[
\mu = e^{r(T-t)} \frac{\omega - \omega_0 - t}{\omega - \omega_0 - T} \mu + e^{r(T-t)} P \left[ \frac{r(\omega - \omega_0 - 2T) - 2}{r^2(\omega - \omega_0 - T)} \right] + \frac{c - r - \hat{\sigma} \sqrt{(T-t)}}{\sigma},
\]

where \( X^{\pi^*}(t) \) is the unique solution of the SDE (3.13) with the initial value \( x_0 \), and \( \mu \equiv E_{t,x} X^{\pi^*}(T) \) and \( \hat{\sigma} \equiv \sqrt{\text{Var}_{t,x} X^{\pi^*}(T)} \).

5. Conclusion

In this paper, we study the optimal investment strategy of the DC pension plan during the accumulation phase. During the accumulation phase, the member contributes a predetermined amount of money as premiums and the management of the pension plan invests the accumulated premiums in the equities and the bonds to increase the
The efficient frontier of the DC pension fund under the optimal investment policy. $c = 0.05, r = 0.02, \sigma = 1, P = 1, \omega = 100, \omega_0 = 20, x_0 = 1, t = 0, T = 40$.

fund value. In practice, most of DC pension plans have the return of premiums clauses to protect the rights of the plan members who die during the accumulation phase. Under the clauses, the members withdraw their premiums when they die and the difference between the return and the accumulation is distributed among the survival members. We formalize the problem into the continuous-time optimal control problem. The management of the pension plan chooses the optimal investment strategy, i.e., the proportions invested in the equities and the bonds to maximize the mean-variance utility of the pension member at the time of retirement. Using the HJB methods, we establish the optimal control strategy and the efficient frontier of the pension member. The impacts of risk aversion coefficient on the optimal investment strategy are that, when the fund size is large, the management prefers to put higher proportions in the bonds to avoid the risk. While the fund size is small, the management needs to take the risk of the equity market to meet the pension member’s old-care expenses. The member with higher risk averse level has smaller proportion allocated in the equities, vice verse.
Acknowledgements. This work is supported by Project 11071136 of NSFC, and Project of Humanities and Social Sciences (Project No.11YJC790056). We would like to thank the institutions for the generous financial support. Special thanks also go to the participants of the seminar on Stochastic Analysis, Insurance Economics and Mathematical Finance at Department of Mathematical Sciences, Tsinghua University for their feedback and useful conversations.

References


