Some results on condition numbers of the scaled total least squares problem

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\textbf{Abstract}

Under the Golub–Van Loan condition for the existence and uniqueness of the scaled total least squares (STLS) solution, a first order perturbation estimate for the STLS solution and upper bounds for condition numbers of a STLS problem have been derived by Zhou et al. recently. In this paper, a different perturbation analysis approach for the STLS solution is presented. The analyticity of the solution to the perturbed STLS problem is explored and a new expression for the first order perturbation estimate is derived. Based on this perturbation estimate, for some STLS problems with linear structure we further study the structured condition numbers and derive estimates for them. Numerical experiments show that the structured condition numbers can be markedly less than their unstructured counterparts.

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1. Introduction

For given \(A \in \mathbb{R}^{m \times n} (m > n)\), \(b \in \mathbb{R}^{m}\), the STLS problem is formulated as (see [12])
\[
\min ||[E \ r]||_F, \quad \text{subject to} \quad \lambda b - r \in \mathcal{R}(A + E),
\]
where \(\lambda\) is a real positive parameter, \(||\cdot||_F\) denotes the Frobenius norm of a matrix and \(\mathcal{R}(\cdot)\) denotes the range space. Suppose that \([E_{STLS} \ r_{STLS}]\) solves the above problem. Then \(x = x_{STLS}\) that satisfies the...
equation \((A + E_{\text{STLS}})\lambda \mathbf{x} = \lambda \mathbf{b} - r_{\text{STLS}}\) is called the STLS solution of \((1)\). A theoretically equivalent but different formulation of the STLS problem can be found in [14]. The STLS problem unifies ordinary least squares (LS), total least squares (TLS) and data least squares (DLS) problems, see [12–14]. Specifically, \(x_{\text{STLS}}\) becomes the TLS solution when \(\lambda = 1\), \(x_{\text{STLS}}\) converges to the LS solution as \(\lambda \to 0\), and \(x_{\text{STLS}}\) converges to the DLS solution as \(\lambda \to \infty\).

Condition numbers measure the worst-case sensitivity of a solution to a problem to small perturbations in the input data. The condition numbers of LS problems have been studied widely. For the STLS problem, recently, Zhou et al. in [26] have derived upper bounds for several kinds of condition numbers based on a first order perturbation estimate for the STLS solution under the well-known Golub–Van Loan condition for the existence and uniqueness of the STLS solution. As shown in the numerical experiments in [26], their perturbation estimate is more realistic than those presented in [5,6,11,22,23], and their bounds for condition numbers are sharp.

In this paper, we consider two kinds of condition numbers, called normwise condition number and componentwise condition number in the terminology of Higham [8] and Rump [15,16]. When a STLS problem is structured, we consider the corresponding structured normwise condition number and structured componentwise condition number. Under the same assumptions as those in [26], we present a different perturbation analysis approach for the STLS solution. It is based on the perturbation analysis results of the singular value decomposition (SVD) [19] and the optimization formulation of the STLS problem [6,12]. In our perturbation analysis results, we show the analyticity of the solution to the perturbed STLS problem and derive a new expression for the first order perturbation estimate for the STLS solution. Based on this perturbation estimate, we further study the structured condition numbers for those STLS problems with some linear structure and derive estimates for them. Actually, like LS problems, large, sparse or structured TLS problems arise in many signal and image processing applications, e.g., spectral parameter estimates in the field of nuclear magnetic resonance (NMR) [9,20,21]. When \(A\) is large, sparse or structured, some methods for solving TLS problems with data matrix \([A \ b]\) have been studied, see [2,10,25] and the references therein. These methods are easy to modify to solve the STLS problems [12]. To our knowledge, however, structured condition numbers of the structured STLS problems have not been considered. In this paper, we study this problem. Numerical experiments show that the structured condition numbers of the STLS problem can be markedly less than their unstructured counterparts.

The paper is organized as follows. In Section 2, we give some preliminaries necessary. In Section 3, we present our perturbation analysis results of the STLS solution. The structured condition numbers of the STLS problem are considered in Section 4. In Section 5, we present numerical experiments to compare the structured condition numbers with their unstructured counterparts. We end the paper with some concluding remarks in Section 6. The detail concerning the proof of Theorem 3.2 is given in Appendix.

Throughout the paper, for given positive integers \(m, n\), \(\mathbb{R}^n\) denotes the space of \(n\)-dimensional real column vectors, \(\mathbb{R}^{m \times n}\) denotes the space of \(m \times n\) real matrices. \(\| \cdot \|, \| \cdot \|_{\infty}\text{, and } \| \cdot \|_F\) denote 2-norm, \(\infty\)-norm and Frobenius norm of their arguments, respectively. Given a matrix \(A\), \(A[:, i]\) is a Matlab notation that denotes the \(ith\) column of \(A\), \(\sigma_i(A)\) denotes the \(ith\) largest singular value of \(A\), \(|A|\) is taken absolute entrywise, the matrix inequality \(|A| \leq |B|\) holds entrywise, and \(A^T\) denotes the pseudo-inverse of \(A\). For a vector \(a\), \(a[i]\) denotes the \(ith\) component of \(a\), \(\text{diag}(a)\) is a diagonal matrix whose diagonals are given by components of \(a\), \(I_n\) denotes the \(n \times n\) identity matrix, \(O_{mn}\) denotes the \(m \times n\) zero matrix, whereas \(O\) denotes a zero matrix whose order is clear from the context. For matrices \(A = [a_1, \ldots, a_n] = [A_j] \in \mathbb{R}^{m \times n}\) and \(B, A \otimes B = [A_jB]\) is the Kronecker product of \(A\) and \(B\), the linear operator \(\text{vec} : \mathbb{R}^{m \times n} \to \mathbb{R}^{mn}\) is defined by \(\text{vec}(A) = [a_1^T, \ldots, a_n^T]^T\). Moreover, \(\nabla\) and \(\nabla^2\) denote the gradient vector and Hessian matrix operators, respectively.

2. Preliminaries

Throughout the paper, we let \(\tilde{U}^T A \tilde{V} = \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)\) be the thin SVD of \(A \in \mathbb{R}^{m \times n}\), where \(\tilde{\sigma}_1 \geq \cdots \geq \tilde{\sigma}_n, \tilde{U} \in \mathbb{R}^{m \times n}, \tilde{U}^T \tilde{U} = I_n, \tilde{V} \in \mathbb{R}^{n \times n}, \tilde{V}^T \tilde{V} = I_n\). Let \(U^T [A \lambda \mathbf{b}] V = \text{diag}(\sigma_1, \ldots, \sigma_{n+1})\) be the
thin SVD of \( A \lambda b \in \mathbb{R}^{m \times (n+1)} \), where \( \sigma_1 \geq \cdots \geq \sigma_{n+1} U = [u_1, \ldots, u_{n+1}] \in \mathbb{R}^{m \times (n+1)}, U^T U = I_{n+1}, V = [v_1, \ldots, v_{n+1}] \in \mathbb{R}^{(n+1) \times (n+1)}, V^T V = I_{n+1} \).

Note that the formulation of STLS problem (1) can be regarded as a TLS problem with data \( A \lambda b \). The known TLS theory and algorithms can be applied directly to the STLS problem [12]. Actually, \( \lambda x_{STLS}^* \) is called the TLS solution of (1) in [12]. The following result presents an existence and uniqueness condition for the STLS solution that is due to the work of Golub and Van Loan [6].

**Theorem 2.1.** If

\[ \sigma_{n+1} < \hat{\sigma}_n \]  

then the STLS problem (1) has a unique solution \( x_{STLS}^* \). Moreover,

\[ x_{STLS}^* = \left( A^T A - \sigma_{n+1}^2 I_n \right)^{-1} A^T b \]

\[ = -\frac{1}{\lambda} \left[ \begin{array}{c} v_{n+1}[1] \\ v_{n+1}[n+1] \\ \vdots \\ v_{n+1}[n+1] \end{array} \right]^T. \]

The following optimization formulation of the STLS problem is important in the paper. It is shown [6,12] that the STLS solution

\[ x_{STLS}^* = \arg \min_x f(x) = \arg \min_x \|Ax - b\|^2 \]

and \( \sigma_{n+1}^2 \) is the global minimum of \( f(x) \), i.e.,

\[ \sigma_{n+1}^2 = \frac{\|Ax_{STLS}^* - b\|^2}{\lambda^{-2} + \|x_{STLS}^*\|^2}. \]

Note that condition (2) implies that \( \sigma_n(A) > 0 \). As stated in [1, p. 178], if \( \sigma_{n+1}([A \lambda b]) = 0 \), then \( \lambda b \in \mathcal{R}(A) \). At this time, the system of equations \( Ax = \lambda b \) is compatible, and we can take \( [E R] = 0 \). As in [26], throughout the paper, we assume that

\[ 0 < \sigma_{n+1} < \hat{\sigma}_n. \]

Following Theorem 2.1, under (7), the STLS problem (1) has a unique STLS solution.

### 3. A perturbation estimate of the STLS solution

Let \( \tilde{A} = A + \Delta A, \tilde{b} = b + \Delta b \), where \( \Delta A \) and \( \Delta b \) denote the perturbations in \( A \) and \( b \), respectively. Consider the perturbed STLS problem

\[ \min \| [E R] \|_F \quad \text{subject to} \quad \lambda \tilde{b} - r \in \mathcal{R}(\tilde{A} + E). \]

It is clear that when \( [\Delta A \lambda \Delta b] \) is sufficiently small, the perturbed STLS problem (8) has a unique solution as well. Denote by \( \tilde{x}_{STLS}^* \) the STLS solution of the perturbed problem, and by

\[ \Delta x := \tilde{x}_{STLS}^* - x_{STLS}^*. \]

The normwise and componentwise condition numbers of the STLS problem are defined in [26] as follows:

**Definition 1**

\[ \kappa_{STLS} := \lim_{\epsilon \to 0} \sup_{\|\Delta A\|_F \leq \epsilon, \|\lambda b\|_F \leq \epsilon} \frac{||\Delta x||}{||\|\Delta a||_F ||x_{STLS}||}, \]

\[ \mu_{STLS} := \lim_{\epsilon \to 0} \sup_{\|\Delta A\| \leq \epsilon, \|\lambda b\| \leq \epsilon} \frac{||\Delta x||_{\infty}}{||x_{STLS}||_{\infty}}. \]
Based on (3), the authors of [26] make a first order perturbation analysis for the STLS solution and derive sharp bounds for the condition numbers shown in the following:

\[
\kappa_{\text{STLS}} \leq \frac{\|M + N\| \|[A, \lambda b]\|_F}{\|x_{\text{STLS}}\|},
\]

(11)

\[
\mu_{\text{STLS}} \leq \frac{\|[M + N] \text{vec}([|A|, \lambda |b|])\|_\infty}{\|x_{\text{STLS}}\|_\infty},
\]

(12)

where

\[
M := \left[ P \otimes b^T - x_{\text{STLS}}^T \otimes (PA^T) - P \otimes (A_{\text{STLS}}x)^T \lambda^{-1}PA^T \right],
\]

\[
N := 2\sigma_{n+1}y \left( v_{n+1}^T \otimes u_{n+1}^T \right),
\]

\[
P = (A^T A - \sigma_{n+1}^2 I_n)^{-1}
\]

and \( y = P x_{\text{STLS}} \).

Below we present a different perturbation analysis approach, which depends strongly on perturbation analysis results of SVD.

Classical contributions to the perturbation analysis of SVD can be found in [17–19], etc. The following lemma is an immediate result of Corollary 2.2 of [19].

**Lemma 3.1.** Let \( \sigma_{\text{min}} \) be the smallest nonzero and simple singular value of a matrix \( X \) with \( v_{\text{min}} \) being its corresponding right singular vector, and let \( \bar{\sigma}_{\text{min}} \) be the smallest singular value of the perturbed matrix \( \bar{X} = X + \Delta X \). If the perturbation \( \Delta X \) is small enough, then \( \bar{\sigma}_{\text{min}} \) is also a simple singular value of \( \bar{X} \), and the corresponding right singular vector \( \bar{v}_{\text{min}} \) is a real analytic function of \( \text{vec}(\Delta X) \) in some neighborhood of the origin.

Based on Lemma 3.1, we get the following result.

**Theorem 3.1.** Suppose the STLS problem (1) satisfies (7). Then the perturbed STLS problem (8) has a unique STLS solution \( x(\Delta A, \Delta b) \) such that \( x(\Delta A, \Delta b) \) is a real analytic function of \( \text{vec}(\Delta X) \) in some neighborhood of the origin.

**Proof.** Note that

\[
\|[\Delta A, \lambda \Delta b]\|_F \leq \max(1, \lambda)\|[\Delta A, \Delta b]\|_F.
\]

From the assumption (7), it follows that if \( \|[\Delta A, \Delta b]\|_F \) is small enough then the perturbed problem (8) satisfies (2) and thus has a unique STLS solution.

Now note that (7) implies that \( \sigma_{n+1} \) is a simple and nonzero singular value of \([A, \lambda b]\) by the interlacing property [24, p. 103] for eigenvalues of symmetric matrices. Let \( \Delta X = [\Delta A, \lambda \Delta b] \), \( \bar{X} = [A, \lambda b] + \Delta X \). A direct application of Lemma 3.1 to \([A, \lambda b]\) means that the minimum right singular vector \( \bar{v}_{n+1} \) of \( \bar{X} \) is a real analytic function of \( \text{vec}(\Delta X) \) in some neighborhood of the origin. Noticing that if \([A + \Delta A, \lambda(b + \Delta b)]\) satisfies (2) then \( \bar{v}_{n+1}[n + 1] \neq 0 \) [6], it follows that the quotients

\[
\frac{\bar{v}_{n+1}[i]}{\bar{v}_{n+1}[n + 1]}, \quad i = 1, \ldots, n
\]

are all real analytic provided that \( \|[\Delta X]\|_F \) is small enough. Thus, based on (4), regarded as a function of \( \text{vec}([\Delta A, \Delta b]) \), the perturbed STLS solution is real analytic in some neighborhood of the origin.

We now present one of our main results. We will put its tedious proof in Appendix.

**Theorem 3.2.** Under the assumption of Theorem 3.1, denote by \( x^* = x_{\text{STLS}} \) the solution of the STLS problem (1), and define \( r = Ax^* - b \), \( G(x) = [x^T - 1] \otimes I_n \). If \( \|[\Delta A, \Delta b]\|_F \) is small enough, then the perturbed problem (8) has a unique STLS solution \( x(\Delta A, \Delta b) \). Moreover,
\[ x(\Delta A, \Delta b) = x_{\text{STLS}} + K \begin{bmatrix} \text{vec}(\Delta A) \\ \Delta b \end{bmatrix} + O \left( \|\Delta A \Delta b\|_F^2 \right), \]  

where
\[ K = \left( A^T A - \sigma_{n+1}^2 I_n \right)^{-1} \begin{bmatrix} 2A^T r \frac{r^T G(x^*)}{\|r\|} - A^T G(x^*) - [I_n \otimes r^T O] \end{bmatrix}. \]  

Now let us show the relation between \( M, N \) and \( K \). We use the same notations as those in Theorem 3.2. Note that
\[ v_{n+1} = \frac{[\lambda x^T - 1]^T}{\|\lambda x^T - 1\|^T}, \quad [A \lambda b]v_{n+1} = \sigma_{n+1} u_{n+1}, \]
see [6] for a proof. A direct calculus gives us
\[ u_{n+1}^T = \frac{1}{\sigma_{n+1}} \frac{r^T}{\sqrt{\lambda^{-2} + \|x^*\|^2}}. \]

It follows that
\[ N = 2\sigma_{n+1} P x^* u_{n+1}^T (v_{n+1}^T \otimes I_m) = P \frac{2x^* r^T G(x^*)}{\lambda^{-2} + \|x^*\|^2} D^{-1}, \]
\[ = 2P A^T \frac{r^T}{\|r\|} G(x^*) D^{-1}, \]
where we used (A8) in the third equality and \( D = \begin{bmatrix} I_{mn} & 0 \\ 0 & \lambda I_m \end{bmatrix} \). Meantime, collecting terms in \( M \) and making a simple calculation, we have
\[ M = -PA^T G(x^*) D^{-1} - P \left[ I_n \otimes r^T O \right]. \]

Comparing with (14), we get the desired equality
\[ (M + N)D = K. \]  

Essentially, Theorem 3.2 establishes the same perturbation estimate for \( \Delta x \) as that in Lemma 3.2 of [26]. As will be seen, however, our new form of the Jacobian \( K \) makes it more convenient to study the structured condition numbers.

In order to facilitate the comparison between structured condition numbers and their unstructured counterparts, we rewrite (11) and (12) in terms of \( K \) as follows:

**Theorem 3.3**

\[ \kappa_{\text{STLS}} \leq \frac{\left\| K D^{-1} \right\|}{\|x_{\text{STLS}}\|} \|A \lambda b\|_F =: \kappa^K_{\text{STLS}}, \]  
\[ \mu_{\text{STLS}} \leq \frac{\left\| K \text{vec}(\|A\| b)\|\right\|_\infty}{\|x_{\text{STLS}}\|_\infty} =: \mu^K_{\text{STLS}}. \]

4. **Structured normwise and componentwise condition numbers**

Suppose that \( \mathcal{L} \subseteq \mathbb{R}^{m \times n} \) is a linear subspace which consists of a class of structured matrices. Specifically, there are \( q (q \leq mn) \) linearly independent matrices \( S_1, \ldots, S_q \in \mathcal{L} \), such that for any \( A \in \mathcal{L} \) we have
A = \sum_{i=1}^{q} p_i S_i, \quad (18)

where \( p = [p_1, \ldots, p_q]^T \in \mathbb{R}^q \). An equivalent formulation of (18) is

\[ \text{vec}(A) = \Phi^{\text{struct}} p, \quad (19) \]

where \( \Phi^{\text{struct}} = [\text{vec}(S_1), \ldots, \text{vec}(S_q)] \). From now on we use (19) to describe the linear structure of \( A \).

We say a matrix \( \Delta A \) has the same structure as that of \( A \) if \( \text{vec}(\Delta A) = \Phi^{\text{struct}} \epsilon \) for some \( \epsilon \in \mathbb{R}^q \). This idea came from [7,15].

For simplicity, for \( \mathcal{L} \) with which we deal in this paper we always assume that each element of \( A \in \mathcal{L} \) depends on a single component of \( p \). Several kinds of structured matrices fall into this category, such as Toeplitz, Hankel, and circulant matrices.

The following theorem will be useful later.

**Theorem 4.1.** \( \Phi^{\text{struct}} \) has the following properties: (a) It is of full column rank. (b) There is at most one nonzero entry in each row and there is no zero column. (c) It is column orthogonal.

**Proof.** (a) is obtained by the definition of \( \Phi^{\text{struct}} \). The first part of (b) is obtained by the assumption that each element of \( A \in \mathcal{L} \) depends on a single component of \( p \) and the second part is obtained by (a). (c) is obtained by the first part of (b). \( \square \)

For a general matrix \( A \in \mathbb{R}^{m \times n} \) without exhibiting any structure, (19) is also valid. In fact, we can take \( \Phi^{\text{struct}} = I_m \), \( p = \text{vec}(A) \).

For the STLS problem (1) we now assume that \( A \) has some linear structure as defined in (19) and \( b \) does not exhibit any such structure. For convenience, we express \( \text{vec}([A \ b]) \) as

\[ \text{vec}([A \ b]) = \Phi^{\text{struct}}_{A,b} s, \quad (20) \]

where \( \Phi^{\text{struct}}_{A,b} = \begin{bmatrix} \Phi^{\text{struct}} & l_m \end{bmatrix}. s = [p^T, b^T]^T \). The expression (20) and the analysis below could be modified to work for a structured \( b \).

For the perturbed STLS problem (8) we now restrict the perturbation matrices \([\Delta A \ \Delta b]\) to have the same structure as that of \([A \ b]\); that is,

\[ \text{vec}([\Delta A \ \Delta b]) = \Phi^{\text{struct}}_{A,b} \epsilon, \quad (21) \]

where \( \epsilon \in \mathbb{R}^{q+m} \). Obviously, \([A + \Delta A \ b + \Delta b]\) will have the same structure as that of \([A \ b]\). With the structured perturbations \([\Delta A \ \Delta b]\) we define the structured normwise and componentwise condition numbers of the STLS problem as

\[ \kappa^{s}_{\text{STLS}} := \lim_{\epsilon \to 0} \sup_{\|\Delta A\|_{\infty} \leq \epsilon \|A\|_{\infty}} \frac{\|\Delta x\|_{\infty}}{\|\epsilon\|_{\text{STLS}}}, \quad (22) \]

\[ \mu^{s}_{\text{STLS}} := \lim_{\epsilon \to 0} \sup_{\|\Delta A\|_{\infty} \leq \epsilon \|A\|_{\infty}} \frac{\|\Delta x\|_{\infty}}{\|\epsilon\|_{\text{STLS}}}, \quad (23) \]

respectively, where \( \Delta x = \bar{x}_{\text{STLS}} - x_{\text{STLS}} \) and \( \bar{x}_{\text{STLS}} \) is the solution of the structured perturbed STLS problem.

Next we show how to derive estimates for the structured condition numbers.

Since Theorem 3.2 is still valid when the perturbations \([\Delta A \ \Delta b]\) are structured, based on (21), (13) becomes

\[ x(\Delta A, \Delta b) = x_{\text{STLS}} + K\Phi^{\text{struct}}_{A,b} \epsilon + O(\|\epsilon\|^2). \quad (24) \]

Note that \( x(\Delta A, \Delta b) \) is just a function of \( \epsilon \). For simplicity, denote it by \( x(\epsilon) \). Then

\[ x(\epsilon) = x_{\text{STLS}} + K\Phi^{\text{struct}}_{A,b} \epsilon + O(\|\epsilon\|^2), \quad (25) \]
where $x_{\text{STLS}} = x(0)$.  
Recall that $G(x) = [x^T - 1] \otimes I_n$ is such that 

$$Ax - b = G(x) \text{vec}([A \ b]).$$

It follows from (20) that 

$$Ax - b = G(x) \Phi_{\text{struct}}^A b.$$

Define $H(x) = G(x) \Phi_{\text{struct}}^A b$. Then 

$$\frac{\partial}{\partial x_i} H = \left( \frac{\partial}{\partial x_i} G \right) \Phi_{\text{struct}}^A b, \quad i = 1, \ldots, n.$$

Thus, from (14), noticing that $\left[ I_n \otimes r^T O \right] = \begin{bmatrix} r^T \frac{\partial}{\partial x_1} G \\ \vdots \\ r^T \frac{\partial}{\partial x_n} G \end{bmatrix}$, we get 

$$K \Phi_{\text{struct}}^A b = \left( A^T A - \sigma^2 n + 1 \right)^{-1} \begin{bmatrix} 2A^T r r^T \frac{\partial}{\partial x_1} H(x^*) - A^T H(x^*) \\ \vdots \\ r^T A \frac{\partial}{\partial x_n} H \end{bmatrix},$$  
(26)

where $x^* = x_{\text{STLS}}, r = Ax^* - b$. For simplicity, we set 

$$K^s = K \Phi_{\text{struct}}^A b.$$  
(27)

From (26) and (14) it is shown that the difference between $K^s$ and $K$ is that $G$ is replaced by $H$. We note that $H(x)$ has $q + m$ columns whereas $G(x)$ has $nm + m$ columns. When $q \ll nm$, the scale of $H(x)$ is much smaller than that of $G(x)$. In practice, it is not difficult to derive the analytic expressions for entries of $H(x)$. Thus, it is convenient to construct $K^s$ via (26). 

Before deriving the structured condition numbers of the STLS problem, we let $w = [\| \Phi_{\text{struct}}^A : 1 \|, \ldots, \| \Phi_{\text{struct}}^A : q \|]^T$ and 

$$D_\lambda = \begin{bmatrix} \text{diag}(w) \\ \lambda I_m \end{bmatrix}.$$  
(28)

Then $D_\lambda$ is invertible and $\| D_\lambda s \| = \| [A \lambda b] \|_F$ by the second part of (b) and (c) in Theorem 4.1, respectively.

Now we are in a position to give the following result.

**Theorem 4.2.** Suppose that $[A \ b]$ is structured as (20). Then 

$$\kappa_{\text{STLS}}^s \leq \frac{\| K^s D_\lambda^{-1} \|_F}{\| x_{\text{STLS}} \|} : = \kappa_{\text{STLS}}^s K,$$  
(29)

$$\mu_{\text{STLS}}^s \leq \frac{\| K^s \|_\infty}{\| x_{\text{STLS}} \|_\infty} : = \mu_{\text{STLS}}^s K.$$  
(30)

**Proof.** Suppose that $[\Delta A \ \Delta b]$ satisfies (21) and $\| [\Delta A \ \lambda \Delta b] \|_F \leq \varepsilon \| [A \ \lambda b] \|_F$ for small $\varepsilon > 0$. Note that 

$$\| [\Delta A \ \lambda \Delta b] \|_F = \| D_\lambda \varepsilon \|$$

and 

$$\| \varepsilon \| \leq \| D_\lambda^{-1} \| D_\lambda \varepsilon \| \leq \varepsilon \| D_\lambda^{-1} \| \| [A \lambda b] \|_F.$$
From (25) we get

\[ \frac{\|\Delta x\|}{\|x_{STLS}\|} \leq \epsilon \frac{\|K^sD_\lambda^{-1}\| \|\langle A, \lambda b \rangle\|_F}{\|x_{STLS}\|} + O(\epsilon^2). \] (31)

Thus by the definition of \( \kappa^s_{STLS} \) we get (29) immediately.

Now suppose that \( \begin{bmatrix} \Delta A \\ \Delta b \end{bmatrix} \) satisfies (21) and \( \|\begin{bmatrix} \Delta A \\ \Delta b \end{bmatrix}\| \leq \epsilon \|A \ b\| \) for small \( \epsilon > 0 \), that is \( |\Phi_{\text{struct}} e| \leq \epsilon |\Phi_{\text{struct}} s| \). By (b) in Theorem 4.1, we have equivalently

\[ |\epsilon| \leq |s| \] (32).

Note that \( \epsilon[i] = 0 \) if \( s[i] = 0 \). Motivated by the proof of Theorem 4.1 of [26], we derive from (32) that

\[ \epsilon = D_s D_\lambda^T e, \quad \|D_s^T e\|_\infty \leq \epsilon, \]

where \( D_s = \text{diag}(s) \), and that

\[ \|\epsilon\|_\infty \leq \|s\|_\infty. \]

Utilizing (25) again we get

\[ \frac{\|\Delta x\|_\infty}{\|x_{STLS}\|_\infty} \leq \epsilon \frac{\|K^sD_\lambda\|_\infty}{\|x_{STLS}\|_\infty} + O(\epsilon^2) = \epsilon \frac{\|K^s\|s\|_\infty}{\|x_{STLS}\|_\infty} + O(\epsilon^2). \] (33)

Thus by the definition of \( \mu^s_{STLS} \) we get (30). \( \square \)

As implied in [15, 16], the ratio between the structured and unstructured condition numbers of the STLS problem may be explored with the help of \( H(x) \). We will investigate this problem in the future research. Now we have the following result.

**Theorem 4.3.** For a STLS problem with structure (20), we have

\[ \kappa^s_{STLS} \leq \kappa^K_{STLS} \quad \text{and} \quad \mu^s_{STLS} \leq \mu^K_{STLS}. \]

**Proof.** By the definition (28) of \( D_\lambda \) and the column orthogonality of \( \Phi_{\text{struct}} A, b \), recalling \( D = \begin{bmatrix} I_{mn} & \lambda I_m \end{bmatrix} \) we derive

\[ \left\| D\Phi_{\text{struct}} A, b D_\lambda^{-1} \right\| = 1 \]

and

\[ \|K^sD_\lambda^{-1}\| = \|KD_\lambda^{-1} D\Phi_{\text{struct}} A, b D_\lambda^{-1}\| \leq \|KD\|^{-1}. \]

Thus, we get \( \kappa^s_{STLS} \leq \kappa^K_{STLS} \). On the other hand, noticing that there is at most one nonzero entry in each row of \( \Phi_{\text{struct}} A, b \), we get

\[ \|K|\text{vec}([A] \ b)|\|_\infty = \|K|\Phi_{\text{struct}} s|\|_\infty = \|K|\Phi_{\text{struct}} s|\|_\infty \]

and

\[ \|K^s\|s\|_\infty \leq \|K|\Phi_{\text{struct}} s|\|_\infty = \|K|\text{vec}([A] \ b)|\|_\infty. \]

Thus, we have \( \mu^s_{STLS} \leq \mu^K_{STLS}. \) \( \square \)

Under the condition that \( A \) is of full column rank, for the LS problem min\( \|Ax - b\|_2 \), the authors in [3] derive the following componentwise condition number, called the mixed condition number in [3]:

\[ \mu_{LS} = \frac{\left\| -\left(x_{LS}^T \otimes A^T\right) + \left(A^T A\right)^{-1} \otimes \left(b - Ax_{LS}\right)^T \right\|_\infty}{\|x_{LS}\|_\infty}. \]
Furthermore, based on the work of [3], the authors in [4] derive the following structured componentwise condition number for \( A \in \mathcal{L} \) with \( \mathcal{L} \) a class of linear structured matrices:

\[
\mu_{k}^{sL} = \frac{\|(-x_{LS}^T \otimes A^\dagger) + (A^T A)^{-1} \otimes (b - Ax_{LS})^T V\| \|\|a\| + |A^\dagger| \|b\|\|_{\infty}}{\|x_{LS}\|_{\infty}},
\]

where \( V = [\text{vec}(S_1), \ldots, \text{vec}(S_k)] \), \( S_1, \ldots, S_k \) is a basis of \( \mathcal{L} \), and \( a = [a_1, \ldots, a_k]^T \in \mathbb{R}^k \) is such that \( A = \sum_{i=1}^{k} a_i S_i \).

As we know, the solution of the STLS problem (1) approaches the LS solution of \( \min\|Ax - b\|_2 \) as \( \lambda \to 0 \). In [26], the authors show that the componentwise condition number of the STLS problem approaches the componentwise condition number of the LS problem as \( \lambda \to 0 \). Next we show that it is also the case for the structured componentwise condition number.

Note that \( \sigma_{n+1}([A \lambda b]) \to 0 \) when \( \lambda \to 0 \). It follows that if \( \sigma_{n+1}([A \lambda b]) < \sigma_n(A) \) is valid for some \( \lambda > 0 \), then \( A^T A - \sigma_{n+1}^2([A \lambda b]) I_n \) is positive definite for \( \lambda \) small enough. Thus, \( (A^T A - \sigma_{n+1}^2([A \lambda b]) I_n)^{-1} \to (A^T A)^{-1} \) as \( \lambda \to 0 \). On the other hand, as \( \lambda \to 0 \), \( x_{STLS} \to x_{LS} \), \( r \to r_{LS} = Ax_{LS} - b \), and

\[
K = -(x_{LS}^T - 1) \otimes A^\dagger - (A^T A)^{-1} (I_n \otimes r_{LS}^T) O,
\]

So, we get

\[
\begin{aligned}
K^s &= K \phi^{\text{struct}}_{A,b} \\
\to &\quad -(x_{LS}^T - 1) \otimes A^\dagger \phi^{\text{struct}}_{A,b} - (A^T A)^{-1} (I_n \otimes r_{LS}^T) \phi^{\text{struct}}_{A,b} \\
= &\quad [-(x_{LS}^T \otimes A^\dagger \phi^{\text{struct}}_{A,b} - (A^T A)^{-1} \otimes r_{LS}^T \phi^{\text{struct}}_{A,b})^\dagger],
\end{aligned}
\]

\[
\mu_{STLS}^{sk} \to \left\|\frac{x_{LS}^T \otimes A^\dagger \phi^{\text{struct}} + (A^T A)^{-1} \otimes r_{LS}^T \phi^{\text{struct}}}{\|x_{LS}\|_{\infty}}\right\|_{\infty} \|\|a\| + |A^\dagger| \|b\|\|_{\infty}
\]

The last equality holds by the definition of \( \phi^{\text{struct}} \).

5. Numerical experiments

We report numerical experiments to illustrate that our structured condition numbers can be considerably smaller than their unstructured counterparts and can measure the sensitivity much more precisely. The STLS problem to be considered in the following is modified from the third test problem in [10]. The original one is just a TLS problem. Below we calculate the unstructured condition numbers and the structured counterparts of the STLS problem in Theorems 3.3 and 4.2. All the experiments are carried out using Matlab 7.6.

Example. The Toeplitz matrix used in this example comes from an application in signal restoration [10]. Specifically, an \( m \times (m - 2\omega) \) convolution matrix \( \bar{T} \) is constructed to have entries in the first column given by

\[
t_{i,1} = \begin{cases} \sqrt{2\pi \omega^2 \exp \left[ \frac{-(\omega - i + 1)^2}{2\omega^2} \right]} & i = 1, 2, \ldots, 2\omega + 1, \\ 0 & \text{otherwise}, \end{cases}
\]

and entries in the first row given by

\[
t_{1,j} = \begin{cases} t_{1,1} & \text{if } j = 1, \\ 0 & \text{otherwise}, \end{cases}
\]
where \( \alpha = 1.25, \omega = 8 \) and \( m = 100 \). A Toeplitz matrix \( A \) and a right-hand side vector \( b \) are constructed as \( A = \overline{T} + E \) and \( b = \overline{g} + e \), where \( \overline{g} = [1, \ldots, 1]^T \), \( E \) is a random Toeplitz matrix with the same structure as \( \overline{T} \) and \( e \) is a random vector. The entries in \( E \) and \( e \) are generated randomly from a normal distribution with mean zero and variance one, and scaled so that

\[
\|e\| = \gamma \|\overline{g}\|, \quad \|E\| = \gamma \|\overline{T}\|. \quad \gamma = 0.001.
\]

The structure of \( A \) is exploited by taking \( \Phi_{\text{struct}} = [\Phi_1^T, \ldots, \Phi_{m-2\omega}^T]^T \), where \( \Phi_i = \begin{bmatrix} I_{2\omega+1} \\ O_{m-2\omega-1, 2\omega+1} \end{bmatrix} \), \( \Phi_i = Z_m \Phi_i (i \geq 1) \) with \( Z_m \) a lower shift matrix of order \( m \), and \( p = [A_{1,1}, \ldots, A_{2\omega+1,1}]^T \), where \( A_{ij} \) denotes the \((i,j)\) entry of \( A \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\lambda & \|\Delta x\|_{\infty} & \epsilon \lambda \mu_{\text{STLS}}^{\text{Sk}} & \epsilon \lambda \mu_{\text{STLS}}^{K} & \|\Delta x\|_{\infty} & \epsilon \mu_{\text{STLS}}^{\text{Sk}} \\
\hline
1 & 6.3388e-8 & 8.7377e-7 & 2.9324e-3 & 6.3692e-8 & 1.1513e-6 & 1.0321e-4 \\
0.05 & 6.3619e-8 & 6.1265e-6 & 1.0991e-3 & 6.3923e-8 & 1.1513e-6 & 1.0321e-4 \\
0.005 & 6.3691e-8 & 6.0804e-5 & 1.0972e-3 & 6.3995e-8 & 1.1523e-6 & 1.0325e-4 \\
0.0005 & 8.4046e-8 & 6.0804e-4 & 1.2490e-3 & 8.4350e-8 & 1.2514e-6 & 1.0325e-4 \\
\hline
\end{array}
\]

In the above table, for each parameter \( \lambda \) we present relative perturbations \( \|\Delta x\|_{\infty} \) of the STLS solution, and their approximate upper bounds \( \epsilon \lambda \mu_{\text{STLS}}^{\text{Sk}} \), \( \epsilon \lambda \mu_{\text{STLS}}^{K} \) and \( \epsilon \mu_{\text{STLS}}^{\text{Sk}} \), \( \epsilon \mu_{\text{STLS}}^{K} \), respectively. The perturbation \( [\Delta A \ \Delta b] \) has the same structure as that of \([A \ b]\) and is constructed by a vector \( \epsilon \), \( \epsilon[i] = \eta[i] s[i] \), where \( s = [p^T, b^T]^T \) and \( \eta[i] \) are random variables uniformly distributed on the interval \((0, 10^{-10})\). \( \epsilon, \lambda \) and \( \epsilon \) are taken as \( \|\Delta A \ \Delta b\|_F \) and \( 10^{-10} \), respectively.

As shown in the table, \( \epsilon \lambda \mu_{\text{STLS}}^{\text{Sk}} \) (resp., \( \epsilon \mu_{\text{STLS}}^{\text{Sk}} \)) provides a tighter perturbation bound than \( \epsilon \lambda \mu_{\text{STLS}}^{K} \) (resp., \( \epsilon \mu_{\text{STLS}}^{K} \)) and the improvement can be of two orders of magnitude or more. As upper bounds for relative perturbations of the STLS solution, both \( \epsilon \lambda \mu_{\text{STLS}}^{\text{Sk}} \) and \( \epsilon \mu_{\text{STLS}}^{\text{Sk}} \) are sharp when \( \lambda \geq 0.05 \), however, \( \epsilon \mu_{\text{STLS}}^{\text{Sk}} \) behaves better when \( \lambda \) is smaller.

6. Concluding remarks

In this paper, we have derived a new expression for the first order estimate of the perturbation in the STLS solution. Based on this estimate, we have studied the structured condition numbers for those STLS problems with some linear structures. Numerical experiments show that the structured condition numbers can be markedly smaller than their unstructured counterparts. Whether there exists any phenomenon that the unstructured componentwise condition number of a STLS problem is \( O(1/\epsilon) \), whereas the structured counterpart is \( O(1) \), as behaved for some structured linear systems (see [16]), is worthy of further study.

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Appendix A. Proof of Theorem 3.2

When \( \|\Delta A \ \Delta b\|_F \) is small enough, the existence and uniqueness of solution of the perturbed STLS problem is obtained easily. Furthermore, by Theorem 3.1, \( x(\Delta A, \Delta b) \) is real analytic in some neighborhood of the origin. Thus, the Taylor series of \( x(\Delta A, \Delta b) \) with center the origin converges provided that \( \|\Delta A \ \Delta b\|_F \) is sufficiently small. As a result, to prove (13) it suffices to prove \( \nabla_{\text{vec}}([\Delta A \ \Delta b]) x(0) \), the Jacobian of \( x(\Delta A, \Delta b) \) at the origin, equal to \( K \).
For convenience, we now denote
\[ \epsilon = \text{vec}(\Delta A \Delta b) \]
and \( x(\epsilon) \) the STLS solution of the perturbed problem (8).

For \( x_{\text{STLS}} \) we have (5), for \( x(\epsilon) \) we have similarly
\[
x(\epsilon) = \arg \min_x \| (A + \Delta A)x - (b + \Delta b) \|_2^2.
\]
Specially, \( x(0) = x_{\text{STLS}} \). Following the notation in (5) we define
\[
f(x, \epsilon) = \| (A + \Delta A)x - (b + \Delta b) \|_2^2.
\]
Recall that \( \tilde{A} = A + \Delta A \) and \( \tilde{b} = b + \Delta b \). Define \( \tilde{r} = \tilde{A}x - \tilde{b} \). Then we derive that
\[
\frac{1}{2} \nabla f(x, \epsilon) = \frac{1}{\lambda - 2 + \| x \|^2} \tilde{A}^T \tilde{A} - \frac{\| \tilde{r} \|^2 I_n}{(\lambda - 2 + \| x \|^2)^2} x^T
\]
and
\[
\frac{1}{2} \nabla^2 f(x, \epsilon) = \frac{\tilde{A}^T \tilde{A}}{\lambda - 2 + \| x \|^2} + \frac{4 \| \tilde{r} \|^2}{(\lambda - 2 + \| x \|^2)^2} xx^T - \frac{\| \tilde{r} \|^2 I_n}{(\lambda - 2 + \| x \|^2)^2} \frac{1}{(\lambda - 2 + \| x \|^2)^2} (2 x \tilde{r}^T \tilde{A} + 2 \tilde{A}^T \tilde{r} x^T).
\]
Particularly, since \( x^* = x_{\text{STLS}} \) is the minimum point of \( f(x) \), for \( \epsilon = 0 \) and \( x = x^* \), they must satisfy
\[
\nabla f(x, \epsilon) = 0.
\]
Hence,
\[
\nabla f(x^*, 0) = 0.
\]
Based on (A1), this yields
\[
A^T r = \frac{\| r \|^2}{\lambda - 2 + \| x^* \|^2} x^*, \tag{A3}
\]
where \( r = Ax^* - b \). It is seen from (A3) that \( A^T r x^*T \) is a symmetric matrix. Now substituting \( \epsilon = 0 \) and \( x = x^* \) into (A2), by simplification we get
\[
\frac{1}{2} \nabla^2 f(x^*, 0) = \frac{\tilde{A}^T \tilde{A}}{\lambda - 2 + \| x^* \|^2} - \frac{\| r \|^2 I_n}{\lambda - 2 + \| x^* \|^2} \frac{1}{(\lambda - 2 + \| x^* \|^2)^2} (\tilde{A}^T \tilde{A} - \frac{\| r \|^2 I_n}{\lambda - 2 + \| x^* \|^2} x^T).
\]
Here we used (A3) and the symmetry of \( A^T r x^*T \). Based on (6), equivalently we get
\[
\frac{1}{2} \nabla^2 f(x^*, 0) = \frac{1}{\lambda - 2 + \| x^* \|^2} (\tilde{A}^T \tilde{A} - \sigma_{n+1}^2 I_n). \tag{A4}
\]
By (2), we know that \( \nabla^2 f(x^*, 0) \) is positive definite.
Note that
\[
\tilde{r} = \tilde{A}x - \tilde{b} = [x^T - 1] \otimes I_n \text{vec}(\tilde{A} \tilde{b}) = G(x) \text{vec}(\tilde{A} \tilde{b})).
\]
Denote $\tilde{s} = \text{vec}([A b]) + \varepsilon$, and $\tilde{a}_i = \tilde{A}[:, i]$. We then get

$$\frac{\partial G}{\partial x_i} \tilde{s} = \tilde{a}_i, \quad i = 1, \ldots, n$$

and

$$\frac{1}{2} \nabla_x f(x, \varepsilon) = \frac{1}{\lambda^{-2} + \|x\|^2} \left[ \tilde{s}^T G^T \frac{\partial G}{\partial x_1} \tilde{s}, \ldots, \tilde{s}^T G^T \frac{\partial G}{\partial x_n} \tilde{s} \right] - \frac{\tilde{s}^T G^T \tilde{s}}{(\lambda^{-2} + \|x\|^2)^2} x^T.$$

From

$$\frac{1}{2} \nabla_{\varepsilon}^2 f(x, \varepsilon) = \frac{1}{\lambda^{-2} + \|x\|^2} \left[ \nabla_{\varepsilon} \left( \tilde{s}^T G^T \frac{\partial G}{\partial x_1} \tilde{s} \right) \right] - \frac{1}{(\lambda^{-2} + \|x\|^2)^2} \left[ \nabla_{\varepsilon} \left( \tilde{s}^T G^T \tilde{s} \right) x_1 \right],$$

we obtain

$$\frac{1}{2} \nabla_{\varepsilon}^2 f(x, \varepsilon) = -\frac{2 x^T r^T G}{(\lambda^{-2} + \|x\|^2)^2} + \frac{1}{\lambda^{-2} + \|x\|^2} \left( \begin{bmatrix} \tilde{a}_1^T G & \cdots & \tilde{a}_n^T G \end{bmatrix} + \begin{bmatrix} r^T \frac{\partial}{\partial x_1} G \\ \vdots \\ r^T \frac{\partial}{\partial x_n} G \end{bmatrix} \right).$$

(A5)

Now from the fact that $x(\varepsilon)$ is the minimum point of $f(x, \varepsilon)$ for $\varepsilon$ near zero, it holds that

$$\nabla f(x(\varepsilon), \varepsilon) = 0.$$

Differentiation by $\varepsilon$ yields

$$\nabla_{\varepsilon}^2 f(x(\varepsilon), \varepsilon) |_{\varepsilon x} + \nabla^2_{\varepsilon x} f(x(\varepsilon), \varepsilon) = 0.$$  

(A6)

For $\varepsilon$ such that the perturbed problem (8) satisfies (2), that is, $\sigma_{n+1}([\tilde{A}, \lambda \tilde{b}]) < \sigma_n(\tilde{A})$, following the line of proving that $\nabla_{\varepsilon}^2 f(x^*, 0)$ is positive definite, it is known that $\nabla_{\varepsilon}^2 f[x(\varepsilon), \varepsilon]$ is also positive definite. From (A6) we get

$$\nabla_{\varepsilon} x(\varepsilon) = -\nabla_{\varepsilon}^2 f[x(\varepsilon), \varepsilon]^{-1} \nabla_{\varepsilon x} f[x(\varepsilon), \varepsilon].$$

Substituting $\varepsilon = 0$ and $x(0) = x_{\text{STLS}} = x^+$ into the above equation, we get

$$\nabla_{\varepsilon} x(0) = -\nabla_{\varepsilon}^2 f[x^*, 0]^{-1} \nabla_{\varepsilon x} f[x^*, 0],$$

$$= \left( A^T A - \sigma_{n+1}^2 I_n \right)^{-1} \left( \frac{2 x^T r^T G(x^*)}{\lambda^{-2} + \|x^*\|^2} - A^T G(x^*) - \begin{bmatrix} r^T \frac{\partial}{\partial x_1} G \\ \vdots \\ r^T \frac{\partial}{\partial x_n} G \end{bmatrix} \right),$$

$$= \left( A^T A - \sigma_{n+1}^2 I_n \right)^{-1} \left( \frac{2 x^+ r^T G(x^*)}{\lambda^{-2} + \|x^+\|^2} - A^T G(x^*) - [I_n \otimes r^T \cdot 0] \right).$$

(A7)

In the latter equality we used the fact that

$$\begin{bmatrix} r^T \frac{\partial}{\partial x_1} G \\ \vdots \\ r^T \frac{\partial}{\partial x_n} G \end{bmatrix} = I_n \otimes r^T \begin{bmatrix} \frac{\partial}{\partial x_1} G \\ \vdots \\ \frac{\partial}{\partial x_n} G \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial}{\partial x_1} G \\ \vdots \\ \frac{\partial}{\partial x_n} G \end{bmatrix} \begin{bmatrix} I_m \end{bmatrix} = [I_m \cdot I_m].$$

As we got (A4) based on (6), we get a concise expression for the first term of the summation in (A7) by (A3)
\[
\frac{2x^*r^T G(x^*)}{\lambda^{-2} + \|x^*\|^2} = 2A^T r^T G(x^*) = 2A^T \frac{r}{\|r\|} r^T G(x^*) \quad \text{(A8)}
\]

Thus, the proof of the theorem is completed. □

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