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Bihamiltonian Systems of Hydrodynamic Type and Reciprocal Transformations

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Abstract. We prove that under certain linear reciprocal transformation, an evolutionary PDE of hydrodynamic type that admits a bihamiltonian structure is transformed to a system of the same type which is still bihamiltonian.

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1. Introduction

Systems of hydrodynamic type are a class of quasilinear evolutionary PDEs of the form

$$\mathbf{u}_t = V\mathbf{u}_x, \quad \mathbf{u} = (u^1, \dots, u^n)^T, \ \ V = (V^i_j(\mathbf{u}))_{n\times n}. \tag{1.1}$$

Assume that the above system has two conservation laws

$$\frac{\partial a(u)}{\partial t} = \frac{\partial b(u)}{\partial x}, \quad \frac{\partial p(u)}{\partial t} = \frac{\partial q(u)}{\partial x} \tag{1.2}$$

with $a(u)q(u) - p(u)b(u) \neq 0$, then we can perform a change of the independent variables

$$(x, t) \mapsto (y(x, t, u(x, t)), s(x, t, u(x, t)) \tag{1.3}$$

by the following defining relations

$$\mathrm{d}y = a(u)\mathrm{d}x + b(u)\mathrm{d}t, \quad \mathrm{d}s = p(u)\mathrm{d}x + q(u)\mathrm{d}t. \tag{1.4}$$

Such a change of the independent variables is called a reciprocal transformation of the system (1.1). It originates from the study of gas dynamics, see [10] and references therein. Under a reciprocal transformation the system (1.1) remains to be a system of hydrodynamic type

$$\mathbf{u}_s = (a\,V - b\,I)(q\,I - p\,V)^{-1}\mathbf{u}_y. \tag{1.5}$$

Here I denotes the $n\times n$ unity matrix and the matrix $qI-pV$ is assumed to be nondegenerate.

The system (1.1) is called a Hamiltonian system of hydrodynamic type if it has the representation

$$\mathbf{u}_t = J\,\nabla h(\mathbf{u}), \tag{1.6}$$

where $J=(J^{ij})$ is a Hamiltonian operator of hydrodynamic type

$$J^{ij}=\eta^{ij}(\mathbf{u})\frac{\mathrm{d}}{\mathrm{d}x}-\eta^{is}(\mathbf{u})\Gamma^j_{sk}(\mathbf{u})u^k_x,\quad 1\leqslant i,j\leqslant n, \tag{1.7}$$

the matrix $\eta=(\eta^{ij})$ is nondegenerate and symmetric on certain open subset U of $\mathbb{R}^n$. Here and henceforth summations over repeated upper and lower indices are assumed. It was proved by Dubrovin and Novikov [4,5] that J is a Hamiltonian operator if and only if the pseudo-Riemannian metric $(\eta_{ij})=\eta^{-1}$ is flat, and Γ^j_{sk} coincide with the Christoffel symbols of the Levi–Civita connection of (η_{ij}). So we can assume that the dependent variables $u^1,\ldots,u^n$ of the system (1.6) are the *flat coordinates* of the metric (η_{ij}), i.e., $\eta_{ij}(u)$ are constants and $\Gamma^j_{sk}=0$. In what follows we will also call a nondegenerate symmetric bilinear form on $T^*_{\mathbf{u}}U$, such as the one given by $\eta=(\eta^{ij})$, a metric.

A natural question is whether under a reciprocal transformation a system of the form (1.6) remains to be a Hamiltonian system of hydrodynamic type. In the special case when the reciprocal transformation is linear in x,t, i.e., when $a(u),b(u),p(u),q(u)$ are constants, Tsarev gave an affirmative answer to the above question [11]. In fact, it was shown by Pavlov [9] that under the linear reciprocal transformation

$$y=ax+bt,\ \ s=px+qt,\quad aq-bp\neq 0, \tag{1.8}$$

the Hamiltonian system (1.6) with $J^{ij}=\eta^{ij}\partial_x$ is transformed to the following Hamiltonian system of hydrodynamic type:

$$\mathbf{v}_s=\bar{J}\,\nabla\bar{h}(\mathbf{v}),\quad \bar{J}^{ij}=\eta^{ij}\,\partial_y. \tag{1.9}$$

Here the new dependent variables v^i and the function $\bar{h}(\mathbf{v})$ are defined by

$$\mathbf{v}=(v^1,\ldots,v^n)^T=\eta\nabla(qh_0-ph),\quad h_0=\frac{1}{2}\eta_{ij}\,u^iu^j, \tag{1.10}$$

$$\bar{h}(\mathbf{v})=a\left(qh-\frac{1}{2}p\,\eta^{ij}\frac{\partial h}{\partial u^i}\frac{\partial h}{\partial u^j}\right)-b\left(qh_0-p\left(u^i\frac{\partial h}{\partial u^i}-h\right)\right). \tag{1.11}$$

In the general cases, the above transformation property of a Hamiltonian system of hydrodynamic type no longer holds true as it was shown by Ferapontov and Pavlov in [8]. Although under a reciprocal transformation (1.3), (1.4) the transformed system of (1.6) can still be represented as a Hamiltonian system, the transformed Hamiltonian operator becomes nonlocal, it contains terms with integral operator ∂_y^{-1}.

In this paper, we study the properties of a bihamiltonian system of hydrodynamic type under the action of a reciprocal transformation. We show that under a linear reciprocal transformation of the form (1.8), a bihamiltonian system (or a hierarchy of bihamiltonian systems) keeps to have a bihamiltonian structure of hydrodynamic type.

The main motivation of this work comes from the program of classification for certain class of bihamiltonian evolutionary PDEs that was initiated by Dubrovin and the second author in [6]. The dispersionless limits of such evolutionary PDEs are bihamiltonian systems of hydrodynamic type. Typical examples of this class of evolutionary PDEs include the KdV equation and the interpolated Toda lattice equation, their dispersionless limits will be considered in the examples of Section 4. One of the important problems related to this classification program is whether a bihamiltonian system of this class remains to be bihamiltonian after a linear reciprocal transformation. The present work is a first step toward answering this problem.

This letter is organized as follows. We first formulate the main results in Section 2, then give their proofs in Section 3. In Section 4 we present two examples to illustrate the main results and the last section is a conclusion.

2. The Main Results

A bihamiltonian system of hydrodynamic type is a system of the form (1.1) that admits two compatible Hamiltonian structures of hydrodynamic type, i.e., it has the following representation:

$$\frac{\partial \mathbf{u}}{\partial t} = J_1 \nabla h(\mathbf{u}), \quad J_1 \nabla h(\mathbf{u}) \equiv J_2 \nabla f(\mathbf{u}) \equiv V(\mathbf{u})\mathbf{u}_x. \tag{2.1}$$

Here J_1, J_2 are two Hamiltonian operators of hydrodynamic type

$$J_1^{ij} = \eta^{ij}\,\partial_x, \quad J_2^{ij} = g^{ij}(\mathbf{u})\,\partial_x - g^{ik}(\mathbf{u})\Gamma_{kl}^j(\mathbf{u})u_x^l. \tag{2.2}$$

The symmetric matrices $\eta = (\eta^{ij}), g = (g^{ij})$ have the properties that η is constant and nondegenerate, g is nondegenerate on certain open subset U of $\mathbb{R}^n$, and $\det(g - \lambda\eta)$ does not vanish identically for any constant parameter λ. Compatibility of these two Hamiltonian operators means that any linear combination $J_2 - \lambda J_1$ also gives a Hamiltonian operator of the same type. Note that we have chosen the flat coordinates $u^1, \ldots, u^n$ of the metric η as the dependent variables of the above system.

Let us consider the effect of the linear reciprocal transformation (1.8) on the bihamiltonian property of the system (2.1). To this end, we also take into account the flow of translation along x. It is also a bihamiltonian system with respect to the above bihamiltonian structure

$$\frac{\partial \mathbf{u}}{\partial t_0} = J_1 \nabla h_0(\mathbf{u}), \quad J_1 \nabla h_0(\mathbf{u}) \equiv J_2 \nabla f_0(\mathbf{u}) \equiv \frac{\partial \mathbf{u}}{\partial x}. \tag{2.3}$$

Here the functions h_0, f_0 are defined by

$$h_0(\mathbf{u}) = \frac{1}{2}\eta_{ij}\,u^i u^j, \quad f_0(\mathbf{u}) = \frac{1}{2}\hat{g}_{ij}\,\hat{u}^i\hat{u}^j, \quad (\eta_{ij}) = (\eta^{ij})^{-1}, \ (\hat{g}_{ij}) = (\hat{g}^{ij})^{-1}$$

with $\hat{g}^{ij}$ being the components of the metric g under its flat coordinates $\hat{u}^1, \ldots, \hat{u}^n$.

We introduce the new dependent variables $\mathbf{v} = (v^1, \ldots, v^n)^T$ by the relation (1.10), then the Jacobian is given by

$$Q := \left(\frac{\partial v^i}{\partial u^j}\right) = qI - pV. \tag{2.4}$$

Assume that Q is nondegenerate on $U \subset \mathbb{R}^n$, denote

$$W := Q^{-1}. \tag{2.5}$$

After the reciprocal transformation (1.8), the systems (2.1) and (2.3) are transformed to

$$\frac{\partial \mathbf{v}}{\partial s} = (aV - bI)W\mathbf{v}_y, \tag{2.6}$$

$$\frac{\partial \mathbf{v}}{\partial t_0} = (aq - bp)W\mathbf{v}_y. \tag{2.7}$$

Note that when the transformation (1.8) satisfies the condition $a = q = 0, b = p = 1$, the flows $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t_0}$ coincide.

Define two metrics $\bar{\eta} = (\bar{\eta}^{ij}(\mathbf{v})), \bar{g} = (\bar{g}^{ij}(\mathbf{v}))$ whose components in the local coordinates $v^1, \ldots, v^n$ are given by the following formulae

$$\bar{\eta}^{ij}(\mathbf{v}) = \eta^{ij}, \quad \bar{g}^{ij}(\mathbf{v}) = g^{ij}(\mathbf{u}), \quad i, j = 1, \ldots, n, \tag{2.8}$$

where $\mathbf{u}$ and $\mathbf{v}$ are related by (1.10).

THEOREM 2.1. *The metrics* $\bar{\eta}$, $\bar{g}$ *are flat, the corresponding Hamiltonian operators* $\bar{J}_1, \bar{J}_2$ *with components*

$$\bar{J}_1^{ij} = \bar{\eta}^{ij}\,\partial_y, \quad \bar{J}_2^{ij} = \bar{g}^{ij}(\mathbf{v})\,\partial_y + \bar{\Gamma}_k^{ij}(\mathbf{v})v_y^k \tag{2.9}$$

are compatible. Here $\bar{\Gamma}_k^{ij}(\mathbf{v}) = -\bar{g}^{il}\bar{\Gamma}_{lk}^j(\mathbf{v})$ *with* $\bar{\Gamma}_{lk}^j(\mathbf{v})$ *being the Christoffel symbols of the Levi–Civita connection of the metric* $\bar{g}$.

In other words, the metrics $\bar{\eta}$, $\bar{g}$ form a flat pencil [2].

THEOREM 2.2. *The systems* (2.6) *is bihamiltonian with respect to the bihamiltonian structure* $\bar{J}_1, \bar{J}_2$, *i.e., it has the representation*

$$\frac{\partial \mathbf{v}}{\partial s} = \bar{J}_1\nabla\bar{h}(\mathbf{v}) \equiv \bar{J}_2\nabla\bar{f}(\mathbf{v}), \tag{2.10}$$

where the functions $\bar{h}, \bar{f}$ *are defined by*

$$\frac{\partial \bar{h}(\mathbf{v})}{\partial v^i} = \frac{\partial (ah(\mathbf{u}) - bh_0(\mathbf{u}))}{\partial u^i}, \quad \frac{\partial \bar{f}(\mathbf{v})}{\partial v^i} = \frac{\partial (af(\mathbf{u}) - bf_0(\mathbf{u}))}{\partial u^i}, \quad 1 \leqslant i \leqslant n. \tag{2.11}$$

Now let us assume that there is given another bihamiltonian system

$$\frac{\partial \mathbf{u}}{\partial t_1} = J_1 \nabla h_1(\mathbf{u}), \quad J_1 \nabla h_1(\mathbf{u}) \equiv J_2 \nabla f_1(\mathbf{u}) \equiv A(\mathbf{u})\mathbf{u}_x, \tag{2.12}$$

we also assume that this flow commutes with the flow given by (2.1). By using Corollary 4.2 of [3], we know that the commutativity of these two flows holds true automatically when the bihamiltonian structure J_1, J_2 is semisimple, i.e., when the characteristic polynomial $\det(g^{ij} - \lambda \eta^{ij})$ has pairwise distinct roots.

THEOREM 2.3. *Under the reciprocal transformation* (1.8), *the system* (2.12) *is transformed to the form*

$$\frac{\partial \mathbf{v}}{\partial t_1} = (aq - bp) A W \mathbf{v}_y, \tag{2.13}$$

it is bihamiltonian with respect to the bihamiltonian structure $\bar{J}_1, \bar{J}_2$ *and has the representation*

$$\frac{\partial \mathbf{v}}{\partial t_1} = \bar{J}_1 \nabla \bar{h}_1(\mathbf{v}) \equiv \bar{J}_2 \nabla \bar{f}_1(\mathbf{v}). \tag{2.14}$$

Here the functions $\bar{h}_1(\mathbf{v}), \bar{f}_1(\mathbf{v})$ *are defined by*

$$\frac{\partial \bar{h}_1(\mathbf{v})}{\partial v^i} = (aq - bp)\frac{\partial h_1(\mathbf{u})}{\partial u^i}, \quad \frac{\partial \bar{f}_1(\mathbf{v})}{\partial v^i} = (aq - bp)\frac{\partial f_1(\mathbf{u})}{\partial u^i}, \quad 1 \leqslant i \leqslant n. \tag{2.15}$$

Remark 2.1. Associated to a bihamiltonian structure of hydrodynamic type there is usually a hierarchy of bihamiltonian evolutionary PDEs

$$\frac{\partial \mathbf{u}}{\partial t_j} = B_j(\mathbf{u})\mathbf{u}_x, \quad j \geqslant 0. \tag{2.16}$$

The independent variable x can be viewed as the spatial variable, t_j as the time variables, and the flows of the hierarchy mutually commute. From Theorems 2.1, 2.3 it follows that the class of bihamiltonian hierarchies of systems of hydrodynamic type is invariant with respect to reciprocal transformations of the form (1.8).

3. Proof of the Main Results

To prove the theorems of the last section, let us first prove some lemmas.

LEMMA 3.1. *The Christoffel symbols* $\Gamma^k_{ij}(\mathbf{u})$ *and* $\bar\Gamma^k_{ij}(\mathbf{v})$, *expressed respectively in the local coordinates* $\mathbf{u}$ *and* $\mathbf{v}$, *of the Levi–Civita connections of the metrics* $g, \bar g$ *satisfy the following relations*

$$\Gamma^k_{ij}(\mathbf{u}) = \bar\Gamma^k_{il}(\mathbf{v}) Q^l_j, \quad 1 \leqslant i, j, k \leqslant n, \tag{3.1}$$

where the matrix Q *is defined in* (2.4).

Proof. Denote by ∇_i the covariant derivative along $\frac{\partial}{\partial u^i}$ of the Levi–Civita connection of the metric g. Then from the second Hamiltonian structure of the system (2.1) we get

$$V^i_j = g^{ik} \nabla_k \nabla_j f(\mathbf{u}), \quad 1 \leqslant i, j \leqslant n. \tag{3.2}$$

From these identities and the flatness of the metric g it follows that

$$\nabla_k V^i_j = \nabla_j V^i_k \tag{3.3}$$

for any fixed indices $1 \leqslant i, j, k \leqslant n$. From the first Hamiltonian structure of the system (2.1) we also have

$$V^i_j = \eta^{ik} \frac{\partial^2 h(\mathbf{u})}{\partial u^k \partial u^j}, \tag{3.4}$$

which lead to the identities

$$\frac{\partial V^i_j}{\partial u^k} = \frac{\partial V^i_k}{\partial u^j} = \eta^{il} \frac{\partial^3 h(\mathbf{u})}{\partial u^l \partial u^j \partial u^k}, \tag{3.5}$$

so by using (3.3) and (2.4) we obtain

$$\Gamma^i_{jl} V^l_k = \Gamma^i_{kl} V^l_j, \quad \Gamma^i_{jl} Q^l_k = \Gamma^i_{kl} Q^l_j, \quad 1 \leqslant i, j, k \leqslant n. \tag{3.6}$$

From these relations and the identity $\nabla_k g_{ij} = 0$ we obtain

$$\begin{aligned}
\partial_{u^i} g_{sj} - \partial_{u^s} g_{ij} = g_{sk} \Gamma^k_{ij} - g_{ik} \Gamma^k_{sj} &= \left(g_{sk} \Gamma^k_{mr} W^r_i - g_{ik} \Gamma^k_{mr} W^r_s \right) Q^m_j \\
&= \left[(\partial_{u^r} g_{sm} - g_{mk} \Gamma^k_{rs}) W^r_i - (\partial_{u^r} g_{im} - g_{mk} \Gamma^k_{ri}) W^r_s \right] Q^m_j \\
&= \left(\partial_{u^r} g_{sm} W^r_i - \partial_{u^r} g_{im} W^r_s \right) Q^m_j \\
&= \left(\partial_{v^i} g_{sm} - \partial_{v^s} g_{im} \right) Q^m_j .
\end{aligned} \tag{3.7}$$

Substituting these identities into the formula for Γ^k_{ij} we get

$$\begin{aligned}
\Gamma^k_{ij}(\mathbf{u}) &= \frac{1}{2} g^{ks} \left(\partial_{u^j} g_{si} + \partial_{u^i} g_{sj} - \partial_{u^s} g_{ij} \right) \\
&= \frac{1}{2} g^{ks} \left(\partial_{v^m} g_{si} + \partial_{v^i} g_{sm} - \partial_{v^s} g_{im} \right) Q^m_j = \bar\Gamma^k_{il}(\mathbf{v}) Q^l_j .
\end{aligned}$$

The lemma is proved. □

LEMMA 3.2. *The metric $\bar g$ is flat.*

Proof. Denote by $\bar\nabla_i$ the covariant derivative along $\frac{\partial}{\partial v^i}$ of the Levi–Civita connection of the metric $\bar g$, and by $R, \bar R$ the curvature tensors of the metric $g, \bar g$, respectively

$$(\nabla_i\nabla_j-\nabla_j\nabla_i)\partial_{u^k}=R_{ijk}{}^s\,\partial_{u^s},\quad (\bar\nabla_i\bar\nabla_j-\bar\nabla_j\bar\nabla_i)\partial_{v^k}=\bar R_{ijk}{}^s\,\partial_{v^s}. \tag{3.8}$$

By using (3.5) and (3.1) we get

$$\begin{aligned}R_{ijk}{}^s&=\partial_{u^i}\Gamma^s_{jk}-\partial_{u^j}\Gamma^s_{ik}+\Gamma^s_{im}\Gamma^m_{jk}-\Gamma^s_{jm}\Gamma^m_{ik}\\&=\partial_{u^i}\left(\bar\Gamma^s_{kl}Q^l_j\right)-\partial_{u^j}\left(\bar\Gamma^s_{kl}Q^l_i\right)+\bar\Gamma^s_{lm}Q^l_i\bar\Gamma^m_{rk}Q^r_j-\bar\Gamma^s_{lm}Q^l_j\bar\Gamma^m_{rk}Q^r_i\\&=\left(\partial_{v^m}\bar\Gamma^s_{kl}-\partial_{v^l}\bar\Gamma^s_{km}+\bar\Gamma^s_{mr}\bar\Gamma^r_{lk}-\bar\Gamma^s_{lr}\bar\Gamma^r_{mk}\right)Q^m_iQ^l_j=\bar R_{mlk}{}^sQ^m_iQ^l_j,\end{aligned}$$

thus $\bar R_{ijk}{}^s=0$ and we proved the lemma. □

Proof of Theorem 2.3. We need to prove that the pair of metrics $\bar\eta$ and $\bar g$ form a flat pencil, i.e., for any constant λ satisfying $\det(\bar g^{ij}-\lambda\,\bar\eta^{ij})\neq 0$, the metric $\bar g_\lambda=\bar g-\lambda\,\bar\eta$ is flat, and the contravariant components of its Levi–Civita connection coincide with those of $\bar g$ in the local coordinates $v^1,\dots,v^n$ (note that they are flat coordinates for the metric $\bar\eta$).

In Appendix D of [2], Dubrovin gave a necessary and sufficient condition for a pair of flat metrics to form a flat pencil. To explain this condition, let us first recall some notations that are introduced in [2]. We denote as above by ∇ the Levi–Civita connection of the metric g and by $\Gamma^{ij}_k=-g^{is}\Gamma^j_{sk}(\mathbf{u})$ its contravariant components in the flat coordinates $u^1,\dots,u^n$ of the metric η. We also denote

$$\partial_i=\partial_{u^i},\quad \partial^i=\eta^{ik}\partial_{u^i},\quad \nabla_i=\nabla_{u^i},\quad \Delta^{ijk}=\eta^{is}\Gamma^{jk}_s(\mathbf{u}). \tag{3.9}$$

Then the necessary and sufficient condition for the pair of metrics η, g to form a flat pencil is the existence of a vector field $\xi=\xi^i\frac{\partial}{\partial u^i}$, such that the identities

$$\Delta^{ijk}=\partial^i\partial^j\xi^k=\Delta^{jik},\quad g^{ij}=\partial^i\xi^j+\partial^j\xi^i+c^{ij} \tag{3.10}$$

hold true for certain constant symmetric tensor c^{ij}, and

$$\Delta^{ij}_s\Delta^{sk}_l=\Delta^{ik}_s\Delta^{sj}_l,\quad (g^{im}\eta^{jl}-\eta^{im}g^{jl})\partial_m\partial_l\xi^k=0, \tag{3.11}$$

where $\Delta^{ij}_k=\eta_{ks}\Delta^{sij}$.

The pair of metrics η, g related to the theorem form a flat pencil, we have to use this fact to prove that the pair of metrics $\bar\eta,\bar g$ also fulfils the above condition. By using the flat coordinates $v^1,\dots,v^n$ of the metric $\bar\eta$ we introduce the following notations that are similar to those given in (3.9):

$$\bar\partial_i=\partial_{v^i},\quad \bar\partial^i=\bar\eta^{ik}\partial_{v^i},\quad \bar\nabla_i=\bar\nabla_{v^i},\quad \bar\Delta^{ijk}=\bar\eta^{is}\bar\Gamma^{jk}_s(\mathbf{v}). \tag{3.12}$$

Here $\bar{\nabla}$ denotes the Levi–Civita connection of the metric $\bar{g}$ and $\bar{\Gamma}^{ij}_k = -\bar{g}^{is}\bar{\Gamma}^j_{sk}(\mathbf{v})$ denote the contravariant components of the connection. From (3.2), (3.4) it follows that

$$\eta V^T = V\eta, \quad gV^T = Vg. \tag{3.13}$$

By using the definition (2.4) of Q we get

$$\eta Q^T = Q\eta, \quad gQ^T = Qg. \tag{3.14}$$

Then from (3.1), the first formula of (3.14) and the definition (2.8) we get

$$\Delta^{ijk} = \eta^{is}\Gamma^{jk}_s = \eta^{is}\bar{\Gamma}^{jk}_l Q^l_s = \bar{\eta}^{ls}\bar{\Gamma}^{jk}_l Q^i_s = \bar{\Delta}^{sjk} Q^i_s, \tag{3.15}$$

thus by using (2.4) and (3.10) we obtain

$$\bar{\Delta}^{ijk} = W^i_s \Delta^{sjk} = W^i_m \partial^m \partial^j \xi^k = \bar{\partial}^i \partial^j \xi^k. \tag{3.16}$$

Similarly, from (3.1) and the second formula of (3.14) we have

$$\Delta^{ijk} = -\eta^{is} g^{jt}\Gamma^k_{st} = -\eta^{is} g^{jt}\bar{\Gamma}^k_{sl} Q^l_t = -\eta^{is} g^{lt}\bar{\Gamma}^k_{sl} Q^j_t = \bar{\Delta}^{itk} Q^j_t, \tag{3.17}$$

and consequently

$$\bar{\Delta}^{ijk} = W^j_s \Delta^{isk} = W^j_s \partial^s \partial^i \xi^k = \bar{\partial}^j \partial^i \xi^k. \tag{3.18}$$

Thus we arrive at the identities

$$\bar{\partial}^i \partial^j \xi^k = \bar{\partial}^j \partial^i \xi^k, \tag{3.19}$$

which implies the existence of a vector field $\bar{\xi} = \bar{\xi}^i \frac{\partial}{\partial v^i}$ such that

$$\partial^i \xi^k = \bar{\partial}^i \bar{\xi}^k, \quad \leqslant i, k \leqslant n. \tag{3.20}$$

So by using (3.10), (3.16) and the last formulae we obtain

$$\bar{\Delta}^{ijk} = \bar{\partial}^i \bar{\partial}^j \bar{\xi}^k, \quad \bar{g}^{ij} = \bar{\partial}^i \bar{\xi}^j + \bar{\partial}^j \bar{\xi}^i + c^{ij}. \tag{3.21}$$

Now we are only left to prove the analogue of (3.11) for the metrics $\bar{\eta}, \bar{g}$. From (3.13), (3.14), (3.16) and (3.18) we have

$$\bar{\Delta}^{ij}_k = \bar{\eta}_{ks}\bar{\Delta}^{sij} = \eta_{ks} W^s_t \Delta^{tij} = \Delta^{ij}_s W^s_k, \quad \bar{\Delta}^{ij}_k = \bar{\eta}_{ks}\bar{\Delta}^{sij} = \eta_{ks} W^i_t \Delta^{stj} = W^i_s \Delta^{sj}_k.$$

They yield

$$\bar{\Delta}^{ij}_s \bar{\Delta}^{sk}_l - \bar{\Delta}^{ik}_s \bar{\Delta}^{sj}_l = W^i_m(\Delta^{mj}_s \Delta^{sk}_r - \Delta^{mk}_s \Delta^{sj}_r) W^r_l = 0.$$

Finally, by using (3.13), (3.14), (3.16) and (3.18) we obtain

$$
\begin{aligned}
(\bar{g}^{is}\bar{\eta}^{jt}-\bar{\eta}^{is}\bar{g}^{jt})\bar{\partial}_s\bar{\partial}_t\bar{\xi}^k &= \bar{g}^{is}\bar{\eta}_{ls}\bar{\Delta}^{ljk}-\bar{g}^{jt}\bar{\eta}_{lt}\bar{\Delta}^{ilk}\\
&= g^{is}\eta_{ls}W^l_m\Delta^{mjk}-g^{jt}\eta_{lt}W^i_m\Delta^{mlk}\\
&= g^{is}\eta_{lm}W^l_s\Delta^{mjk}-g^{jt}\eta_{lt}W^i_m\Delta^{mlk}\\
&= g^{is}W^l_s\eta^{jm}\partial_l\partial_m\xi^k-g^{jm}W^i_s\eta^{sl}\partial_m\partial_l\xi^k\\
&= W^i_s(g^{sl}\eta^{jm}-\eta^{sl}g^{jm})\partial_l\partial_m\xi^k=0.
\end{aligned}
$$

Thus the theorem is proved. □

Proof of Theorem 2.3. Let us first prove the existence of the functions $\bar{h}_1(\mathbf{v})$, $\bar{f}_1(\mathbf{v})$ that satisfy the equations (2.15). From the commutativity condition of the flows $\frac{\partial}{\partial t}, \frac{\partial}{\partial t_1}$

$$
\frac{\partial}{\partial t_1}\left(\frac{\partial \mathbf{u}}{\partial t}\right)=\frac{\partial}{\partial t_1}\left(\frac{\partial \mathbf{u}}{\partial t}\right), \tag{3.22}
$$

it follows that

$$
AV=VA, \tag{3.23}
$$

where the matrices V, A are defined respectively in (2.1), (2.12). By using the first Hamiltonian structure of the system (2.12) we obtain

$$
\eta^{ik}\frac{\partial^2 h_1}{\partial u^k\partial u^m}V^m_j=V^i_k\eta^{km}\frac{\partial^2 h_1}{\partial u^m\partial u^j}, \tag{3.24}
$$

together with the identities given in (3.13) they lead to

$$
\frac{\partial^2 h_1}{\partial u^i\partial u^k}W^k_j=\frac{\partial^2 h_1}{\partial u^j\partial u^k}W^k_i. \tag{3.25}
$$

These identities imply the existence of the function $\bar{h}_1(\mathbf{v})$ that satisfies the first equation of (2.15). To prove the existence of the function $\bar{f}_1(\mathbf{v})$, let us use the second Hamiltonian structure of the system (2.12) to obtain

$$
A^i_j=g^{ik}\frac{\partial^2 f_1}{\partial u^k\partial u^j}-g^{ik}\Gamma^m_{kj}\frac{\partial f_1}{\partial u^m}. \tag{3.26}
$$

By using (3.13) and (3.6) we have

$$
\begin{aligned}
A^i_lV^l_j &= g^{ik}V^l_j\frac{\partial^2 f_1}{\partial u^k\partial u^l}-g^{ik}\Gamma^m_{kl}V^l_j\frac{\partial f_1}{\partial u^m}=g^{ik}\frac{\partial^2 f_1}{\partial u^k\partial u^l}V^l_j-g^{ik}\Gamma^m_{jl}V^l_k\frac{\partial f_1}{\partial u^m}\\
&= g^{ik}\frac{\partial^2 f_1}{\partial u^k\partial u^l}V^l_j-g^{lk}\Gamma^m_{jl}V^i_k\frac{\partial f_1}{\partial u^m}.
\end{aligned} \tag{3.27}
$$

Thus from the identity (3.23) and the formula

$$
V^i_kA^k_j=V^i_kg^{kl}\frac{\partial^2 f_1}{\partial u^l\partial u^j}-V^i_kg^{kl}\Gamma^m_{lj}\frac{\partial f_1}{\partial u^m}, \tag{3.28}
$$

it follows that

$$V_k^i g^{kl} \frac{\partial^2 f_1}{\partial u^l \partial u^j} = g^{ik} \frac{\partial^2 f_1}{\partial u^k \partial u^l} V_j^l. \tag{3.29}$$

So we arrive at the identities

$$\frac{\partial^2 f_1}{\partial u^i \partial u^k} W_j^k = \frac{\partial^2 f_1}{\partial u^j \partial u^k} W_i^k, \tag{3.30}$$

which imply the existence of a function $\bar{f}_1(\mathbf{v})$ that satisfies the second equation of (2.15).

Now by using (2.15) and (3.1) we have

$$\begin{aligned} \bar{J}_2^{ik} \frac{\partial \bar{f}_1(\mathbf{v})}{\partial v^k} &= \bar{g}^{ik} \partial_y \left(\frac{\partial \bar{f}_1}{\partial v^k} \right) + \bar{\Gamma}_l^{ik} v_y^l \frac{\partial \bar{f}_1}{\partial v^k} \\ &= (aq - bp) \left[g^{ik} \frac{\partial^2 f_1}{\partial u^k \partial u^m} W_l^m v_y^l + W_l^m \Gamma_m^{ik} v_y^l \frac{\partial f_1}{\partial u^k} \right] \\ &= (aq - bp) A_m^i W_l^m v_y^l. \end{aligned} \tag{3.31}$$

It follows that the system (2.13) can be represented in the form

$$\frac{\partial \mathbf{v}}{\partial t_1} = \bar{J}_2 \nabla \bar{f}_1(\mathbf{v}). \tag{3.32}$$

Similarly, one can show that the system (2.13) also has the expression

$$\frac{\partial \mathbf{v}}{\partial t_1} = \bar{J}_1 \nabla \bar{h}_1(\mathbf{v}), \tag{3.33}$$

thus it is bihamiltonian and the theorem is proved. □

Proof of Theorem 2.2. For the system (2.6) we have

$$\frac{\partial \mathbf{v}}{\partial s} = \frac{\partial \mathbf{v}}{\partial t_0} \frac{\partial x}{\partial s} + \frac{\partial \mathbf{v}}{\partial t} \frac{\partial t}{\partial s} = (aq - bp)^{-1} \left(a \frac{\partial \mathbf{v}}{\partial t} - b \frac{\partial \mathbf{v}}{\partial t_0} \right). \tag{3.34}$$

The theorem follows immediately from this formula and Theorem 2.3. □

4. Two Examples

We now give two examples to illustrate the above results.

EXAMPLE 4.1. For a given positive integer m, consider the system

$$u_t = (m+1)\, u^m u_x. \tag{4.1}$$

It is the $(m+1)$th flow of the dispersionless KdV hierarchy, and has the following bihamiltonian structure

$$u_t = J_1 \nabla h(u) \equiv J_2 \nabla f(u), \tag{4.2}$$

where

$$J_1 = \partial_x, \quad J_2 = u\partial_x + \frac{u_x}{2}, \quad h(u) = \frac{u^{m+2}}{m+2}, \; f(u) = \frac{2}{2m+1}\, u^{m+1}. \tag{4.3}$$

Consider the simplest linear reciprocal transformation

$$y = t, \; s = -x. \tag{4.4}$$

We introduce the new dependent variable v as in (1.10) by $v = u^{m+1}$. Then after the above reciprocal transformation (4.1) is converted to the form

$$v_s = -\frac{1}{m+1}\, v^{-\frac{m}{m+1}}\, v_y. \tag{4.5}$$

By using Theorems 2.1 and 2.2, we know that (4.5) is also a bihamiltonian system

$$v_s = \bar{J}_1 \nabla \bar{h}(v) \equiv \bar{J}_2 \nabla \bar{f}(v), \tag{4.6}$$

$$\bar{J}_1 = \partial_y, \quad \bar{J}_2 = v^{\frac{1}{m+1}}\, \partial_y + \frac{1}{2(m+1)} v^{-\frac{m}{m+1}}\, v_y, \tag{4.7}$$

$$\bar{h}(v) = -\frac{m+1}{m+2}\, v^{\frac{m+2}{m+1}}, \quad \bar{f}(v) = -2v. \tag{4.8}$$

Let us consider the $(k+1)$th flow of the dispersionless KdV hierarchy

$$u_{t_1} = (k+1)u^k u_x \tag{4.9}$$

which is also a bihamiltonian system

$$u_{t_1} = J_1 \nabla h_1(u) \equiv J_2 \nabla f_1(u), \quad h_1 = \frac{u^{k+2}}{k+2}, \; f_1 = \frac{2}{2k+1}\, u^{k+1}. \tag{4.10}$$

Under the reciprocal transformation (4.4) it is transformed to the following bihamiltonian system:

$$v_{t_1} = \frac{k+1}{m+1} v^{\frac{k-m}{m+1}}\, v_y \equiv \bar{J}_1 \nabla \bar{h}_1(v) \equiv \bar{J}_2 \nabla \bar{f}_1(v), \tag{4.11}$$

where

$$\bar{h}_1(v) = \frac{m+1}{m+k+2}\, v^{\frac{k+m+2}{m+1}}, \quad \bar{f}_1(v) = \frac{2(k+1)(m+1)}{(2k+1)(m+k+1)}\, v^{\frac{m+k+1}{m+1}}. \tag{4.12}$$

EXAMPLE 4.2. Consider the long wave limit (also called dispersionless limit)

$$u_{tt} = (\mathrm{e}^u)_{xx} \tag{4.13}$$

of the interpolated Toda equation [1,2]

$$\epsilon^2 u_{tt} = \mathrm{e}^{u(x+\epsilon)} + \mathrm{e}^{u(x-\epsilon)} - 2\mathrm{e}^{u(x)}. \tag{4.14}$$

It has the following bihamiltonian representation

$$\begin{pmatrix} w \\ u \end{pmatrix}_t = J_1 \nabla h(w,u) \equiv J_2 \nabla f(w,u), \tag{4.15}$$

where

$$J_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 2\mathrm{e}^u \partial_x + \mathrm{e}^u u_x & w\partial_x \\ w\partial_x + w_x & 2\partial_x \end{pmatrix},$$

$$h(w,u) = \mathrm{e}^u + \frac{w^2}{2}, \quad f(w,u) = w.$$

Let us perform the reciprocal transformation (4.4) and define the new dependent variables $\bar{w}, \bar{u}$ according to (1.10)

$$\bar{w} = \mathrm{e}^u, \quad \bar{u} = w. \tag{4.16}$$

Then (4.15) is transformed to the following equation:

$$\bar{w}_s = -\bar{u}_y, \quad \bar{u}_s = -\frac{1}{\bar{w}} \bar{w}_y. \tag{4.17}$$

By using Theorems 2.1 and 2.2 we know that it is bihamiltonian

$$\begin{pmatrix} \bar{w} \\ \bar{u} \end{pmatrix}_s = \bar{J}_1 \nabla \bar{h}(\bar{w},\bar{u}) \equiv \bar{J}_2 \nabla \bar{f}(\bar{w},\bar{u}),$$

where

$$\bar{J}_1 = \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix}, \quad \bar{J}_2 = \begin{pmatrix} 2\bar{w}\partial_y + \bar{w}_y & \bar{u}\partial_y \\ \bar{u}\partial_y + \bar{u}_y & 2\partial_y \end{pmatrix}, \tag{4.18}$$

$$\bar{h}(\bar{w},\bar{u}) = -\bar{w}\log\bar{w} + \bar{w} - \frac{\bar{u}^2}{2}, \tag{4.19}$$

$$\bar{f}(\bar{w},\bar{u}) = -\frac{\bar{u}\log\bar{w}}{2} + \bar{u} - \sqrt{-4\bar{w}+\bar{u}^2}\ \ arctanh\left(\frac{\bar{u}}{\sqrt{-4\bar{w}+\bar{u}^2}}\right). \tag{4.20}$$

Now we consider the flow

$$w_{t_1} = \mathrm{e}^u w_x + w\mathrm{e}^u u_x, \quad u_{t_1} = w w_x + \mathrm{e}^u u_x \tag{4.21}$$

which belongs to the dispersionless Toda hierarchy [1]. It has the bihamiltonian structure

$$\begin{pmatrix} w \\ u \end{pmatrix}_{t_1} = J_1 \nabla h_1(w,u) \equiv J_2 \nabla f_1(w,u)$$

with

$$h_1(w,u) = \mathrm{e}^u w + \frac{w^3}{6}, \quad f_1(w,u) = \frac{1}{2}\left(\mathrm{e}^u + \frac{w^2}{2}\right).$$

After the reciprocal transformation (4.4) it is transformed to

$$\bar{w}_{t_1} = \bar{u}\bar{w}_y + \bar{w}\bar{u}_y, \quad \bar{u}_{t_1} = \bar{u}\bar{u}_y + \bar{w}_y. \tag{4.22}$$

By using Theorem 2.3, we know that the above system is also bihamiltonian

$$\begin{pmatrix} \bar{w} \\ \bar{u} \end{pmatrix}_{t_1} = \bar{J}_1 \nabla \bar{h}_1(\bar{w}, \bar{u}) \equiv \bar{J}_2 \nabla \bar{f}_1(\bar{w}, \bar{u}),$$

where

$$\bar{h}_1(\bar{w}, \bar{u}) = \frac{1}{2}(\bar{u}^2\bar{w} + \bar{w}^2), \quad \bar{f}_1(\bar{w}, \bar{u}) = \frac{\bar{u}\bar{w}}{2}.$$

5. Conclusion

We have considered the effect of a linear reciprocal transformation on the bihamiltonian structure of a bihamiltonian system of hydrodynamic type. In the case when the bihamiltonian structure is related to a Frobenius manifold, there are some special linear reciprocal transformations that correspond to the Legendre transformations between Frobenius manifolds. Such transformations were introduced and studied by Boris Dubrovin in the setting of Frobenius manifold theory [2]. The bihamiltonian structure and the reciprocal transformation that are considered in Example 4.2 just belong to this case.

As we mentioned in the Introduction (Section 1), we are interested in the problem of whether a more general class of bihamiltonian systems remain to be bihamiltonian after certain linear reciprocal transformations. Such bihamiltonian systems are the objects of study of the classification program that was initiated in [6], their dispersionless limits are bihamiltonian systems of hydrodynamic type. A positive answer to this problem was given in [1] for the example of the extended Toda lattice hierarchy, a hierarchy of bihamiltonian evolutionary PDEs. It was shown there that after certain linear reciprocal transformation this hierarchy is transformed to the extended nonlinear Schrödinger hierarchy which is also bihamiltonian. Results regarding the general case for this problem will appear in [7] and other publications.

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