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A Two-component Generalization of the Camassa-Holm Equation and its Solutions

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Abstract. An explicit reciprocal transformation between a two-component generalization of the Camassa–Holm equation, called the 2-CH system, and the first negative flow of the AKNS hierarchy is established. This transformation enables one to obtain solutions of the 2-CH system from those of the first negative flow of the AKNS hierarchy. Interesting examples of peakon and multi-kink solutions of the 2-CH system are presented.

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Key words. Camassa–Holm equation, AKNS hierarchy, reciprocal transformation.

1. Introduction

The Camassa–Holm equation, which was derived physically as a shallow water wave equation by Camassa and Holm in [8,9], takes the form

$$u_t + \kappa\, u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

where $u = u(x,t)$ is the fluid velocity in the x direction and the constant κ is related to the critical shallow water wave speed. The subscripts x, t of u denote the partial derivatives of the function u w.r.t. x, t, for example, $u_t = \partial_t u$, $u_{xxt} = \partial_t \partial_x \partial_x u$. Similar notations will be used frequently later in this paper. This equation first appeared in the work of Fuchssteiner and Fokas [20] on their theory of hereditary symmetries of soliton equations. As it was shown by Camassa and Holm [8], Equation (1.1) shares most of the important properties of an integrable system of K dV type, for example, the existence of Lax pair formalism, the bi-hamiltonian structure and the applicability of the inverse scattering method to its initial value problem. When $\kappa > 0$, the Camassa–Holm Equation (1.1) has smooth solitary waves. It has a peculiar property that when $\kappa \to 0$ these solutions become piecewise smooth and have corners at their crests, such solutions are weak solutions of (1.1) with $\kappa = 0$ and are called "peakons". It was also proved in [12,13] that such solitary waves and peakons are stable. Since the works of Camassa and Holm, this equation has become a well-known example of integrable systems and has been studied from various point of views in, for example, [1,3,10,14,18,19,22, 24,30–32] and references therein.

We consider in the present paper the following two-component generalization of the Camassa–Holm equation:

$$m_t + u\, m_x + 2m\, u_x - \rho\, \rho_x = 0, \tag{1.2}$$
$$\rho_t + (\rho\, u)_x = 0. \tag{1.3}$$

Here $m = u - u_{xx} + \frac{1}{2}\kappa$. Under the constraint $\rho = 0$, this system is reduced to the Camassa–Holm equation (1.1). Such a generalization is based on the following consideration. We note the fact that both bihamiltonian structures of the Camassa–Holm hierarchy and that of the KdV hierarchy are deformations of the following bihamiltonian structure of hydrodynamic type:

$$\begin{aligned}&\{u(x), u(y)\}_1 = \delta'(x-y),\\ &\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}\,u'(x)\delta(x-y).\end{aligned} \tag{1.4}$$

This fact also implies that the dispersionless limit of the Camassa–Holm hierarchy coincides with that of the KdV hierarchy. It was shown in [16,27] that deformations of the bihamiltonian structure (1.4) that depend polynomially on the variables $u_x, u_{xx}, \ldots$ are uniquely characterized, up to Miura type transformations, by a function $c(u)$, this function is called the central invariant of the deformed bihamiltonian structure. For the KdV hierarchy the central invariant is a nonzero constant, while for the Camassa–Holm hierarchy the central invariant is given by a nonzero constant multiplied by u. So the bihamiltonian structures of these integrable hierarchies are representatives of two different classes of deformations of (1.4). One of the main features of the integrable hierarchies that correspond to bihamiltonian structures with constant central invariants is the existence of tau functions [15], this property no longer holds true for integrable hierarchies that correspond to bihamiltonian structures with nonconstant central invariants. In fact, many of the well-known integrable hierarchies of evolutionary PDEs with one spatial variable possess bihamiltonian structures that are deformations of bihamiltonian structure of hydrodynamic type with constant central invariants, and the existence of tau functions plays an important role in the study of these integrable hierarchies. The Camassa–Holm hierarchy is an exceptional example of integrable systems which does not possess tau functions. Now let us consider the following bihamiltonian structure of hydrodynamic type:

$$\begin{aligned}&\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0,\\ &\{w_1(x), w_2(y)\}_1 = \delta'(x-y),\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\{w_1(x), w_1(y)\}_2 = 2\delta'(x-y),\\ &\{w_1(x), w_2(y))\}_2 = w_1(x)\delta'(x-y) + w_1'(x)\delta(x-y),\\ &\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)]\,\delta(x-y).\end{aligned} \tag{1.6}$$

It was shown in [27] that the deformation with constant central invariants $c_1 = c_2 = \frac{1}{24}$ leads to the nonlinear Schrödinger hierarchy, which can be converted to the AKNS hierarchy [2] after an appropriate transformation; while the one with central invariants $c_1 = \frac{1}{24}(w_1 + 2\sqrt{w_2})^2, c_2 = \frac{1}{24}(w_1 - 2\sqrt{w_2})^2$ leads to a bihamiltonian integrable hierarchy, which has (1.2), (1.3) as its first nontrivial flow under the change of coordinates

$$w_1 = 2(u - u_x), \quad w_2 = -\rho^2 + (u - u_x)^2, \tag{1.7}$$

the rescaling $t \mapsto -2t$ and the Galilean transformation $x \mapsto x + \frac{1}{2}\kappa t,\ t \mapsto t,\ u \mapsto u - \frac{1}{2}\kappa,\ \rho \mapsto \rho$. So from the point of view of deformations of bihamiltonian structures of hydrodynamic type, the systems (1.1) and (1.2), (1.3) have the same property, i.e. both of their bihamiltonian structures have nonconstant central invariants. Note that the bihamiltonian structure of the system (1.2), (1.3), as it was shown in [27], is obtained from (1.5), (1.6) by the addition of the deformation term $-\delta''(x-y)$ to the bracket $\{w_1(x), w_2(y)\}_1$. We will call the system (1.2), (1.3) the two-component Camassa–Holm (2-CH) system henceforth. This system was also derived independently by Falqui [17] by using the bihamiltonian approach.

The main result of the present paper is the establishment of a reciprocal transformation between the 2-CH system and the first negative flow of the AKNS hierarchy. Recall that the Camassa–Holm equation (1.1) has a similar relation with the first negative flow of the K dV hierarchy, the corresponding reciprocal transformation (also called a hodograph transformation) was found by Fuchssteiner in [19]. Several attempts have been made to obtain solutions of the Camassa–Holm equation (1.1) with $\kappa > 0$ from that of the first negative flow of the KdV hierarchy by using this reciprocal transformation in, for example, [10, 22, 24–26, 32] (for the limiting case $\kappa = 0$, different approach of solving the Camassa–Holm equation is needed, see for example [7, 11]). However, since the inverse of this reciprocal transformation involves the solving of a nonlinear ODE of second order, only particular solutions like the multi-soliton solutions were obtained in explicit forms by using this approach. The advantage of the reciprocal transformation between the 2-CH system and the first negative flow of the AKNS hierarchy is that it gives an explicit correspondence between solutions of these two systems, this correspondence is presented in Theorems 2.1, 2.3 and Theorems 3.1, 3.2. In Sections 2 and 3 we first give the construction of the reciprocal transformation, then in Sec. 4 we show some interesting examples of solutions of the 2-CH system that are obtained from solutions of the first negative flow of the AKNS hierarchy by using the reciprocal transformation, they include peakon and multi-kink solutions.

2. A Reciprocal Transformation for the 2-CH System

The 2-CH system is equivalent to the compatibility conditions of the following linear systems:

$$\phi_{xx}+\left(-\frac{1}{4}+m\,\lambda-\rho^2\,\lambda^2\right)\phi=0, \tag{2.1}$$

$$\phi_t=-\left(\frac{1}{2\lambda}+u\right)\phi_x+\frac{u_x}{2}\phi. \tag{2.2}$$

Here $m=u-u_{xx}+\frac{1}{2}\kappa$. Since the term $\frac{1}{2}\kappa$ can be canceled by a Galilean transformation, we assume $\kappa=0$ in this and the next section. The linear equation (2.1) is known as the Schrödinger spectral problem with energy dependent potential. Antonowicz and Fordy [5] considered the more general spectral problem

$$\phi_{xx}+(c+u_1\,\lambda+\cdots+u_n\,\lambda^n)\phi=0 \tag{2.3}$$

and associated to it $n+1$ compatible local Hamiltonian structures. Here c is a given constant. In [5] it was also shown that (2.3) can be transformed to the spectral problem

$$\psi_{xx}+(v_0+v_1\,\lambda+\cdots+v_{n-1}\,\lambda^{n-1})\psi=\lambda^n\,\psi. \tag{2.4}$$

We will use similar transformations below in order to relate the 2-CH system with the first negative flow of the AKNS hierarchy. The relations of the spectral problem (2.4) and its generalizations to multi-Hamiltonian structures and integrable systems were studied in [4,6,28], and in the particular case of $n=2$ in [23]. The bihamiltonian structure of the 2-CH system (1.2), (1.3) was also given in [5] from the spectral problem (2.3) with $n=2$. In [16] the bihamiltonian structures related to the generalized spectral problems of [6] were considered from the point of view of deformations of bihamiltonian structures of hydrodynamic type, their central invariants are in general nonconstants.

Since when ρ vanishes, the 2-CH system (1.2), (1.3) degenerates to the Camassa–Holm equation (1.1), we assume hereafter $\rho\neq 0$. Equation (1.3) shows that the 1-form

$$\omega=\rho\,\mathrm{d}x-\rho\,u\,\mathrm{d}t \tag{2.5}$$

is closed, so it define a reciprocal transformation $(x,t)\mapsto(y,s)$ by the relation

$$\mathrm{d}y=\rho\,\mathrm{d}x-\rho\,u\,\mathrm{d}t,\quad \mathrm{d}s=\mathrm{d}t, \tag{2.6}$$

and we have

$$\frac{\partial}{\partial x}=\rho\,\frac{\partial}{\partial y},\quad \frac{\partial}{\partial t}=\frac{\partial}{\partial s}-\rho\,u\,\frac{\partial}{\partial y}. \tag{2.7}$$

Denote $\varphi=\sqrt{\rho}\,\phi$, then the spectral problem (2.1), (2.2) is converted to the following one

$$\varphi_{yy}+\left(-\lambda^2+P\,\lambda+Q\right)\varphi=0, \tag{2.8}$$

$$\varphi_s+\frac{\rho}{2\lambda}\varphi_y-\frac{\rho_y}{4\lambda}\varphi=0, \tag{2.9}$$

where

$$P=\frac{m}{\rho^2},\quad Q=-\frac{1}{4\rho^2}-\frac{\rho_{yy}}{2\rho}+\frac{\rho_y^2}{4\rho^2}. \tag{2.10}$$

Now let us consider the isospectral problem (2.8), (2.9). The compatibility conditions read

$$P_s=\rho_y,\quad Q_s+\frac{1}{2}\rho\, P_y+P\,\rho_y=0,\quad \frac{1}{2}\rho\, Q_y+Q\rho_y+\frac{1}{4}\rho_{yyy}=0. \tag{2.11}$$

By integrating the third equation of (2.11) and comparing the resulting equation with (2.10) we obtain

$$\rho^2\, Q+\frac{1}{2}\rho\,\rho_{yy}-\frac{1}{4}\rho_y^2=C=-\frac{1}{4}. \tag{2.12}$$

From the first equation of (2.11) we know that there exists a function $f(y,s)$ such that

$$P=\frac{\partial f(y,s)}{\partial y},\quad \rho=\frac{\partial f(y,s)}{\partial s}. \tag{2.13}$$

Substituting the expressions of P, Q, ρ that are given by (2.13) and the second formula of (2.10) into the second equation of (2.11), we arrive at the following equation for f:

$$\frac{f_{ss}}{2f_s^3}+f_y f_{ys}-\frac{f_{ss}f_{ys}^2}{2f_s^3}+\frac{f_{ys}f_{yss}}{2f_s^2}+\frac{1}{2}f_s f_{yy}+\frac{f_{ss}f_{yys}}{2f_s^2}-\frac{f_{yyss}}{2f_s}=0. \tag{2.14}$$

THEOREM 2.1. *Let f be a solution of the Equation (2.14), and*

$$u=f_y f_s^2+\frac{f_{ss}f_{ys}}{f_s}-f_{yss},\quad \rho=f_s. \tag{2.15}$$

If $x(y,s)$ is a solution of the following system of ODEs:

$$\frac{\partial x}{\partial y}=\frac{1}{\rho},\quad \frac{\partial x}{\partial s}=u, \tag{2.16}$$

then $(u(y,t),\rho(y,t),x(y,t))$ is a parametric solution of the 2-CH system (1.2), (1.3).

Remark. *We say that the triple $(u(y,t),\rho(y,t),x(y,t))$ is a parametric solution of the 2-CH system if the functions $\bar{u}(x,t)=u(y(x,t),t)$, $\bar{\rho}(x,t)=\rho(y(x,t),t)$ satisfy the system (1.2), (1.3), here $y=y(x,t)$ is the inverse function of $x=x(y,t)$. For simplicity, we will use the same symbol u,ρ to denote the functions $u(y(x,t),t)$, $\rho(y(x,t),t)$ as functions of x and t.*

Proof. Due to the definition of the reciprocal transformation, we only need to verify the validity of the equation

$$u - u_{xx} = m = \rho^2 P = f_s^2 f_y. \tag{2.17}$$

Denote by E the L.H.S of Equation (2.14). By using the definition (2.15) of the function u we obtain through a straightforward computation

$$u - \rho\,(\rho\, u_y)_y - f_s^2 f_y + 2 f_s^3 E_y + 4 f_s^2 f_{ys} E = 0$$

which yields (2.17). The theorem is proved. □

DEFINITION 2.2. *A function $f = f(y,s)$ is called a primary solution of the 2-CH system (1.2), (1.3) if it satisfies the Equation (2.14).*

Given a solution $(u(x,t), \rho(x,t))$ of the 2-CH system (1.2), (1.3), the formulae (2.10) and (2.13) determines a primary solution $f(y,s)$, we call it the primary solution that is associated to $(u(x,t), \rho(x,t))$. On the other hand, any primary solution $f(y,s)$ yields a solution of the 2-CH system in a parametric form through the formulae (2.15), (2.16). In the next theorem it will be shown that from a primary solution $f(y,s)$ one can construct another solution of the 2-CH system. This solution is still in parametric form, however, in this case the function $x = x(y,s)$ is given explicitly in terms of $f(y,s)$ without the need of integration.

THEOREM 2.3. *Let $f(y,s)$ be a solution of the Equation (2.14). Define the functions $x = x(y,s), u = u(y,s), \rho = \rho(y,s)$ by*

$$x = f(s,y), \quad u = \frac{\partial x}{\partial s}, \quad \frac{1}{\rho} = \frac{\partial x}{\partial y}. \tag{2.18}$$

Then $(u(y,t), \rho(y,t), x(y,t))$ is a parametric solution of the 2-CH system (1.2), (1.3).

Proof. Substituting $u = \frac{\partial x}{\partial s}$, $\frac{1}{\rho} = \frac{\partial x}{\partial y}$ into the Equations (1.2), (1.3) we know, by using the relation (2.7), that $(u(y,t), \rho(y,t), x(y,t))$ gives a solution to the 2-CH system if and only if the function $x(y,s)$ satisfies

$$x_{ss} + \frac{2x_s x_{ys}}{x_y} + \frac{x_{yy}}{x_y^4} - \frac{x_{ys}^2 x_{yy}}{x_y^4} + \frac{x_{yss} x_{yy}}{x_y^3} + \frac{x_{ys} x_{yys}}{x_y^3} - \frac{x_{yyss}}{x_y^2} = 0. \tag{2.19}$$

This equation follows immediately from the fact that the function $f(y,s)$ satisfies (2.14). The theorem is proved. □

Let us note that for the parametric solution (2.18), the associated primary solution $\bar{f}(y,s)$ that is determined by the formulae (2.10), (2.13) is in general different from the original primary solution $f(y,s)$. This procedure yields a Bäcklund transformation $f(y,s)\mapsto \bar{f}(y,s)$ for the Equation (2.14). We will consider in detail such a class of Bäcklund transformation for the Equation (2.14) and the 2-CH system in another publication. In the next section, we will show how to construct primary solutions of the 2-CH system from solutions of the first negative flow of the AKNS hierarchy.

3. Relations to the First Negative Flow of the AKNS Hierarchy

The AKNS spectral problem is given by

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_y=\begin{pmatrix}\lambda & -q\\ r & -\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{3.1}$$

The first negative flow of the ANKS hierarchy is equivalent to the compatibility conditions of (3.1) with the linear system

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_s=\frac{1}{4\lambda}\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{3.2}$$

This flow can be represented in the form

$$q_s=\frac{1}{2}b,\quad r_s=\frac{1}{2}c,\quad b_y=2aq,\quad c_y=2ar,\quad a_y+br+cq=0. \tag{3.3}$$

By using the last three equations of (3.3) we obtain

$$a^2+bc=\varepsilon^2, \tag{3.4}$$

where ε is a constant. We assume that $\varepsilon\neq 0$.

THEOREM 3.1. *Let (a,b,c,q,r) be a solution of the Equations (3.3) with $\varepsilon^2=1$, then any function $f(y,s)$ satisfying*

$$2a=b\,\mathrm{e}^{-f}-c\,\mathrm{e}^{f} \tag{3.5}$$

gives a primary solution of the 2-CH system.

Proof. Assume that the function $f=f(y,s)$ satisfies (3.5). Let us first prove the following formula:

$$f_y=q\,\mathrm{e}^{-f}+r\,\mathrm{e}^{f}. \tag{3.6}$$

Due to (3.3) and (3.5) we have

$$\begin{aligned}0&=(2a-b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f})_y=2a_y-b_y\,\mathrm{e}^{-f}+c_y\,\mathrm{e}^{f}+(b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f})f_y\\ &=-2(br+cq)-2aq\,\mathrm{e}^{-f}+2ar\,\mathrm{e}^{f}+(b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f})f_y\\ &=-2(br+cq)-(b\,\mathrm{e}^{-f}-c\,\mathrm{e}^{f})(q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f})+(b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f})f_y\\ &=(b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f})(f_y-q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f}).\end{aligned} \tag{3.7}$$

The formula (3.6) then follows from (3.7) and the fact that

$$b\,\mathrm{e}^{-f}+c\,\mathrm{e}^{f}=\sqrt{(b\,\mathrm{e}^{-f}-c\,\mathrm{e}^{f})^2+4\,b\,c}=\sqrt{4(a^2+b\,c)}=\pm 2\varepsilon\neq 0. \tag{3.8}$$

Differentiating both sides of (3.6) w.r.t. y and s we obtain respectively

$$\begin{aligned} f_{yy} &= (q\,\mathrm{e}^{-f}+r\,\mathrm{e}^{f})_y = q_y\,\mathrm{e}^{-f}+r_y\,\mathrm{e}^{f}-(q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f})f_y \\ &= q_y\,\mathrm{e}^{-f}+r_y\,\mathrm{e}^{f}-(q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f})(q\,\mathrm{e}^{-f}+r\,\mathrm{e}^{f}) \\ &= q_y\,\mathrm{e}^{-f}+r_y\,\mathrm{e}^{f}-q^2\,\mathrm{e}^{-2f}+r^2\,\mathrm{e}^{2f}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} f_{ys} &= (q\,\mathrm{e}^{-f}+r\,\mathrm{e}^{f})_s = q_s\,\mathrm{e}^{-f}+r_s\,\mathrm{e}^{f}-(q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f})f_s \\ &= \frac{1}{2}b\,\mathrm{e}^{-f}+\frac{1}{2}c\,\mathrm{e}^{f}-(q\,\mathrm{e}^{-f}-r\,\mathrm{e}^{f})f_s. \end{aligned} \tag{3.10}$$

For any solution $(\phi_1=\phi_1(y,s;\lambda), \phi_2=\phi_2(y,s;\lambda))$ of the systems (3.1), (3.2) define

$$\varphi=\mathrm{e}^{-\frac{f}{2}}\phi_1+\mathrm{e}^{\frac{f}{2}}\phi_2, \quad P=f_y, \quad \rho=f_s, \tag{3.11}$$

$$Q=-\frac{3}{4}q^2\,\mathrm{e}^{-2f}-\frac{1}{2}q\,r-\frac{3}{4}r^2\,\mathrm{e}^{2f}+\frac{1}{2}q_y\,\mathrm{e}^{-f}-\frac{1}{2}r_y\,\mathrm{e}^{f}. \tag{3.12}$$

By using Equations (3.1)–(3.10) and the fact that $\varepsilon^2=1$, we can show through a straightforward and lengthy computation that the functions φ, P, Q, ρ satisfy the Equations (2.8), (2.9) and (2.12). The theorem then follows from the derivation of Equation (2.14) that is given in the last section. □

Remark. The main point in the proof of the above theorem is the establishment of the gauge transformation defined by the first formula of (3.11) between the iso-spectral problems (2.8), (2.9) and (3.1), (3.2). Such a gauge transformation also yields the correspondence between the positive flows of the AKNS hierarchy and certain flows that are related to the spectral problem (2.8).

The following theorem gives an explicit way of constructing a solution of the first negative flow of the AKNS hierarchy from a primary solution of the 2-CH system.

THEOREM 3.2. *If f is a primary solution of the 2-CH system (1.2), (1.3), then we can construct a solution of the first negative flow of the AKNS hierarchy by the following formulae:*

$$q=\frac{\mathrm{e}^{f}}{2}\left(f_y+\frac{\varepsilon-f_{ys}}{f_s}\right), \quad r=\frac{\mathrm{e}^{-f}}{2}\left(f_y-\frac{\varepsilon-f_{ys}}{f_s}\right), \quad b=2q_s, \quad c=2r_s, \quad a=\frac{b\,\mathrm{e}^{-f}-c\,\mathrm{e}^{f}}{2},$$

where $\varepsilon=1$ or $\varepsilon=-1$.

Proof. Since f is a primary solution of the 2-CH system, we have $E=0$ where E is defined as in the proof of Theorem 2.1. Then by a straightforward computation, we obatin

$$b_y - 2aq = 2e^f E = 0, \;\; c_y - 2ar = 2e^{-f} E = 0, \;\; a^2 + bc = \varepsilon^2 = 1.$$

The theorem is proved. □

By using the freedom in the choice of signs of the parameter ε we obtain the following corollary.

COROLLARY 3.3. *We have the following two Bäcklund transformations for the equation (2.14):*

$$f \mapsto B_\varepsilon f = f + \log\left(\frac{f_{ss} f_{ys} + f_s^3 f_y - f_s f_{yss} + \varepsilon(f_{ss} - f_s^2)}{f_{ss} f_{ys} + f_s^3 f_y - f_s f_{yss} + \varepsilon(f_{ss} + f_s^2)}\right), \;\; \varepsilon = 1 \text{ or } -1. \tag{3.13}$$

Proof. Since f is a primary solution of the 2-CH system, by using Theorem 3.2 we obtain two solutions of the first negative flow of the AKNS hierarchy. We denote them by $(a_\varepsilon, b_\varepsilon, c_\varepsilon, q_\varepsilon, r_\varepsilon)$, $\varepsilon = 1, -1$. Due to Theorem 3.1, each of these two solutions yields two primary solutions of the 2-CH system

$$f_{\varepsilon,\gamma} = \log\frac{\gamma - a_\varepsilon}{c_\varepsilon} = \log\frac{b_\varepsilon}{\gamma + a_\varepsilon}, \quad \varepsilon, \gamma = \pm 1. \tag{3.14}$$

It is easy to see that $f_{\varepsilon,\gamma} = f$ when $\gamma = \varepsilon$. The corollary is then proved if we identify $B_\varepsilon f$ with $f_{\varepsilon,-\varepsilon}$. □

The Bäcklund transformations given by the above corollary can also be represented in terms of the dependent variables q, r of the first negative flow of the AKNS hierarchy, due to their long expressions we do not present the formulae here. Instead, let us illustrate the procedure of obtaining solutions of the system (3.3) starting from the following trivial solution:

$$q = \sum_{i=1}^{n} e^{\zeta_i y + \frac{\gamma}{\zeta_i} s + \xi_i}, \;\; r = 0, \;\; b = 2q_s, \;\; c = 0, \;\; a = \gamma, \tag{3.15}$$

where $n \in \mathbb{N}$, ζ_i, ξ_i, γ are arbitrary constants with ζ_i non-vanishing and pairwise distinct. According to Theorem 3.1 we know that

$$f_0 = \log\frac{b}{\gamma + a} = \log\frac{q_s}{\gamma} \tag{3.16}$$

is a primary solution of the 2-CH system. By using the above corollary, we obtain two solutions of the Equation (2.14), however, only one of them makes sense. By repeating this procedure, we obtain a sequence of primary solutions $f_0, f_1, f_2, \ldots$

of the 2-CH system. Then by using Theorem 2.1 or 2.3 and Theorem 3.2, we obtain a sequence of solutions of the 2-CH system and the first negative flow of the AKNS hierarchy. Such class of solutions of the 2-CH system have very nice properties, we will study them in more detail in the next section.

4. Particular Solutions of the 2-CH System

We are now ready to present in this section some examples of solutions of the 2-CH system, they include the peakon and multi-kink solutions.

EXAMPLE 1. Let us first try, without referring to the reciprocal transformation constructed above, to find travelling wave solutions of the 2-CH system. Assume $u=h(x+vt)$, $\rho=g(x+vt)$, where v is a constant. Then the Equations (1.2) and (1.3) become

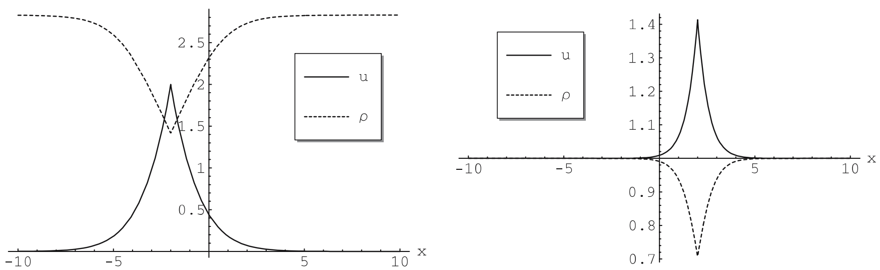
$$v(h'-h''')+3hh'-2h'h''-hh'''+\kappa h'-gg'=0, \tag{4.1}$$
$$vg'+gh'+hg'=0. \tag{4.2}$$

We can solve the second equation directly to obtain $g=\frac{A}{v+h}$, where A is a constant. Then the Equation (4.1) becomes an ODE for h. This ODE can be solved by a standard method. By carefully choosing the integration constant, we obtain the following solution:

$$u=\chi-\sqrt{\chi^2-v^2},\quad \rho=\sqrt{vK}\left(1+\sqrt{\frac{\chi-v}{\chi+v}}\right),\quad \chi=(v-K)\cosh(x+vt)+K, \tag{4.3}$$

where $K=-\frac{1}{4}\kappa$. If $v>0, K>0$, this is a travelling peakon solution, see Figure 1.

EXAMPLE 2. The first negative flow of the AKNS hierarchy has an important reduction. Under the assumption



(a) (4.3) with $v=2, K=1, t=1$. **(b)** (4.10) with $p_1=1, x_0=2$.

Figure 1. Travelling peakon and stationary peakon.

$$q = -\frac{w_y}{2}, \quad r = \frac{w_y}{2}, \quad b = -\sinh w, \quad c = \sinh w, \quad a = \cosh w, \tag{4.4}$$

it is reduced to the sinh-Gordon equation $w_{ys} = \sinh w$.

Now let us employ the results of the previous sections to obtain a stationary peakon solution of the 2-CH system (1.2) and (1.3) with $\kappa = 0$. Due to Theorem 3.1, a solution of the sinh-Gordon equation leads to a primary solution of the 2-CH system

$$f(y,s) = \varepsilon \log\left(-\tanh\frac{w}{2}\right), \text{ where } \varepsilon = \pm 1. \tag{4.5}$$

Then the equations in (2.13) become

$$P = \varepsilon \frac{w_y}{w_{ys}}, \quad \rho = \varepsilon \frac{w_s}{w_{ys}}. \tag{4.6}$$

By using Theorem 2.1, we find a parametric solution of the 2-CH system with $\kappa = 0$,

$$x(y,s) = \varepsilon \log w_s + x_0, \quad u(y,s) = \varepsilon \frac{w_{ss}}{w_s}, \quad \rho = \varepsilon \frac{w_s}{w_{ys}}, \tag{4.7}$$

where x_0 is an arbitrary constant.

Now let us choose a kink solution of the sinh-Gordon equation

$$w(y,s) = 4\tanh^{-1}\left(e^{p_1 y + \frac{s}{p_1} + q_1}\right) \tag{4.8}$$

with some constants $p_1 \neq 0, q_1$ and substitute it into (4.7), we obtain a stationary solution of the 2-CH system,

$$u = \frac{\sqrt{1 + e^{2\varepsilon(x - x_0)}}}{p_1}, \quad \rho = \frac{1}{p_1\sqrt{1 + e^{2\varepsilon(x - x_0)}}}. \tag{4.9}$$

Then it is easy to see that the following u, ρ defined by

$$u = \frac{\sqrt{1 + e^{-2|x - x_0|}}}{p_1}, \quad \rho = \frac{1}{p_1\sqrt{1 + e^{-2|x - x_0|}}} \tag{4.10}$$

give a stationary peakon solution of the 2-CH system, see Figure 1. Note that a peakon solution with constant speed $-\frac{1}{2}\kappa$ for the 2-CH system without the assumption of $\kappa = 0$ can be obtained by the Galilean transformation $x \mapsto \tilde{x} = x + \frac{1}{2}\kappa t, \; t \mapsto \tilde{t} = t, \; u \mapsto \tilde{u} = u - \frac{1}{2}\kappa, \; \rho \mapsto \tilde{\rho} = \rho$.

EXAMPLE 3. In this example we give some explicit expressions of kink and 2-kink interaction solutions of the 2-CH system with $\kappa = 0$. These solutions are derived from the particular trivial solutions of the first negative flow of the AKNS hierarchy given in the end of the last section and by using the Bäcklund transformations of Corollary 3.3. It's easy to see that there are only constant solutions when $n = 1$. We consider here the cases when $n = 2$ and $n = 3$.

Let us first assume $n=2$. Denote $\xi_i = p_i\, y + \frac{s}{p_i} + q_i$, then we have the following solution for the first negative flow of the AKNS hierarchy

$$q = p_1 \mathrm{e}^{\xi_1} + p_2 \mathrm{e}^{\xi_2}, \;\; r=0, \;\; b=2(\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2}), \;\; c=0, \;\; a=1, \tag{4.11}$$

where $p_1 \neq p_2$. By applying Theorem 3.1 and the Bäcklund transformations of Corollary 3.3 we arrive at the following two primary solutions of the 2-CH system:

$$f_0 = \log\left(\mathrm{e}^{\xi_1}+\mathrm{e}^{\xi_2}\right), \;\; f_1 = \log\left(\frac{(p_1-p_2)^2 \mathrm{e}^{\xi_1+\xi_2}}{p_1^2 \mathrm{e}^{\xi_1} + p_2^2 \mathrm{e}^{\xi_2}}\right). \tag{4.12}$$

Note that a further application of the Bäcklund transformations of Corollary 3.3 leads to $f_2=\log(0)$. So in this case we can only obtain two primary solutions. By using Theorem 2.3 we obtain two solutions of the 2-CH system which have the form (2.18) with the function $x(y,s)$ given respectively by

$$x_0 = \log\left(\mathrm{e}^{\tilde{\xi}_1}+\mathrm{e}^{\tilde{\xi}_2}\right), \;\; x_1 = \log\left(\frac{(p_1-p_2)^2 \mathrm{e}^{\tilde{\xi}_1+\tilde{\xi}_2}}{p_1^2 \mathrm{e}^{\tilde{\xi}_1} + p_2^2 \mathrm{e}^{\tilde{\xi}_2}}\right), \tag{4.13}$$

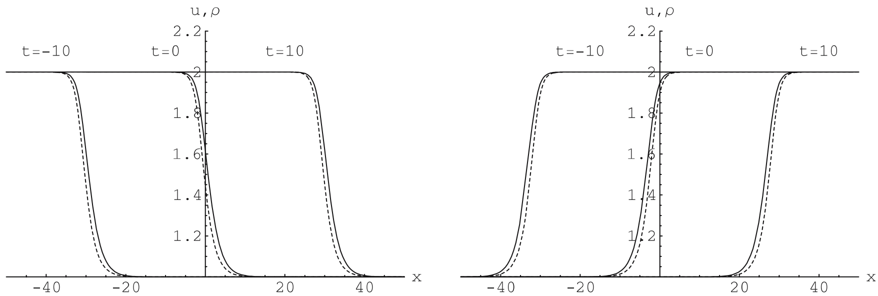
where $\tilde{\xi}_i = p_i\, s + \frac{y}{p_i} + q_i$. The solution obtained from x_0 (respectively, x_1) is an antikink (respectively, kink), see Figure 2 where the profiles of ρ are represented by the dashed curves.

Now let us consider the case when $n=3$, in a similar way to the previous case we obtain three solutions of the 2-CH system which have the form (2.18) with the function $x(y,s)$ given respectively by

$$x_0 = \log\left(\mathrm{e}^{\tilde{\xi}_1}+\mathrm{e}^{\tilde{\xi}_2}+\mathrm{e}^{\tilde{\xi}_3}\right),$$

$$x_1 = \log\left(\frac{p_1^2(p_2-p_3)^2\mathrm{e}^{\tilde{\xi}_2+\tilde{\xi}_3} + p_2^2(p_3-p_1)^2\mathrm{e}^{\tilde{\xi}_3+\tilde{\xi}_1} + p_3^2(p_1-p_2)^2\mathrm{e}^{\tilde{\xi}_1+\tilde{\xi}_2}}{p_2^2 p_3^2 \mathrm{e}^{\tilde{\xi}_1} + p_3^2 p_1^2 \mathrm{e}^{\tilde{\xi}_2} + p_1^2 p_2^2 \mathrm{e}^{\tilde{\xi}_3}}\right),$$

$$x_2 = \log\left(\frac{(p_1-p_2)^2(p_2-p_3)^2(p_3-p_1)^2\mathrm{e}^{\tilde{\xi}_1+\tilde{\xi}_2+\tilde{\xi}_3}}{p_1^4(p_2-p_3)^2\mathrm{e}^{\tilde{\xi}_2+\tilde{\xi}_3} + p_2^4(p_3-p_1)^2\mathrm{e}^{\tilde{\xi}_3+\tilde{\xi}_1} + p_3^4(p_1-p_2)^2\mathrm{e}^{\tilde{\xi}_1+\tilde{\xi}_2}}\right),$$



(a) u, ρ obtained from x_0.

(b) u, ρ obtained from x_1.

Figure 2. Kink and antikink. ((4.13) with $p_1=1, p_2=2, q_1=0, q_2=0$.)

where p_1, p_2, p_3 are pairwise distinct. These solutions describe the antikink–antikink, antikink–kink and kink–kink interactions, respectively, their figures can be found in arXiv: nlin.SI/0501028.

In general, for any $n \in \mathbb{N}$ we expect to arrive at in this way n solutions of the 2-CH system, each of which is a $(n-1)$-kink solution if we choose the parameters p_i, q_i in an appropriate way. They correspond to the interactions of k antikinks and $n-1-k$ kinks with $k=0,\dots,n-1$. Besides the cases of $n=2,3$, we can also check this assertion for the case when $n=4$. We will leave the analysis of the general case to a subsequent publication.

5. Conclusion

We have constructed an explicit reciprocal transformation between the 2-CH system (1.2), (1.3) and the first negative flow of the AKNS hierarchy (3.3) with $\varepsilon^2=1$. The primary solution $f(y,s)$ satisfying the Equation (2.14) plays a crucial role in this construction. This transformation comprises of two steps, the first step is the correspondence between solutions of the 2-CH system and the primary solutions satisfying (2.14), it is given by the formulae (2.10) and (2.13) and Theorems 2.1 and 2.3. The second step is the correspondence between solutions of the first negative flow of the AKNS hierarchy and the primary solutions of the 2-CH system, it is given by Theorems 3.1 and 3.2. These correspondences are presented in simple and explicit forms, they enable us to obtain solutions of the 2-CH system from those of the first negative flow of the AKNS hierarchy, which includes in particular the well known sine-Grodon and sinh-Grodon equations.

In terms of the primary solutions we also obtained in Sections 2 and 3 two kinds of Bäcklund transformations for the 2-CH system or the first negative flow of the AKNS hierarchy, and showed in Section 4 that the Bäcklund transformations given in this section lead to interesting multi-kink solutions of the 2-CH system. It would be interesting to express in terms of the primary solutions of the 2-CH system the Bäcklund transformations of the AKNS hierarchy that are well known in the literatures, see for example [21,29].

For the travelling peakon solution of the 2-CH system that is given in the first example of Section 4, we assume that the constant $K>0$. When $K=0$ this solution degenerates to a peakon solution of the Camassa–Holm equation (1.1). We will return to analyze in a subsequent paper the various properties of particular solutions of the 2-CH system, including the problem of existence of multi-peakon solutions. Although the 2-CH system that we considered here was derived from the problem of classification of deformations of bihamiltonian structures of hydrodynamic type, we do expect that it would find for itself physically important applications.

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