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On Segal–Wilson's Construction for the τ-Functions of the Constrained KP Hierarchies

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Abstract. In this Letter, we study the constrained KP hierarchies by employing Segal–Wilson's theory on the τ-functions of the KP hierarchy. We first describe the elements of the Grassmannian which correspond to solutions of the constrained KP hierarchy, and then we show how to construct its rational and soliton solutions from these elements of the Grassmannian.

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1. Introduction

In recent years, it was found that many interesting integrable systems in $1+1$-dimensions, apart from the well-known KdV hierarchy and other l-reduced KP hierarchies [1], can be constrained from the KP hierarchy. Among these integrable systems, the most well known are the AKNS and Yajima–Oikawa hierarchies [2–5]. These integrable systems can be put into the following form [4, 5]:

$$L_{t_n} = [B_n, L], \tag{1.1a}$$

$$q_{i,t_n} = B_n q_i, \tag{1.1b}$$

$$r_{i,t_n} = -B_n^* r_i, \quad i = 1, 2, \ldots, m; \quad n \geqslant 2, \tag{1.1c}$$

$$L^k = B_k + \sum_{i=1}^{m} q_i \partial^{-1} r_i, \tag{1.1d}$$

where the micro-differential operator L is defined as

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots, \tag{1.2}$$

u_i's, q_i's and r_i's are functions of the variable $t = (t_1, t_2, \ldots)$, $\partial = \partial/\partial x$ with $x = t_1$, B_n is the differential part of the micro-differential operator L^n, B_n^* denotes the adjoint of B_n, $\partial^{-1} r_i$ is defined as

$$\partial^{-1} r_i = r_i \partial^{-1} - r_{i,x} \partial^{-2} + r_{i,xx} \partial^{-3} - \cdots, \tag{1.3}$$

and k, m are given positive integers. The hierarchy of equations in (1.1) can be represented in terms of the dynamical variables $u_2, u_3, \dots, u_k$ and $q_i, r_i (i = 1, 2, \dots, m)$. We call this hierarchy of equations the vector k-constrained KP hierarchy [4, 6]. For example, when $k = 1$, this hierarchy of equations coincides with the generalized AKNS hierarchy [4–6], and when $k = 2$ it is the generalized Yajima–Oikawa hierarchy with the following first flow:

$$q_{i,t_2} = q_{i,xx} + 2u_2 q_i, \tag{1.4a}$$

$$u_{2,t_2} = \sum_{i=1}^{m} (q_i r_i)_x, \tag{1.4b}$$

$$r_{i,t_2} = -r_{i,xx} - 2u_2 r_i, \tag{1.4c}$$

$$i = 1, 2, \dots, m$$

The vector k-constrained KP hierarchy (1.1) was shown to possess bi-Hamiltonian structure in [4, 5, 7], and bilinear representation in [6, 8]. In [6], the vector k-constrained KP hierarchy was solved by using its bilinear representation and the free fermion operators.

The purpose of this Letter is to interpret the vector k-constrained KP hierarchy from the viewpoint of Segal–Wilson's theory on the τ-functions of the KP hierarchy [9]. We can see that such a description of the vector k-constrained KP hierarchy is more simple and clearer in some aspects than by using the bilinear construction [6, 8]. We first recall some relevent basic facts from the Segal–Wilson's theory in the next section, in Section 3 we interpret the vector k-constrained KP hierarchy by using Segal–Wilson's theory, and in Section 4 we show how to construct rational and soliton solutions for the vector k-constrained KP hierarchy by employing the result of Section 3.

2. On Segal–Wilson's Theory [9]

Let H be the space of all square integrable complex-valued functions on the denote this component of $\mathrm{Gr}(H)$ by Gr as in [9]. An element W of Gr is said to be transversal if $p_+|_W: W \to H_+$ is a bijection.

Let Γ_+ be the set of all nonvanishing holomorphic functions g in the circle $|\lambda| \leqslant 1$ which satisfy $g(0) = 1$ and can be uniquely written in the form

$$g(t,\lambda) = \mathrm{e}^{\sum_{i=1}^{\infty} t_i \lambda^i}, \tag{2.1}$$

where $\{t_i\}_{i \geqslant 1}$ is a set of real numbers. We denote $\Gamma_+^W = \{g \in \Gamma_+ | g^{-1}W$ is transversal$\}$, then Γ_+^W is a dense subset of Γ_+. From (2.1), we see that for any $g \in \Gamma_+$ there is a unique set $\{t_i\}_{i \geqslant 1}$ corresponding to it, so in what follows, when we say that $t = (t_1, t_2, \ldots)$ belongs to Γ_+ (or Γ_+^W) we mean that the corresponding g belongs to Γ_+ (or Γ_+^W).

For each $W \in$ Gr there is defined a holomorphic function $\tau_W(t)$ for $t = (t_1, t_2, \ldots) \in \Gamma_+$ which is called the τ-function corresponding to W. For each $W \in$ Gr there is also a unique function (called a Baker function or wave function) $\psi = \psi_W(t,\lambda)$, defined for $t \in \Gamma_+^W$ and $\lambda \in S^1$ such that

(a) $\psi(t,\cdot) \in W$ for each fixed $t \in \Gamma_+^W$,

(b) ψ has the form $\psi = g(t,\lambda)(1 + \sum_{i=1}^{\infty} a_i(t)\lambda^{-i})$, where $g(t,\lambda) = \mathrm{e}^{\sum_{i=1}^{\infty} t_i \lambda^i}$.

It has been shown that for any integer $n \geqslant 2$ there is a unique differential operator

$$B_n = \sum_{i=0}^{n} b_{ni}\partial^i, \quad b_{nn} = 1, \tag{2.2}$$

such that

$$\frac{\partial \psi}{\partial t_n} = B_n \psi \tag{2.3}$$

and the τ-function has the following relation with the wave function

$$\begin{aligned} \psi_W(t,\lambda) &= \frac{\tau_W\left(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, \ldots\right)}{\tau_W(t)} g(t,\lambda) \\ &= \left(\sum_{j=0}^{\infty} \frac{p_j(-\tilde{\partial})\tau_W(t)}{\tau_W(t)} \lambda^{-j}\right) g(t,\lambda), \end{aligned} \tag{2.4}$$

where

$$\tilde{\partial} = \left(\frac{\partial}{\partial x}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots\right),$$

and $p_j(t)$'s are the Schur polynomials defined by

$$e^{\sum_{i=1}^{\infty} t_i \lambda^i} = \sum_{i=0}^{\infty} p_i(t)\lambda^i. \tag{2.5}$$

Let $P = 1 + \Sigma_{i=1}^{\infty} a_i(t)\partial^{-i}$, define $L = P\partial P^{-1}$, then

$$B_n = (L^n)_+, \qquad L_{t_n} = [B_n, L].$$

So $\tau_W(t)$ is a τ-function for the KP hierarchy.

3. Interpretation of the Vector k-Constrained KP Hierarchy (1.1) in Terms of Segal–Wilson's Theory

Now we want to know what kind of conditions should be imposed on $W \in \mathrm{Gr}$ such that we can deduce from it a solution for the vector k-constrained KP hierarchy. We have the following proposition:

PROPOSITION. *Let $W \in \mathrm{Gr}, k \geqslant 1$ be a positive integer, $\psi = \psi_W(t, \lambda)$ be the wave function corresponding to W. If there are m $(m \geqslant 1)$ smooth functions $q_1(t), q_2(t), \ldots, q_m(t)$ defined on Γ_+^W and m elements $\Phi_1(\lambda), \Phi_2(\lambda), \ldots, \Phi_m(\lambda)$ of H which are independent on t such that*

$$\text{(a)} \det Q = \det \begin{pmatrix} q_1 & q_2 & \cdots & q_m \\ q_{1,x} & q_{2,x} & \cdots & q_{m,x} \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{(m-1)} & q_2^{(m-1)} & \cdots & q_m^{(m-1)} \end{pmatrix} \neq 0, \quad q_i^{(l)} = \frac{\partial^l q_i}{\partial x^l}, \tag{3.1}$$

(b) *If $\sum_{i=1}^m h_i \Phi_i(\lambda) \in W$ for some numbers $h_1, h_2, \ldots, h_m$ which are independent on λ, then $h_i = 0$ for $i = 1, 2, \ldots, m$,*

$$\text{(c)} \ \lambda^k \psi - \sum_{i=1}^m q_i \Phi_i \in W, \tag{3.2}$$

then we can deduce from W a solution for the vector k-constrained KP hierarchy (1.1).

Proof. We prove the proposition through the following four steps:

(1) We first prove that

$$q_{i,t_n} = B_n q_i, \quad i = 1, 2, \ldots, m; n \geqslant 2, \tag{3.3}$$

where B_n is defined as in Section 2 for the given $W \in \mathrm{Gr}$. For this purpose, we consider the following system of linear equations for the unknowns $\varphi_1, \varphi_2, \ldots, \varphi_m$:

$$\sum_{i=1}^m q_i \varphi_i = -(B_k \psi - \lambda^k \psi), \tag{3.4a}$$

$$\sum_{i=1}^{m} q_{i,x}\varphi_i = -(B_k\psi - \lambda^k\psi)^{(1)} + d_{11}\psi, \tag{3.4b}$$

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$$\sum_{i=1}^{m} q_i^{(m-1)}\varphi_i = -(B_k\psi - \lambda^k\psi)^{(m-1)} + \sum_{l=1}^{m-1} d_{m-1l}\psi^{(l-1)}, \tag{3.4c}$$

where for a function $f(t)$ we denoted $f^{(l)} = \partial^l f/\partial x^l, d_{ij}$'s are some functions of t which do not depend on λ, and they are determined by the condition that the right-hand sides of the lth linear equation in (3.4) can be written in the form $g(w_{l1}\lambda^{-1} + w_{l2}\lambda^{-2} + \cdots)$, where $g = e^{\Sigma_{i=1}^{\infty} t_i\lambda^i}, l = 1, 2, \ldots, m$. Since ψ has the form $g(1 + a_1\lambda^{-1} + \cdots)$, and $B_k\psi - \lambda^k\psi$ has the form $g(d_1\lambda^{-1} + d_2\lambda^{-2} + \cdots)$, we see that d_{ij}'s are uniquely determined by the above-mentioned condition. Thus, by using condition (3.1), we obtain a unique solution $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$ for given $t \in \Gamma_+^W$ and $\lambda \in S^1$.

Let us denote

$$R = -(B_k\psi - \lambda^k\psi, (B_k\psi - \lambda^k\psi)^{(1)}, \ldots, (B_k - \lambda^k\psi)^{(m-1)})^T, \tag{3.5}$$

$$H = (0, d_{11}\psi, d_{21}\psi + d_{22}\psi_x, \ldots, \sum_{i=1}^{m-1} d_{m-1i}\psi^{(i-1)})^T. \tag{3.6}$$

Then from condition (3.2) we see that the components of $(Q^{-1}R)_x$ belong to W, and we simply denote this fact by $(Q^{-1}R)_x \in W$, we shall use similar notations in what follows. So Equation (3.4) implies that $\varphi_{i,x} \in W (i = 1, 2, \ldots, m)$. On the other hand, from the definition of d_{ij}'s we know that $\varphi_{i,x}$ has the form $g(c_{i0} + c_{i1}\lambda^{-1} + c_{i2}\lambda^{-2} + \cdots)$. Since $g^{-1}W$ is transversal, we thus conclude that

$$\varphi_{i,x} = r_i\psi, \tag{3.7}$$

where $r_i(t) = c_{i0}$.

Condition (3.2) leads to

$$(B_k\psi - \lambda^k\psi)_{t_n} + \sum_{i=1}^{m} q_{i,t_n}\Phi_i \in W, \tag{3.8}$$

so from Equations (3.4a) and

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0, \tag{3.9}$$

we have

$$B_{n,t_k}\psi - \sum_{i=1}^{m} B_n q_i\varphi_i + \sum_{i=1}^{m} q_{i,t_n}\Phi_i \in W. \tag{3.10}$$

From (3.2) and (3.4), we can see that $\varphi_i - \Phi_i \in W$. Thus, by using this fact and (3.7), (3.10), we have

$$\sum_{i=1}^{m}((B_n q_i) - q_{i,t_n})\Phi_i \in W, \tag{3.11}$$

where $(B_n q_i)$ is understood as a function of t obtained by acting B_n on q_i. So from condition (b) of the proposition, we see that q_i satisfies Equation (3.3).

(2) Secondly we prove that for a given positive integer $n \geqslant 2$ there are differential operators $A_{n1}, A_{n2}, \ldots, A_{nm}$ of order less than n, satisfying

$$\varphi_{i,t_n} = A_{ni}\psi, \quad i = 1, 2, \ldots, m. \tag{3.12}$$

From the linear equations in (3.4), we have

$$Q\varphi_{t_n} = -Q_{t_n}\varphi + R_{t_n} + H_{t_n}, \tag{3.13}$$

where $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)^T$, Q is defined as in (3.1), and R, H defined by (3.5), (3.6). By using Equations (3.4a) and (3.9), we have

$$\begin{aligned} R_{t_n} = & -\left(B_{n,t_k}\psi - \sum_{i=1}^{m} B_n q_i\varphi_i, \left(B_{n,t_k}\psi - \sum_{i=1}^{m} B_n q_i\varphi_i\right)^{(1)}, \ldots, \right. \\ & \left. \left(B_{n,t_k}\psi - \sum_{i=1}^{m} B_n q_i\varphi_i\right)^{(m-1)}\right)^T \end{aligned} \tag{3.14}$$

Thus, from (2.3), (3.7) and the fact

$$-Q_{t_n}\varphi + \left(\sum_{i=1}^{m}(B_n q_i)\varphi_i, \sum_{i=1}^{m}(B_n q_i)^{(1)}\varphi_i, \ldots, \sum_{i=1}^{m}(B_n q_i)^{(m-1)}\varphi_i\right) = 0, \tag{3.15}$$

we know that there are differential operators $\tilde{A}_{n1}, \tilde{A}_{n2}, \ldots, \tilde{A}_{nm}$ such that

$$-Q_{t_n}\varphi + R_{t_n} + H_{t_n} = (\tilde{A}_{n1}, \tilde{A}_{n2}, \ldots, \tilde{A}_{nm})^T\psi. \tag{3.16}$$

From the definition of d_{ij}'s, we see that the lth component of $-Q_{t_n}\varphi + R_{t_n} + H_{t_n}$ can be written in the form $g(e_{l1}\lambda^{n-1} + e_{l2}\lambda^{n-2} + \cdots)$. So the order of the operator $\tilde{A}_{ni}(i = 1, 2, \ldots, m)$ must be less than n. Now let us define

$$A_n = (A_{n1}, A_{n2}, \ldots, A_{nm})^T = Q^{-1}(\tilde{A}_{n1}, \tilde{A}_{n2}, \ldots, \tilde{A}_{nm})^T, \tag{3.17}$$

then we see that the operators $A_{ni}(i = 1, 2, \ldots, m)$ satisfy the equations in (3.12), and their orders are less than n.

(3) Thirdly, we prove that

$$r_{i,t_n} = -B_n^* r_i, \quad i = 1, 2, \ldots, m. \tag{3.18}$$

Define

$$C_{ni} = \sum_{l=0}^{n-1} \sum_{s=l+1}^{n} (-1)^{l+s+1} (b_{ns} r_i)^{(s-l-1)} \partial^l,$$
$$i = 1, 2, \ldots, m; n \geqslant 2, \tag{3.19}$$

where b_{ns}'s are the coefficients of the differential operator B_n as given in (2.2). Then we can see that C_{ni} is the unique differential operator which satisfies

$$C_{ni,x} + C_{ni}\partial = r_i B_n - (B_n^* r_i), \tag{3.20}$$

where $(B_n^* r_i)$ is understood as the function obtained by acting B_n^* on r_i. So from $\varphi_{i,x} = r_i \psi$, we obtain

$$\begin{aligned} (\varphi_{i,x})_{t_n} &= r_{i,t_n}\psi + r_i B_n \psi = (r_{i,t_n} + (B_n^* r_i))\psi + (r_i B_n - (B_n^* r_i))\psi \\ &= (r_{i,t_n} + (B_n^* r_i))\psi + (C_{ni,x} + C_{ni}\partial)\psi. \end{aligned}$$

Thus by using the equations in (3.12), we have

$$(r_{i,t_n} + (B_n^* r_i))\psi + ((C_{ni} - A_{ni})_x + (C_{ni} - A_{ni})\partial)\psi = 0. \tag{3.21}$$

Let us denote

$$C_{ni} - A_{ni} = \sum_{l=0}^{n-1} f_{il}\partial^l \quad \text{and}$$
$$\psi = (1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots)g(t, \lambda) = \tilde{P}g(t, \lambda),$$

where $\tilde{P} = 1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots$, then Equation (3.21) can be written as

$$(r_{i,t_n} + (B_n^* r_i))\tilde{P} + \sum_{l=0}^{n-1}\sum_{s=0}^{l} f_{il,x}\binom{l}{s}(\partial^{l-s}\tilde{P})\lambda^s$$
$$+ \sum_{l=0}^{n-1}\sum_{s=0}^{l+1} f_{il}\binom{l+1}{s}(\partial^{l+1-s}\tilde{P})\lambda^s = 0. \tag{3.22}$$

The coefficient of λ^n on the left-hand side of Equation (3.22) is f_{in-1}, so we have $f_{in-1} = 0$. Then from this fact we know that the coefficient of λ^{n-1} on the left-hand side of Equation (3.22) is f_{in-2}, thus $f_{in-2} = 0$, in this way we can prove that $f_{il} = 0$ for $l = 0, 1, \ldots, n-1; n \geqslant 2$. Thus, $A_{ni} = C_{ni}$, and from Equation (3.21) we see that $r_i (i = 1, 2, \ldots, m)$ satisfies Equation (3.18).

(4) Let L be the micro-differential operator defined in Section 2, we need to prove that L satisfies Equation (1.1d). From the linear equations in (3.4), we see that φ_i has the form

$$\varphi_i(t, \lambda) = (h_{i1}\lambda^{-1} + h_{i2}\lambda^{-2} + \cdots) g(t, \lambda),$$

where $g(t, \lambda) = e^{\sum_{i=1}^{\infty} t_i \lambda^i}$. Then by using Equation (3.7), we obtain

$$h_{il} = \sum_{s=0}^{l-1} (-1)^{l-s-1} (r_i a_s)^{(l-s-1)}, \tag{3.23}$$

where $a_0 = 1$ and a_i's are defined by

$$\psi(t, \lambda) = (1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots) g(t, \lambda).$$

Since

$$B_k \psi - \lambda^k \psi = \frac{\partial \psi}{\partial t_k} - \lambda^k \psi = \left(\sum_{l=1}^{\infty} \frac{\partial a_l}{\partial t_k} \lambda^{-l} \right) g(t, \lambda),$$

from Equation (3.4a) we obtain that

$$\frac{\partial a_l}{\partial t_k} = -\sum_{i=1}^{m} q_i h_{il} = \sum_{i=1}^{m} \sum_{s=0}^{l-1} (-1)^{(l-s)} q_i (r_i a_s)^{(l-s-1)}. \tag{3.24}$$

Equation (3.24) is equivalent to

$$\frac{\partial P}{\partial t_k} = -\sum_{i=1}^{m} q_i \partial^{-1} r_i P, \tag{3.25}$$

where $P = 1 + a_1 \partial^{-1} + a_2 \partial^{-2} + \cdots$. From [9], we know that P also satisfies the following equation:

$$\frac{\partial P}{\partial t_k} = B_k P - P \partial^k. \tag{3.26}$$

So from Equations (3.25) and (3.26) we see that $L = P \partial P^{-1}$ satisfies Equation (1.1d).

We have proved that from the given $W \in \mathrm{Gr}$ satisfying the condition of the proposition, we can find a unique set of functions $r_1, r_2, \ldots, r_m$ such that $L, q_i, r_i, i = 1, 2, \ldots, m$ satisfy the vector k-constrained KP hierarchy (1.1). The proposition is now proved. □

We now give two remarks on conditions (a) and (b) of the above proposition.

Remark 3.1. Let us assume that $\Phi_1, \Phi_2, \dots, \Phi_m$ do not meet the condition (b) while condition (c) is satisfied. In the case when all Φ_i's belong to W, the τ-function corresponding to W is a τ-function for the k-reduced KP hierarchy [1, 9]. In other cases, we can find a maximal subset of $\{\Phi_i\}_{1\leqslant i\leqslant m}$ with elements which satisfy condition (b). We assume for simplicity that this subset is $\{\Phi_i\}_{1\leqslant i\leqslant m_1}$. Then, for any $m_1+1 \leqslant j \leqslant m$, we can find some numbers $a_{j1}, a_{j2}, \dots, a_{jm_1}$ such that $\Phi_j - \sum_{l=1}^{m_1} a_{jl}\Phi_l \in W$. Let us define

$$\tilde{q}_i = q_i + \sum_{l=m_1+1}^{m} a_{li}q_l, \quad \text{for } i = 1, 2, \dots, m_1,$$

then $\tilde{q}_i(t), \Phi_i(\lambda), i = 1, 2, \dots, m_1$ still satisfy condition (c) of the proposition. If $\tilde{q}_i(t), i = 1, 2, \dots, m_1$ also satisfy condition (a), then we can deduce from W a solution $L, \tilde{q}_i(t), \tilde{r}_i(t), 1 \leqslant i \leqslant m_1$ for Equation (1.1) with m replaced by m_1. If we further define $\tilde{q}_i(t) = \tilde{r}_i(t) = 0$ for $m_1+1 \leqslant i \leqslant m$, then $L, \tilde{q}_i(t), \tilde{r}_i(t), 1 \leqslant i \leqslant m$ constitute a solution for Equation (1.1).

Remark 3.2. We now assume that conditions (b) and (c) are satisfied. We further assume that on a subset Ω of Γ_+^W the functions $q_1(t), q_2(t), \dots, q_{m_1}(t)$ satisfy condition (a) for certain $m_1 \leqslant m$, and $\det M_j = 0$ for $m_1 + 1 \leqslant j \leqslant m$, where M_j is the $(m_1+1) \times (m_1+1)$ matrix with the first m_1 columns $(q_i, q_{i,x}, \dots, q_i^{(m_1)})^T, i = 1, 2, \dots, m_1$, and the last column $(q_j, q_{j,x}, \dots, q_j^{(m_1)})^T$. Then we can express $q_j (m_1+1 \leqslant j \leqslant m)$ in the form $q_j(t) = \sum_{l=1}^{m_1} \omega_{jl} q_l(t)$, where the ω_{jl}'s do not depend on x. Define $\tilde{\Phi}_i = \Phi_i + \sum_{l=m_1+1}^{m} \omega_{li}\Phi_l$ for $1 \leqslant i \leqslant m_1$, then $\tilde{\Phi}_1, \tilde{\Phi}_2, \dots, \tilde{\Phi}_{m_1}$ and $q_1, q_2, \dots, q_{m_1}$ still satisfy conditions (b) and (c). By using a similar procedure to that given in the above proof of the proposition, we can find functions $r_i(t), i = 1, 2, \dots, m_1$ defined on Ω such that $L, q_i(t), r_i(t), i = 1, 2, \dots, m_1$ satisfy Equation (1.1) with m replaced by m_1.

Let us denote

$$q_i(t) = \frac{\rho_i(t)}{\tau_W(t)}, \qquad r_i(t) = \frac{\sigma_i(t)}{\tau_W(t)}, \quad i = 1, 2, \dots, m, \tag{3.27}$$

then, from [6, 8], we know that

$$\varphi_i(t, \lambda) = \frac{\sigma_i\left(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, \dots\right)}{\lambda\tau_W(t)} \mathrm{e}^{\sum_{i=1}^{\infty} t_i\lambda^i}, \tag{3.28}$$

and the vector k-constrained KP hierarchy (1.1) can be put into bilinear equations in terms of $\tau_W(t), \rho_i(t), \sigma_i(t), i = 1, 2, \dots, m$. So we may say that if $W \in \mathrm{Gr}$

satisfies the conditions of the above proposition or the conditions given in the above remarks, then the corresponding $\tau_W(t)$ is also a τ-function for the vector k-constrained KP hierarchy (1.1).

4. Some Examples

In this section we show through some examples how to construct rational and soliton solutions for the vector k-constrained KP hierarchy by making use of the result of Section 3.

We first consider the rational solutions. Let $\{s_i | i \geqslant 0\}$ be a set of integers with the property that $s_i < s_{i+1}$ for $i \geqslant 0$ and there exists a positive integer γ such that $s_i = i$ for all $i \geqslant \gamma$. Denote the subspace of H spanned by $\{\lambda^{s_i}\}_{i\geqslant 0}$ by W, then $W \in \mathrm{Gr}$. The corresponding τ-function can be expressed as follows [9]:

$$\tau_W(t) = \det\,(p_{\mu_i - i + j}(-t))_{0\leqslant i,j \leqslant \gamma-1}, \tag{4.1}$$

where $\mu_i = i - s_i, p_j(t)'s$ for $j \geqslant 0$ are the Schur polynomials defined by (2.5) and $p_j(t) = 0$ when $j < 0$. The above τ-function is just the τ-function of the KP hierarchy corresponding to the Young diagram $(\mu_1, \mu_2, \ldots, \mu_{\gamma-1})$ [1, 9].

Since when $i \geqslant \gamma, s_i = i$, we see that for any given positive integer $k \geqslant 1$, there exist integers $\nu_1, \nu_2, \ldots, \nu_l$ such that $\lambda^k W/V \subset W$, where $l \leqslant \gamma$, V is the subspace of H spanned by $\{\lambda^{\nu_i}\}_{i=1}^{l}$. Thus, from the proposition and the remarks of Section 3, we see that $\tau_W(t)$ defined by (4.1) is also a τ-function of the vector k-constrained KP hierarchy (1.1) for certain $m \leqslant l$.

EXAMPLE 4.1. Let $s_0 = -1, s_1 = 0, s_i = i$ for $i \geqslant 2$. Then from (2.4) and (4.1) we have

$$\tau_W(t) = \tau = \tfrac{1}{2}t_1^2 + t_2, \tag{4.2}$$

$$\begin{aligned}\psi_W(t,\lambda) &= \psi = \left(1 - \frac{\tau_x}{\tau}\lambda^{-1}\right) \mathrm{e}^{\Sigma_{i=1}^{\infty} t_i \lambda^i} \\ &= -\frac{\tau_x}{\tau}\lambda^{-1} + \left(-\frac{\tau_x}{\tau}p_1(t) + 1\right) + \\ &\quad + \left(-\frac{\tau_x}{\tau}p_3(t) + p_2(t)\right)\lambda^2 + \cdots.\end{aligned} \tag{4.3}$$

We first consider Equation (1.1) with $k = 1$. Choose

$$q_1(t) = \frac{\tau_x}{\tau}p_1(t) - 1, \quad \Phi_1 = -\lambda, \tag{4.4}$$

then $\lambda\psi - q_1(t)\Phi_1 \in W$, and from the proof of the proposition of Section 3, we have

$$\varphi_1(t,\lambda) = -q_1^{-1}(B_1\psi - \lambda\psi) \quad \text{and}$$
$$\varphi_{1,x} = (c_{10} + c_{11}\lambda^{-1} + c_{12}\lambda^{-2} + \cdots)\,\mathrm{e}^{\Sigma_{i=1}^{\infty} t_i\lambda^i},$$

where

$$c_0 = q_1^{-1}\left(\frac{\tau_x}{\tau}\right)_x = -\frac{1}{\tau}.$$

So we have

$$r_1 = c_{10} = -\frac{1}{\tau}. \tag{4.5}$$

Thus, we have obtained a solution of Equation (1.1) with $k = 1, m = 1$. We note that the functions $u_i, i \geqslant 2$ is determined by $\tau = \tau_W(t)$, for example, $u_2 = \partial^2/\partial x^2 \ln \tau = q_1 r_1$.

We now consider Equation (1.1) with $k = 2$. We choose

$$q_1(t) = \frac{\tau_x}{\tau}, \qquad \Phi_1 = -\lambda, \tag{4.6}$$

then $\lambda^2\psi - q_1\Phi_1 \in W$, and we find that

$$r_1(t) = -\frac{1}{\tau}. \tag{4.7}$$

So we obtained a solution of Equation (1.1) with $k = 2, m = 1$.

EXAMPLE 4.2. Let W be spanned by $\{\lambda^{s_i}\}_{i\geqslant 0}$, where $s_0 = -2, s_1 = 0, s_i = i$ for $i \geqslant 2$, then $W \in \mathrm{Gr}$ and the corresponding τ-function and the wave function are

$$\tau_W(t) = \tau = t_3 - \tfrac{1}{3}t_1^3, \tag{4.8}$$

$$\psi_W(t,\lambda) = \psi = \left(1 - \frac{\tau_x}{\tau}\lambda^{-1} + \frac{\tau_{xx}}{2\tau}\lambda^{-2}\right)\mathrm{e}^{\Sigma_{i=1}^{\infty} t_i\lambda^i}. \tag{4.9}$$

We consider the equations in (1.1) with $k = 1$. Define

$$q_1 = \frac{t_1}{\tau}, \qquad q_2 = \frac{\frac{1}{6}t_1^3 - t_1t_2 + t_3}{\tau}, \tag{4.10}$$

$$\Phi_1 = -\lambda^{-1}, \qquad \Phi_2 = \lambda, \tag{4.11}$$

then $\lambda\psi - q_1\Phi_1 - q_2\Phi_2 \in W$. To find the functions r_1, r_2, we need to solve the following system of linear equations for φ_1, φ_2:

$$q_1\varphi_1 + q_2\varphi_2 = -(B_1\psi - \lambda\psi), \tag{4.12a}$$

$$q_{1,x}\varphi_1 + q_{2,x}\varphi_2 = -(B_1\psi - \lambda\psi)_x + d_{11}\psi, \tag{4.12b}$$

where $d_{11} = -(\tau_x/\tau)_x$. Then $\varphi_{i,x}(i = 1,2)$ can be put into the form $(c_{i0} + c_{i1}\lambda^{-1} + \cdots)\,\mathrm{e}^{\Sigma_{l=1}^{\infty} t_l\lambda^l}$, and $r_i = c_{i0}$. So we find

$$r_1 = -\frac{p_3(t)}{\tau} = -\frac{t_3 + t_1t_2 + \frac{1}{6}t_1^3}{\tau}, \quad r_2 = -\frac{t_1}{\tau}. \tag{4.13}$$

Thus, we have obtained a solution of Equation (1.1) with $k = 1, m = 2$.

We now consider soliton solutions of Equation (1.1). Let $\{\alpha_i, \beta_i\}_{i=1}^N$ be a set of nonzero complex numbers, $0 < |\alpha_i|, |\beta_i| < 1$, with $\alpha_i^k \neq \beta_j^k, \alpha_l \neq \alpha_s, \beta_l \neq \beta_s$ for $l \neq s$, and also let $\{a_{ij}\}_{i,j=1}^N$ be a set of complex numbers, where k is a given positive integer. Then denote W to be the closure of the space of functions f which are holomorphic in the unit disc except for a pole of order less than N at the origin, and satisfy the condition

$$f(\beta_i) = \sum_{j=1}^{N} a_{ji} f(\alpha_j), \quad i = 1, 2, \ldots, N. \tag{4.14}$$

Then $W \in \mathrm{Gr}$, and the corresponding τ-function can be put into the following form [9]:

$$\tau_W(t) = \tau = \det(\mu_{ij})_{1 \leqslant i,j \leqslant N}, \tag{4.15}$$

where

$$\mu_{ij} = \beta_i^{-j}\,\mathrm{e}^{\eta_i} - \sum_{l=1}^{N} a_{li}\alpha_l^{-j}\,\mathrm{e}^{\xi_l}, \qquad \eta_i = \mathrm{e}^{\Sigma_{l=1}^{\infty} t_l\beta_i^l},$$
$$\xi_i = \mathrm{e}^{\Sigma_{l=1}^{\infty} t_l\alpha_i^l}, \quad i, j = 1, 2, \ldots, N, \tag{4.16}$$

and the wave function has the form

$$\psi_W(t,\lambda) = \psi = (1 + a_1\lambda^{-1} + \cdots + a_N\lambda^{-N})\,\mathrm{e}^{\Sigma_{i=1}^{\infty} t_i\lambda^i} \tag{4.17}$$

where

$$a_l = \frac{\det(c_{ij}^l)_{1 \leqslant i,j \leqslant N}}{\tau}, \quad l = 1, 2, \ldots, N, \tag{4.18}$$

with $c^l_{ij} = \mu_{ij}$ for $j \neq l$ and $c^l_{il} = \sum_{s=1}^N a_{si}e^{\xi_s} - e^{\eta_i}$. When $\alpha_{ij} = 0$ for $i \neq j$ and $\alpha_{ii} \neq 0$, the τ-function given by (4.15) leads to the N-soliton solution for the KP hierarchy [1, 9].

We now assume that a_{ij}'s have the following form:

$$a_{ij} = \sum_{l=1}^{m} \frac{d_i^{(l)}\, e_j^{(l)}}{\alpha_i^k - \beta_j^k}, \quad i, j = 1, 2, \ldots, N, \tag{4.19}$$

where $e^{(l)} = (e_1^{(l)}, e_2^{(l)}, \ldots, e_N^{(l)})^T, l = 1, 2, \ldots, m$ are linearly independent vectors, and so are $d^{(l)} = (d_1^{(l)}, d_2^{(l)}, \ldots, d_N^{(l)})^T, l = 1, 2, \ldots, m$, then $\tau(t)$ given by (4.15) is also a τ-function for Equation (1.1). To prove this assertion, let us choose

$$q_l(t) = \sum_{i=1}^{N} d_i^{(l)} \psi_W(t, \alpha_i), \qquad \Phi_l = \sum_{s=1}^{N} b_s^{(l)} \lambda^{-s}, \quad l = 1, 2, \ldots, m, \tag{4.20}$$

where the constants $b_s^{(l)}$ are determined by the following linear equations:

$$\sum_{s=1}^{N} \left(\sum_{j=1}^{N} a_{ji}\alpha_j^{-s} - \beta_i^{-s} \right) b_s^{(l)} = e_i^{(l)}, \quad i = 1, 2, \ldots, N. \tag{4.21}$$

The coefficient matrix of the above linear equations for a fixed l is nonsingular for almost all $\alpha_i, \beta_i, a_{ij}$, so Φ_i's given in the form of (4.20) are uniquely determined, and they are also linearly independent since $e^{(1)}, e^{(2)}, \ldots, e^{(m)}$ are linearly independent. Then it is easy to see that $\lambda^k\psi - \sum_{l=1}^m q_l\Phi_l \in W$, so by using the result of Section 3 we see that the assertion is true.

EXAMPLE 4.3. Let us take $N = 2, a_{ij} = d_i e_j/(\alpha_i^k - \beta_j^k)$. Then the τ-function defined by (4.15) has the following expression:

$$\begin{aligned} \tau(t) \;=\; & \beta_1^{-2}\beta_2^{-2}(\beta_1 - \beta_2)\, e^{\eta_1+\eta_2} \times \\ & \times \left\{ 1 + \sum_{i,j=1}^{2} \frac{\tilde{a}_{ij}\beta_j}{\alpha_i - \beta_j} e^{\xi_i - \eta_j} + (\tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21}) \times \right. \\ & \left. \times \frac{\beta_1\beta_2(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_1 - \beta_2)(\beta_1 - \alpha_2)} e^{\xi_1+\xi_2-\eta_1-\eta_2} \right\}, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \tilde{a}_{11} &= \frac{\beta_1(\alpha_1 - \beta_1)(\beta_2 - \alpha_1)}{\alpha_1^2(\beta_1 - \beta_2)} a_{11}, \\ \tilde{a}_{12} &= \frac{\beta_2(\alpha_1 - \beta_2)(\alpha_1 - \beta_1)}{\alpha_1^2(\beta_1 - \beta_2)} a_{12}, \end{aligned} \tag{4.23a}$$

$$\tilde{a}_{21} = \frac{\beta_1(\alpha_2 - \beta_1)(\beta_2 - \alpha_2)}{\alpha_2^2(\beta_1 - \beta_2)} a_{21},$$
$$\tilde{a}_{22} = \frac{\beta_2(\alpha_2 - \beta_2)(\alpha_2 - \beta_1)}{\alpha_2^2(\beta_1 - \beta_2)} a_{22}, \tag{4.23b}$$

and $q_1 = \rho/\tau, r_1 = \sigma/\tau$, where ρ and σ are defined by

$$\rho = c_1 \left\{ \sum_{i=1}^{2} \tilde{d}_i \mathrm{e}^{\xi_i} + \sum_{j=1}^{2} \frac{(\tilde{d}_1 \tilde{a}_{2j} - \tilde{d}_2 \tilde{a}_{1j})\beta_j(\alpha_1 - \alpha_2)}{(\alpha_1 - \beta_j)(\alpha_2 - \beta_j)} \mathrm{e}^{\xi_1 + \xi_2 - \eta_j} \right\}, \tag{4.24a}$$

$$\sigma = c_2 \left\{ \sum_{i=1}^{2} \tilde{e}_i \beta_i \mathrm{e}^{-\eta_i} + \sum_{j=1}^{2} \frac{(\tilde{e}_1 \tilde{a}_{j2} - \tilde{e}_2 \tilde{a}_{j1})\beta_1\beta_2(\beta_2 - \beta_1)}{(\alpha_j - \beta_2)(\alpha_j - \beta_1)} \mathrm{e}^{\xi_j - \eta_1 - \eta_2} \right\}, \tag{4.24b}$$

where

$$c_1 = (\beta_1 - \beta_2)(\alpha_1 - \beta_1)(\alpha_1 - \beta_2)\beta_1^{-2}\beta_2^{-2}\alpha_1^{-2}\,\mathrm{e}^{\eta_1+\eta_2},$$
$$c_2 = \beta_1^{-2}\beta_2^{-2}\,\mathrm{e}^{\eta_1+\eta_2},$$
$$\tilde{d}_1 = d_1, \quad \tilde{d}_2 = \frac{(\alpha_2 - \beta_1)(\beta_2 - \alpha_2)\alpha_1^2}{(\alpha_1 - \beta_1)(\beta_2 - \alpha_1)\alpha_2^2} d_2, \quad \tilde{\mathrm{e}}_1 = -\beta_1 \mathrm{e}_1, \quad \tilde{\mathrm{e}}_2 = \beta_2 \mathrm{e}_2.$$

Thus, we have obtained a double soliton solution for Equation (1.1) with $m = 1$. This double soliton solution of the vector k-constrained KP hierarchy is equivalent to the one given in [6].

Remark 4.1. Relations similar to (4.19) also appeared when we constructed soliton solutions for the vector k-constrained KP hierarchy by using the bilinear construction in [6].

Remark 4.2. In the special case when $N = m$, we can see from the proposition of Section 3 that the τ-function defined by (4.15) is also a τ-function of Equation (1.1), where the a_{ij}'s do not need to satisfy condition (4.19). Thus, any τ-function defined by (4.15) is also a τ-function for Equation (1.1) for an appropriate m. This result was also obtained in [6] by using the bilinear construction.

Remark 4.3. Our construction of the soliton solutions of the vector k-constrained KP hierarchy (1.1) in this section was mainly motivated by [10], where the soliton solutions for the vector nonlinear Schrödinger equation, the vector model of Yajima–Oikawa type, etc., are constructed by imposing some self-consistency conditions on the potential u of the time-dependent Schödinger equation $iv_{l,t} = v_{l,xx} - uv_l, l = 1, 2, \ldots, m$, and relations similar to (4.14), (4.19) were given.

5. Conclusion

We have given an interpretation for the vector k-constrained KP hierarchy in terms of Segal–Wilson's theory on the τ-functions of the KP hierarchy, and have shown how to construct its rational and soliton by using this interpretation. We see that all the τ-functions given by (4.1) and (4.15) are also τ-functions for the vector k-constrained KP hierarchy (1.1) with appropriate m. This result is more easily seen by using this interpretation than by using the bilinear construction given in [6]. We hope that this interpretation for the vector k-constrained KP hierarchy would be useful when we consider its algebro-geometrical solutions.

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