

On Quasi-triviality and Integrability of a Class of Scalar Evolutionary PDEs

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Abstract

For certain class of perturbations of the equation $u_t = f(u)u_x$, we prove the existence of change of coordinates, called quasi-Miura transformations, that reduce these perturbed equations to the unperturbed ones. As an application, we propose a criterion for the integrability of these equations.

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1 Introduction

The notion of quasi-triviality for certain class of evolutionary PDEs was introduced in [8]. Let us first explain this notation by looking at the example of the Kortweg-de Vries (KdV) equation

$$u_t = uu_x + \frac{\varepsilon^2}{12} u_{xxx}, \quad (1.1)$$

where $u = u(x, t)$ and ε is the dispersion parameter. By the definition of [8] this equation is quasi-trivial, it means that we can perform a change of the dependent variable

$$v = u + \frac{\varepsilon^2}{24} \partial_x^2 \left[-\log u_x + \frac{\varepsilon^2}{240} \left(5 \frac{u^{(4)}}{u_x^2} - 9 \frac{u_{xx} u_{xxx}}{u_x^3} + 4 \frac{u_{xx}^3}{u_x^4} \right) + O(\varepsilon^4) \right] \quad (1.2)$$

such that in the new dependent variable the KdV equation is formally reduced to the dispersionless equation

$$v_t = v v_x. \quad (1.3)$$

Here the transformation (1.2) is understood as a formal power series of ε . The inverse transformation has the form

$$u = v + \frac{\varepsilon^2}{24} \partial_x^2 \left[\log v_x + \frac{\varepsilon^2}{240} \left(5 \frac{v^{(4)}}{v_x^2} - 21 \frac{v_{xx} v_{xxx}}{v_x^3} + 16 \frac{v_{xx}^3}{v_x^4} \right) + O(\varepsilon^4) \right]. \quad (1.4)$$

This property of the KdV equation was first observed in [1] (see also [8]). It corresponds to the genus expansion formula for the free energy of the 2D topological gravity [5, 10, 14] which provides one of the important links between 2D topological field theory and integrable hierarchies [7, 8, 10, 15, 21]. In [8, 9] it was proved that such

a property of quasi-triviality is also shared by a wide class of bihamiltonian integrable hierarchies, and it plays an important role in the study of the problem of classification of bihamiltonian integrable hierarchies. In a very recent paper [6], Dubrovin showed that at the approximation up to ε^4 any Hamiltonian perturbation of the equation (1.3) is quasi-trivial.

In the present paper we consider, without the assumption of possession of Hamiltonian structures, the property of quasi-triviality for a class of generalized scalar evolutionary PDEs of the form

$$u_t = f(u)u_x + \varepsilon(f_1(u)u_{xx} + f_2(u)u_x^2) + \varepsilon^2(f_3(u)u_{xxx} + f_4(u)u_xu_{xx} + f_5(u)u_x^3) + \dots, \quad f'(u) \neq 0. \quad (1.5)$$

Here the r.h.s. of the equation is a power series of the parameter ε , the coefficients of ε^k are graded homogeneous polynomials of degree $k+1$ of the variables u_x, u_{xx}, \dots with $\deg \partial_x^k u = k$, and the coefficients of these polynomials are assumed to be smooth functions of u . Note that when the power series of ε is a polynomial, (1.5) becomes a usual evolutionary PDE.

We are going to prove the quasi-triviality of the equation (1.5) according to the following definition of [8]:

Definition 1.1 *The generalized evolutionary PDE (1.5) is called quasi-trivial if there exists a quasi-Miura transformation of the form*

$$u = v + \sum_{k \geq 1} \varepsilon^k F_k(v, v_x, \dots, \partial_x^{m_k} v) \quad (1.6)$$

that formally reduces it to the equation

$$v_t = f(v)v_x. \quad (1.7)$$

Here F_k , $k \geq 1$ are smooth functions and m_k are some positive integers.

We call a quasi-Miura transformation of the form (1.6) that transforms the equation (1.5) to its leading term equation (1.7) the *reducing transformation* of (1.5). The transformation (1.6) has an inverse of the same form. Note that in the original definition of quasi-triviality given in [8, 9], the coefficients F_k of the quasi-Miura transformation are required to be rational in the variables u_x, u_{xx}, \dots , here we slightly generalize this definition, see Sec. 4 for the explicit description of the quasi-Miura transformations that we will encounter in this paper.

The main motivation of this work originates from our attempt to generalize the classification scheme given in [8] for a class of bihamiltonian evolutionary PDEs, we expect that the requirement of bihamiltonian property can be replaced by a weaker one. The first step along this line is to find a more general class of evolutionary PDEs that possess the quasi-triviality property, since this property of the equations plays an important role in the classification scheme [8, 9, 16]. Another motivation of our study comes from the application of the reducing transformation to the perturbative study of solutions of the equation (1.5), see for example [6] in which such reducing transformations are used to study the critical behavior of solutions of the perturbed equations.

A direct consequence of the quasi-triviality property is the existence of infinitely many flows of the form

$$u_s = h(u)u_x + \sum_{k \geq 2} \varepsilon^{k-1} W_k(u, u_x, \dots, \partial_x^{n_k} u) \quad (1.8)$$

that commute with the flow (1.5), here $h(u)$ is an arbitrary smooth function. By imposing the conditions of polynomial dependence of the functions W_k on the variables $u_x, \dots, \partial^{n_k} u$, we propose a criterion for the formal integrability of the equation (1.5).

The plan of the paper is as follows: In Sec. 2 and Sec. 3 we introduce some basic notations including the Definition 3.9 of formal integrability for the equations of the form (1.5), and prove an important property of the formally integrable equations in Theorem 3.10. In Sec. 4 we prove the quasi-triviality of the equations of the form (1.5) and summarize the main results in Theorem 4.3, 4.5. In Sec. 5 we describe a criterion of formal integrability and illustrate this notation by some examples. Finally, in the conclusion we discuss the generalization of the quasi-triviality property to certain systems of evolutionary PDEs.

2 Miura type transformations

We first introduce some notations that will be used in this paper, see [8] for more detailed expositions. Let $u(x)$ be a smooth function of a real variable x and denote $u_0 = u(x)$, $u_s = \partial_x^s u(x)$, $s \geq 1$. We define the ring \mathcal{R} of differential polynomials of $u(x)$ as

$$\mathcal{R} = C^\infty(u_0)[u_1, u_2, \dots]. \quad (2.1)$$

It is a graded ring with $\deg u_i = i$, $i \geq 1$ and $\deg h(u_0) = 0$ for any smooth function h . We denote by \mathcal{A} the ring of formal power series of an indeterminate ε of the form

$$f = \sum_{i \geq 0} f_i(u_0, \dots, u_i) \varepsilon^i \quad (2.2)$$

where $f_i \in \mathcal{R}$ are homogeneous differential polynomials of degree i .

The derivations of \mathcal{A} form a Lie algebra

$$\mathfrak{g} = \left\{ \hat{X} = \sum_{s \geq 0} X_s \frac{\partial}{\partial u_s} \mid X_s \in \mathcal{A} \right\} \quad (2.3)$$

with Lie bracket

$$\left[\sum_{s \geq 0} X_s \frac{\partial}{\partial u_s}, \sum_{s \geq 0} Y_s \frac{\partial}{\partial u_s} \right] = \sum_{s \geq 0} Z_s \frac{\partial}{\partial u_s}, \quad \text{where } Z_s = \sum_{t \geq 0} \left(X_t \frac{\partial Y_s}{\partial u_t} - Y_t \frac{\partial X_s}{\partial u_t} \right). \quad (2.4)$$

We regard the ring \mathcal{A} as the coordinate ring of an infinite dimensional manifold, and the Lie algebra \mathfrak{g} as the Lie algebra of vector fields of this manifold.

Introduce the differential operator $\partial_x \in \mathfrak{g}$ by

$$\partial_x = \sum_{s \geq 0} u_{s+1} \frac{\partial}{\partial u_s}. \quad (2.5)$$

An element $\hat{X} \in \mathfrak{g}$ is called an evolutionary vector field if $[\hat{X}, \partial_x] = 0$, which implies

$$\hat{X} = \sum_{s \geq 0} (\partial_x^s X) \frac{\partial}{\partial u_s} \quad \text{for certain } X \in \mathcal{A}. \quad (2.6)$$

The function X is called the component of the evolutionary vector field \hat{X} . Denote

$$\mathcal{E} = \{\text{the vector space of evolutionary vector fields}\}. \quad (2.7)$$

Then we readily have the following proposition:

Proposition 2.1 *The vector space \mathcal{E} is a Lie subalgebra of \mathfrak{g} with the center $\{a\partial_x | a \in \mathbb{R}\}$, and its Lie bracket can be expressed as*

$$[\hat{X}, \hat{Y}] = \hat{Z}, \text{ where } Z = \hat{X}(Y) - \hat{Y}(X), \forall \hat{X}, \hat{Y} \in \mathcal{E}. \quad (2.8)$$

Since the map $X \mapsto \hat{X}$ is a bijection between \mathcal{A} and \mathcal{E} , we can pullback the Lie bracket of \mathcal{E} to \mathcal{A} . Then we obtain a Lie bracket on \mathcal{A}

$$[X, Y] = \hat{X}(Y) - \hat{Y}(X) = \sum_{s \geq 0} \left((\partial_x^s X) \frac{\partial Y}{\partial u_s} - (\partial_x^s Y) \frac{\partial X}{\partial u_s} \right). \quad (2.9)$$

Henceforth, we will also call an element X of \mathcal{A} a vector field.

The subalgebra \mathcal{B} of the Lie algebra $(\mathcal{A}, [\cdot, \cdot])$ defined by

$$\mathcal{B} = \{X = \sum_{i \geq 1} f_i(u_0, \dots, u_i) \varepsilon^i \in \mathcal{A} \mid \deg f_i = i\} \quad (2.10)$$

corresponds to the class of evolutionary PDEs that we will study in this paper. Namely, if we express X in the form

$$X = \varepsilon f(u) u_x + \varepsilon^2 (f_1(u) u_{xx} + f_2(u) u_x^2) + \varepsilon^3 (f_3(u) u_{xxx} + f_4(u) u_x u_{xx} + f_5(u) u_x^3) + \dots, \quad (2.11)$$

and assume that it satisfies the condition $f'(u) \neq 0$, then

$$u_t = \varepsilon^{-1} X \quad (2.12)$$

is just the equation (1.5). A vector field $X \in \mathcal{B}$ that satisfies the above condition will be called generic. We call $\varepsilon f(u) u_x$ the leading term of the vector field X , and the equation (1.7) the leading term equation of (2.12).

In the next section, we will consider the normal forms of the generalized evolutionary PDEs of the form (1.5) under the Miura type transformations.

Definition 2.2 ([8]) *A Miura type transformation is a transformation on the ring \mathcal{A} which has the form*

$$u \mapsto \tilde{u} = X_0(u) + \varepsilon X_1(u) u_1 + \varepsilon^2 (X_2(u) u_2 + X_3(u) u_1^2) + \dots, \quad \frac{\partial X_0(u)}{\partial u} \neq 0. \quad (2.13)$$

A Miura type transformation is an automorphism of the ring \mathcal{A} , and it induces an automorphism of the Lie algebra \mathcal{B} . It is easy to see that all such transformations form a group which is called the Miura group [8]. It is the semi-direct product of two subgroups. The first subgroup is the diffeomorphism group of \mathbb{R}

$$u \mapsto \tilde{u} = f(u), \text{ where } \frac{\partial f}{\partial u} \neq 0. \quad (2.14)$$

The second subgroup is formed by the Miura type transformations with $X_0(u) = u$. One can prove that any Miura type transformation with $X_0(u) = u$ can be expressed as

$$g : u \mapsto \tilde{u} = e^{\hat{Y}} u = u + \hat{Y}(u) + \frac{1}{2} \hat{Y}(\hat{Y}(u)) + \frac{1}{6} \hat{Y}(\hat{Y}(\hat{Y}(u))) + \dots \quad (2.15)$$

where $Y = \varepsilon Y_1(u) u_1 + \varepsilon^2 (Y_2(u) u_2 + Y_3(u) u_1^2) + \dots \in \mathcal{B}$. The automorphism of \mathcal{B} induced by this Miura type transformation g is given by

$$X \mapsto g(X) = e^{-\text{ad}_Y} X = X - [Y, X] + \frac{1}{2} [Y, [Y, X]] - \frac{1}{6} [Y, [Y, [Y, X]]] + \dots \quad (2.16)$$

Here the vector field $g(X)$ is obtained by first expressing the vector field X in the new coordinate \tilde{u} , then redenote \tilde{u} by u . We call the transformations of these two subgroups the Miura type transformations of the first and second kind respectively.

3 Formal symmetries and integrability

In this section, we first give the definition of formal symmetries for a generalized evolutionary PDE of the form (1.5) (or equivalently, (2.12) for a generic vector field $X \in \mathcal{B}$), then based on the properties of the formal symmetries we introduce the notion of formal integrability. We show that the property of formal integrability of an equation of the form (1.5) is equivalent to the existence of a unique reduced form of the equation under Miura type transformations.

Definition 3.1 *Given a generic vector field $X \in \mathcal{B}$ of the form (2.11), a formal symmetry of the equation (1.5) is a flow of the form*

$$\varepsilon \partial_s u = Y, \quad Y \in \mathcal{B} \quad (3.1)$$

which commutes with (1.5). We will also call Y a formal symmetry of the vector field X .

This definition is adopted from the usual definition of generalized infinitesimal symmetries of an evolutionary PDE. It differs from the usual one at the following two points. Firstly, even when (1.5) is a usual evolutionary PDE whose right hand side is truncated, i.e. a polynomial of the finite number of variables u_1, \dots, u_m for certain positive integer m , its formal symmetries are not necessarily truncated. While in the usual definition an infinitesimal symmetry depend only on finite number of the variables u_i , $i \geq 1$. Secondly, the right hand side of a formal symmetry (3.1) is required to have the same form (2.11) as of the vector field X (except the condition $f'(u) \neq 0$). These features of the formal symmetries will play a crucial role in our discussion of formal integrability of the equations of the form (1.5).

Before the exposition of properties of the formal symmetries, let us first do some preparations. Let \mathcal{P} be the set of all ordered sequences of integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with the properties

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0, \quad |\lambda| = \sum_{i \geq 1} \lambda_i < \infty. \quad (3.2)$$

We call $|\lambda|$ the degree of λ , and denote by \mathcal{P}^d the subset of degree d elements of \mathcal{P} . Each $\lambda \in \mathcal{P}$ is associated to a unique monomial

$$u_\lambda = \prod_{i \geq 1} u_{\lambda_i} \in \mathcal{R}. \quad (3.3)$$

Given two elements $\lambda, \mu \in \mathcal{P}^d$, we say $\lambda > \mu$ (resp. $\lambda < \mu$) if the first non-zero entry of the sequence $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \lambda_3 - \mu_3, \dots)$ is greater (resp. less) than 0. By using this ordering of \mathcal{P}^d , we can define in a natural way the highest order term of a homogeneous differential polynomial $X \in \mathcal{R}$ of degree d . For example, the highest order term of

$$X = X_1(u)u_4u_1 + X_2(u)u_3u_2 + X_3(u)u_2u_2u_1 \quad (3.4)$$

is $X_1(u)u_4u_1$, and we denote the sum of all other terms in X by *l.o.t.*

Lemma 3.2 *Let $X \in \mathcal{R}$ be homogeneous of degree d of the form*

$$X = f(u) \prod_{k=1}^m (u_k)^{\alpha_k} + \text{l.o.t.}, \quad (3.5)$$

then we have

$$[uu_1, X] = \left(\sum_{k=2}^m \alpha_k + d - 1 \right) f(u) u_1 \prod_{k=1}^m (u_k)^{\alpha_k} + l.o.t. \quad (3.6)$$

Proof. After the substitution of (3.5) into (2.9), we can prove the lemma by a straightforward computation. \square

Lemma 3.3 Assume $X \in \mathcal{R}$ be given as in the above lemma. Then $[uu_1, X] = 0$ if and only if $d = 1$, i.e. X has the form $X = f(u)u_1$.

Proof. Let X have the form (3.5) and satisfies $[uu_1, X] = 0$, then from Lemma 3.2 we obtain $\alpha_1 = 1, \alpha_2 = \dots = \alpha_m = 0$, so $X = f(u)u_1$. Conversely, it is obvious that the identity $[uu_1, X] = 0$ holds true for any vector field X of the form $f(u)u_1$. The lemma is proved. \square

Lemma 3.4 For a homogeneous differential polynomial $Y \in \mathcal{R}$ of degree $d > 2$, if the highest order term of Y has a factor u_1 , then we can find a homogeneous differential polynomial $X \in \mathcal{R}$ of degree $d - 1$ such that $[u_0 u_1, X]$ has the same highest order term as that of Y .

Proof. When $d \geq 2$, we always have $\left(\sum_{k=2}^m \alpha_k + d - 1 \right) \neq 0$. So the lemma immediately follows from Lemma 3.2. \square

Theorem 3.5 Let $Y \in \mathcal{B}$ be a symmetry of a generic vector field $X \in \mathcal{B}$ and $Y \neq au_x$, $a \in \mathbb{R}$. Then Y is also generic and is determined by its leading terms.

Proof. We only need to prove that symmetries are determined by their leading terms. Let $Y, Z \in \mathcal{B}$ be two symmetries of the generic vector field $X \in \mathcal{B}$ and have the same leading terms. Then from the identity

$$[X, Y - Z] = [X, Y] - [X, Z] = 0 \quad (3.7)$$

it follows that $W = Y - Z$ is also a symmetry of X . Since Y, Z have the same leading terms, the vector field W has the following expression

$$W = \sum_{d \geq 2} \varepsilon^d W_d(u, u_1, \dots, u_d). \quad (3.8)$$

Denote

$$X = \sum_{d \geq 1} \varepsilon^d X_d(u, \dots, u_d) \quad (3.9)$$

Then from (3.7) we know that the vector field $\varepsilon^2 W_2$ is a symmetry of the vector field $\varepsilon X_1(u, u_1)$. According to Lemma 3.3, symmetries of $\varepsilon X_1(u, u_1)$ must be of degree 1, so we have $W_2 = 0$. Similarly, we can prove that all the W_m vanish and consequently, $Y = Z$. The theorem is proved. \square

Corollary 3.6 Let $X \in \mathcal{B}$ be generic, Y_1, Y_2 be two symmetries of X , then $[Y_1, Y_2] = 0$.

Proof. By using the Jacobi identity we know that $[Y_1, Y_2]$ is also a symmetry of X . But the leading terms of $[Y_1, Y_2]$ vanish, so from Theorem 3.5 it follows that $[Y_1, Y_2] = 0$. \square

Remark 3.7 Let us denote $X \sim Y$ if $[X, Y] = 0$, then the above corollary shows that \sim is an equivalence relation on the set of generic vector fields of \mathcal{B} .

Theorem 3.8 For any generic $X \in \mathcal{B}$, there exists a Miura type transformation g such that

$$g(X) = \varepsilon u u_1 + \varepsilon^2 (f_{(2)} u_2 + f_{(1,1)} u_1^2) + \sum_{d \geq 3} \varepsilon^d \left(\sum_{\lambda \in \mathcal{P}_1^d} f_\lambda(u) u_\lambda \right), \quad (3.10)$$

where \mathcal{P}_1^d is the set of partitions of degree d whose nonzero entries are greater than 1.

Proof. Let $\varepsilon f(u) u_1$ be the leading term of the vector field X . We can use a Miura type transformation of the first kind $\tilde{u} = f(u)$ to transform X to the following form

$$g(X) = \varepsilon u u_1 + \sum_{d \geq 2} \varepsilon^d \left(\sum_{\lambda \in \mathcal{P}^d} f_\lambda(u) u_\lambda \right). \quad (3.11)$$

Then by using Lemma 3.4, we can find a series of Miura type transformations of the second kind to eliminate the terms with factor u_1 step by step (except for the terms of degree 2). The theorem is proved. \square

The expression (3.10) is called a reduced form of the vector field X . Note that such a reduced form may not be unique.

Definition 3.9 A generalized evolutionary PDE of the form (1.5) that corresponds to a generic vector field $X \in \mathcal{B}$ of the form (2.11) is called formally integrable, if any vector field of the form $Y = \varepsilon h(u) u_1 \in \mathcal{B}$ can be extended to a symmetry

$$\tilde{Y} = \varepsilon h(u) u_1 + \sum_{k \geq 2} \varepsilon^k Y_k \in \mathcal{B} \quad (3.12)$$

of this equation.

By definition, a formally integrable equation of the form (1.5) must have infinitely many formal symmetries. Thus this definition is an adaption of the commonly used one of integrability from the point of view of symmetries in soliton theory, see for example [11, 12, 17, 18, 19, 20] and references therein, to the present class of generalized evolutionary PDEs. In [11, 12], it was conjectured by Fokas that the existence of a single time independent non-Lie point symmetry implies the existence of infinitely many for a scalar evolutionary PDE. We will reformulate this conjecture for our class of equations and their formal symmetries in Sec. 5. In [17, 18, 20] Mikhailov, Shabat and their collaborators formulated certain necessary conditions for the existence of a higher order symmetry for a system of evolutionary PDEs. These conditions are expressed in terms of the so called canonical conservation laws, they yield an effective algorithm of testing the existence of a higher order symmetry for certain class of evolutionary PDEs of lower order. In [19], Sanders and Wang gave a symbolic algorithm of checking the existence of symmetries for certain class of scalar evolutionary PDE, such equations depend polynomially on the dependent variable and its x -derivatives, and satisfy certain homogeneity conditions. We will give in Sec. 5 an alternative way of checking the formal integrability of the generalized evolutionary PDEs, our approach is base on the quasi-triviality of such equations which will be proved in the next section.

The following theorem relates the uniqueness of the reduced form of a vector field to the integrability of the corresponding PDE.

Theorem 3.10 *A generic vector field $X \in \mathcal{B}$ of the form (2.11) has a unique reduced form if and only if the corresponding equation (1.5) is formally integrable.*

Proof. Without the loss of generality, we assume that X is already in a reduced form that is given by the right hand side of (3.10).

Let us first assume that the reduced form of X is unique. Given a vector field

$$\varepsilon Y_1 = \varepsilon h(u) u_1 \in \mathcal{B},$$

consider the following vector field

$$g_1(X) := e^{\text{ad}_{\varepsilon Y_1}} X = \varepsilon u_0 u_1 + \varepsilon^2 (f_{(2)} u_2 + f_{(1,1)} u_1^2) + \dots, \quad (3.13)$$

where g_1 corresponds to the Miura type transformation

$$g_1 : u \mapsto e^{-\varepsilon \hat{Y}_1} u. \quad (3.14)$$

From the proof of Theorem 3.8, we know that there exists a Miura transformation

$$g_2 : u \mapsto e^{\hat{Z}} u, \quad Z = \varepsilon^2 Z_2 + \varepsilon^3 Z_3 + \dots \in \mathcal{B} \quad (3.15)$$

which transforms $g_1(X)$ to a reduced form. It follows from our assumption on the uniqueness of reduced form for the vector field X that

$$X = e^{-\text{ad}_Z} g_1(X).$$

On the other hand, $e^{-\text{ad}_Z} e^{\text{ad}_{\varepsilon Y_1}} X$ can be rewritten as $e^{\text{ad}_Y} X$ with a vector field Y of the form

$$Y = \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots \in \mathcal{B}.$$

From the equality $X = e^{\text{ad}_Y} X$ it then follows that the vector field X has a symmetry Y with the leading term εY_1 . Thus by our definition the equation $\varepsilon u_t = X$ is integrable.

Now let us assume that the equation $\varepsilon u_t = X$ is integrable. If X has another reduced form \tilde{X} , then it must be related to X by a Miura type transformation

$$\tilde{X} = e^{\text{ad}_Y} X, \quad Y = \varepsilon Y_1 + \varepsilon^2 Y_2 + \dots \in \mathcal{B}. \quad (3.16)$$

From the definition of integrability, we can find a symmetry Z of the vector field X that has the same leading term as that of the vector field Y , i.e.,

$$Z = \varepsilon Y_1 + \sum_{k \geq 2} \varepsilon^k Z_k \in \mathcal{B}. \quad (3.17)$$

So we can express \tilde{X} as

$$\tilde{X} = e^{\text{ad}_Y} e^{-\text{ad}_Z} X = e^{\text{ad}_W} X. \quad (3.18)$$

Here the vector field W has the form $W = \varepsilon^2 W_2 + \dots$. Since both X and \tilde{X} are expressed in reduced forms, by using Lemma 3.2 we know that $W = 0$. Thus $\tilde{X} = X$ and we proved the uniqueness of reduced form of the vector field X . The theorem is proved. \square

4 Quasi-triviality

We now proceed to consider the quasi-triviality of a generalized evolutionary PDE of the form (1.5) corresponding to a generic vector field X of the form (2.11). Define a map from the set of infinite series to itself

$$R : (y_1, y_2, \dots) \mapsto (z_1, z_2, \dots) \quad (4.1)$$

in the following recursive way

$$z_1 = \frac{1}{y_1}, \quad z_n = \frac{1}{y_1} \sum_{k \geq 1} y_{k+1} \frac{\partial z_{n-1}}{\partial y^k}, \quad n \geq 2. \quad (4.2)$$

When y_k are given by the k -th order derivatives of a single variable smooth function A , then the above define z_k are just the k -th order derivatives of the inverse function of A . By using this observation it is easy to see that R is an involution, i.e. R^2 is the identity map.

Lemma 4.1 *The equation $[uu_1, f(u, u_1, \dots)] = 0$ for the unknown function f has the following general solution*

$$f = \frac{1}{\phi_1} c(u, \phi_2, \phi_3, \dots), \quad (4.3)$$

where c is an arbitrary smooth function and $(\phi_1, \phi_2, \dots) = R(u_1, u_2, \dots)$.

Proof. A solution of the equation $[uu_1, f(u, u_x, \dots)] = 0$ corresponds to a flow $\frac{\partial u}{\partial s} = f$ that commutes with the flow $\frac{\partial u}{\partial t} = uu_x$. By performing a transformation

$$(x, t, s, u(x, t, s)) \mapsto (u, t, s, x(u, t, s)), \quad (4.4)$$

we can rewrite these two flows into the form

$$\frac{\partial x}{\partial s} = g(u, x_u, x_{uu}, \dots), \quad \frac{\partial x}{\partial t} = -u \quad (4.5)$$

Here $g = -f(u, \frac{1}{x_u}, \frac{1}{x_u} \partial_u \frac{1}{x_u}, \dots) \frac{\partial x}{\partial u}$. The condition of commutativity of the flows is given by

$$\frac{\partial g}{\partial t} = 0.$$

Due to the fact that $\frac{\partial}{\partial t} \frac{\partial^k x}{\partial u^k} = \frac{\partial^k}{\partial u^k} \frac{\partial x}{\partial t} = -\frac{\partial^k u}{\partial u^k} = -\delta_{k,1}$, we know that the general solution of the above equation has the form $g = -c(u, x_{uu}, x_{uuu}, \dots)$. for certain smooth function c . So f must have the form (4.3), and the lemma is proved. \square

Proposition 4.2 *For any given smooth function $F(u, u_1, \dots)$, the equation $[uu_1, f] = F$ for the unknown function $f = f(u, u_1, \dots)$ has a particular solution*

$$f = -u_1 \left(\int^{\phi_1} \phi_1 F(u, \tilde{u}_1, \tilde{u}_2, \dots) d\phi_1 \right) \Big|_{(\phi_1, \phi_2, \dots) = R(u_1, u_2, \dots)}, \quad (4.6)$$

where $(\tilde{u}_1, \tilde{u}_2, \dots) = R(\phi_1, \phi_2, \dots)$.

Proof. Assume that we have a solution of the form $f = \frac{1}{\phi_1} c(u, \phi_1, \phi_2, \phi_3, \dots)$. By using the above theorem we know that the function c must satisfy the following equation

$$\frac{\partial c}{\partial \phi_1} = -\phi_1 F(u, u_1, u_2, \dots) = -\phi_1 F(u, \frac{1}{\phi_1}, -\frac{\phi_2}{\phi_1^3}, \dots). \quad (4.7)$$

Here in the second equality we used the substitution

$$(u_1, u_2, \dots) = R(\phi_1, \phi_2, \dots) = (\frac{1}{\phi_1}, -\frac{\phi_2}{\phi_1^3}, \dots). \quad (4.8)$$

By integrating (4.7) with respect to ϕ_1 and making the substitution

$$(\phi_1, \phi_2, \dots) = R(u_1, u_2, \dots)$$

we finish the proof of the lemma. \square

Now we are ready to prove the existence of a reducing transformation for the generalized evolutionary PDEs.

Theorem 4.3 *Given a generalized evolutionary PDE of the form*

$$u_t = uu_x + \varepsilon (f_1(u)u_2 + f_2(u)u_1^2) + \sum_{k \geq 3} \varepsilon^{k-1} S_k, \quad (4.9)$$

with $S_k \in \mathcal{R}$ being homogeneous differential polynomials of degree k , we have

i) there exists a unique transformation of the form

$$u \mapsto v = e^{\hat{X}} u, \quad (4.10)$$

$$X = \sum_{k \geq 1} \varepsilon^k u_1^{-L_k} \sum_{m=0}^{M_k} Y_{k,m}(u, u_1, \dots, u_{N_k}) (\log u_1)^m \quad (4.11)$$

that reduces the equation (4.9) to $v_t = vv_x$. Here $Y_{k,m} \in \mathcal{R}$ are homogeneous differential polynomials of degree $L_k + k$, and L_k, M_k, N_k are integers which only depend on k .

ii) The transformation (4.10), (4.11) also reduces any symmetry

$$u_s = h(u)u_1 + \sum_{k \geq 2} \varepsilon^{k-1} Q_k, \quad Q_k \in \mathcal{R}, \quad \deg Q_k = k \quad (4.12)$$

of the equation (4.9) to the form $v_s = h(v)v_x$.

Proof. Let the reducing transformation takes the following form

$$u \mapsto v = e^{\hat{X}}(u) = u + \hat{X}(u) + \frac{1}{2}\hat{X}(\hat{X}(u)) + \frac{1}{6}\hat{X}(\hat{X}(\hat{X}(u))) + \dots \quad (4.13)$$

where $X = \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + \varepsilon^4 X_4 + \dots$. It will eliminate all the perturbations, so we have

$$e^{\text{adx}}(\varepsilon S_1) = \varepsilon S_1 + \varepsilon^2 S_2 + \varepsilon^3 S_3 + \varepsilon^4 S_4 + \varepsilon^5 S_5 + \dots \quad (4.14)$$

where $S_1 = uu_1$, $S_2 = f_1(u)u_2 + f_2(u)u_1^2$. The coefficients of ε^k give us the following equations

$$\begin{aligned} S_2 &= [X_1, uu_1], \\ S_3 &= [X_2, uu_1] + \frac{1}{2}[X_1, [X_1, uu_1]], \\ S_4 &= [X_3, uu_1] + \frac{1}{2}[X_2, [X_1, uu_1]] + \frac{1}{2}[X_1, [X_2, uu_1]] + \frac{1}{6}[X_1, [X_1, [X_1, uu_1]]] \end{aligned}$$

and so on. By using Proposition 4.2, we can solve these equations and obtain $X_1, X_2, X_3, X_4, \dots$ recursively, and the homogeneity condition on $Y_{k,m}$ guarantees the uniqueness of the solution $X_k, k \geq 1$. Thus we proved the first part of the theorem.

To prove the second result of the theorem, let us note that after the transformation (4.10), (4.11) the equation (4.12) is transformed to

$$u_s = e^{-\text{ad}_X}(h(u)u_1 + \sum_{k \geq 2} \varepsilon^{k-1} Q_k) = h(u)u_x + \sum_{k \geq 2} \varepsilon^{k-1} \tilde{Q}_k(u, u_1, \dots, u_{m_k}) \quad (4.15)$$

here for the simplicity of notations we keep to use the symbol u instead of v , and \tilde{Q}_k are homogeneous polynomials of $\log u_1, u_1, \frac{1}{u_1}, u_2, \dots, u_{m_k}$ of degree k (note that $\deg \log u_1 = 0$), where m_k is a positive integer depending on k . Since the transformation (4.10), (4.11) reduces the equation (4.9) to the form $u_t = uu_x$ we know that

$$[uu_x, h(u)u_x + \sum_{k \geq 2} \varepsilon^{k-1} \tilde{Q}_k(u, u_1, \dots, u_k)] = 0 \quad (4.16)$$

By using Lemma 4.1 and the fact that $\deg Q_k = k > 1$, we see that the functions \tilde{Q}_k must vanish. The theorem is proved. \square

Remark 4.4 We conjecture that the integers L_k, M_k, N_k has the following expressions:

$$L_k = 3k - 2, M_k = 2 \left\lfloor \frac{k-1}{2} \right\rfloor + \delta_{0, \lfloor \frac{k-1}{2} \rfloor}, N_k = k + 1.$$

We hope that they can be proved by a careful analysis of the above procedure of the construction of the reducing transformation.

By using Theorem 4.3 we are readily led to the following corollary:

Theorem 4.5 Any generalized evolutionary PDE of the form (1.5) is quasi-trivial.

As we already noted in the introduction, the notion of quasi-Miura transformation that was introduced in [8] (see also [9]) requires that the functions $F_k(u, u_x, u_{xx}, \dots)$ depend rationally on the variables u_1, u_2, \dots . Here we drop this rationality condition and still call (1.6) a quasi-Miura transformation. For the equation (1.5), when $f_2 f' - f_1 f'' = 0$ the reducing transformation (1.6) depend rationally on the variables $u_k, k \geq 1$.

Example 4.6 (KdV Equation) The KdV equation (1.1) has the reducing transformation (1.2), (1.4).

Example 4.7 (Camassa-Holm Equation) We consider another important equation in soliton theory, the Camassa-Holm equation [3, 4, 13]

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \quad (4.17)$$

Let us perform the rescaling $t \mapsto -3\varepsilon t, x \mapsto \varepsilon x$, then the above equation can be put into the following form:

$$\begin{aligned} \varepsilon u_t &= (1 - \varepsilon \partial_x^2)^{-1} \left(\varepsilon uu_1 - \varepsilon^3 \left(\frac{2}{3} u_1 u_2 + \frac{1}{3} uu_3 \right) \right) \\ &= \varepsilon uu_1 + \varepsilon^3 \left(\frac{7u_1 u_2}{3} + \frac{2uu_3}{3} \right) + \varepsilon^5 \left(\frac{23u_2 u_3}{3} + \frac{11u_1 u_4}{3} + \frac{2uu_5}{3} \right) + \dots \end{aligned} \quad (4.18)$$

It has the reducing transformation

$$\begin{aligned}
v \mapsto u = v + \varepsilon^2 & \left(\frac{7v_2}{6} - \frac{v v_2^2}{3v_1^2} + \frac{v v_3}{3v_1} \right) + \varepsilon^4 \left(\frac{6v_2^3}{5v_1^2} - \frac{202v v_2^4}{45v_1^4} \right. \\
& + \frac{32v^2 v_2^5}{9v_1^6} - \frac{181v_2 v_3}{90v_1} + \frac{398v v_2^2 v_3}{45v_1^3} - \frac{70v^2 v_2^3 v_3}{9v_1^5} - \frac{191v v_3^2}{90v_1^2} \\
& + \frac{19v^2 v_2 v_3^2}{6v_1^4} + \frac{143v_4}{72} - \frac{133v v_2 v_4}{45v_1^2} + \frac{34v^2 v_2^2 v_4}{15v_1^4} - \frac{73v^2 v_3 v_4}{90v_1^3} \\
& \left. + \frac{13v v_5}{18v_1} - \frac{41v^2 v_2 v_5}{90v_1^3} + \frac{v^2 v_6}{18v_1^2} \right) + \dots
\end{aligned}$$

Both the KdV equation and the Camassa-Holm equation have semisimple bihamiltonian structures, their quasi-triviality have been proved in [8] and [9]. The following example of the Burgers equation does not possess hamiltonian structures.

Example 4.8 (Burgers Equation) *The Burgers equation*

$$u_t = uu_x + \varepsilon u_{xx} \quad (4.19)$$

has the reducing transformation

$$\begin{aligned}
v \mapsto u = v + \varepsilon \frac{v_2}{v_1} + \varepsilon^2 & \left(\frac{2v_2^3}{v_1^4} - \frac{7v_2 v_3}{3v_1^3} + \frac{v_4}{2v_1^2} \right) + \varepsilon^3 \left(\frac{24v_2^5}{v_1^7} \right. \\
& - \frac{46v_2^3 v_3}{v_1^6} + \frac{16v_2 v_3^2}{v_1^5} + \frac{34v_2^2 v_4}{3v_1^5} - \frac{10v_3 v_4}{3v_1^4} - \frac{11v_2 v_5}{6v_1^4} + \frac{v_6}{6v_1^3} \Big) \\
& + \varepsilon^4 \left(\frac{568v_2^7}{v_1^{10}} - \frac{4544v_2^5 v_3}{3v_1^9} + \frac{1086v_2^3 v_3^2}{v_1^8} - \frac{179v_2 v_3^3}{v_1^7} + \frac{1154v_2^4 v_4}{3v_1^8} \right. \\
& - \frac{380v_2^2 v_3 v_4}{v_1^7} + \frac{221v_3^2 v_4}{6v_1^6} + \frac{26v_2 v_4^2}{v_1^6} - \frac{70v_2^3 v_5}{v_1^7} + \frac{731v_2 v_3 v_5}{18v_1^6} \\
& \left. - \frac{101v_4 v_5}{30v_1^5} + \frac{83v_2^2 v_6}{9v_1^6} - \frac{13v_3 v_6}{6v_1^5} - \frac{5v_2 v_7}{6v_1^5} + \frac{v_8}{24v_1^4} \right) + \dots
\end{aligned}$$

Example 4.9 *Our last example is a class of equations parameterized by a smooth function f*

$$u_{t_f} = u_1 f(u + \varepsilon u_1) = \sum_{n \geq 0} \varepsilon^n \frac{f^{(n)}(u)}{n!} u_1^{n+1}. \quad (4.20)$$

It's easy to verify that ∂_{t_f} and ∂_{t_g} commute for arbitrary smooth functions f and g . The reducing transformation of this equation has an explicit form

$$v \mapsto u = e^{\varepsilon \hat{X}}(v) = v + \varepsilon \hat{X}(v) + \frac{\varepsilon^2}{2} \hat{X}(\hat{X}(v)) + \frac{\varepsilon^3}{6} \hat{X}(\hat{X}(\hat{X}(v))) + \dots, \quad (4.21)$$

where $X = v_1 \log(v_1)$. This equation is quite different from those of the above three examples, because its reducing transformation contains $\log(v_1)$, while that of the KdV, the Camassa-Holm and the Burgers equation are rational in jet variables v_1, v_2, \dots

5 Testing of integrability

Given a generalized evolutionary PDE of the form (1.5), let

$$g : u \mapsto v$$

be the quasi-Miura transformation (1.6) that reduces it to the form (1.7). For any smooth function $h(v)$, the equation (1.7) has a symmetry

$$v_s = h(v)v_x. \quad (5.1)$$

Rewrite this flow in the u coordinates by using the quasi-Miura transformation g^{-1} , we have

$$\varepsilon u_s = \varepsilon h(u)u_x + \sum_{k \geq 2} \varepsilon^k W_k(u, u_1, \dots, u_k), \quad (5.2)$$

where in general W_k are not polynomials of the variables u_1, u_2, \dots

In order to compute these W_k in a more direct way, we first perform a change of the dependent variable

$$u \mapsto f(u) \quad (5.3)$$

to transform the equation (1.5) into the form

$$\varepsilon u_t = \tilde{X}, \quad \tilde{X} = \varepsilon u u_x + \sum_{k \geq 2} \varepsilon^k \tilde{X}_k(u, \dots, u_k). \quad (5.4)$$

Consider the equation $[\tilde{X}, Y] = 0$ for the unknown vector field

$$Y = \varepsilon h \circ f^{-1}(u)u_x + \sum_{k \geq 2} \varepsilon^k Y_k(u, \dots, u_k).$$

By using Proposition 4.2 we can solve this equation recursively to obtain Y . Under certain homogeneity condition that is similar to the one given in Theorem 4.3 the solution Y is unique. By performing the transformation $u \mapsto f^{-1}(u)$ the vector field Y is transformed to the form

$$\varepsilon h(u)u_x + \sum_{k \geq 2} \varepsilon^k \tilde{Y}_k(u, \dots, u_k).$$

Then we have $W_k = \tilde{Y}_k$.

Although the above flow (5.2) commutes with the flow given by the equation (1.5), in general it does not meet the requirement of being a formal symmetry of (1.5) according to our Definition 3.1. This is because in general the right hand side of the equation (5.2) does not belong to \mathcal{B} , i.e., the functions $W_k(u, u_1, \dots, u_k)$ are not polynomials of the variables u_1, u_2, \dots, u_k .

Now let us come back to our Definition 3.9 of formal integrability for a generalized evolutionary PDE of the form (1.5). By using the second result of Theorem 4.3 and the quasi-triviality of the equation (1.5), we know that if a vector field $\varepsilon h(u)u_x$ can be extended to a formal symmetry of the equation (1.5), then this formal symmetry must coincide with (5.2). Thus we have the following:

Criterion of formal integrability. *A generalized evolutionary PDE of the form (1.5) is formally integrable iff for any smooth function $h(u)$, the functions W_k that appear in the right hand side of (5.2) are homogeneous differential polynomials of u_1, u_2, \dots, u_k of degree k .*

Let us consider the formal integrability of the four equations considered in the last section. The formal integrability of the equation given in the fourth example is trivial, because we can write down all its symmetries explicitly. For the other three equations, we have the following proposition:

Proposition 5.1 *The KdV equation (1.1), the Camassa-Holm equation (4.17) and the Burgers equation (4.19) are formally integrable.*

Proof. According to Proposition 4.2 and the above construction, we know that the flow (5.2) for the KdV equation must take the following form

$$u_s = h(u)u_x + \sum_{k \geq 1} \varepsilon^{2k} \frac{1}{u_x^{m_k}} \sum_{\lambda \in \mathcal{P}^{m_k+2k+1}} C_{k,\lambda} h^{(D_{k,\lambda})}(u)u_\lambda, \quad (5.5)$$

where m_k are integers, $C_{k,\lambda}$ are rational numbers and $D_{k,\lambda}$ are positive integers. To prove the theorem, we only need to show that for each monomial

$$u_x^{-m_k} C_{k,\lambda} h^{(D_{k,\lambda})}(u)u_\lambda \quad (5.6)$$

that appears in the above expression, either $C_{k,\lambda} = 0$ or it can be reduced to the form

$$C_{k,\lambda} h^{(D_{k,\lambda})}(u)u_{\lambda'}, \quad \lambda' \in \mathcal{P}.$$

To this end, let us take $h(u) = u^{D_{k,\lambda}}$. From the classical theory of the KdV equation, we know that the flow (5.5) belongs to the KdV hierarchy and so its right hand side is a truncated differential polynomial. So the monomial (5.6) has the required property and we proved the formal integrability of the KdV equation.

In a similar way, we can prove the formal integrability of the Camassa-Holm equation and Burgers equation. The only technical point we should note is that the analogue of (5.5) for the Camassa-Holm equation need to be slightly modified. We omit the details here. The proposition is proved. \square

The following two examples illustrate a procedure to identify the formally integrable equations among those which possess certain particular forms.

Example 5.2 *Consider the formally integrable equations among those of the form*

$$u_t = uu_x + \varepsilon(f_1(u)u_{xx} + f_2(u)u_x^2), \quad f_1(u) \neq 0. \quad (5.7)$$

By using the criterion of formal integrability we easily obtain

$$f_1(u) = au + b, \quad f_2(u) = -\frac{1}{2}a, \quad (5.8)$$

where a, b are arbitrary constants. When $a = 0, b = 1$ the above equation is just the well known Burgers equation (4.19). If $a \neq 0$, we may assume without loss of generality that $a = 1$. The Galilean transformation $x \mapsto x - bt, t \mapsto t, u \mapsto u - b$ converts (5.7) to the form

$$u_t = uu_x + \varepsilon \left(uu_{xx} - \frac{1}{2}u_x^2 \right) \quad (5.9)$$

Let $u = w^2$, the equation (5.9) becomes

$$w_t = w^2(w_x + \varepsilon w_{xx}) \quad (5.10)$$

This equation is also linearizable under the change of the independent variables and is called C-integrable, see [2] and the equation (3.37) there.

The equation (5.10) admits a recursion operator

$$R = \varepsilon w \partial_x + w + w^2 (w_x + \varepsilon w_{xx}) \partial_x^{-1} \frac{1}{w^2}. \quad (5.11)$$

One can prove that R is a hereditary strong symmetry [13]. So we obtain a hierarchy of symmetries of the equation (5.10) which can be expressed as

$$w_{t_0} = -w_x, w_{t_1} = 0, w_{t_{2+k}} = R^k (w^2 (w_x + \varepsilon w_{xx})) = (k+1)w^{k+2}w_x + \dots. \quad (5.12)$$

To see that all the right hand sides of these symmetries are differential polynomials of the variables w_x, w_{xx}, \dots , we introduce a generating function

$$F = \sum_{k=0}^{\infty} \frac{w_{t_{k+2}}}{\lambda^{k+2}} = \frac{w_{t_2}}{\lambda^2} + \frac{w_{t_3}}{\lambda^3} + \dots.$$

It satisfies the following equation

$$RF = \lambda \left(F - \frac{w_{t_2}}{\lambda^2} \right). \quad (5.13)$$

Let $w_{t_k} = w^2 \partial_x h_k$, $H = \sum_{k \geq 2} \frac{h_k}{\lambda^k}$, the equation (5.13) becomes a linear ODE of H , its solution is

$$H = \frac{1}{w} \sum_{k=0}^{\infty} \varepsilon^k (A \partial_x)^k \left(A \frac{w + \varepsilon w_x}{\lambda} \right), \text{ where } A = \sum_{l=1}^{\infty} \frac{w^l}{\lambda^l}.$$

This implies that all the right hand sides of the flows given in (5.12) are homogeneous differential polynomials. By using a similar argument as given in the proof of Proposition 5.1, one can prove that the equation (5.10) is formally integrable.

The integrable hierarchy (5.12) has the conversation law

$$\left(-\frac{1}{w} \right)_{t_k} = \partial_x h_k. \quad (5.14)$$

It defines a reciprocal transformation

$$dy = \frac{1}{w} dx - \sum_{k \geq 2} h_k dt_k \quad (5.15)$$

which transform the whole hierarchy to the Burgers hierarchy (up to a rescaling). In the next example we will give more detailed explanations on the reciprocal transformation.

In the above example, the conditions (5.8) can in fact be derived by requiring the existence of a function $h(u)$ with $h''(u) \neq 0$ such that the resulting functions W_k are differential polynomials. Similar situation also occurs in other examples that we computed, such as the one which will be presented below. Based on these examples, we reformulate the conjecture of Fokas [11, 12] on the existence of infinitely many symmetries of a scalar evolutionary PDE for the class of equation considered in this paper as follows:

Conjecture 5.3 A generalized evolutionary PDE of the form (1.5) is formally integrable iff there exists a smooth function $h(u)$ satisfying

$$h'' - \frac{h'}{f'} f'' \neq 0, \quad (5.16)$$

such that the flow (5.2) that is obtained from the equation (5.1) by the reducing transformation of (1.5) gives a symmetry of (1.5), i.e., the functions W_k are homogeneous differential polynomials of degree k .

Example 5.4 Consider the equation of the form

$$u_t = uu_x + \varepsilon^2 (g_1(u)u_{xxx} + g_2(u)u_x u_{xx} + g_3(u)u_x^3), \quad g_1(u) \neq 0. \quad (5.17)$$

Lemma 5.5 If the equation (5.17) is formally integrable, then the functions g_1, g_2, g_3 must satisfy the equations

$$\begin{aligned} 9g_1^2 g_2'' - 6g_1 g_2 g_1'' - 9g_1 g_1' g_2' \\ - 18g_1 g_2 g_2' + 8g_2 g_1'^2 + 12g_2^2 g_1' + 4g_2^3 = 0, \end{aligned} \quad (5.18)$$

$$\begin{aligned} 12g_1^2 g_1''' - 8g_1 g_1' g_1'' - 12g_1 g_2 g_1'' \\ - 3g_1 g_1' g_2' + 4g_1'^3 + 6g_2 g_1'^2 + 2g_2^2 g_1' = 0 \end{aligned} \quad (5.19)$$

and the relation

$$g_3 = \frac{1}{72g_1} \left(6g_2^2 - 30g_2 g_1' - 4g_1'^2 + 27g_1 g_2' + 12g_1 g_1'' \right). \quad (5.20)$$

Proof. By using the criterion of formal integrability, we easily arrive at the result of the lemma from the polynomiality property of the first few W_k 's. The lemma is proved. \square

Lemma 5.6 If the conditions (5.19), (5.20) hold true and $g_1(u)$ is not a constant, then there exists a reciprocal transformation

$$dy = f(u)dx + \rho(u, u_x, u_{xx})dt, \quad ds = dt, \quad (5.21)$$

such that u satisfies an equation of the following form:

$$u_s = \tilde{f}(u)u_y + \varepsilon^2 (u_{yyy} + \tilde{g}_2(u)u_{yy}u_y + \tilde{g}_3(u)u_y^3). \quad (5.22)$$

Proof. To define the reciprocal transformation (5.21), $f(u)$ must be the density of a conservation law, i.e. there exists a function $\rho(u, u_x, u_{xx})$ such that

$$(f(u))_t = (\rho(u, u_x, u_{xx}))_x.$$

By using the equation (5.17), it's easy to see that $f(u)$ must satisfy the following equation:

$$g_1 f''' + 2g_1' f'' + g_1'' f' - g_2 f'' - g_2' f' + 2g_3 f' = 0, \quad (5.23)$$

and the flux ρ is given by

$$\rho = uf - F + \varepsilon^2 \left(g_1 f' u_{xx} + \frac{1}{2} (g_2 f' - g_1 f'' - g_1' f') u_x^2 \right), \quad (5.24)$$

where $F(u) = \int f(u) du$.

Suppose we have a function $f(u)$ satisfying (5.23) and a function ρ given by (5.24), then the reciprocal transformation (5.21) is well defined and it converts the equation (5.17) to the following one

$$u_s = F(u) u_y + \varepsilon^2 (g_1(u) f(u)^3 u_{yyy} + \tilde{g}_2(u) u_{yy} u_y + \tilde{g}_3(u) u_y^3). \quad (5.25)$$

To complete the proof, we only need to show that $f = g_1^{\frac{1}{3}}$ is a solution of the equation (5.23). In fact, after the substitutions of $f = g_1^{\frac{1}{3}}$ and (5.20) into the equation (5.23), we obtain an equation which is equivalent to (5.19). The lemma is proved. \square

Now let us perform a Miura type transformation of the first kind

$$u \mapsto \tilde{u} = \tilde{f}(u)$$

to the equation (5.22), we obtain an equation of the following form

$$\tilde{u}_s = \tilde{u} \tilde{u}_y + \varepsilon^2 (\tilde{u}_{yyy} + \tilde{g}_2(\tilde{u}) \tilde{u}_{yy} \tilde{u}_y + \tilde{g}_3(\tilde{u}) \tilde{u}_y^3). \quad (5.26)$$

Since the above reciprocal and Miura type transformations keep the formal integrability property, the functions $\tilde{g}_1 = 1, \tilde{g}_2, \tilde{g}_3$ also satisfy the conditions (5.18)-(5.20), i.e.

$$\tilde{g}_2'' - 2\tilde{g}_2 \tilde{g}_2' + \frac{4}{9} \tilde{g}_2^3 = 0, \quad (5.27)$$

$$\tilde{g}_3 = \frac{1}{24} (2\tilde{g}_2^2 + 9\tilde{g}_2'). \quad (5.28)$$

Lemma 5.7 Suppose the functions $\tilde{g}_2(\tilde{u}), \tilde{g}_3(\tilde{u})$ satisfy the conditions (5.27), (5.28), then there exists a Miura type transformation

$$u \mapsto \bar{u} = \tilde{u} + \varepsilon f_0(\tilde{u}) \tilde{u}_y + \varepsilon^2 (f_1(\tilde{u}) \tilde{u}_{yy} + f_2(\tilde{u}) \tilde{u}_y^2) \quad (5.29)$$

which transforms the equation (5.26) into the KdV equation

$$\bar{u}_s = \bar{u} \bar{u}_y + \varepsilon^2 \bar{u}_{yyy}.$$

Proof. The ODE (5.27) has the following general solution

$$\tilde{g}_2(\tilde{u}) = -\frac{3f'(\tilde{u})}{2f(\tilde{u})}, \quad f(\tilde{u}) = a_0 + a_1 \tilde{u} + a_2 \tilde{u}^2,$$

where a_0, a_1, a_2 are arbitrary constants. Let's take

$$f_0 = \frac{\sqrt{\frac{3}{2}} \sqrt{4a_0 a_2 - a_1^2}}{\sqrt{f} \sqrt{f' + 2\sqrt{a_2} \sqrt{f}}}, \quad f_1 = \frac{3\sqrt{a_2}}{2\sqrt{f}}, \quad f_2 = f_1' - \frac{1}{6} f_1^2.$$

The lemma is proved after a straightforward verification. \square

The above three Lemmas show that the equation (5.17) is formally integrable if the functions g_1, g_2, g_3 satisfy (5.18)-(5.20), and modulo reciprocal and Miura type transformations there is only one formally integrable equation, namely, the KdV equation. So our formal integrability in fact leads to integrable equations in the common sense.

To see some concrete formally integrable equations of the form (5.17) that are in different guises of the KdV equation, let us further impose the additional conditions

on the functions g_1, g_2, g_3 by requiring that the right hand side of (5.17) is graded homogeneous with respect to the following grading

$$\deg u = 1, \deg \partial_x^m u = 1 - m, \deg \varepsilon = 1 + \delta, \quad (5.30)$$

where δ is certain constant. Since $g_1(u) \neq 0$, we can take

$$g_1(u) = u^{1-2\delta}, \quad g_2(u) = \alpha u^{-2\delta}, \quad \alpha \text{ is constant.} \quad (5.31)$$

Then the equations (5.18), (5.19) have the following solutions (α, δ) :

$$\begin{aligned} & (0, \frac{1}{2}), (-\frac{3}{2}, \frac{1}{2}), (-3, \frac{1}{2}), (-2, 0), (-1, 0), (0, -\frac{1}{4}), \\ & (-\frac{3}{4}, -\frac{1}{4}), (0, -\frac{1}{2}), (-6, 2). \end{aligned}$$

The associated equations are given by

$$(0, \frac{1}{2}) : u_t = u u_x + \varepsilon^2 u_{xxx}, \quad (5.32)$$

$$(-\frac{3}{2}, \frac{1}{2}) : w_t = w^2 w_x + \varepsilon^2 w_{xxx}, \quad w^2 = u, \quad (5.33)$$

$$(-3, \frac{1}{2}) : u_t = u u_x + \varepsilon^2 \left(u_{xxx} - \frac{3}{u} u_x u_{xx} + \frac{15}{8u^2} u_x^3 \right), \quad (5.34)$$

$$(-2, 0) : w_t = w^3 w_x + \varepsilon^2 w^3 w_{xxx}, \quad w^3 = u, \quad (5.35)$$

$$(-1, 0) : w_t = w^3 w_x + \varepsilon^2 (3w^2 w_x w_{xx} + w^3 w_{xxx}), \quad w^3 = u, \quad (5.36)$$

$$(0, -\frac{1}{4}) : w_t = w^2 w_x + \varepsilon^2 (3w^2 w_x w_{xx} + w^3 w_{xxx}), \quad w^2 = u, \quad (5.37)$$

$$(-\frac{3}{4}, -\frac{1}{4}) : w_t = w^2 w_x + \varepsilon^2 \left(\frac{3}{2} w^2 w_x w_{xx} + w^3 w_{xxx} \right), \quad w^2 = u, \quad (5.38)$$

$$(0, -\frac{1}{2}) : u_t = u u_x + \varepsilon^2 \left(u^2 u_{xxx} + \frac{1}{9} u_x^3 \right), \quad (5.39)$$

$$(-6, 2) : w_t = \frac{w_x}{w} + \varepsilon^2 w^3 w_{xxx}, \quad \frac{1}{w} = u. \quad (5.40)$$

The first two equations (5.32) and (5.33) are the KdV and mKdV equation respectively. The third one (5.34) is equivalent to the KdV equation (5.32) by the following Miura type transformation

$$u \mapsto \tilde{u} = u + \varepsilon^2 \left(\frac{3u_{xx}}{2u} - \frac{15u_x^2}{8u^2} \right), \quad (5.41)$$

where u satisfies (5.34) and \tilde{u} satisfies (5.32).

The equations (5.32)-(5.34) have the following pairs of conserved quantities respectively:

$$\int u dx, \int u^2 dx; \int w dx, \int w^2 dx; \int \sqrt{u} dx, \int \frac{1}{\sqrt{u}} dx.$$

By using these conserved quantities we can define six reciprocal transformations, they relate the remaining six equations (5.35)-(5.40) to the first three equations (5.32)-(5.34). Let us consider in detail the reciprocal transformation that is defined by the

second conserved quantity $\int u^2(x,t)dx$ of the equation (5.32). The conservation law is given by

$$(u^2)_t = \rho_x, \text{ where } \rho = \frac{2}{3}u^3 + \varepsilon^2(2uu_{xx} - u_x^2). \quad (5.42)$$

Thus we can define the following reciprocal transformation:

$$dy = u^2 dx + \rho dt, \quad ds = dt, \quad (5.43)$$

its inverse is given by

$$dx = \frac{1}{u^2} dy - \left(\frac{2}{3}u + \varepsilon^2 \left(\frac{2u_{xx}}{u} - \frac{u_x^2}{u^2} \right) \right) ds, \quad dt = ds. \quad (5.44)$$

From the definition (5.43) we obtain $\partial_x = u^2 \partial_y$, so

$$u_x = u^2 u_y, \quad u_{xx} = u^2 (u^2 u_{yy} + 2u u_y^2). \quad (5.45)$$

Now the reciprocal transformation (5.44) can be rewrite as

$$dx = \frac{1}{u^2} dy - \left(\frac{2}{3}u + \varepsilon^2 (2u^3 u_{yy} + 3u^2 u_y^2) \right) ds, \quad dt = ds. \quad (5.46)$$

So in terms of the new independent variables y, s , the function u satisfies the following equation:

$$u_s = \frac{u^3}{3} u_y + \varepsilon^2 (u^6 u_{yyy} + 6u^5 u_y u_{yy} + 3u^4 u_y^3). \quad (5.47)$$

By performing the rescaling $s \mapsto \tilde{s} = s/3, \varepsilon \mapsto \tilde{\varepsilon} = \sqrt{3}\varepsilon$ and the Miura transformation of first kind $u \mapsto \tilde{u} = u^3$, the above equation is transformed to

$$\tilde{u}_{\tilde{s}} = \tilde{u} \tilde{u}_y + \tilde{\varepsilon}^2 \left(\tilde{u}^2 \tilde{u}_{yyy} + \frac{1}{9} \tilde{u}_y^3 \right), \quad (5.48)$$

which coincides with the equation (5.39).

For the other cases, we only point out the following facts:

1. The equation (5.32) is transformed to the equation (5.37) by the reciprocal transformation defined by the conserved quantity $\int u dx$.
2. The equation (5.33) is transformed to the equations (5.36), (5.38) respectively by the reciprocal transformations defined by conserved quantities $\int w dx$ and $\int w^2 dx$.
3. The equation (5.34) is transformed to the equations (5.35), (5.40) respectively by the reciprocal transformations defined by conserved quantities $\int \sqrt{u} dx$ and $\int 1/\sqrt{u} dx$.

Note that the equation (5.36) and (5.37) are two cases of the S-integrable equation (3.55) given in [2]

The analysis of the above two examples shows that the formal integrability condition is rather rigid. Modulo the Miura type transformations and reciprocal transformations, the Burgers equation and the KdV equation are the only integrable equations among the two classes of equations of the form (5.7) and (5.17). The following conjecture generalize the results of the first example:

Conjecture 5.8 *All the formally integrable equations written in the reduced form*

$$u_t = uu_x + \varepsilon(f_2(u)u_2 + f_{11}(u)u_1^2) + \varepsilon^2 f_3(u)u_3 + \varepsilon^3(f_4(u)u_4 + f_{22}(u)u_2^2) + \dots \quad (5.49)$$

with $f_2(u) \neq 0$ are parameterized by the functions $f_2(u)$ and $f_{11}(u)$. In other words, the formal integrability condition for the equation (5.49) with $f_2(u) \neq 0$ uniquely determines all the coefficient functions $f_\lambda(u)$ with $|\lambda| > 2$ through the functions $f_2(u)$ and $f_{11}(u)$, for example

$$\begin{aligned} f_3 &= \frac{4}{3} f_2 f_{11} + \frac{2}{3} f_2 f_2', \\ f_4 &= \frac{5}{3} f_2^2 f_{11}' + \frac{1}{3} f_2^2 f_2'' + 2 f_2 f_{11}^2 + \frac{1}{3} f_2 (f_2')^2 + \frac{5}{3} f_{11} f_2 f_2', \\ f_{22} &= -\frac{10}{3} f_{11}^3 - f_{11}^2 f_2' + \frac{1}{3} f_{11} (f_2')^2 - \frac{29}{3} f_2 f_{11} f_{11}' - \frac{10}{3} f_2 f_{11} f_2'' \\ &\quad - 4 f_2 f_2' f_{11}' - f_2 f_2' f_2'' - 3 f_2^2 f_{11}'' - f_2^2 f_2''', \dots \end{aligned}$$

Moreover, modulo certain reciprocal transformation a formal integrable equation of the above form is equivalent, under a Miura type transformation, to a formal symmetry of the Burgers equation (4.19).

The parameter space for the class of integrable equations of the form (5.49) with $f_2(u) = 0$ is unknown yet. It seems that there exist infinitely many function parameters in this space. We hope that the additional condition of possessing a hamiltonian structure will restrict this parameter space to a manageable one. We will consider this problem in subsequent publications.

6 Conclusion

We have proved the quasi-triviality of a generalized evolutionary PDE of the form (1.5) and proposed a criterion for its integrability. Due to Proposition 4.2, the proof of Theorem 4.3 gives a constructive algorithm to obtain an explicit expression of the quasi-Miura transformation that reduces the equation (1.5) to its leading term equation (1.7). Also due to Proposition 4.2, the explicit expression of the flows of the form (5.2) that commute with (1.5) can be obtained in a direct way as explained in the beginning of Sec. 5, thus the algorithm of testing the formal integrability of an equation of the form (1.5) can be encoded into a simple computer program.

Now it is natural to ask the question whether the quasi-triviality property can be generalized to systems of generalized evolutionary PDEs of the form

$$u_t^i = \sum_{j=1}^n \lambda_j^i(\mathbf{u}) u_x^j + \sum_{k \geq 1} \varepsilon^k Q_k^i(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k+1)}), \quad i = 1, \dots, n \geq 2, \quad (6.1)$$

where $\mathbf{u} = (u^1, \dots, u^n)$, λ_j^i are smooth functions of \mathbf{u} and Q_k^i are homogeneous polynomials of $\partial_x^m u^l$, $l = 1, \dots, n$, $m = 1, \dots, k + 1$ with degree $k + 1$, here we define $\deg \partial_x^m u^l = m$. It is proved in [9] that if the above system has a semisimple bihamiltonian structure then it is quasi-trivial, i.e., there exists a quasi-Miura transformation of the form

$$u^i \mapsto v^i = u^i + \sum_{k \geq 1} \varepsilon^k F_k^i(\mathbf{u}, \mathbf{u}_x, \dots), \quad i = 1, \dots, n \quad (6.2)$$

that reduces the above system to a system given by its leading terms

$$v_t^i = \sum_{j=1}^n \lambda_j^i(\mathbf{v}) v_x^j. \quad (6.3)$$

We expect that the condition of possessing a semisimple bihamiltonian structure could be replaced by a much weaker one in order to ensure the quasi-triviality of the systems of the above form.

The definition of formal integrability for equations of the form (1.5) can be directly generalized to systems of equations of the form (6.1), we plan to study the problem of its quasi-triviality and integrability in subsequent publications.

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