

On the Drinfeld-Sokolov Hierarchies of D type

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Abstract

We extend the notion of pseudo-differential operators that are used to represent the Gelfand-Dickey hierarchies, and obtain a similar representation for the full Drinfeld-Sokolov hierarchies of D_n type. By using such pseudo-differential operators we introduce the tau functions of these bihamiltonian hierarchies, and prove that these hierarchies are equivalent to the integrable hierarchies defined by Date-Jimbo-Kashiwara-Miwa and Kac-Wakimoto from the basic representation of the Kac-Moody algebra $D_n^{(1)}$.

Key words: pseudo-differential operator, Drinfeld-Sokolov hierarchy, tau function, bilinear equation, BKP hierarchy

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1 Introduction

For every affine Lie algebra \mathfrak{g} and a choice of a vertex c_m of the extended Dynkin diagram, Drinfeld and Sokolov constructed in [7] a hierarchy of integrable systems which generalizes the prototypical soliton equation—the Korteweg-de Vries equation. This construction provides a big class of integrable hierarchies that have important applications in different areas of mathematical physics. In particular, the integrable hierarchies that are associated to the affine Lie algebras of ADE type are shown to be closely related to 2D topological field theory and Gromov-Witten invariants, see [8, 11, 13, 14, 21, 22, 28, 34] and references therein. In establishing such relationships the tau functions of the integrable hierarchies play a crucial role, they correspond to the partition functions of topological field theory models. The unknown functions of the hierarchies are related to some special two point correlation functions.

The definition of the tau functions for the Drinfeld-Sokolov hierarchies and their generalizations [23] was given in [16, 24, 30] by using the dressing operators of the hierarchies. In terms of the tau functions such integrable hierarchies and their generalizations are represented as systems of Hirota bilinear equations, they can also be constructed by using the representation theoretical approach to soliton equations developed by Date, Jimbo, Kashiwara, Miwa [5, 4] and by Kac, Wakimoto [27, 26]. In this approach the systems of Hirota bilinear equations are constructed from an integrable highest weight representation of \mathfrak{g} and its vertex operator realization, the tau functions that satisfy these equations are elements of the orbit of the highest weight vector of the representation under the action of the affine Lie group. Note that tau functions of the Drinfeld-Sokolov hierarchies are also defined in [12, 33] via certain symmetry (called tau-symmetry in [11]) of the Hamiltonian densities of the hierarchies represented in forms of modified

KdV type. Here the unknown functions of the Drinfeld-Sokolov hierarchies in forms of modified KdV type and in that of KdV type are related by Miura type transformations.

For general Drinfeld-Sokolov hierarchies there are no canonical choices for their unknown functions, and the definition of the tau functions given in [12, 16, 24, 30] in terms of the dressing operators is in certain sense implicit. However, in the particular case when the affine Lie algebra is $A_n^{(1)}$ the Drinfeld-Sokolov hierarchy coincides with the Gelfand-Dickey hierarchy [18], the unknown functions can be taken as the coefficients of a differential operator

$$L = D^{n+1} + u^n D^{n-1} + \dots + u^2 D + u^1, \quad D = \frac{d}{dx},$$

and the integrable hierarchy can be represented in the form

$$\frac{\partial L}{\partial t_k} = [(L^{\frac{k}{n+1}})_+, L], \quad k = \mathbb{Z}_+ \setminus (n+1)\mathbb{Z}_+. \quad (1.1)$$

Here u^i are functions of the spatial variable x and the time variables t_1, t_2, \dots . This integrable hierarchy has the Hamiltonian structure

$$\frac{\partial u^i}{\partial t_k} = \{u^i(x), H_{k+n+1}\},$$

where the Poisson bracket is defined by

$$\{F, G\} = \int \text{res} \left(\left[\frac{\delta F}{\delta L}, L \right] \frac{\delta G}{\delta L} \right) dx$$

for local functionals F, G , and the densities of the Hamiltonians $H_k = \int h_k(u, u_x, \dots) dx$ can be chosen as

$$h_k = \frac{n+1}{k} \text{res} L^{\frac{k}{n+1}}.$$

The advantage of such a choice of the Hamiltonian densities lies in the fact that they satisfy the tau symmetry condition

$$\frac{k}{n+1} \frac{\partial h_k}{\partial t_l} = \frac{l}{n+1} \frac{\partial h_l}{\partial t_k}.$$

Due to this property of the densities the tau function of the Gelfand-Dickey hierarchy can be introduced, as it was done in [5, 11, 15, 33], by the equations

$$\frac{\partial^2 \log \tau}{\partial x \partial t_k} = \frac{k}{n+1} h_k, \quad k = \mathbb{Z}_+ \setminus (n+1)\mathbb{Z}_+. \quad (1.2)$$

Note that the Hamiltonians for the general Drinfeld-Sokolov hierarchies are also given in [7], however the densities given there do not satisfy the tau

symmetry condition. In order to fulfill such a condition these densities should be modified by adding certain terms which are total x -derivatives of some differential polynomials of the unknown functions.

In the above formalism of the Drinfeld-Sokolov hierarchy associated to the affine Lie algebra $A_n^{(1)}$, the integrable hierarchy and the relation of its unknown functions with the tau function are given in a relatively explicit form. The purpose of the present paper is to give a similar representation for the Drinfeld-Sokolov hierarchy associated to the affine Lie algebra $D_n^{(1)}$ and the vertex c_0 of the Dynkin diagram. Such a formalism is helpful for people to have a clear picture of the relation of integrable systems with Gromov-Witten invariants and topological field theory models associated to ADE singularities [13, 14, 17, 20, 21, 22, 35]. In fact, Drinfeld and Sokolov already represented in [7] part of the integrable hierarchy in terms of a pseudo-differential operator of the form

$$L = D^{2n-2} + \sum_{i=1}^{n-1} D^{-1} (u^i D^{2i-1} + D^{2i-1} u^i) + D^{-1} \rho D^{-1} \rho, \quad (1.3)$$

where the functions $u^1, \dots, u^{n-1}, u^n = \rho$ serve as the unknown functions of the hierarchy. The integrable systems of the hierarchy can be labeled by the elements of a chosen base

$$\{\Lambda^j \in \mathfrak{g}^j, \Gamma^j \in \mathfrak{g}^{j(n-1)} \mid j \in 2\mathbb{Z} + 1\} \quad (1.4)$$

of the principal Heisenberg subalgebra of $D_n^{(1)}$ (see Sec. 4 for the definition of these symbols). Denote by P the fractional power $L^{\frac{1}{2n-2}}$ of L which is a pseudo-differential operator of the form

$$P = D + w_1 D^{-1} + w_2 D^{-2} + \dots,$$

then the part of the integrable hierarchy that corresponds to the elements Λ^j can be represented as [7]

$$\frac{\partial L}{\partial t_k} = [(P^k)_+, L], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (1.5)$$

The other part that corresponds to the elements Γ^j can not be represented in this way by using only the pseudo-differential operators L, P .

Inspired by the Lax pair representations of the dispersionless integrable hierarchy that appear in 2D topological field theory [8, 32], we attempt to represent the flows corresponding to the elements Γ^j by the square root Q of L which takes the form

$$Q = D^{-1} \rho + \sum_{k \geq 0} w_k D^k.$$

However, this operator is not a pseudo-differential operator in the usual sense, because it contains infinitely many terms with positive powers of D , so one cannot compute the square of Q . We note that in the dispersionless case, with D replaced by its symbol p , one can define the square of Q , and define the dispersionless hierarchy by using L, P and Q .

We are to show in this paper that there exists a new kind of pseudo-differential operators which are allowed to contain infinitely many terms with positive power of D such as Q , so we can define the square root of the pseudo-differential operator L in the space of such operators. Then by using the pseudo-differential operators L and Q we can get the Lax pair representation of the remaining part of the integrable hierarchy and define its tau function in a way that one does for the Gelfand-Dickey hierarchy, see Theorem 4.11. By using this new kind of pseudo-differential operators, we also find a Lax pair representation of the two-component BKP hierarchy (see [3], c.f. [31]). We show that the Drinfeld-Sokolov hierarchy of D_n type is the $(2n - 2, 2)$ -reduction of the two-component BKP hierarchy [4]. In this way we also prove that the square root of the tau function satisfies the Hirota bilinear equations that are constructed in [4] from the basic representation of the affine Lie algebra $D_n^{(1)}$, see (5.21), (5.22) and Theorem 5.2. These bilinear equations are equivalent to those given by Kac and Wakimoto [27] and by Givental and Milanov [22], see Theorem 6.1.

In order to obtain the above mentioned results, we first extend, in Section 2, the usual definition of the ring of pseudo-differential operators. Then in Section 3 we define a hierarchy of integrable systems and its tau function by using the pseudo-differential operator L of the form (1.3) and its fractional powers P, Q . In Section 4 we show that the constructed hierarchy coincides with the Drinfeld-Sokolov hierarchy associated to the affine Lie algebra $D_n^{(1)}$ and the vertex c_0 of its Dynkin diagram. In Section 5 we give a Lax pair representation of the two-component BKP hierarchy, its tau function, and its $(2n - 2, 2)$ -reductions. In the final section we give some concluding remarks.

2 Pseudo-differential operators

In this section we generalize the concept of pseudo-differential operators and list some useful properties of them.

2.1 Definitions

Let \mathcal{A} be a commutative ring with unity, and $D : \mathcal{A} \rightarrow \mathcal{A}$ be a derivation. The algebra of pseudo-differential operators over \mathcal{A} is defined to be

$$\mathcal{D}^- = \left\{ \sum_{i < \infty} f_i D^i \mid f_i \in \mathcal{A} \right\}.$$

This is a complete topological ring, whose topological basis is given by the following filtration

$$\cdots \subset \mathcal{D}_{(d-1)}^- \subset \mathcal{D}_{(d)}^- \subset \mathcal{D}_{(d+1)}^- \subset \cdots, \quad \mathcal{D}_{(d)}^- = \left\{ \sum_{i \leq d} f_i D^i \mid f_i \in \mathcal{A} \right\}.$$

The product of two pseudo-differential operators $A = \sum_{i \leq k} f_i D^i \in \mathcal{D}^-$ and $B = \sum_{j \leq l} g_j D^j \in \mathcal{D}^-$ is defined by

$$A \cdot B = \sum_{i \leq k} \sum_{j \leq l} \sum_{r \geq 0} \binom{i}{r} f_i D^r (g_j) D^{i+j-r} \in \mathcal{D}^-. \quad (2.1)$$

One can see that for every $s \in \mathbb{Z}$, the coefficient of D^s in (2.1) is a finite sum of elements of \mathcal{A} , so the above product is well defined.

In our formalism of the Drinfeld-Sokolov hierarchy of D_n type below, one need not only operators in \mathcal{D}^- but also operators in the following larger abelian group

$$\mathcal{D} = \left\{ \sum_{i \in \mathbb{Z}} f_i D^i \mid f_i \in \mathcal{A} \right\}.$$

However, it is impossible to extend the product (2.1) to \mathcal{D} because when expanding the product of two elements of \mathcal{D} one meets summations of infinitely many elements of \mathcal{A} , which are not well defined unless \mathcal{A} possesses certain topology.

Now we assume that on \mathcal{A} there is a gradation

$$\mathcal{A} = \prod_{i \geq 0} \mathcal{A}_i, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$$

such that \mathcal{A} is topologically complete w.r.t. the induced decreasing filtration

$$\mathcal{A} = \mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{(d-1)} \supset \mathcal{A}_{(d)} \supset \mathcal{A}_{(d+1)} \supset \cdots, \quad \mathcal{A}_{(d)} = \prod_{i \geq d} \mathcal{A}_i.$$

Let $D : \mathcal{A} \rightarrow \mathcal{A}$ be a derivation of degree one, i.e. $D(\mathcal{A}_i) \subset \mathcal{A}_{i+1}$. An operator $A \in \mathcal{D}^- \subset \mathcal{D}$ is said to be homogeneous if there exists an integer $k \in \mathbb{Z}$ such that

$$A = \sum_{i \leq k} f_i D^i, \quad f_i \in \mathcal{A}_{k-i},$$

and the integer k is called the degree of A . We denote by \mathcal{D}_k the subgroup that consists of all homogeneous pseudo-differential operators of degree k , then the abelian group \mathcal{D} has the following decomposition

$$\mathcal{D} = \prod_{k \in \mathbb{Z}} \mathcal{D}_k.$$

We introduce the following subgroups of \mathcal{D} :

$$\mathcal{D}_{(d)}^+ = \prod_{k \geq d} \mathcal{D}_k, \quad \mathcal{D}^+ = \bigcup_{d \in \mathbb{Z}} \mathcal{D}_{(d)}^+.$$

Then \mathcal{D}^+ is topologically complete w.r.t. the filtration

$$\cdots \supset \mathcal{D}_{(d-1)}^+ \supset \mathcal{D}_{(d)}^+ \supset \mathcal{D}_{(d+1)}^+ \supset \cdots .$$

For any $A \in \mathcal{D}_k$ and $B \in \mathcal{D}_l$, it is easy to see that their product defined by (2.1) belongs to \mathcal{D}_{k+l} , so we can extend this product to \mathcal{D}^+ such that \mathcal{D}^+ becomes a ring.

Definition 2.1 *Elements of \mathcal{D}^- (resp. \mathcal{D}^+) are called pseudo-differential operators of the first type (resp. the second type) over \mathcal{A} . The intersection of \mathcal{D}^- and \mathcal{D}^+ in \mathcal{D} is denoted by*

$$\mathcal{D}^b = \mathcal{D}^- \cap \mathcal{D}^+,$$

and its elements are called bounded pseudo-differential operators.

Sometimes to indicate the ring \mathcal{A} and the derivation D , we will use the notations $\mathcal{D}^\pm(\mathcal{A}, D)$ instead of \mathcal{D}^\pm .

The general form of $A \in \mathcal{D}$ reads

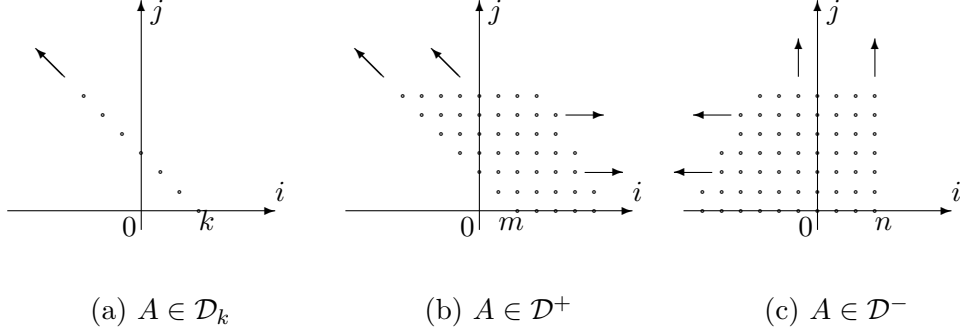
$$A = \sum_{i \in \mathbb{Z}} \sum_{j \geq 0} a_{i,j} D^i, \quad a_{i,j} \in \mathcal{A}_j. \quad (2.2)$$

The following lemma is obvious.

Lemma 2.2 *Suppose $A \in \mathcal{D}$ is given as in (2.2), then*

- i) $A \in \mathcal{D}_k$ iff the coefficients $a_{i,j}$ are supported on the ray $\{(i, j) \mid i + j = k, j \geq 0\}$;*
- ii) $A \in \mathcal{D}^+$ iff there exists $m \in \mathbb{Z}$ such that $a_{i,j}$ are supported on the domain $\{(i, j) \mid j \geq \max\{0, m - i\}\}$;*
- iii) $A \in \mathcal{D}^-$ iff there exists $n \in \mathbb{Z}$ such that $a_{i,j}$ are supported on the domain $\{(i, j) \mid i \leq n, j \geq 0\}$.*

This lemma has a graphic interpretation as follows:



From this interpretations it is easy to see the following alternative expressions of the elements of $A \in \mathcal{D}^\pm$.

- i) If $A \in \mathcal{D}^+$, then there exists $m \in \mathbb{Z}$ and $a_{i,j} \in \mathcal{A}_j$ such that A can be written as the following two forms:

$$A = \sum_{i \in \mathbb{Z}} \left(\sum_{j \geq \max\{0, m-i\}} a_{i,j} \right) D^i, \quad (2.3)$$

$$A = \sum_{j \geq 0} \left(\sum_{i \geq m-j} a_{i,j} D^i \right). \quad (2.4)$$

- ii) If $A \in \mathcal{D}^-$, then there exists $n \in \mathbb{Z}$ and $a_{i,j} \in \mathcal{A}_j$ such that A can be written as follows:

$$A = \sum_{i \leq n} \left(\sum_{j \geq 0} a_{i,j} \right) D^i, \quad (2.5)$$

$$A = \sum_{j \geq 0} \left(\sum_{i \leq n} a_{i,j} D^i \right). \quad (2.6)$$

We call the expressions (2.3) and (2.5) the *normal expansion* of A , while the expressions (2.4) and (2.6) the *dispersion expansion* of A .

Properties of pseudo-differential operators of the first type are well known. Similar to the operators in \mathcal{D}^- we can also define the adjoint operator, the residue, the positive part and the negative part of a pseudo-differential operator of the second type. Let $A \in \mathcal{D}^+$ be given by (2.2), then

$$A^* = \sum_{i \in \mathbb{Z}} \sum_{j \geq 0} (-1)^i D^i \cdot a_{i,j}, \quad \text{res } A = \sum_{j \geq 0} a_{-1,j},$$

$$A_+ = \sum_{i \geq 0} \sum_{j \geq 0} a_{i,j} D^i, \quad A_- = \sum_{i < 0} \sum_{j \geq 0} a_{i,j} D^i.$$

Clearly $A^*, A_+, A_- \in \mathcal{D}^+$ and $\text{res } A \in \mathcal{A}$. In particular, if $A \in \mathcal{D}^\pm$, then $A_\mp \in \mathcal{D}^b$.

An operator $A \in \mathcal{D}^\pm$ is called a *differential operator* if its negative part A_- vanishes. Note that every differential operator in \mathcal{D}^- is of finite order, while the ones in \mathcal{D}^+ may be not. The differential operators in \mathcal{D}^\pm form subrings of \mathcal{D}^\pm respectively, and they act on \mathcal{A} in the obvious way. Given a differential operator $A \in \mathcal{D}^\pm$, we denote by $A(f)$ the action of A on $f \in \mathcal{A}$.

Let us introduce some other notations to be used latter. Elements of the quotient space $\mathcal{F} = \mathcal{A}/D(\mathcal{A})$ are called *local functionals*, and they are represented in the form

$$\int f dx = f + D(\mathcal{A}), \quad f \in \mathcal{A}.$$

Introduce the map

$$\langle \rangle : \mathcal{D} \rightarrow \mathcal{F}, \quad A \mapsto \langle A \rangle = \int \text{res } A dx.$$

We then define the pairing

$$\langle A, B \rangle = \langle AB \rangle \tag{2.7}$$

on each of the following four spaces:

$$\mathcal{D}^+ \times \mathcal{D}^+, \quad \mathcal{D}^- \times \mathcal{D}^-, \quad \mathcal{D}^b \times \mathcal{D}, \quad \mathcal{D} \times \mathcal{D}^b.$$

It is easy to see that this pairing is symmetric and is nondegenerate on each of the above spaces.

2.2 Properties of pseudo-differential operators

In this subsection we present some useful properties of pseudo-differential operators. We assume henceforth that \mathcal{A} is an integral domain.

Lemma 2.3 *Let $A, B \in \mathcal{D}^\pm$. If the commutator $[A^m, B] = 0$ for some positive integer m , then $[A, B] = 0$.*

Proof The \mathcal{D}^- case is well known, we only prove the \mathcal{D}^+ case. Suppose $C = [A, B] \neq 0$. We take the dispersion expansions

$$A = \sum_{j \geq a} \sum_{i \geq k_j} A_{i,j} D^i, \quad C = \sum_{j \geq c} \sum_{i \geq l_j} C_{i,j} D^i,$$

such that neither $A_{k_a, a}$ nor $C_{l_c, c}$ vanishes, then the coefficient of $D^{(m-1)k_a+l_c}$ in

$$[A^m, B] = [A, B]A^{m-1} + A[A, B]A^{m-2} + \cdots + A^{m-1}[A, B]$$

reads

$$mA_{k_a, a}^{m-1}C_{l_c, c} + \cdots,$$

where \cdots denote the terms with higher degrees in \mathcal{A} . This contradicts with $[A^m, B] = 0$. The lemma is proved. \square

Let $\rho \in \mathcal{A}$ be an invertible element, we consider the operator

$$Q = D^{-1}\rho + Q_+ \in \mathcal{D}^+, \quad (2.8)$$

where Q_+ is a differential operator in \mathcal{D}^+ . Such an operator Q is invertible, whose inverse reads

$$\begin{aligned} Q^{-1} &= (D^{-1}\rho(1 + \rho^{-1}DQ_+))^{-1} \\ &= (1 - \rho^{-1}DQ_+ + \rho^{-1}DQ_+\rho^{-1}DQ_+ - \cdots) \rho^{-1}D. \end{aligned} \quad (2.9)$$

Note that Q^{-1} is a differential operator in \mathcal{D}^+ .

Lemma 2.4 *Let $Q \in \mathcal{D}^+$ be given in (2.8), then D can be uniquely expressed as the following form*

$$D = \sum_{i \geq 1} h_i Q^{-i}, \quad h_i \in \mathcal{A}. \quad (2.10)$$

Moreover, $m h_m - \text{res } Q^m \in D(\mathcal{A})$ for every $m \geq 1$.

Proof The first assertion follows from a simple induction. We are going to prove the second one by using the following fact

$$\text{res } Q^m = (DQ^m)_+ - D(Q^m)_+.$$

The first assertion shows that

$$(DQ^m)_+ = \left(\sum_{i \geq 1} h_i Q^{m-i} \right)_+ = \sum_{i \geq 1} h_i (Q^{m-i})_+.$$

We assume $(Q^m)_+ = \sum_{i \geq 0} a_{m,i} Q^{-i}$ with $a_{m,i} \in \mathcal{A}$, then

$$\begin{aligned} D(Q^m)_+ &= \sum_{i \geq 0} a'_{m,i} Q^{-i} + \sum_{i \geq 0} a_{m,i} \sum_{j \geq 1} h_j Q^{-i-j} \\ &= \sum_{i \geq 0} a'_{m,i} Q^{-i} + \sum_{j \geq 1} h_j (Q^m)_+ Q^{-j}, \end{aligned}$$

where $a'_{m,i} = D(a_{m,i})$.

By using the above three formulae, one can obtain

$$\sum_{m \geq 1} (\text{res } Q^m) Q^{-m} = \sum_{i \geq 1} \sum_{m=1-i}^0 h_i(Q^m)_+ Q^{-m-i} - \sum_{m \geq 1} \sum_{i \geq 0} a'_{m,i} Q^{-i-m}.$$

Note that $(Q^m)_+ = Q^m$ when $m \leq 0$, so by comparing the coefficients of Q^{-m} we have

$$m h_m - \text{res } Q^m = \sum_{i=0}^{m-1} a'_{m-i,i}.$$

The lemma is proved. \square

Lemma 2.5 *Let A be a pseudo-differential operator in \mathcal{D}^+ , and $\rho \in \mathcal{A}$ be an invertible element of degree zero. Then there exists a unique pseudo-differential operator $B \in \mathcal{D}^+$ such that $A = \rho B D + D B \rho$. Furthermore, if $A^* = \pm A$, then $B^* = \mp B$.*

Proof Without loss of generality, we can assume that A is homogeneous of degree k , i.e., $A = \sum_{i \leq k} a_i D^i$, $a_i \in \mathcal{A}_{k-i}$. Suppose $B = \sum_{i \leq k-1} b_i D^i$, then one can determine b_{k-1}, b_{k-2}, \dots recursively by $A = \rho B D + D B \rho$. So we derive the first part of the lemma.

If $A^* = \pm A$, then

$$\rho(B^* \pm B)D + D(B^* \pm B)\rho = 0,$$

hence $B^* \pm B = 0$ due to the uniqueness in the first part. The lemma is proved. \square

3 An integrable hierarchy represented by pseudo-differential operators

In this section we are to construct a hierarchy of evolutionary partial differential equations starting from a pseudo-differential operator L . This hierarchy possesses a bihamiltonian structure which coincides with that of the Drinfeld-Sokolov hierarchy of D_n type, moreover, it possesses a tau function.

3.1 Construction of the hierarchy

Let M be an open ball in \mathbb{C}^n with coordinates (u^1, u^2, \dots, u^n) . We define the algebra \mathcal{A} of differential polynomials on M to be

$$\mathcal{A} = \mathcal{O}(M)[[u^{i,s} \mid i = 1, \dots, n, s = 1, 2, \dots]].$$

There is a gradation on \mathcal{A} defined by

$$\deg f = 0 \text{ for } f \in C^\infty(M), \quad \deg u^{i,s} = s,$$

then it is easy to see that \mathcal{A} is topologically complete. We introduce a derivation D of degree one over \mathcal{A} as follows

$$D : \mathcal{A} \rightarrow \mathcal{A}, \quad D = \sum_{s \geq 0} \sum_{i=1}^n u^{i,s+1} \frac{\partial}{\partial u^{i,s}},$$

where $u^{i,0} = u^i$. Now let us construct the algebras \mathcal{D}^\pm starting from \mathcal{A} and D as we did in the last section.

Let L be the pseudo-differential operator given in (1.3). Obviously L belongs to $\mathcal{D}^b = \mathcal{D}^- \cap \mathcal{D}^+$ and satisfies $L^* = DLL^{-1}$. Here we re-denote the coordinate u^n by ρ , and will use this notation frequently in what follows.

Firstly, we regard L as an element of \mathcal{D}^- , then by using properties of the usual pseudo-differential operators we have the following lemma.

Lemma 3.1 *There exists a unique pseudo-differential operator $P \in \mathcal{D}^-$ of the form*

$$P = D + u_1 D^{-1} + u_2 D^{-2} + \dots \quad (3.1)$$

such that $P^{2n-2} = L$. Moreover, the operator P satisfies $[P, L] = 0$ and

$$P^* = -DPD^{-1}. \quad (3.2)$$

In [5], Date, Jimbo, Kashiwara and Miwa proved the following lemma.

Lemma 3.2 ([5]) *The constraint (3.2) to an operator P of the form (3.1) is equivalent to the condition that for every $k \in \mathbb{Z}_+^{\text{odd}}$ the free term of $(P^k)_+$ vanishes, i.e. $(P^k)_+(1) = 0$.*

The above two lemmas imply that the following equations

$$\frac{\partial L}{\partial t_k} = [(P^k)_+, L], \quad k \in \mathbb{Z}_+^{\text{odd}} \quad (3.3)$$

are well defined, and they give evolutionary partial differential equations of u^1, \dots, u^n . In particular, $D = \frac{d}{dx}$ with $x = t_1$, and by taking residue of $D \left(\frac{\partial L}{\partial t_k} - [(P^k)_+, L] \right)$ one has

$$\frac{\partial \rho}{\partial t_k} = -(P^k)_+^*(\rho). \quad (3.4)$$

The flows in (3.3) first appeared in [7] as part of the Drinfeld-Sokolov hierarchy of D_n type.

Note that the Drinfeld-Sokolov hierarchy of D_n type contains n series of commuting flows, but only $n - 1$ series of the flows are given in (3.3), so in this sense the equations (3.3) do not form a complete integrable hierarchy. One main result in the present paper is that the n th series of flows of the Drinfeld-Sokolov hierarchy of D_n type can be represented by the square root of L regarded as an element of \mathcal{D}^+ .

Lemma 3.3 *There exists a unique pseudo-differential operator $Q \in \mathcal{D}^+$ of the following form*

$$Q = D^{-1}\rho + \sum_{m \geq 0} Q_m D \quad (3.5)$$

such that $Q^2 = L$. Here Q_m are homogeneous differential operators in \mathcal{D}^b with degree $2m$, and satisfy $Q_m^* = Q_m$. Moreover, the operator Q satisfies

$$Q^* = -DQD^{-1}, \quad (3.6)$$

$$-Q_+^*(\rho) = \frac{1}{2}DL_+(1). \quad (3.7)$$

Proof By substituting (1.3) and (3.5) into $DQ^2 = DL$ and comparing the homogeneous terms, we can obtain

$$\rho Q_m D + DQ_m \rho = A_m, \quad m = 0, 1, 2, \dots \quad (3.8)$$

Here A_m are differential operators depending on $L, Q_0, Q_1, \dots, Q_{m-1}$ and satisfy $A_m + A_m^* = 0$. Then according to Lemma 2.5, Q_m can be determined by induction, and they satisfy $Q_m^* = Q_m$.

The symmetry property (3.6) is trivial. To show (3.7), we consider the free terms on both hand sides of (3.8):

$$DQ_m(\rho) = \begin{cases} u^{m+1, 2m+1}, & m = 0, 1, \dots, n-2, \\ 0, & m \geq n-1. \end{cases}$$

Hence

$$-Q_+^*(\rho) = \sum_{m \geq 0} DQ_m(\rho) = \sum_{m=0}^{n-2} u^{m+1, 2m+1} = \frac{1}{2}DL_+(1).$$

The lemma is proved. \square

According to Lemmas 2.3 and 3.3, the following evolutionary equations are well defined:

$$\frac{\partial L}{\partial \hat{t}_k} = [-(Q^k)_-, L] = [(Q^k)_+, L], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.9)$$

In particular, we have

$$\frac{\partial \rho}{\partial \hat{t}_k} = -(Q^k)_+^*(\rho), \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.10)$$

When $k = 1$ we obtain $\partial \rho / \partial \hat{t}_1 = \frac{1}{2} DL_+(1)$, this flow is linearly independent with $\partial \rho / \partial t_{2i-1}$ ($1 \leq i \leq n-1$), so from the bihamiltonian recursion relation (see below) we see that the equations given in (3.3) are linearly independent with that defined in (3.9).

Theorem 3.4 *The flows in (3.3), (3.9) commute with each other.*

Proof The commutativity of these flows follows from the following equivalent representations of (3.3), (3.9):

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial P}{\partial \hat{t}_k} = [-(Q^k)_-, P], \quad (3.11)$$

$$\frac{\partial Q}{\partial t_k} = [(P^k)_+, Q], \quad \frac{\partial Q}{\partial \hat{t}_k} = [-(Q^k)_-, Q], \quad (3.12)$$

which can be verified as Lemma 2.3. The theorem is proved. \square

The dispersionless limit of the flows $\frac{\partial}{\partial \hat{t}_k}$ was first given by Takasaki in [32], but the dispersionful one was not given there. Following [32], we call the flows (3.3) and (3.9) the *positive* and the *negative* flows respectively. The above theorem shows that the positive and the negative flows form an integrable hierarchy. We will show that it is equivalent to the Drinfeld-Sokolov hierarchy of D_n type.

3.2 Bihamiltonian structure and tau structure

In this subsection we show that the hierarchy (3.3), (3.9) carries a bihamiltonian structure, and the densities of the Hamiltonians can be chosen to satisfy the tau symmetry condition. We then define the tau function of the hierarchy by using this tau symmetry following the approach of [11].

Let $\mathcal{L} = DL$, it has the form

$$\mathcal{L} = D^{2n-1} + \sum_{i=1}^{n-1} (u^i D^{2i-1} + D^{2i-1} u^i) + \rho D^{-1} \rho. \quad (3.13)$$

Given a local functional $F = \int f dx \in \mathcal{A}/D(\mathcal{A})$, we define its variational derivative w.r.t. \mathcal{L} to be an element $X = \delta F / \delta \mathcal{L} \in \mathcal{D}$ such that

$$\delta F = \langle X, \delta \mathcal{L} \rangle, \quad X = X^*. \quad (3.14)$$

The existence of such an element can be verified by taking

$$X = \frac{1}{2} \sum_{i=0}^{n-1} \left(D^{-2i} \frac{\delta F}{\delta v^i(x)} + \frac{\delta F}{\delta v^i(x)} D^{-2i} \right), \quad (3.15)$$

where $v^0 = \rho^2$ and v^1, \dots, v^{n-1} are determined by representing the operator \mathcal{L} in the following form

$$\mathcal{L} = D^{2n-1} + \sum_{i=1}^{n-1} v^i D^{2i-1} + \sum_{i=1}^{n-1} \tilde{v}^i D^{2i-2} + \rho D^{-1} \rho.$$

Note that the new coordinates v^1, \dots, v^{n-1} are related to u^1, \dots, u^{n-1} by a Miura-type transformation, and the functions \tilde{v}^i determined by the condition $\mathcal{L} + \mathcal{L}^* = 0$ are linear functions of the derivatives of v^1, \dots, v^{n-1} .

On the other hand, the variational derivative X defined in (3.14) is determined up to the addition of a kernel part Z that satisfies

$$Z_+(\rho) = 0, \quad Z_- = \sum_{i \leq n} (w_i D^{-2i} + D^{-2i} w_i), \quad w_i \in \mathcal{A}.$$

The following compatible Poisson brackets are given in Proposition 8.3 of [7] (see also [10]) for the bihamiltonian structure of the Drinfeld-Sokolov hierarchy of D_n type:

$$\{F, G\}_1(\mathcal{L}) = \langle X, (DY_+ \mathcal{L})_- - (\mathcal{L}Y_+ D)_- + (\mathcal{L}Y_- D)_+ - (DY_- \mathcal{L})_+ \rangle, \quad (3.16)$$

$$\{F, G\}_2(\mathcal{L}) = \langle X, (\mathcal{L}Y)_+ \mathcal{L} - \mathcal{L}(Y \mathcal{L})_+ \rangle, \quad (3.17)$$

where F and G are two arbitrary local functionals, and

$$X = \frac{\delta F}{\delta \mathcal{L}}, \quad Y = \frac{\delta G}{\delta \mathcal{L}}.$$

Note that in the above formulae of the Poisson brackets the second component in the pairing $\langle \cdot, \cdot \rangle$ belongs to \mathcal{D}^b for any $Y \in \mathcal{D}$, so from the definition of $\langle \cdot, \cdot \rangle$ given in (2.7) we see that the first component X is not restricted to the space \mathcal{D}^+ or \mathcal{D}^- . One can show by a direct computation that the definition of these Poisson brackets is independent of the choice of the kernel parts of X and Y , so they are well defined.

Theorem 3.5 *The hierarchy (3.3), (3.9) has the following bihamiltonian representation:*

$$\frac{\partial F}{\partial t_k} = \{F, H_{k+2n-2}\}_1 = \{F, H_k\}_2, \quad (3.18)$$

$$\frac{\partial F}{\partial \hat{t}_k} = \{F, \hat{H}_{k+2}\}_1 = \{F, \hat{H}_k\}_2. \quad (3.19)$$

Here $F \in \mathcal{F}$ is any local functional, and the Hamiltonians are given by

$$H_k = \frac{2n-2}{k} \langle P^k \rangle, \quad \hat{H}_k = \frac{2}{k} \langle Q^k \rangle, \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.20)$$

Proof Let us start with the computation of the variational derivatives of the Hamiltonians H_k . By using the identity $P^{2n-2} = L$ (see Lemma 3.1) and the symmetric property of the pairing $\langle \cdot, \cdot \rangle$ we have

$$\begin{aligned} \delta H_k &= (2n-2) \langle P^{k-1}, \delta P \rangle = (2n-2) \langle P^{k-2n+2}, P^{2n-3} \delta P \rangle \\ &= \langle P^{k-2n+2}, \delta L \rangle = \langle P^{k-2n+2} D^{-1}, \delta \mathcal{L} \rangle = \langle Y_k, \delta \mathcal{L} \rangle, \end{aligned} \quad (3.21)$$

where $Y_k = P^{k-2n+2} D^{-1} \in \mathcal{D}$. From (3.2) it follows that $Y_k^* = Y_k$, so we can take

$$\frac{\delta H_k}{\delta \mathcal{L}} = Y_k = P^{k-2n+2} D^{-1}. \quad (3.22)$$

To show (3.18), we first note due to Lemma 3.2 the validity of

$$\begin{aligned} D(P^k)_+ D^{-1} &= D(P^k D^{-1})_+ = (DP^k D^{-1})_+, \\ (P^k D^{-1} D)_- &= (P^k D^{-1})_- D \end{aligned} \quad (3.23)$$

for any $k \in \mathbb{Z}_+^{\text{odd}}$. So from (3.3) we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_k} &= D(P^k)_+ L - DL(P^k)_+ = (DP^k D^{-1})_+ \mathcal{L} - \mathcal{L}(P^k)_+ \\ &= (\mathcal{L} Y_k)_+ \mathcal{L} - \mathcal{L}(Y_k \mathcal{L})_+. \end{aligned}$$

On the other hand, by using the commutativity between L and P (see Lemma 3.1) we can also represent $\frac{\partial \mathcal{L}}{\partial t_k}$ in the following form:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_k} &= D(P^k)_+ L - DL(P^k)_+ \\ &= \left(D(P^k)_+ L - DL(P^k)_+ \right)_+ + \left(D(P^k)_+ L - DL(P^k)_+ \right)_- \\ &= \left(-D(P^k)_- L + DL(P^k)_- \right)_+ + \left(D(P^k)_+ L - DL(P^k)_+ \right)_- \\ &= (\mathcal{L}(Y_{k+2n-2})_- D - D(Y_{k+2n-2})_- \mathcal{L})_+ \\ &\quad + (D(Y_{k+2n-2})_+ \mathcal{L} - \mathcal{L}(Y_{k+2n-2})_+ D)_- \end{aligned}$$

Now the equivalence of the flows (3.3) with (3.18) follows from the above identities together with the relation

$$\frac{\partial F}{\partial t_k} = \left\langle \frac{\delta F}{\delta \mathcal{L}}, \frac{\partial \mathcal{L}}{\partial t_k} \right\rangle.$$

By using the property (3.6) of the operator Q we know that for any $k \in \mathbb{Z}_+^{\text{odd}}$ the free term of Q^k vanishes, then a similar argument as above

leads to the equivalence of the flows (3.9) with (3.19). The theorem is proved. \square

By using the formula (3.22) and

$$\frac{\delta \hat{H}_k}{\delta \mathcal{L}} = Q^{k-2} D^{-1},$$

we obtain the following proposition.

Proposition 3.6 *The local functionals $H_1, H_3, \dots, H_{2n-3}$ and \hat{H}_1 are linearly independent Casimirs of the first Poisson bracket $\{, \}_1$.*

We now verify that the above defined densities of the Hamiltonians satisfy the tau symmetry condition, and we can thus define the tau function for the integrable hierarchy (3.3), (3.9). To this end let us introduce a series of rescaled time variables

$$T^{\alpha,p} = \begin{cases} \frac{(2n-2)\Gamma(p+1+\frac{2\alpha-1}{2n-2})}{\Gamma(\frac{2\alpha-1}{2n-2})} t_{(2n-2)p+2\alpha-1}, & \alpha = 1, \dots, n-1, \\ \frac{2\Gamma(p+1+\frac{1}{2})}{\Gamma(\frac{1}{2})} \hat{t}_{2p+1}, & \alpha = n \end{cases}$$

with $p = 0, 1, 2, \dots$. Then the Hamiltonian equations (3.18), (3.19) read

$$\frac{\partial F}{\partial T^{\alpha,p}} = \{F, H_{\alpha,p}\}_1 = \left(p + \frac{1}{2} + \mu_\alpha\right)^{-1} \{F, H_{\alpha,p-1}\}_2,$$

where the densities of the Hamiltonians $H_{\alpha,p}$ are given by

$$h_{\alpha,p-1} = \begin{cases} \frac{\Gamma(\frac{2\alpha-1}{2n-2})}{(2n-2)\Gamma(p+1+\frac{2\alpha-1}{2n-2})} \text{res } P^{(2n-2)p+2\alpha-1}, & \alpha = 1, \dots, n-1, \\ \frac{\Gamma(\frac{1}{2})}{2\Gamma(p+1+\frac{1}{2})} \text{res } Q^{2p+1}, & \alpha = n, \end{cases}$$

and the constants μ_α are the spectrum of the underlying Frobenius manifold [8, 10], they have the expressions

$$\mu_\alpha = \begin{cases} \frac{2\alpha-n}{2n-2}, & \alpha = 1, \dots, n-1, \\ 0, & \alpha = n. \end{cases}$$

Then we have tau symmetry

$$\frac{\partial h_{\alpha,p-1}}{\partial T^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial T^{\alpha,p}},$$

and the differential polynomials

$$\Omega_{\alpha,p;\beta,q} = \partial_x^{-1} \frac{\partial h_{\alpha,p-1}}{\partial T^{\beta,q}}, \quad \alpha, \beta = 1, 2, \dots, n; \quad p, q \geq 0$$

have the property

$$\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}.$$

Hence the chosen $h_{\alpha,p}$ give a tau structure, in the sense of [11], of the bihamiltonian structure of the integrable hierarchy (3.3), (3.9). This tau structure defines the tau function $\hat{\tau}$ of the integrable hierarchy by

$$\frac{\partial^2 \log \hat{\tau}}{\partial T^{\alpha,p} \partial T^{\beta,q}} = \Omega_{\alpha,p;\beta,q}. \quad (3.24)$$

4 Drinfeld-Sokolov hierarchies and pseudo-differential operators

In this section we first recall some facts about the Drinfeld-Sokolov hierarchies associated to untwisted affine Lie algebras, see details in [7]. Then we consider the Drinfeld-Sokolov hierarchy of D_n type and identify it with the hierarchy (3.3), (3.9) constructed in the last section.

4.1 Definition of the Drinfeld-Sokolov hierarchies

Let \mathfrak{g} be an untwisted affine Lie algebra, and $\{e_i, f_i, h_i \mid i = 0, 1, 2, \dots, n\}$ be a set of Weyl generators of \mathfrak{g} . In Drinfeld and Sokolov's construction, the central element c is not used, so we always assume $c = 0$. We need to use the following two gradations on \mathfrak{g} [7, 26]:

i) the principal/canonical gradation

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}^j, \quad \deg e_i = -\deg f_i = 1, \quad i = 0, 1, \dots, n;$$

ii) the homogeneous/standard gradation

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \deg e_i = -\deg f_i = \delta_{i0}, \quad i = 0, 1, \dots, n.$$

We will use notations such as $\mathfrak{g}^{<0} = \sum_{i < 0} \mathfrak{g}^i$ below.

In [7] Drinfeld and Sokolov assigned a standard gradation to any chosen vertex c_i of the Dynkin diagram of \mathfrak{g} and used the standard gradation to construct an integrable hierarchy. As mentioned in the beginning of the present paper, we only consider the case that the vertex is chosen to be c_0

which is the special one added to the Dynkin diagram of the corresponding simple Lie algebra. Integrable hierarchies that associated to different choices of the vertices are related by Miura type transformations.

Denote by E (resp. E_+) the set of exponents (resp. positive exponents) of \mathfrak{g} . Let \mathfrak{s} be the Heisenberg subalgebra associated to the principal gradation, which is defined to be the centralizer of $\Lambda = \sum_{i=0}^n e_i$. One can fix a basis $\lambda_j \in \mathfrak{g}^j$ ($j \in E$) of \mathfrak{s} .

Let $C^\infty(\mathbb{R}, W)$ be the set of smooth functions from \mathbb{R} to a linear space W . We consider operators of the form

$$\mathcal{L} = D + \Lambda + q, \quad q \in C^\infty(\mathbb{R}, \mathfrak{g}_0 \cap \mathfrak{g}^{\leq 0}), \quad (4.1)$$

where $D = \frac{d}{dx}$, and x is the coordinate on \mathbb{R} .

Proposition 4.1 ([7]) *There exists an element $U \in C^\infty(\mathbb{R}, \mathfrak{g}^{<0})$ such that the operator $\mathcal{L}_0 = e^{-\text{ad}_U} \mathcal{L}$ has the form*

$$\mathcal{L}_0 = D + \Lambda + H, \quad H \in C^\infty(\mathbb{R}, \mathfrak{s} \cap \mathfrak{g}^{<0}), \quad (4.2)$$

and for different choices of U , the map H differs by the addition of the total x -derivative of a differential polynomial of q .

We fix a U as given in the above proposition, and introduce a map

$$\varphi : C^\infty(\mathbb{R}, \mathfrak{g}) \rightarrow C^\infty(\mathbb{R}, \mathfrak{g}), \quad A \mapsto e^{\text{ad}_U} A. \quad (4.3)$$

The Drinfeld-Sokolov hierarchy is a hierarchy of partial differential equations of gauge equivalence classes of \mathcal{L} defined by

$$\frac{\partial \mathcal{L}}{\partial t_j} = [\varphi(\lambda_j)^+, \mathcal{L}], \quad j \in E_+. \quad (4.4)$$

Here $\varphi(\lambda_j)^+$ stands for the projection of $\varphi(\lambda_j)$ onto $C^\infty(\mathbb{R}, \mathfrak{g}^{\geq 0})$, and the gauge transformations of \mathcal{L} read

$$\mathcal{L} \mapsto e^{\text{ad}_N} \mathcal{L}, \quad N \in C^\infty(\mathbb{R}, \mathfrak{g}_0 \cap \mathfrak{g}^{<0}). \quad (4.5)$$

Theorem 4.2 ([7]) *The Drinfeld-Sokolov hierarchy carries a bihamiltonian structure, and the Hamiltonian densities are given by the expansion coefficients of the map H of (4.2) in the basis $\{\lambda_{-j} \mid j \in E_+\}$.*

For the classical untwisted affine Lie algebras, Drinfeld and Sokolov proposed a way to represent their hierarchies via certain scalar pseudo-differential operators over \mathcal{A} , the algebra of gauge invariant differential polynomials of q in (4.1). They gave such representations for the full hierarchies of the $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ types by using pseudo-differential operators of the first

type. However, for the $D_n^{(1)}$ case, as pointed out by Drinfeld and Sokolov, the pseudo-differential operators in \mathcal{D}^- are not enough to represent the full hierarchy. Our purpose of introducing the space \mathcal{D}^+ in the present paper is to represent the full Drinfeld-Sokolov hierarchy of D_n type in terms of scalar pseudo-differential operators.

The following lemma tells how to construct scalar pseudo-differential operators from the operator \mathcal{L} .

Lemma 4.3 ([7]) *Let \mathcal{R} be a ring with unity. We consider matrices of the form*

$$R = \begin{pmatrix} \alpha^t & a \\ R_1 & \beta \end{pmatrix} \in \mathcal{R}^{m \times m},$$

in which the block $R_1 \in \mathcal{R}^{(m-1) \times (m-1)}$ is invertible, α, β are $(m-1)$ -dimensional column vectors, and the superscript t means the transpose of matrices. Define $\Delta(R) = a - \alpha^t R_1^{-1} \beta$, then the following statements are true.

i) Suppose x_1, x_2, \dots, x_m, y belong to some \mathcal{R} -module such that

$$R \cdot (x_1, x_2, \dots, x_m)^t = (y, 0, \dots, 0)^t,$$

then $\Delta(R) \cdot x_m = y$.

ii) For any upper triangular matrix $\tilde{N} \in \mathcal{R}^{m \times m}$ with unity on the main diagonal one has $\Delta(\tilde{N}R\tilde{N}^{-1}) = \Delta(R)$.

iii) Given an anti-isomorphism $*$ of \mathcal{R} , one can define an anti-isomorphism T of $\mathcal{R}^{m \times m}$ by $(R^T)_{ij} = R_{m+1-j, m+1-i}^*$. It satisfies $\Delta(R^T) = \Delta(R)^*$.

4.2 Positive flows of the Drinfeld-Sokolov hierarchy of D_n type

In this subsection, we recall the approach given in [7] that represents part of the Drinfeld-Sokolov hierarchy of D_n type, the positive flows (3.3), by using pseudo-differential operators.

We first recall the matrix realization of the affine Lie algebra \mathfrak{g} of $D_n^{(1)}$ type [26, 7]. Denote by $e_{i,j}$ the $2n \times 2n$ matrix that takes value 1 at the (i, j) -entry and zero elsewhere, then one can realize \mathfrak{g} by choosing the Weyl generators as follows:

$$e_0 = \frac{\lambda}{2}(e_{1,2n-1} + e_{2,2n}), \quad e_n = \frac{1}{2}(e_{n+1,n-1} + e_{n+2,n}), \quad (4.6)$$

$$e_i = e_{i+1,i} + e_{2n+1-i,2n-i}, \quad 1 \leq i \leq n-1, \quad (4.7)$$

$$f_0 = \frac{2}{\lambda}(e_{2n-1,1} + e_{2n,2}), \quad f_n = 2(e_{n-1,n+1} + e_{n,n+2}), \quad (4.8)$$

$$f_i = e_{i,i+1} + e_{2n-i,2n+1-i}, \quad 1 \leq i \leq n-1, \quad (4.9)$$

$$h_i = [e_i, f_i], \quad 0 \leq i \leq n. \quad (4.10)$$

In particular, the associated simple Lie algebra \mathfrak{g}_0 of D_n type is realized as

$$\mathfrak{g}_0 = \left\{ A \in \mathbb{C}^{2n \times 2n} \mid A = -SA^T S^{-1} \right\}, \quad (4.11)$$

where S is the following matrix

$$S = \sum_{i=1}^n (-1)^{i-1} (e_{i,i} + e_{2n+1-i,2n+1-i}),$$

and $A^T = (a_{l+1-j, k+1-i})$ for any $k \times l$ matrix $A = (a_{ij})$. Note that in this realization the algebra \mathfrak{g} is just $\mathfrak{g}_0 \otimes \mathbb{C}[\lambda, \lambda^{-1}]$.

The set of exponents of \mathfrak{g} is given by

$$E = \{1, 3, 5, \dots, 2n-3\} \cup \{(n-1)'\} + (2n-2)\mathbb{Z},$$

where $(n-1)'$ indicates that when n is even the multiplicity of each exponent congruent to $n-1$ modulo $2n-2$ is 2. A basis of the principal Heisenberg subalgebra \mathfrak{s} can be chosen as

$$\{-\Lambda^k \in \mathfrak{g}^k, \Gamma^k \in \mathfrak{g}^{k(n-1)} \mid k \in 2\mathbb{Z} + 1\},$$

where $\Lambda = \sum_{i=0}^n e_i$, and

$$\begin{aligned} \Gamma = & \kappa \left(e_{n,1} - \frac{1}{2}e_{n+1,1} - \frac{\lambda}{2}e_{n,2n} + \frac{\lambda}{4}e_{n+1,2n} \right. \\ & \left. + (-1)^n (e_{2n,n+1} - \frac{1}{2}e_{2n,n} - \frac{\lambda}{2}e_{1,n+1} + \frac{\lambda}{4}e_{1,n}) \right) \end{aligned} \quad (4.12)$$

with $\kappa = 1$ when n is even and $\sqrt{-1}$ when n is odd. Here Λ^j and Γ^j are define to be the j -th power of Λ and Γ respectively for $j > 0$, while for $j < 0$

$$\Lambda^j = (\lambda^{-1}\Lambda^{2n-3})^{-j}, \quad \Gamma^j = (\lambda^{-1}\Gamma)^{-j}. \quad (4.13)$$

We now rewrite the Drinfeld-Sokolov hierarchy of D_n type (4.4) into the form

$$\frac{\partial \mathcal{L}}{\partial t_k} = [\varphi(-\Lambda^k)^+, \mathcal{L}], \quad \frac{\partial \mathcal{L}}{\partial \hat{t}_k} = [\varphi(\Gamma^k)^+, \mathcal{L}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (4.14)$$

We call the flows $\frac{\partial}{\partial t_k}$ and $\frac{\partial}{\partial \hat{t}_k}$ the positive and the negative flows of the Drinfeld-Sokolov hierarchy of D_n type respectively. We will show that these

flows coincide with the positive and negative flows (3.3) and (3.9) defined by the pseudo-differential operator L .

It is shown in [7] that in the orbit of gauge transformations of \mathcal{L} , one can find a canonical representative \mathcal{L}^{can} of the form

$$\mathcal{L}^{\text{can}} = D + \Lambda + q^{\text{can}},$$

where q^{can} reads

$$q^{\text{can}} = \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(q_j (e_{1,2j} + e_{2n+1-2j,2n}) + q_{n-j} (e_{1,2n+1-2j} + e_{2j,2n}) \right) + \hat{q} \quad (4.15)$$

with

$$\hat{q} = \begin{cases} \frac{1}{2}(q_{n/2} + \rho)(e_{1,n} + e_{n+1,2n}) + (q_{n/2} - \rho)(e_{1,n+1} + e_{n,2n}), & n \text{ even,} \\ -\sqrt{-1}\rho(\frac{1}{2}e_{1,n} - e_{1,n+1} + e_{n,2n} - \frac{1}{2}e_{n+1,2n}), & n \text{ odd.} \end{cases}$$

The coefficients q_1, \dots, q_{n-1} and ρ are gauge invariant differential polynomials of q that appears in (4.1). They serve as coordinates of the orbit space of gauge transformations, and we will use them as unknown functions of the Drinfeld-Sokolov hierarchy.

Let \mathcal{A} be the algebra of differential polynomials of q_1, \dots, q_{n-1} and ρ , denote $\mathcal{A}^- = \mathcal{A}((\lambda^{-1}))$, we introduce a free \mathcal{A}^- -module

$$V = (\mathcal{A}^-)^{2n} = \left\{ \sum_{i < \infty} \alpha_i \lambda^i \mid \alpha_i \in \mathcal{A}^{2n} \right\}.$$

Let us fix a basis $\{\hat{\psi}_{2n}, \psi_{2n-1}, \dots, \psi_1\}$ of V , where $\hat{\psi}_{2n} = \frac{\lambda}{2}\psi_1 + \psi_{2n}$, and ψ_i is the column vector whose i -th entry is 1 and others are zero.

In the notions of Lemma 4.3, we let $\mathcal{R} = \mathcal{D}^-$ and denote by \mathcal{R}_+ the sub-algebra of \mathcal{R} consisting of differential operators. We define an \mathcal{R}_+ -module structure on V by

$$D \cdot \alpha = \mathcal{L}^{\text{can}} \alpha, \quad \alpha \in V. \quad (4.16)$$

Note that $\mathcal{L}^{\text{can}}|_{\lambda=0} \in \mathcal{R}_+^{2n \times 2n}$, let

$$R = (\mathcal{L}^{\text{can}}|_{\lambda=0})^T = -\text{diag}(D, D, \dots, D) + \Lambda|_{\lambda=0} + (q^{\text{can}})^T,$$

then it is straightforward to verify that

$$R \cdot (\hat{\psi}_{2n}, \psi_{2n-1}, \dots, \psi_1)^t = (-\lambda \psi_2, 0, \dots, 0)^t = (-\lambda D \cdot \psi_1, 0, \dots, 0)^t. \quad (4.17)$$

Denote $\mathcal{L} = -\Delta(R)$, where Δ is the operation defined in Lemma 4.3, then $\mathcal{L}^* = -\mathcal{L}$ by using (4.11) and the third part of Lemma 4.3. It is

easy to see that \mathcal{L} has the form (3.13). This observation gives a Miura-type transformation between u^1, \dots, u^n and $q_1, \dots, q_{n-1}, \rho$, so the algebra \mathcal{A} defined above coincides with the one that is given in the last section. Moreover, the second part of Lemma 4.3 implies that \mathcal{L} is invariant w.r.t. the gauge transformations (4.5), thus the Drinfeld-Sokolov hierarchy can be represented by the operator \mathcal{L} , or equivalently by $L = D^{-1}\mathcal{L}$.

Note that the operator $\mathcal{L} \notin \mathcal{R}_+$, since V is only an \mathcal{R}_+ -module \mathcal{L} cannot act on V , and the first part of Lemma 4.3 cannot be applied directly. To resolve this problem, Drinfeld and Sokolov decomposed V into two subspaces such that D^- can act on one of them, then the first part of Lemma 4.3 can be applied. In this way, the positive flows of the Drinfeld-Sokolov hierarchy (4.14) are represented in the form (3.3) as the positive flows given by the pseudo-differential operator L of the form (1.3).

In the matrix realization of \mathfrak{g} , the elements Λ and Γ are $2n \times 2n$ matrices with entries in $\mathbb{C}[\lambda]$, so they can act on the space V . One can verify that the following decomposition holds true

$$V = V_1 \oplus V_2, \quad V_1 = \text{Im } \Lambda = \text{Ker } \Gamma, \quad V_2 = \text{Ker } \Lambda = \text{Im } \Gamma.$$

Denote $T = e^U$, where U is the matrix appeared in Proposition 4.1 with $\mathcal{L} = \mathcal{L}^{\text{can}}$, then we also have

$$V = V'_1 \oplus V'_2, \quad V'_1 = TV_1, \quad V'_2 = TV_2. \quad (4.18)$$

Since the operator $\lambda^{-1}\Lambda^{2n-2}$ is the identity operator when restricted to V_1 , let $\mathcal{P} = \varphi(\lambda^{-1}\Lambda^{2n-2})$ with φ being defined in (4.3), then \mathcal{P} is the projection from V to V'_1 . We denote the projection of $\alpha \in V$ in V'_1 by $\alpha' = \mathcal{P}\alpha$, and define the action

$$\begin{aligned} D^{-1} \cdot \alpha' &= (\mathcal{L}^{\text{can}})^{-1} \alpha' = T(\Lambda - (\Lambda - \mathcal{L}_0))^{-1} T^{-1} \alpha' \\ &= T(\Lambda^{-1} + \Lambda^{-1}(\Lambda - \mathcal{L}_0)\Lambda^{-1} + (\Lambda^{-1}(\Lambda - \mathcal{L}_0))^2 \Lambda^{-1} + \dots) T^{-1} \alpha'. \end{aligned}$$

Here the operator \mathcal{L}_0 defined in (4.2) now reads

$$\mathcal{L}_0 = e^{-U} \mathcal{L}^{\text{can}} e^U = D + \Lambda + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} f_k \Lambda^{-k} + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} g_k \Gamma^{-k} \quad (4.19)$$

with $f_k, g_k \in \mathcal{A}$ and the negative powers Λ, Γ defined in (4.13). Note that $\text{Im } \Lambda^{-1} \subset \text{Im } \Lambda$, $\text{Im } \Gamma^{-1} \subset \text{ker } \Lambda$, then $D^{-1} \cdot \alpha' \in V'_1$, so V'_1 becomes an \mathcal{R} -module.

It follows from $[\mathcal{L}_0, \Lambda] = 0$ that $[\mathcal{P}, \mathcal{L}^{\text{can}}] = 0$, then by acting \mathcal{P} on both sides of (4.17) one has

$$R \cdot (\hat{\psi}'_{2n}, \psi'_{2n-1}, \dots, \psi'_1)^t = (-\lambda D \cdot \psi'_1, 0, \dots, 0)^t.$$

Now the first part of Lemma 4.3 can be employed to prove the following lemma.

Lemma 4.4 ([7]) *Let $\mathcal{L} = -\Delta(R)$, $L = D^{-1}\mathcal{L}$, then L takes the form (1.3). Define $P = L^{\frac{1}{2n-2}} \in \mathcal{D}^-$ as in Lemma 3.1, then for any $i \in \mathbb{Z}$ the following equalities hold true*

$$\varphi(\Lambda^i)\psi'_1 = P^i \cdot \psi'_1, \quad (4.20)$$

$$(\varphi(\Lambda^{2i+1})^+\psi_1)' = (P^{2i+1})_+ \cdot \psi'_1. \quad (4.21)$$

By using the second equality, one can represent the positive flows $\frac{\partial}{\partial t_k}$ of the Drinfeld-Sokolov hierarchy (4.14) in the form (3.3). We are to explain in the next subsection that the negative flows of (4.14) can be represented as (3.9).

The first equality of the above lemma gives the following result.

Proposition 4.5 ([7]) *Let f_k be the coefficients that appear in (4.19), then $f_k + \frac{1}{k}\text{res } P^k \in D(\mathcal{A})$ for all $k \in \mathbb{Z}_+^{\text{odd}}$.*

From Theorem 4.2 and (3.20) we know that this proposition relates the densities of the Hamiltonians of the positive flows of the Drinfeld-Sokolov hierarchy with that of the positive flow (3.3) defined in the last section.

4.3 Negative flows of the Drinfeld-Sokolov hierarchy of D_n type

In the last subsection, the pseudo-differential operator representation for the positive flows of the Drinfeld-Sokolov hierarchy of D_n type is obtained by introducing a \mathcal{D}^- -module structure on the space V'_1 and using Lemma 4.3 as was done in [7]. In order to obtain a similar representation for the negative flows, we try to assign a \mathcal{D}^+ -module structure to V'_2 . However, it seems that there is no such a structure on V'_2 , so we first extend the space V'_2 to a larger one V''_2 which admits a \mathcal{D}^+ -module structure, then we employ Lemma 4.3 and obtain the pseudo-differential operator representation for the negative flows of the Drinfeld-Sokolov hierarchy of D_n type.

Recall that V_2 as an $\mathcal{A}((\lambda^{-1}))$ -module with is spanned by the following two vectors:

$$\hat{\psi}_1 = \frac{1}{2}\psi_1 - \frac{1}{\lambda}\psi_{2n}, \quad \hat{\psi}_2 = \Gamma\hat{\psi}_1 = \kappa \left(\psi_n - \frac{1}{2}\psi_{n+1} \right). \quad (4.22)$$

The action of Γ restricted to V_2 satisfies $\Gamma^2 = \lambda$, so we introduce $\Gamma^{-1} = \lambda^{-1}\Gamma$, see (4.13). It is easy to see that every vector $\alpha \in V_2$ can be uniquely expressed in the form

$$\alpha = \sum_{i \leq m} a_i \Gamma^i \hat{\psi}_1, \quad a_i \in \mathcal{A}, \quad m \in \mathbb{Z}. \quad (4.23)$$

This observation shows that the space V_2 is in fact a rank-one free module of the following algebra

$$\mathcal{D}^-(\mathcal{A}, \Gamma) = \left\{ \sum_{i < \infty} a_i \Gamma^i \mid a_i \in \mathcal{A} \right\}.$$

This is the algebra of “pseudo-differential operators of the first type” (see Sec. 2.1) over the algebra \mathcal{A} with the derivation “ D ” being the following trivial map

$$\Gamma : \mathcal{A} \rightarrow \mathcal{A}, \quad f \mapsto 0,$$

which surely gives a derivation of degree one over \mathcal{A} .

By regarding another trivial map

$$\Gamma^{-1} : \mathcal{A} \rightarrow \mathcal{A}, \quad f \mapsto 0,$$

as a derivation of degree one, one can also define the algebra of “pseudo-differential operators of the second type” with respect to the algebra \mathcal{A} and the derivation Γ^{-1} as

$$\mathcal{D}^+(\mathcal{A}, \Gamma^{-1}) = \left\{ \sum_{j \geq 0} \sum_{i \leq m+j} a_{i,j} \Gamma^i \mid a_{i,j} \in \mathcal{A}_j, \quad m \in \mathbb{Z} \right\}.$$

We denote by \hat{V}_2 the rank-one free module of the algebra $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$ with generator $\hat{\psi}_1$, which has a linear topology induced from that of $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$. It is easy to see that the algebra $\mathcal{D}^-(\mathcal{A}, \Gamma)$ is a subalgebra of $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$ (see Lemma 2.2), hence V_2 is a subspace of \hat{V}_2 .

To define the space V_2'' , we need to extend the space V to certain space \hat{V} that involves \hat{V}_2 as a subspace. Since the space V is defined to be $(\mathcal{A}^-)^{2n}$, in which the algebra $\mathcal{A}^- = \mathcal{A}((\lambda^{-1}))$ can also be defined as $\mathcal{D}^-(\mathcal{A}, \lambda)$ with λ being the trivial derivation, we similarly extend the space V to

$$\hat{V} = \hat{\mathcal{A}}^{2n}, \quad \hat{\mathcal{A}} = \mathcal{D}^+(\mathcal{A}, \lambda^{-1}).$$

The space \hat{V} has a linear topology induced from that of $\hat{\mathcal{A}}$. It is easy to see that the linear transformations $\Lambda, \Gamma, T = e^U : V \rightarrow V$ can be extended naturally to \hat{V} . Then the expression

$$\alpha = \sum_{j \geq 0} \sum_{i \leq m+j} a_{i,j} \Gamma^i \hat{\psi}_1 \in \hat{V}_2 \tag{4.24}$$

is also convergent in \hat{V} according to its topology, hence the space \hat{V}_2 is indeed a subspace of \hat{V} .

Now let us introduce another subspace of \hat{V} :

$$V_2'' = T \hat{V}_2 \subset \hat{V},$$

then V_2' is a subspace of V_2'' . As in the previous subsection we define a map

$$\mathcal{Q} : V \rightarrow V_2', \quad \mathcal{Q} = \varphi(\lambda^{-1}\Gamma^2)$$

with φ defined in (4.3). Then we have the following commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\lambda^{-1}\Gamma^2} & V_2 & \hookrightarrow & \hat{V}_2 \\ \cong \downarrow T & & \cong \downarrow T & & \cong \downarrow T \\ V & \xrightarrow{\mathcal{Q}} & V_2' & \hookrightarrow & V_2'' \end{array}$$

We also denote the composition of \mathcal{Q} and the inclusion $V_2' \hookrightarrow V_2''$ by \mathcal{Q} , and write $\alpha'' = \mathcal{Q}\alpha$ for any vector $\alpha \in V$.

Lemma 4.6 *The space \hat{V}_2 is a free $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$ -module with generator $T^{-1}\psi_1''$.*

Proof To see that $T^{-1}\psi_1''$ is another generator besides $\hat{\psi}_1$, we only need to show that these two vectors are related by the action of a unit of the algebra $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$.

Recall that $T = e^U$, according to the present matrix realization the element U given in Proposition 4.1 has the form $U_0 + O(\lambda^{-1})$ with U_0 being a strictly upper triangular matrix, and that the vector $\hat{\psi}_1$ defined in (4.22) can be represented as

$$\hat{\psi}_1 = \lambda^{-1}\Gamma^2\psi_1,$$

so we have

$$\psi_1'' = \mathcal{Q}\psi_1 = T\lambda^{-1}\Gamma^2T^{-1}\psi_1 = T(\hat{\psi}_1 + O(\lambda^{-1})) \in V_2'.$$

By using the general form (4.23) of elements of V_2 and the identity $\Gamma^{2j+1}|_{V_2} = \lambda^j\Gamma$, one can represent $T^{-1}\psi_1'' \in V_2$ in the following form:

$$T^{-1}\psi_1'' = \left(1 + \sum_{i<0} b_i\Gamma^i\right) \hat{\psi}_1, \quad b_i \in \mathcal{A}. \quad (4.25)$$

Obviously the element $1 + \sum_{i<0} b_i\Gamma^i \in \mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$ is invertible. The lemma is proved. \square

Aiming at a \mathcal{D}^+ -module structure on the space V_2'' such that the action of D coincides with (4.16) when restricted to the subspace V_2' , we need to define the action of $(\mathcal{L}^{\text{can}})^i$ ($i \in \mathbb{Z}$) on the space V_2'' . Note that the operator $\mathcal{L}_0 : V \rightarrow V$ given in (4.19) can be extended to \hat{V} , we denote its restriction on the space \hat{V}_2 by $\hat{\mathcal{L}}_0$, which reads

$$\hat{\mathcal{L}}_0 = \mathcal{L}_0|_{\hat{V}_2} = D + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} g_k \Gamma^{-k}.$$

Here $g_1 \in \mathcal{A}$ is invertible as indicated in [7], so the operator $\hat{\mathcal{L}}_0$ is invertible on \hat{V}_2 , and its inverse is given by

$$\begin{aligned}\hat{\mathcal{L}}_0^{-1} &= (g_1 \Gamma^{-1} (1 + g_1^{-1} \Gamma D + M))^{-1} \\ &= (1 - (g_1^{-1} \Gamma D + M) + (g_1^{-1} \Gamma D + M)^2 - \dots) g_1^{-1} \Gamma,\end{aligned}$$

where $M = g_1^{-1} \sum_{j \geq 1} g_{2j+1} \Gamma^{-2j}$. One can expand the right hand side and obtain

$$\hat{\mathcal{L}}_0^{-1} = \sum_{s \geq 0} \sum_{r \leq s} A_{rs} \Gamma^{r+1}, \quad A_{rs} = \sum_{j=0}^s c_{rsj} D^j, \quad c_{rsj} \in \mathcal{A}_{s-j}, \quad (4.26)$$

in which $A_{00} = c_{000} = g_{10}^{-1}$ with g_{10} being the projection of g_1 onto \mathcal{A}_0 . Note that g_{10}/ρ is a positive constant, where ρ appears in the definition (4.15) of \mathcal{L}^{can} , and we have normalized Γ such that this constant is 1. Since A_{rs} are differential operators of degree s , i.e., $A_{rs}(\mathcal{A}_d) \subset \mathcal{A}_{d+s}$, then by using the expressions (4.26) and (4.24) one can verify that the action of $\hat{\mathcal{L}}_0^{-1}$ on \hat{V}_2 is well defined. Also note that the image $\hat{\mathcal{L}}_0^{-1}(V_2)$ is not contained in V_2 though $\hat{\mathcal{L}}_0(V_2) \subset V_2$, this is why we extend V_2 to \hat{V}_2 .

To go forward, we need to present another expression for vectors in \hat{V}_2 .

Lemma 4.7 *Every vector $\alpha \in \hat{V}_2$ can be uniquely expressed in the form*

$$\alpha = \sum_{j \geq 0} \sum_{i \leq m+j} b_{i,j} \hat{\mathcal{L}}_0^{-i} T^{-1} \psi_1'', \quad b_{i,j} \in \mathcal{A}_j, \quad m \in \mathbb{Z}. \quad (4.27)$$

Proof According to Lemma 4.6, we suppose $\alpha \in \hat{V}_2$ has the form

$$\alpha = \sum_{j \geq k} \sum_{i \leq m+j} a_{i,j} \Gamma^i T^{-1} \psi_1'' + \dots, \quad a_{i,j} \in \mathcal{A}_j,$$

where \dots stands for the terms of the form (4.27). Let us proceed to prove the lemma by induction on the lower bound k of the index j .

First, we have

$$\begin{aligned}\alpha &= \sum_{i \leq m+k} a_{i,k} \Gamma^i T^{-1} \psi_1'' + \sum_{j \geq k+1} \sum_{i \leq m+j} a_{i,j} \Gamma^i T^{-1} \psi_1'' + \dots \\ &= a_{m+k,k} \Gamma^{m+k} T^{-1} \psi_1'' + \sum_{i \leq m-1+k} a_{i,k} \Gamma^i T^{-1} \psi_1'' \\ &\quad + \sum_{j \geq k+1} \sum_{i \leq m+j} a_{i,j} \Gamma^i T^{-1} \psi_1'' + \dots.\end{aligned} \quad (4.28)$$

From the expansion (4.26) it follows that

$$\hat{\mathcal{L}}_0^{-l} = \sum_{s \geq 0} \sum_{r \leq s} A_{rs}^{(l)} \Gamma^{r+l}, \quad A_{rs}^{(l)} \in (\mathcal{D}^-)_+, \quad \deg A_{rs}^{(l)} = s, \quad (4.29)$$

where $A_{00}^{(l)} = g_{10}^{-l}$, hence by using (4.25) we have

$$\begin{aligned}
& \hat{\mathcal{L}}_0^{-l} T^{-1} \psi_1'' - g_{10}^{-l} \Gamma^l T^{-1} \psi_1'' \\
&= \left(\sum_{r \leq -1} A_{r0}^{(l)} \Gamma^{r+l} + \sum_{s \geq 1} \sum_{r \leq s} A_{rs}^{(l)} \Gamma^{r+l} \right) T^{-1} \psi_1'' \\
&= \left(\sum_{r \leq -1} A_{r0}^{(l)} \Gamma^{r+l} + \sum_{s \geq 1} \sum_{r \leq s} A_{rs}^{(l)} \Gamma^{r+l} \right) \left(1 + \sum_{i < 0} b_i \Gamma^i \right) \hat{\psi}_1 \\
&= \left(\sum_{r \leq -1} c_{r,0} \Gamma^{r+l} + \sum_{s \geq 1} \sum_{r \leq s} c_{r,s} \Gamma^{r+l} \right) \hat{\psi}_1 \\
&= \left(\sum_{r \leq -1} \tilde{c}_{r,0} \Gamma^{r+l} + \sum_{s \geq 1} \sum_{r \leq s} \tilde{c}_{r,s} \Gamma^{r+l} \right) T^{-1} \psi_1'', \tag{4.30}
\end{aligned}$$

where $c_{r,s}, \tilde{c}_{r,s} \in \mathcal{A}_s$. The above computation represents the action of the operator $\hat{\mathcal{L}}_0^{-l}$ (see (4.29)) on certain vector in \hat{V}_2 by an element in $\mathcal{D}^+(\mathcal{A}, \Gamma^{-1})$. By using equation (4.30), we can eliminate the term $a_{m+k,k} \Gamma^{m+k} T^{-1} \psi_1''$ in (4.28) and arrive at

$$\alpha = \sum_{i \leq m-1+k} \tilde{a}_{i,k} \Gamma^i T^{-1} \psi_1'' + \sum_{j \geq k+1} \sum_{i \leq m+j} \tilde{a}_{i,j} \Gamma^i T^{-1} \psi_1'' + \dots, \quad \tilde{a}_{i,j} \in \mathcal{A}_j.$$

Then by induction on the upper bound of the index i appearing in the first summation we have

$$\alpha = \sum_{j \geq k+1} \sum_{i \leq m+j} \tilde{a}_{i,j} \Gamma^i T^{-1} \psi_1'' + \dots,$$

which shows that the lower bound of the index j has increased by one. The lemma is proved. \square

Now we are ready to introduce a \mathcal{D}^+ -module structure on the space V_2'' by defining the action

$$D^i \cdot \alpha'' = \varphi(\hat{\mathcal{L}}_0^i) \alpha'', \quad \alpha'' \in V_2'', \quad i \in \mathbb{Z}, \tag{4.31}$$

which extends the action (4.16) on V_2' to an action on V_2'' . Then Lemma 4.7 is equivalent to the following theorem.

Theorem 4.8 *The \mathcal{D}^+ -module V_2'' is a free module with generator ψ_1'' .*

Let us apply Lemma 4.3 to the algebra $\mathcal{R} = \mathcal{D}^+$ and the module V_2'' . By acting the projection operator \mathcal{Q} to both sides of (4.17), we have

$$R \cdot (\hat{\psi}_{2n}'', \psi_{2n-1}'', \dots, \psi_1'')^t = (-\lambda D \cdot \psi_1'', 0, \dots, 0)^t,$$

hence $L \cdot \psi_1'' = \lambda \psi_1''$, where $L = -D^{-1}\Delta(R)$ as given before. According to Lemma 3.3 we introduce a pseudo-differential operator $Q \in \mathcal{D}^+$ such that $L = Q^2$, and consider the action of Q^i on V_2'' for any integer i .

Lemma 4.9 *For any integer i the following equality holds true:*

$$\varphi(\Gamma^i)\psi_1'' = Q^i \cdot \psi_1''. \quad (4.32)$$

Proof We only need to prove the case $i = 1$. Since V_2'' is a free \mathcal{D}^+ -module, there exists an element $A \in \mathcal{D}^+$ such that $\varphi(\Gamma)\psi_1'' = A \cdot \psi_1''$. Note that $[\varphi(\Gamma), \mathcal{L}^{\text{can}}] = 0$, so the action of $\varphi(\Gamma)$ on V_2'' commutes with $D \in \mathcal{D}^+$, hence

$$A^2 \cdot \psi_1'' = \varphi(\Gamma^2)\psi_1'' = \lambda \psi_1'' = L \cdot \psi_1''.$$

By using the freeness of V_2'' , we have $A^2 = L = Q^2$. It follows that $A = \pm Q$.

To show $A = Q$, we only need to compare their leading terms. Equation (4.30) leads to

$$\varphi(\Gamma)\psi_1'' = \varphi(g_{10}\hat{\mathcal{L}}_0^{-1} + \cdots)\psi_1'' = (g_{10}D^{-1} + \cdots) \cdot \psi_1'',$$

which implies that the leading term of $\text{res } A$ is g_{10} . On the other hand g_{10} takes the same sign with $\rho = \text{res } Q$, thus $A = Q$. The lemma is proved. \square

By using Lemmas 2.4 and 4.9, one can prove the following proposition. The argument is almost the same with the one for Proposition 4.5 given in [7], so we omit the details here.

Proposition 4.10 *Let g_k be the coefficients that appear in (4.19), then $g_k - \frac{1}{k}\text{res } Q^k \in D(\mathcal{A})$ for all $k \in \mathbb{Z}_+^{\text{odd}}$.*

This proposition connects the Hamiltonians of the negative flows of the Drinfeld-Sokolov hierarchy of D_n type to those (3.20) corresponding to the negative flows (3.9).

Now we arrive at the main result of the present section.

Theorem 4.11 *The flows (4.14) of the Drinfeld-Sokolov hierarchy of D_n type coincide with the flows of the integrable hierarchy (3.3), (3.9).*

Proof It is shown in [7] that the Drinfeld-Sokolov hierarchy of D_n type has a bihamiltonian structure given by the two Poisson brackets (3.16), (3.17). For the flow (4.4) corresponding to the element λ_j , the Hamiltonian with respect to the second Poisson bracket is given by

$$\mathcal{H}_j = \int (H | \lambda_j) dx, \quad j \in E_+,$$

where H is given in (4.2) and $(\cdot | \cdot)$ is the trace form defined by

$$(G | H) = \text{res}_\lambda (\lambda^{-1} \text{tr}(GH)).$$

We choose a basis (1.4) of the Heisenberg subalgebra \mathfrak{s} as

$$\lambda_k = -\Lambda^k, \quad \lambda_{k(n-1)'} = \Gamma^k, \quad k \in 2\mathbb{Z} + 1.$$

Note that

$$(\Lambda^k | \Lambda^l) = (2n - 2)\delta_{k,-l}, \quad (\Lambda^k | \Gamma^l) = 0, \quad (\Gamma^k | \Gamma^l) = 2\delta_{k,-l},$$

where k, l run over all odd integers, hence by using (4.19) we have

$$\mathcal{H}_k = -(2n - 2) \int f_k dx, \quad \mathcal{H}_{k(n-1)'} = 2 \int g_k dx, \quad k \in \mathbb{Z}_+^{\text{odd}}.$$

They are the Hamiltonians for the positive and negative flows of the Drinfeld-Sokolov hierarchy (4.14) w.r.t. the second Poisson bracket (3.17).

According to Propositions 4.5, 4.10 and Theorem 3.5, these Hamiltonians satisfy

$$\mathcal{H}_k = H_k, \quad \mathcal{H}_{k(n-1)'} = \hat{H}_k, \quad k \in \mathbb{Z}_+^{\text{odd}},$$

where H_k, \hat{H}_k are the Hamiltonians of the integrable hierarchy (3.3), (3.9) with respect to the second Poisson bracket (3.17). So the Drinfeld-Sokolov hierarchy of D_n type (4.14) and the integrable hierarchy (3.3),(3.9) coincide. The theorem is proved. \square

5 The two-component BKP hierarchy and its reductions

In this section we represent the two-component BKP hierarchy that is introduced in [3] via pseudo-differential operators, and show that the hierarchy (3.3), (3.9) is just a reduction, which was considered in [4], of the two-component BKP hierarchy.

5.1 The two-component BKP hierarchy

Let \tilde{M} be an infinite-dimensional manifold with local coordinates

$$(a_1, a_3, a_5, \dots, b_1, b_3, b_5, \dots),$$

and $\tilde{\mathcal{A}}$ be the algebra of differential polynomials on \tilde{M} :

$$\tilde{\mathcal{A}} = C^\infty(\tilde{M})[[a_i^s, b_i^s \mid i \in \mathbb{Z}_+^{\text{odd}}, s \geq 1]].$$

As in Section 3, we assign a gradation on $\tilde{\mathcal{A}}$ such that $\tilde{\mathcal{A}}$ is topologically complete. Define a derivation D by

$$D = \sum_{s \geq 0} \sum_{i \in \mathbb{Z}_+^{\text{odd}}} \left(a_i^{s+1} \frac{\partial}{\partial a_i^s} + b_i^{s+1} \frac{\partial}{\partial b_i^s} \right),$$

then the algebras $\tilde{\mathcal{D}}^\pm = \mathcal{D}^\pm(\tilde{\mathcal{A}}, D)$ of pseudo-differential operators can be constructed as we did in Section 2.1.

Introduce two pseudo-differential operators

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i} \in \tilde{\mathcal{D}}^-, \quad (5.1)$$

$$\Psi = 1 + \sum_{i \geq 1} b_i D^i \in \tilde{\mathcal{D}}^+, \quad (5.2)$$

where $a_2, a_4, a_6, \dots, b_2, b_4, b_6, \dots \in \tilde{\mathcal{A}}$ are determined by the following conditions

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \Psi^* = D\Psi^{-1}D^{-1}. \quad (5.3)$$

Now let us define a pair of operators

$$P = \Phi D \Phi^{-1} \in \tilde{\mathcal{D}}^-, \quad Q = \Psi D^{-1} \Psi^{-1} \in \tilde{\mathcal{D}}^+.$$

Lemma 5.1 *The operators P, Q have the following expressions (c.f. (3.1), (3.5)):*

$$P = D + \sum_{i \geq 1} u_i D^{-i}, \quad Q = D^{-1} \rho + \sum_{i \geq 1} v_i D^i,$$

where $\rho = (\Psi^{-1})^*(1)$. They satisfy

$$P^* = -DPD^{-1}, \quad Q^* = -DQD^{-1}, \quad (5.4)$$

and for any $k \in \mathbb{Z}_+^{\text{odd}}$ we have

$$(P^k)_+(1) = 0, \quad (Q^k)_+(1) = 0. \quad (5.5)$$

Proof The expression of P is obvious. To show that of Q , we consider its negative part:

$$\begin{aligned} Q_- &= (\Psi D^{-1} \Psi^{-1})_- = (D^{-1} \Psi^{-1})_- = \left((D^{-1} \Psi^{-1})^* \right)_-^* \\ &= - \left((\Psi^{-1})^* D^{-1} \right)_-^* = - \left((\Psi^{-1})^*(1) D^{-1} \right)^* = D^{-1} \rho. \end{aligned}$$

The symmetry property (5.4) is obvious, it implies (5.5). The lemma is proved. \square

We define the following evolutionary equations:

$$\frac{\partial \Phi}{\partial t_k} = -(P^k)_- \Phi, \quad \frac{\partial \Psi}{\partial t_k} = ((P^k)_+ - \delta_{k1} Q^{-1}) \Psi, \quad (5.6)$$

$$\frac{\partial \Phi}{\partial \hat{t}_k} = -(Q^k)_- \Phi, \quad \frac{\partial \Psi}{\partial \hat{t}_k} = (Q^k)_+ \Psi, \quad (5.7)$$

where $k \in \mathbb{Z}_+^{\text{odd}}$. According to (5.3) and (5.5), it is easy to see that these flows are well defined, and they yield the Lax equations of the form (3.11), (3.12). By a straightforward calculation one can verify the commutativity of these flows, hence they form an integrable hierarchy indeed. We will show that this hierarchy possesses tau functions, and that these tau functions satisfy the same bilinear equations of the two-component BKP hierarchy defined in [3].

First, let us introduce two wave functions

$$w = w(\mathbf{t}, \hat{\mathbf{t}}; z) = \Phi e^{\xi(\mathbf{t}; z)}, \quad \hat{w} = \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) = \Psi e^{xz + \xi(\hat{\mathbf{t}}; -z^{-1})}, \quad (5.8)$$

where $x = t_1$, the function ξ is defined by

$$\xi(\mathbf{t}; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k, \quad (5.9)$$

and for any $i \in \mathbb{Z}$ the action of D^i on e^{xz} is set to be $D^i e^{xz} = z^i e^{xz}$.

It is easy to see that $Pw = zw$, $Q\hat{w} = z^{-1}\hat{w}$, and that the flows (5.6), (5.7) are equivalent to the following equations

$$\frac{\partial w}{\partial t_k} = (P^k)_+ w, \quad \frac{\partial \hat{w}}{\partial t_k} = (P^k)_+ \hat{w}, \quad (5.10)$$

$$\frac{\partial w}{\partial \hat{t}_k} = -(Q^k)_- w, \quad \frac{\partial \hat{w}}{\partial \hat{t}_k} = -(Q^k)_- \hat{w}. \quad (5.11)$$

Here $(Q^k)_- w$ is understood as $((Q^k)_- \Phi) e^{\xi(\mathbf{t}; z)}$, and $(Q^k)_- \hat{w}$ is defined similarly. The following theorem can be proved as it was done for the KP hierarchy given in [5, 6].

Theorem 5.2 *The hierarchy (5.6), (5.7) is equivalent to the following bilinear equation*

$$\text{res}_z z^{-1} w(\mathbf{t}, \hat{\mathbf{t}}; z) w(\mathbf{t}', \hat{\mathbf{t}}'; -z) = \text{res}_z z^{-1} \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) \hat{w}(\mathbf{t}', \hat{\mathbf{t}}'; -z). \quad (5.12)$$

Here and below the residue of a Laurent series is defined as $\text{res}_z \sum_i f_i z^i = f_{-1}$.

Let ω be the following 1-form

$$\omega = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \left(\text{res } P^k dt_k + \text{res } Q^k d\hat{t}_k \right). \quad (5.13)$$

By using the equations (5.6) and (5.7), one can show that ω is closed, so given any solution of the hierarchy (5.6), (5.7) there exists a function $\tau(\mathbf{t}, \hat{\mathbf{t}})$ such that

$$\omega = d(2 \partial_x \log \tau). \quad (5.14)$$

Moreover, one can fix a tau function such that the wave functions can be written as

$$w(\mathbf{t}, \hat{\mathbf{t}}; z) = \frac{\tau(\dots, t_k - \frac{2}{kz^k}, \dots, \hat{\mathbf{t}})}{\tau(\mathbf{t}, \hat{\mathbf{t}})} e^{\xi(\mathbf{t}; z)}, \quad (5.15)$$

$$\hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) = \frac{\tau(\mathbf{t}, \dots, \hat{t}_k + \frac{2z^k}{k}, \dots)}{\tau(\mathbf{t}, \hat{\mathbf{t}})} e^{\xi(\hat{\mathbf{t}}; -z^{-1})}. \quad (5.16)$$

Introduce a vertex operator X as

$$X(\mathbf{t}; z) = \exp\left(\sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k\right) \exp\left(-\sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{2}{kz^k} \frac{\partial}{\partial t_k}\right),$$

then the bilinear equation (5.12) reads

$$\begin{aligned} & \text{res}_z z^{-1} X(\mathbf{t}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\mathbf{t}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}') \\ &= \text{res}_z z^{-1} X(\hat{\mathbf{t}}; -z^{-1}) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\hat{\mathbf{t}}'; z^{-1}) \tau(\mathbf{t}', \hat{\mathbf{t}}'), \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \text{res}_z z^{-1} X(\mathbf{t}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\mathbf{t}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}') \\ &= \text{res}_z z^{-1} X(\hat{\mathbf{t}}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\hat{\mathbf{t}}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}'). \end{aligned} \quad (5.17)$$

In [3, 25], Date, Jimbo, Kashiwara and Miwa defined the two-component BKP hierarchy from a two-component neutral free fermions realization of the basic representation of an infinite-dimensional Lie algebra \mathfrak{g}_∞ , which corresponds to the Dynkin diagram of D_∞ type [26]. The tau function of their hierarchy satisfies the bilinear equations (5.17) and defines two wave functions as (5.15), (5.16), so the equations (5.6), (5.7) give a representation of the two-component BKP hierarchy in terms of pseudo-differential operators.

Remark 5.3 In [31], Shiota gave a Lax pair representation of the two-component BKP hierarchy as follows. Let $\phi^{(\nu)}$ ($\nu = 0, 1$) be the following pseudo-differential operators of the first type

$$\phi^{(\nu)} = 1 + \sum_{i \geq 1} a_i^{(\nu)} D_\nu^{-i}$$

satisfying $(\phi^{(\nu)})^* = D_\nu (\phi^{(\nu)})^{-1} D_\nu^{-1}$, where D_0, D_1 are two commuting derivations. Let

$$P^{(\nu)} = \phi^{(\nu)} D_\nu (\phi^{(\nu)})^{-1},$$

then the two-component BKP hierarchy can be defined as

$$\frac{\partial \phi^{(\nu)}}{\partial t_k^{(\nu)}} = - \left(P^{(\nu)} \right)_-^k \phi^{(\nu)}, \quad \frac{\partial \phi^{(\nu)}}{\partial t_k^{(1-\nu)}} = \left(P^{(1-\nu)} \right)_+^k \left(\phi^{(\nu)} \right), \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (5.18)$$

Here the right hand side of the second equation means the action of the differential operator $\left(P^{(1-\nu)} \right)_+^k$ on the coefficients of $\phi^{(\nu)}$. It is easy to see that $D_\nu = \frac{\partial}{\partial t_1^{(\nu)}}$. We identify $t_k^{(0)} = t_k$, $t_k^{(1)} = \hat{t}_k$ henceforth.

Introduce the wave functions

$$w^{(\nu)}(\mathbf{t}, \hat{\mathbf{t}}; z^{(\nu)}) = \phi^{(\nu)} e^{\xi^{(\nu)}}, \quad \xi^{(\nu)} = \xi(\mathbf{t}^{(\nu)}; z^{(\nu)})$$

with ξ given in (5.9). The hierarchy (5.18) was shown in [31] to be equivalent to the following bilinear equation

$$\begin{aligned} & \text{res}_{z^{(0)}} \left(z^{(0)} \right)^{-1} w^{(0)}(\mathbf{t}, \hat{\mathbf{t}}; z^{(0)}) w^{(0)}(\mathbf{t}', \hat{\mathbf{t}}'; -z^{(0)}) \\ &= \text{res}_{z^{(1)}} \left(z^{(1)} \right)^{-1} w^{(1)}(\mathbf{t}, \hat{\mathbf{t}}; z^{(1)}) w^{(1)}(\mathbf{t}', \hat{\mathbf{t}}'; -z^{(1)}). \end{aligned} \quad (5.19)$$

By comparing the bilinear equations (5.19) and (5.12), it is easy to see that Shiota's wave functions are related to ours by

$$w^{(0)}(\mathbf{t}, \hat{\mathbf{t}}; z) = w(\mathbf{t}, \hat{\mathbf{t}}; z), \quad w^{(1)}(\mathbf{t}, \hat{\mathbf{t}}; z) = \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; -z^{-1}),$$

from which one can obtain the relations between $a_i^{(0)}, a_i^{(1)}$ and a_i, b_i .

5.2 Reductions of the two-component BKP hierarchy

Given an integer $n \geq 3$, the condition $P^{2n-2} = Q^2$ defines a differential ideal of $\tilde{\mathcal{A}}$, which is denoted by \mathcal{I} . It is easy to see that this ideal is preserved by the flows (5.6), (5.7), so we obtain a reduction of the two-component BKP hierarchy.

Let $L = P^{2n-2} = Q^2$, then according to Lemma 5.1 the operator L has the form (1.3). Hence the algebra \mathcal{A} defined in Section 3.1 is isomorphic to $\tilde{\mathcal{A}}/\mathcal{I}$, and the reduced hierarchy is an integrable hierarchy over \mathcal{A} . It is easy to see that the derivatives of L with respect to t_k, \hat{t}_k are exactly given by (3.3), (3.9). Namely the hierarchy (3.3),(3.9) is the reduction of the two-component BKP hierarchy under the condition $P^{2n-2} = Q^2$.

It can be shown that the condition $P^{2n-2} = Q^2$ reduces the bilinear equations (5.12) to the form

$$\text{res}_z z^{(2n-2)j-1} w(\mathbf{t}, \hat{\mathbf{t}}; z) w(\mathbf{t}', \hat{\mathbf{t}}'; -z) = \text{res}_z z^{-2j-1} \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) \hat{w}(\mathbf{t}', \hat{\mathbf{t}}'; -z) \quad (5.20)$$

with $j \geq 0$, and that conversely the equations (5.20) impose the constraint $P^{2n-2} = Q^2$ to the two-component BKP hierarchy. Hence we establish the

equivalence between the bilinear equations (5.20) and the hierarchy (3.3), (3.9). The proof is lengthy and technical (c.f. the reduction from the KP hierarchy to the Gelfand-Dickey hierarchies in [6]), so we omit the details here. In terms of the tau function, the bilinear equations (5.20) can be expressed as

$$\begin{aligned} & \operatorname{res}_z z^{(2n-2)j-1} X(\mathbf{t}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\mathbf{t}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}') \\ &= \operatorname{res}_z z^{2j-1} X(\hat{\mathbf{t}}; z) \tau(\mathbf{t}, \hat{\mathbf{t}}) X(\hat{\mathbf{t}}'; -z) \tau(\mathbf{t}', \hat{\mathbf{t}}'), \quad j \geq 0. \end{aligned} \quad (5.21)$$

Note that these bilinear equations are precisely the ones obtained from the $(2n-2, 2)$ -reduction of the two-component BKP hierarchy [4, 25].

From the definition (3.24) and (5.14) of the tau functions $\hat{\tau}$ and τ it follows that they are related by

$$\tau^2 = \hat{\tau}. \quad (5.22)$$

Remark 5.4 *In our presentation, it is easy to obtain the $(2n-2, 2)$ -reduction of the two-component BKP hierarchy by using the operators P and Q as above, while it is not straightforward to perform this reduction by using Shiota's operators $P^{(\nu)}$ ($\nu = 0, 1$) [31].*

Remark 5.5 *In [2], Date gave a method to derive the x -parts of the Lax equations of the Drinfeld-Sokolov hierarchies based on the associated bilinear equations. In particular, the Lax operators corresponding to the highest weight Λ_j ($j = 2, \dots, n-2$) of the affine Lie algebra $D_n^{(1)}$ are obtained. They coincide with the Lax operators of the Drinfeld-Sokolov hierarchies of D type corresponding to the j -th vertex of the associated Dynkin diagram [7].*

The x -part of the Lax operators for the Drinfeld-Sokolov hierarchies of D -type that are associated to the vertex c_0 of the Dynkin diagrams can also be obtained, although not presented in [2], by using Date's approach. This case corresponds to the basic representation, i.e. the highest weight Λ_0 , of $D_n^{(1)}$, the associated bilinear equations can be found in [3, 25].

The main difficulty that we are concerned with in the present paper is to obtain the t -parts of the Lax equations of the Drinfeld-Sokolov hierarchies of D type in terms of pseudo-differential operators, especially for the negative flows. In some cases, for example $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, the full hierarchy can be represented by the fractional powers of the Lax operators in \mathcal{D}^- . However for the cases like $D_n^{(1)}$ one has to use the algebras \mathcal{D}^\pm to represent the full hierarchy in terms of pseudo-differential operators.

6 Conclusion

We represent the full Drinfeld-Sokolov hierarchy of D_n type into Lax equations in terms of pseudo-differential operators, which is analogous to the

Gelfand-Dickey hierarchies. We also give a Lax pair representation for the two-component BKP hierarchy, and show that its $(2n-2, 2)$ -reduction yields the Drinfeld-Sokolov hierarchy of D_n type. The key step in our approach is to introduce the concept of pseudo-differential operators of the second type which are defined over a topologically complete differential algebra, so that they are allowed to contain infinitely many terms with positive powers of the derivation D .

The Lax pair representations of the Drinfeld-Sokolov hierarchy of D_n type and the two-component BKP hierarchy are convenient for further studies of the properties of the hierarchies. In [36] it is shown that the two-component BKP hierarchy carries a bihamiltonian structure which is expected to correspond to an infinite-dimensional Frobenius manifold [1]. However it seems to us that this bihamiltonian structure can not be reduced to the one of the Drinfeld-Sokolov hierarchy of D_n type given in Section 3.2 under the constraint $P^{2n-2} = Q^2$.

Note that the bilinear equation (5.17) corresponds to the basic representation of the affine Lie algebra D'_∞ in the notion of [25]. It is shown in [29] that the $(2n-2, 2)$ -reduction (5.21) corresponds to the basic representation of the affine Lie algebra $D_n^{(1)}$. Then according to [26, 27], the bilinear equation (5.21) is equivalent to the Kac-Wakimoto hierarchy constructed from the principal vertex operator realization of the basic representation of the affine Lie algebra $D_n^{(1)}$ [27]. More precisely, the bilinear equation (5.17) can be written as

$$\Omega(v \otimes v) = 0. \quad (6.1)$$

Here

$$\Omega = \sum_{m \in \mathbb{Z}} (-1)^m \left(\phi_m \otimes \phi_{-m} + \hat{\phi}_m \otimes \hat{\phi}_{-m} \right)$$

with $\{\phi_m\}_{m \in \mathbb{Z}}$, $\{\hat{\phi}_m\}_{m \in \mathbb{Z}}$ being two sets of neutral fermions, and v is a nonzero point in the basic module of D'_∞ which corresponds to the tau function via the boson-fermion correspondence. One can verify that $\Omega_2 = \Omega^* \Omega$ is a generalized Casimir operator of $D_n^{(1)}$, then the equation (6.1) is equivalent to

$$\Omega_2(v \otimes v) = 0.$$

This equation is exactly the Kac-Wakimoto bilinear equation via the boson-fermion correspondence, see §7 of [25], §0 of [27] or §14.12 of [26] for more details. By comparing the boson-fermion correspondences, one can obtain the following relation between the time variables $\mathbf{t}, \hat{\mathbf{t}}$ of the Drinfeld-Sokolov hierarchy of D_n type (or the Date-Jimbo-Kashiwara-Miwa hierarchy) and the time variables s_j ($j \in E_+$) of the the Kac-Wakimoto hierarchy:

$$t_k = \sqrt{2} s_k, \quad \hat{t}_k = \sqrt{2n-2} s_{k(n-1)'}$$

In [22], Givental and Milanov proved that the total descendant potential for semisimple Frobenius manifolds associated to a simple singularity satisfies certain hierarchy of Hirota bilinear/quadratic equations, see also [19, 20, 21]. Such a hierarchy of bilinear equations is shown to be equivalent to the corresponding Kac-Wakimoto hierarchy constructed from the principal vertex operator realization of the basic representation of the untwisted affine Lie algebra [22, 35, 17]. So we arrive at the following result.

Theorem 6.1 *Up to a rescaling of the flows, the following integrable hierarchies are equivalent:*

- i) the hierarchy (3.3), (3.9);*
- ii) the Drinfeld-Sokolov hierarchy associated to $D_n^{(1)}$ and the c_0 vertex of its Dynkin diagram;*
- iii) the Date-Jimbo-Kashiwara-Miwa hierarchy constructed from the basic representation of the affine Lie algebra $D_n^{(1)}$;*
- iv) the Kac-Wakimoto hierarchy corresponding to the principal vertex operator realization of the basic representation of the affine Lie algebra $D_n^{(1)}$;*
- v) the Givental-Milanov hierarchy for the simple singularity of D_n type.*

Remark 6.2 *The equivalence between the hierarchies ii) and iv) was also contained in a general result obtained by Hollowood and Miramontes in [24]. The hierarchies iv) and v) was shown to be equivalent in [22] for the case $n = 4$, and in [35] for general n . In [17] Frenkel, Givental and Milanov show that such an equivalence holds true for all ADE singularities.*

Note that the bihamiltonian structure (3.16), (3.17) is of topological type [9, 11, 10], its leading term comes from the Frobenius manifold associated to the Coxeter group of D_n type. In [11] a hierarchy of dispersionless bihamiltonian integrable systems is associated to any semisimple Frobenius manifold, such an integrable hierarchy is called the Principal Hierarchy. It is also shown that there is a so called topological deformation of the Principal Hierarchy which satisfies the condition that its Virasoro symmetries can be represented by the action of some linear operators, called the Virasoro operators, on the tau function of the hierarchy. We expect that the Drinfeld-Sokolov hierarchy associated to $D_n^{(1)}$ and the c_0 vertex of its Dynkin diagram coincides, after a rescaling of the time variables, with the topological deformation of the Principal Hierarchy of the Frobenius manifold that is associated to the Coxeter group of type D_n . We will investigate this aspect of the hierarchy in a subsequent publication.

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