EXTENDABILITY OF CONFORMAL STRUCTURES ON
PUNCTURED SURFACES

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Abstract. For a smooth immersion $f$ from the punctured disk $D \setminus \{0\}$ into $\mathbb{R}^n$ extendable continuously at the puncture, if its mean curvature is square integrable and the measure of $f(D) \cap B_{r_k} = o(r_k)$ for a sequence $r_k \to 0$, we show that the Riemannian surface $(D_r \setminus \{0\}, g)$ where $g$ is the induced metric is conformally equivalent to the unit Euclidean punctured disk, for any $r \in (0, 1)$. For a locally $W^{2,2}$ Lipschitz immersion $f$ from the punctured disk $D_2 \setminus \{0\}$ into $\mathbb{R}^n$, if $\|\nabla f\|_{L^\infty}$ is finite and the second fundamental form of $f$ is in $L^2$, we show that there exists a homeomorphism $\phi : D \to D$ such that $f \circ \phi$ is a branched $W^{2,2}$-conformal immersion from the Euclidean unit disk $D$ into $\mathbb{R}^n$.

1. Introduction

In two dimensional variation, especially conformally invariant, problems in differential geometry, it is important to know whether isloated singularities are removable while preserving conformal properties. In this paper, we shall study the problem of extending conformal structure of an immersion from a punctured 2-dimensional disk across the puncture as a branched conformal immersion. For an immersion of class $C^{2,\alpha}$ from a punctured disk with bounded mean curvature $H$ and has a unique limit point at the puncture, Gulliver showed in [3] that there exists a $C^{1,\alpha}$ conformal mapping from the disk such that its restriction to the punctured disk is a $C^{2,\alpha}$ parametrization of the immersion. The minimal surface case was proved by Osserman in [8]. We shall relax the pointwise bound on $H$ by integral bounds. We shall also be concerned with immersions from a punctured disk with lower regularity. For immersions from a disk (not punctured) of class $C^{1,\alpha}$ (the first fundamental form is $C^\alpha$), the classical theorem of Korn-Lichtenstein asserts existence of isothermal coordinates (a simplified proof was given by Chern [1]). When the induced metric is merely bounded measurable with $g_{11}g_{22} - g_{12}^2 \geq c$ almost everywhere for some positive constant $c$, Morrey’s measurable Riemann mapping theorem states that there is a homeomorphism from the unit disk to a neighborhood of a point in the immersed disk satisfying the conformality conditions almost everywhere. Yet, in this generality, the reparametrized immersion does not admit high regularity. It is advantageous to work with immersions that are in the space $W^{2,2} \cap W^{1,\infty}$ since the $L^2$-norm of the second fundamental form can then be defined.

In Section 2, we show in Theorem 2.1 that for a smooth immersion $f$ from the punctured disk $D \setminus \{0\}$ into $\mathbb{R}^n$ which admits a continuous extension at 0 sending $0 \in D$ to $0 \in \mathbb{R}^n$, if its mean curvature $H$ is square integrable and the length of $f(D) \cap \partial B_{r_k} \to 0$, or more generally, $\mu(f(D) \cap B_{r_k}) = o(r_k)$, along some sequence $r_k \to 0$, then $(D_r \setminus \{0\}, f^* g_{\mathbb{R}^n})$ is

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conformal to \((D \setminus \{0\}, g_{\mathbb{R}^2})\) for any \(0 < r < 1\). The reason of taking a smaller disk is due to lack of control on \(f\) near \(\partial D\). This generalizes the result in [3] for \(H\) is assumed to be pointwise bounded by a uniform constant therein.

In Section 3, we consider \(W^{2,2} \cap W^{1,\infty}\) immersions \(f\) from the punctured disk that admits a continuous extension at the puncture. There are two problems to be resolved. First, we need to find a conformal structure \(\mathcal{C}\) compatible with the induced metric \(g\) by \(f\) away from the singularity. This does not directly follow from Morrey’s result if we require \(C\) to lack of control on \(\partial D\).

First, we need to find a conformal structure \(\mathcal{C}\) compatible with the induced metric \(g\) by a smooth immersion, we can apply Proposition 3.4 to prove Theorem 3.5: For a smooth immersion \(f : D \setminus \{0\} \rightarrow \mathbb{R}^n\) with \(A \in L^2\) and \(\|\nabla f\|_{L^\infty(D \setminus \{0\})} < \infty\), the Riemann surface \((D \setminus \{0\}, g)\) is conformally diffeomorphic to \((D \setminus \{0\}, g_0)\) for any \(0 < \delta < 1\).

In section 3.2, we show that there exists a complex structure \(\mathcal{C}\) on the punctured disk arising from the Lipschitz map \(f\). The coordinate maps defining the complex structure \(\mathcal{C}\) are in \(W^{2,2} \cap W^{1,\infty}\), but the transition functions between coordinate charts are holomorphic. This is done by an approximation argument together with results on mappings with square integrable generalized second fundamental forms [5, 4, 6, 7]. For existence of isothermal coordinates of Lipschitz immersions with \(L^2\)-bounded second fundamental form, an approach via the method of moving frames was given in [9]. Note that as a positive lower bound (almost everywhere) on the metric tensor is imposed, \(\mathcal{C}\) needs not to be extendable across the puncture.

In section 3.3, we derive Theorem 3.10 that can be stated as: For a \(W^{2,2} \cap W^{1,\infty}\) mapping \(f\) from the punctured Euclidean disk (of radius 2) \(D_2 \setminus \{0\}\) into \(\mathbb{R}^n\), if \(f\) is a \(C^0\) immersion with \(df \otimes df > C(K) g_0 > 0\) almost everywhere on any compact set \(K\) in \(D \setminus \{0\}\) and its (generalized) second fundamental form is in \(L^2\), then there exists a homeomorphism \(\phi : D \rightarrow D\) such that \(f \circ \phi^{-1}\) is a branched \(W^{2,2}\)-conformal immersion of the Euclidean disk \(D\) into \(\mathbb{R}^n\) with \(0\) as its only possible branch point. Moreover, the maps \(\phi, \phi^{-1}\) are in \(W^{2,2} \cap W^{1,\infty}\) on \(D \setminus \{0\}\) with \(\phi(0) = 0\) and \((f \circ \phi)^* g_{\mathbb{R}^n} = e^{2u} g_{\mathbb{R}^2}\) on \(D \setminus \{0\}\), where \(u \in W^{1,2}_\text{loc} \cap L^\infty_\text{loc}\) on \(D \setminus \{0\}\).

Theorem 3.10 generalizes to the global case: Let \((\Sigma, h)\) be an oriented surface (not necessarily compact) and let \(S \subset \Sigma\) be a finite set. Suppose \(f \in W^{2,2}_\text{loc}(\Sigma \setminus S, \mathbb{R}^n)\) and \(f\) is a \(C^0\) immersion on \(\Sigma \setminus S\). Assume that (1) \(g = df \otimes df > C(K) h\) almost everywhere on any compact set \(K \subset \Sigma \setminus S\), (2) \(\|\nabla f\|_{L^\infty(\Sigma, h)} < \infty\) and (3) \(A \in L^2(\Sigma)\). Then there is a complex structure \(\mathcal{C}\) on \(\Sigma\) such that \(f : (\Sigma, \mathcal{C}) \rightarrow \mathbb{R}^n\) is a branched \(W^{2,2}\)-conformal immersion with its branch locus contained in \(S\).
The results in this paper remain valid if the ambient space $\mathbb{R}^n$ is replaced by a smooth compact Riemannian manifold $M^n$, via Nash’s isometric embedding theorem.

2. Smooth immersions from a punctured disk

**Theorem 2.1.** Let $D$ be the open unit disk in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}^n$ be a continuous map with $f(0) = 0$ and $f|_{D\setminus\{0\}}$ is a smooth immersion. Set $\mu(\Sigma \cap K) = \mu_g(f^{-1}(K))$ for any $K \subset \mathbb{R}^n$, where $g$ is the induced metric by $f$ on $\Sigma \setminus \{0\}$ where $\Sigma = f(D)$. Assume

1. $\int_{\Sigma \setminus \{0\}} |H|^2 d\mu_g < \infty$, where $H$ is the mean curvature of $\Sigma \setminus \{0\}$;

2. There exist positive numbers $\epsilon_k \to 0$ such that $\frac{\mu(\Sigma \cap B_{\epsilon_k}(0))}{\epsilon_k} \to 0$ as $k \to \infty$.

Then $(D_r \setminus \{0\}, g)$ is conformal to $(D \setminus \{0\}, g_0)$ where $g_0$ is the Euclidean metric on $D$ and $D_r$ is the Euclidean open disk of radius $r$ for any $r \in (0, 1)$.

The whole image set $\Sigma$ may not have the structure of a surface at the puncture. We will need to know that it admits the generalized mean curvature that is in $L^2$.

**Lemma 2.2.** Under the assumptions in Theorem 2.1, $\Sigma_r = f(D_r)$ is a rectifiable integral 2-varifold with generalized mean curvature in $L^2$, for any $0 < r < 1$.

**Proof.** We first show that $\Sigma_r$ has finite total measure. By assumption (2), there is an integer $k_0$ such that $\mu(\Sigma \cap B_{\epsilon_k}(0)) \leq \mu(\Sigma \cap B_{\epsilon_0}(0)) < 1$. The set $\Sigma_r \setminus B_{\epsilon_0}(0)$ is compact and $g$ is bounded there, so it has finite measure. Hence $\mu(\Sigma_r) < \infty$.

For any $y \in \Sigma_r$, let $\theta(y) = \mathcal{H}^0(f^{-1}(y))$ where $\mathcal{H}^0$ is the 0-dimensional Hausdorff measure. By the general area formula, see 8.4 in [11],

$$\int_{\Sigma_r \setminus B_{\epsilon_0}(0)} \theta(y) d\mathcal{H}^2(y) = \mu(\Sigma_r \setminus B_{\epsilon_0}(0)) \leq \mu(\Sigma_r) < \infty.$$ 

Letting $k \to \infty$, we see $\theta(y)$ is integrable on $\Sigma_r \setminus \{0\}$. Further, being the image of a smooth immersion, $\Sigma_r \setminus \{0\}$ is a countable union of embedded surfaces, hence, it is countably 2-rectifiable (Lemma 11.1, [11]). It follows that $(\Sigma, \theta)$ is a rectifiable integral 2-varifold ($p.77$, [11]).

Let $\eta$ be a cut-off function with values between 0 and 1 with $|\eta'| \leq C$, and it equals 1 on $[1, +\infty)$ and 0 on $(-\infty, \frac{1}{2})$. Then

$$\eta_k(x) = \eta\left(\frac{|f(x)|}{2\epsilon}\right), \quad x \in D_r$$

is 0 when $f(x) \in \Sigma_r \cap B_\epsilon(0)$ and equals 1 when $f(x) \in \Sigma_r \setminus B_{2\epsilon}(0)$; $\eta_k$ is continuous on $D_r$ since $f$ is continuous and $\eta_k$ is locally Lipschitz on $D_r \setminus \{0\}$. Moreover, as $g$ is the induced metric on $\Sigma_r \setminus \{0\}$ by the immersion $f|_{D_r \setminus \{0\}}$, we have, by Kato’s inequality, on $D_r \setminus \{0\}$ the estimate

(2.1) $|\nabla g \eta_k| \leq \frac{C}{2\epsilon} |\nabla g f| \leq \frac{C}{2\epsilon} |\nabla g f| = \frac{C}{2\epsilon} \sqrt{2}$.

For any $C^1$ vector field $X$ on $\mathbb{R}^n$, $\eta_k X$ is a $C^1$ vector field along $\Sigma_r$ as $\eta_k$ vanishes on $\Sigma_r \cap B_{\epsilon_k}(0)$. Then

(2.2) $-\int_{\Sigma_r} H \cdot \eta_k X = \int_{\Sigma_r} \text{div}_{\Sigma_r}(\eta_k X) = \int_{\Sigma_r} X \cdot \nabla g \eta_k + \int_{\Sigma_r} \eta_k \text{div}_{\Sigma_r} X.$
Since $\Sigma_r$ is bounded, $X|_{\Sigma_r}$ is bounded in $C^1$. Then
\[
\left| \int_{\Sigma_r} H \cdot \eta_k X \right| \leq C \left( \int_{\Sigma_r \setminus \{0\}} |H|^2 d\mu_g \right)^{1/2} \mu(\Sigma_r \cap B_{2\epsilon_k}(0))^{1/2} \to 0 \text{ as } k \to \infty
\]
by assumptions (1) and (2), and
\[
\left| \int_{\Sigma_r} X \cdot \nabla \eta_k \right| \leq \frac{C}{\epsilon_k} \mu(\Sigma_r \cap (B_{2\epsilon_k}(0))) \to 0 \text{ as } k \to \infty
\]
by (2.1) and assumption (2). Letting $k \to 0$ in (2.2), we have
\[
- \int_{\Sigma_r} H \cdot X = \int_{\Sigma_r} \div X.
\]
As $X$ is arbitrary, $H$ is the generalized mean curvature of $\Sigma_r$ (cf. [11]) and by (1) it is in $L^2(\Sigma_r)$. $\square$

The following rigidity result for the Dirichlet problem of harmonic functions on a punctured disk will be used in the proof of Theorem 2.1 to eliminate the conformal types of annuli.

**Lemma 2.3.** Let $\phi$ be a harmonic function on $(D_r \setminus \{0\}, g)$ where $g$ is as in Theorem 2.1. If
\[
\phi|_{\partial D_r} = 0, \quad \|\phi\|_{L^\infty} < \infty, \quad \text{and} \quad \int_{D_r \setminus \{0\}} |\nabla_g \phi|^2 d\mu_g < \infty,
\]
then $\phi = 0$.

**Proof.** Since $\eta_k$ is 0 in $B_\epsilon(0)$, $\eta_k(f)$ is 0 on $f^{-1}(B_\epsilon(0)) \cap D_r$. Then
\[
\left| \int_{f^{-1}(B_{2\epsilon}(0)) \cap D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g \right| = \left| \int_{f^{-1}(B_{2\epsilon} \setminus B_\epsilon(0)) \cap D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g \right|
\]
\[
= \left| \int_{f^{-1}(B_{2\epsilon} \setminus B_\epsilon(0)) \cap D_r} (\phi \nabla_g \phi \nabla_g \eta_k(f) + \eta_k(f) |\nabla_g \phi|^2) d\mu_g \right|
\]
\[
\leq \|\phi\|_{L^\infty} \left( \int_{f^{-1}(B_{2\epsilon} \setminus B_\epsilon(0)) \cap D_r} |\nabla_g \phi|^2 d\mu_g \right)^{1/2} \left( \mu(\Sigma_r \cap B_{2\epsilon}(0)) \right)^{1/2}
\]
\[
+ \int_{f^{-1}(B_{2\epsilon} \setminus B_\epsilon(0)) \cap D_r} |\nabla_g \phi|^2 d\mu_g,
\]
where we used the following identity
\[
\mu_g(f^{-1}(B_{2\epsilon} \setminus B_\epsilon(0)) \cap D_r) = \mu(\Sigma_r \cap (B_{2\epsilon} \setminus B_\epsilon(0))).
\]
By Lemma 2.2, we can apply Simon’s monotonicity formula for surfaces with square integrable mean curvature [10], see also [5], to assert
\[
\frac{\mu(\Sigma_r \cap B_{2\epsilon}(0))}{\epsilon^2} < C.
\]
Since the Dirichlet energy of $\phi$ is finite over $D_r \setminus \{0\}$ and $\mu_g(f^{-1}(B_{2\epsilon}(0) \cap D_r) \to 0$ as $\epsilon \to 0$, it follows from the continuity of integration that
\[
\lim_{\epsilon \to 0} \int_{f^{-1}(B_{2\epsilon}(0)) \cap D_r} |\nabla_g \phi|^2 d\mu_g = 0,
\]
which in turn implies
\[
\lim_{\epsilon \to 0} \int_{f^{-1}(B_{2\epsilon}(0)) \cap D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g = 0.
\]
Since $\eta_k(f) \phi$ is smooth on $D_r$ and is 0 in a neighborhood of 0, by the harmonicity of $\phi$,
\[
\int_{D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) = 0.
\]
Then we get
\[
\int_{D_r} |\nabla_g \phi|^2 d\mu_g = \lim_{\epsilon \to 0} \int_{D_r \setminus f^{-1}(B_{2\epsilon}(0))} |\nabla_g \phi|^2 d\mu_g
\]
\[
= \lim_{\epsilon \to 0} \int_{D_r \setminus f^{-1}(B_{2\epsilon}(0))} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g
\]
\[
= \lim_{\epsilon \to 0} \int_{D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g - \lim_{\epsilon \to 0} \int_{f^{-1}(B_{2\epsilon}(0)) \cap D_r} \nabla_g \phi \nabla_g (\eta_k(f) \phi) d\mu_g
\]
\[
= 0
\]
Therefore $\phi$ must be a constant and identically 0 as it vanishes on $\partial D_r$. \qed

**Proof of Theorem 2.1:** Since $g$ is positive definite on $D \setminus \{0\}$, by the uniformization theorem for Riemann surfaces, the punctured Riemannian disk $(D \setminus \{0\}, g)$ is conformally equivalent to one and only one of the three: (i) a finite annulus $(D \setminus B_{r_0}, g_0)$ for some $r_0 \in (0, 1)$, (ii) the punctured plane $(\mathbb{C} \setminus \{0\}, g_0)$, (iii) the punctured disk $(D \setminus \{0\}, g_0)$. Here $g_0$ denotes the Euclidean metric. Now we fix a positive number $r < 1$

In Case (i), $(D_r \setminus \{0\}, g)$ is conformal to $(D \setminus B_{r_0})$ for some $r_0 \in (0, 1)$. There exists a nonconstant bounded harmonic function $h$ on $(D \setminus B_{r_0}, g_0)$ with finite energy, which vanishes on either the inner circle $\partial B_{r_0}$ or the outer circle $\partial D$ but not both. Note that both harmonicity and the energy of $h$ are invariant under conformal diffeomorphisms of the 2-dimensional domain. Moreover, any conformal diffeomorphism $\psi$ between $(D_r \setminus \{0\}, g)$ and $(D \setminus B_{r_0}, g_0)$ either maps $\partial D_r \to \partial D$ or $\partial D_r \to \partial B_{r_0}$. Without loss of any generality we may assume that $h \circ \psi$ equals 0 on $\partial D_r$. However, Lemma 2.3 asserts $h \circ \psi$ must be 0. This shows that Case (i) cannot happen.

In Case (ii), there exists a conformal diffeomorphism $\varphi : (D \setminus \{0\}, g) \to (\mathbb{C} \setminus \{0\}, g_0)$. Then $\varphi(D_r \setminus \{0\})$ stays inside or outside the embedded closed curve $\varphi(\partial D_r)$ in $\mathbb{C}$. In the former case, $(D_r \setminus \{0\}, g)$ is conformally diffeomorphic to $(D \setminus \{0\}, g_0)$ by the Riemann mapping theorem since $\partial D_r$ bounds a simply connected domain in $\mathbb{C}$ by the Jordan curve theorem. In the latter case, by using an inversion $\frac{z-p}{|z-p|^2}$ for some point $p$ in the interior of the bounded domain enclosed by $\partial D_r$, we see that $(D_r \setminus \{0\}, g)$ is conformally equivalent to a bounded simply connected domain punctured once, which is conformal to $(D \setminus \{0\}, g_0)$.
In Case (iii), \((D \setminus \{0\}, g)\) is conformally equivalent to a simply connected domain in \(D\) with one puncture, hence conformal to \((D \setminus \{0\}, g_0)\).

The decay rate (2) in Theorem 2.1 on the area of \(\Sigma\) inside small exterior balls can be achieved under an assumption on length and this leads to

**Corollary 2.4.** Let \(f : D \to \mathbb{R}^n\) be a continuous map with \(f(0) = 0\) and \(f|_{D \setminus \{0\}}\) is a smooth immersion. Assume \(\int_{\Sigma \setminus \{0\}} |H|^2 \, d\mu_g < \infty\). If \(\mathcal{H}^1(\Sigma \cap \partial B_\epsilon(0)) \to 0\) as \(\epsilon \to 0\), where \(\mathcal{H}^1\) is the 1-dimensional Hausdorff measure, then the conclusion of Theorem 2.1 holds.

**Proof.** It suffices to show (2) in Theorem 2.1 holds on \(\Sigma\) for any fixed \(r \in (0, 1)\). The position vector \(f\) in \(\mathbb{R}^n\) satisfies

\[
\text{div}_\Sigma f = 2
\]
on \(D \setminus \{0\}\) and

\[
\text{div}_\Sigma f = \text{div}_\Sigma f^T - f \cdot H
\]
where \(f^T\) denotes the tangential part of \(f\) and \(\text{div}_\Sigma\) is the divergence along \(\Sigma\). It follows

\[
(2.3) \quad \text{div}_\Sigma f^T = 2 + f \cdot H.
\]

Let \(\rho(x)\) be the distance from \(x\) to 0 in \(\mathbb{R}^n\). For \(\rho > 0\), \(\mu(\Sigma \cap \partial B_{\rho}(0))\) depends on \(\rho\) continuously. By Sard’s theorem, the regular values of the restriction of \(\rho\) to \(\Sigma\) are dense, we may assume that \(\epsilon_k > 0\) and \(\delta > 0\) are regular values of \(\rho|_\Sigma\) and \(\epsilon_k \to 0\) as \(k \to \infty\); therefore, for any \(0 < r < 1\), the surface

\[
\Sigma_{\delta, \epsilon_k} := \Sigma_r \cap (B_\delta(0) \setminus B_{\epsilon_k}(0))
\]
has compact closure and smooth boundary \(\Gamma_\delta \cup \Gamma_{\epsilon_k}\). Integrating (2.3) over \(\Sigma_{\delta, \epsilon_k}\) leads to

\[
2\mu(\Sigma_{\delta, \epsilon_k}) \leq 2\delta \int_{\Sigma_{\delta, \epsilon_k}} |H| \, d\mu_g + \int_{\Sigma_{\delta, \epsilon_k}} \text{div}_\Sigma f^T \, d\mu_g
\]
\[
\leq 2\delta \mu(\Sigma_{\delta, \epsilon_k})^{1/2} \left( \int_{\Sigma_{\delta, \epsilon_k}} |H|^2 \right)^{1/2} + \int_{\Gamma_\delta \cup \Gamma_{\epsilon_k}} f^T \cdot \nu
\]
\[
\leq \mu(\Sigma_{\delta, \epsilon_k}) + \delta^2 \int_{\Sigma_{\delta, \epsilon_k}} |H|^2 + \int_{\Gamma_\delta \cup \Gamma_{\epsilon_k}} f^T \cdot \nu
\]
where \(\nu\) is the unit outward normal to \(\Sigma_{\delta, \epsilon_k}\) at its boundary \(\Gamma_\delta \cup \Gamma_{\epsilon_k}\). Noting that \(\nu\) is tangent to \(\Sigma_{\delta, \epsilon_k}\), \(f^T \cdot \nu = f \cdot \nu \leq 0\) on \(\Gamma_{\epsilon_k}\). Letting \(k \to \infty\) and using \(|f^T \cdot \nu| \leq |f|\), we see

\[
(2.4) \quad \mu(\Sigma_r \cap B_\delta(0)) \leq \delta^2 \int_{\Sigma \cap B_\delta(0) \setminus \{0\}} |H|^2 + \delta \mu(\Gamma_\delta)
\]
where \(\mu(\Gamma_\delta) = \mu_g(f^{-1}(\partial B_\delta(0)))\) is the total length of the preimage curve of \(\Gamma_\delta\) in \(D\) measured in \(g\). Replacing \(\delta\) by \(\epsilon_k\) in (2.4) and using the assumption that \(\mathcal{H}^1(\Sigma_r \cap \partial B_{\epsilon_k}(0)) \to 0\) as \(k \to \infty\) and \(|H|^2\) is integrable over \(\Sigma \setminus \{0\}\), it is evident that \(\mu(\Sigma_r \cap B_{\epsilon_k}(0))/\epsilon_k \to 0\) as \(k \to \infty\).
3. $W^{2,2}$ Lipschitz immersions from a punctured disk

In this section, we will be mainly concerned with immersions, that are not necessarily smooth, from punctured surfaces into $\mathbb{R}^n$.

3.1. Conformal type of $W^{2,2}$-conformal immersions of $D\setminus\{0\}$. Let $\omega$ be twice the standard Kähler form of $\mathbb{CP}^n$ and denote $W_0^{1,2}(\mathbb{C})$ the space of functions $v \in L^2_{loc}(\mathbb{C})$ with $\nabla v \in L^2(\mathbb{C})$. We set $J(\varphi) = |D\varphi \wedge D\varphi|$ to denote the Jacobi of $\varphi$.

**Theorem 3.1** (Müller-Sverák [7]). Let $\varphi \in W_0^{1,2}(\mathbb{C}, \mathbb{CP}^n)$ satisfy

$$\int_{\mathbb{C}} \varphi^* \omega = 0 \quad \text{and} \quad \int_{\mathbb{C}} J(\varphi) \leq \gamma < 2\pi.$$

Then there is a unique function $v \in W_0^{1,2}(\mathbb{C})$ solving the equation $-\Delta v = *\varphi^* \omega$ in $\mathbb{C}$ with boundary condition $\lim_{z \to \infty} v(z) = 0$. Moreover

$$\|v\|_{L^\infty(\mathbb{C})} + \|\nabla v\|_{L^2(\mathbb{C})} \leq C(\gamma) \int_{\mathbb{C}} |\nabla \varphi|^2.$$

We include the following result in [6] for completeness.

**Corollary 3.2.** Let $\varphi \in W^{1,2}(D, \mathbb{CP}^n)$ satisfy

$$\int_D J(\varphi) \leq \gamma < 2\pi.$$

Then there is a continuous function $v \in W_0^{1,2}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ solving the equation $-\Delta v = *\varphi^* \omega$ in $D$ and satisfying the estimates

$$\|v\|_{L^\infty(D)} + \|\nabla v\|_{L^2(D)} \leq C(\gamma) \int_D |\nabla \varphi|^2.$$

**Proof.** Define the map $\varphi' : \mathbb{C} \to \mathbb{CP}^{n-1}$ by

$$\varphi'(z) = \begin{cases} 
\varphi(z) & \text{if } z \in D \\
\varphi(\frac{1}{z}) & \text{if } z \in \mathbb{C} \setminus \overline{D}
\end{cases}$$

and taking the trace of $\varphi$ on $\partial D$ for $\varphi'$ there. Then $\varphi' \in W_0^{1,2}(\mathbb{C}, \mathbb{CP}^{n-1})$ and

$$\int_{\mathbb{C}} \varphi'^* \omega = 0, \quad \int_{\mathbb{C}} J(\varphi') = 2 \int_D J(\varphi).$$

The desired result then follows from Theorem 3.1. \hfill \Box

**Lemma 3.3.** Let $0 < a < 1$ and $\varphi \in W^{1,2}(D\setminus \overline{D}_a, \mathbb{CP}^n)$. There is a constant $\epsilon_0 > 0$ such that if $\|\nabla \varphi\|_{L^2} < \epsilon_0$, then we can find $v \in L^\infty(D\setminus \overline{D}_a)$ which solves the equation $-\Delta v = *\varphi^* \omega$ in $D\setminus \overline{D}_a$ and satisfies the estimates

$$\|v\|_{L^\infty(D\setminus \overline{D}_a)} \leq C(a) \|\nabla \varphi\|_{L^2(D\setminus \overline{D}_a)}.$$

**Proof.** Let

$$\widetilde{\varphi}(z) = \begin{cases} 
\varphi(\frac{1}{z}) & \text{if } 1 < |z| < \frac{1}{a} \\
\varphi(z) & \text{if } a \leq |z| \leq 1 \\
\varphi(a^2 \frac{1}{z}) & \text{if } a^2 < |z| < a.
\end{cases}$$
It follows $\tilde{\varphi} \in W^{1,2}(D_{\frac{1}{a}} \setminus \overline{D}_a, \mathbb{C}^n)$ as
\[
\int_{D_{\frac{1}{a}} \setminus \overline{D}_a^2} |\nabla \tilde{\varphi}|^2 = 3 \int_{D \setminus \overline{D}_a} |\nabla \varphi|^2, \quad \int_{D_{\frac{1}{a}} \setminus \overline{D}_a^2} |\tilde{\varphi}|^2 \leq C(a) \int_{D \setminus \overline{D}_a} |\varphi|^2
\]
by the conformal invariance of the Dirichlet integral. Cover the annulus $D_{\frac{1}{a}} \setminus \overline{D}_a$ by countably many open disks such that every point is contained in finitely many such disks. Take a partition of unity subordinates to this cover: there exist smooth functions $\rho_i$ on the annulus such that

1. $0 \leq \rho_i \leq 1$, and $\text{supp} \rho_i \subset D_{\frac{1}{a}} \setminus \overline{D}_a^2$.
2. $\sum_i \rho_i(z) = 1$, $\forall z \in D_{\frac{1}{a}} \setminus \overline{D}_a^2$.
3. For any $z \in D_{\frac{1}{a}} \setminus \overline{D}_a^2$, there is a neighborhood $V$ of $z$, such that there are only finitely many $\rho_i$ with $\text{supp} \rho_i \cap V \neq \emptyset$.

By (3), there are only finitely many $\rho_i$ whose support intersects the compact set $\overline{D} \setminus D_a$ and we label them as $\rho_1, \ldots, \rho_m$, and assume $\text{supp} \rho_i \subset D_{\frac{1}{a}}(z_i)$, where $D_{\frac{1}{a}}(z_i) \subset D_{\frac{1}{a}} \setminus \overline{D}_a^2$.

By Corollary 3.2, we can find $v_i \in L^\infty(\mathbb{C}) \cap W^{1,2}(\mathbb{C})$, which solves the equation
\[-\Delta v_i = *\tilde{\varphi}^* \omega, \quad \forall z \in D_{\frac{1}{a}}(z_i),\]
such that
\[
\|v_i\|_{L^\infty(\mathbb{C})} + \|\nabla v_i\|_{L^2(\mathbb{C})} \leq C\|\nabla \tilde{\varphi}\|_{L^2(D_{\frac{1}{a}}(z_i))}.
\]

Then on $D \setminus \overline{D}_a$,
\[-\Delta \sum_{i=1}^m \rho_i v_i = *\tilde{\varphi}^* \omega - 2 \sum_{i=1}^m \nabla \rho_i \nabla v_i - \sum_{i=1}^m v_i \Delta \rho_i.
\]

Let $v'$ be the solution to the Dirichlet problem in $D$:
\[
\begin{cases}
-\Delta v' &= 2 \sum_{i=1}^m \nabla \rho_i \nabla v_i + \sum_{i=1}^m v_i \Delta \rho_i, \\
v'|_{\partial D} &= 0.
\end{cases}
\]

Since
\[
\|\Delta v'\|_{L^2(D)} < C(\max_i \|\Delta \rho_i\|_{C^0}, \max_i \|\nabla \rho_i\|_{C^0}) \sum_i \|v_i\|_{W^{1,2}(D)}^2 < C\|\nabla \varphi\|_{L^2(D \setminus \overline{D}_a)}^2,
\]
the elliptic estimates implies
\[
\|v'\|_{L^\infty(D)} < C\|\nabla \varphi\|_{L^2(D \setminus \overline{D}_a^2)}.
\]

Let $v = \sum_{i=1}^m \rho_i v_i + v'$. Then
\[-\Delta v = *\tilde{\varphi}^* \omega, \quad z \in D \setminus \overline{D}_a
\]
and
\[
\|v\|_{L^\infty(D)} < C\|\nabla \varphi\|_{L^2(D)}
\]
where $C$ depends on $\|\Delta \rho_i\|_{C^0}$ and $\|\nabla \rho_i\|_{C^0}$ for $i = 1, \ldots, m$. $\square$
For a conformal immersion \( f : D \to \mathbb{R}^n \), let \( G \in W^{1,2}(D, \mathbb{C}P^{n-1}) \) be the associated Gauß map. Here we embed the Grassmannian \( G(2, n) \) of oriented 2-planes into \( \mathbb{C}P^{n-1} \) by sending an orthonormal basis \( \{e_1, e_2\} \) to \([(e_1 + ie_2)/\sqrt{2}] \). Then

(3.1) \[ K_g e^{2u} = *G^* \omega \quad \text{and} \quad \int_D |\nabla G|^2 = \frac{1}{2} \int_D |A|^2 \, d\mu_g \]

where \( K_g \) is the Gauß curvature and \( A \) is the second fundamental form of \( f(D) \) (cf. [7]).

**Proposition 3.4.** Let \( a \in (0, 1) \) and \( f \in W^{2,2}_{\text{conf,loc}}(D \setminus \overline{D}_a, \mathbb{R}^n) \). Set \( g = df \otimes df \) and let \( \Gamma \) be the set of closed embedded curves that are nontrivial in \( \pi_1(D \setminus \overline{D}_a) \). Then

\[ \inf_{\gamma \in \Gamma} L_g(\gamma) > 0 \]

where \( L_g(\gamma) \) denotes the length of \( \gamma \) measured in \( g \).

**Proof.** Assume there exists \( \gamma_k \in \Gamma \), such that \( L_g(\gamma_k) \to 0 \) as \( k \to \infty \). We consider \( \gamma_k \) as a smooth map from \([0, 1]\) to \( D \setminus \overline{D}_a \) with \( \gamma_k(0) = \gamma_k(1) \). Let \( g = e^{2u}g_0 \). Then

(3.2) \[ \int_{\gamma_k} e^u = \int_0^1 e^{u(\gamma_k)} \sqrt{(x'_k)^2 + (y'_k)^2} \, dt = L_g(\gamma_k) \to 0 \quad \text{as} \quad k \to \infty. \]

**Claim.** For any \( \epsilon > 0 \), \( \gamma_k \subset (D \setminus \overline{D}_{1-\epsilon}) \cup (D_{a+\epsilon} \setminus \overline{D}_a) \) for sufficiently large \( k \).

Assume the claim is not true. Without loss of generality, we may assume \( \gamma_k(0) \in \overline{D}_{1-\epsilon} \setminus D_{a+\epsilon} \). Since \( f \in W^{2,2}_{\text{conf,loc}}(D \setminus \overline{D}_a, \mathbb{R}^n) \), we can find \( C = C(\epsilon, f) \), such that for any \( \gamma \subset D_{1-\frac{1}{2}} \setminus D_{a+\frac{1}{2}} \),

\[ L_g(\gamma) \geq CL(\gamma), \]

where \( L \) is the length of \( \gamma \) measured in the Euclidean metric \( g_0 \). If \( \gamma_k \subset D_{1-\frac{1}{2}} \setminus D_{a+\frac{1}{2}} \), then

\[ L_g(\gamma_k) \geq CL(\gamma) \geq C\pi \epsilon. \]

If \( \gamma_k \) is not in \( D_{1-\frac{1}{2}} \setminus D_{a+\frac{1}{2}} \), then we can find \( t_0 \), such that \( \gamma_k(t_0) \in \partial D_{1-\frac{1}{2}} \cup \partial D_{a+\frac{1}{2}} \), and

\[ \gamma([0, t_0]) \subset D_{1-\frac{1}{2}} \setminus D_{a+\frac{1}{2}}. \]

Thus,

\[ L_g(\gamma_k) \geq L_g(\gamma_k|_{[0, t_0]}) \geq CL(\gamma_k|_{[0, t_0]}) \geq C|\gamma_k(t_0) - \gamma_k(0)| \geq \frac{C\epsilon}{2}. \]

It contradicts the fact that \( L_g(\gamma_k) \to 0 \). Now the Claim is established.

Without loss of generality, we may assume \( \gamma_k \subset D_{a+\epsilon} \setminus \overline{D}_a \) for sufficiently large \( k \), because \( f(\frac{a}{2}) \) is also in \( W^{2,2}_{\text{conf,loc}}(D \setminus \overline{D}_a, \mathbb{R}^n) \) and the metric induced by \( f(\frac{a}{2}) \) is uniformly equivalent to \( g \). By Lemma 3.3, there exists a smooth bounded function \( v \) on \( \mathbb{C} \) solving the equation

\[ -\Delta v = Ke^{2u} \]

on \( D \setminus \overline{D}_a \), where \( K \) is the Gauß curvature of \( g \) and \( \Delta \) is the Euclidean Laplacian. Noting that \( u \) is also a solution to the same equation, \( u - v \) is a harmonic function on \( D \setminus \overline{D}_a \) with respect to \( g_0 \). Consider the harmonic function \( w \) on \( D \setminus \overline{D}_a \) defined by

\[ w = u - v - \lambda \log |z|, \quad \text{where} \quad \lambda = \frac{1}{2\pi r_0} \int_{\partial D_{r_0}} \frac{\partial(u - v)}{\partial r}, \quad \text{and} \quad r_0 \in (a, 1). \]
Then we can find a holomorphic function $F$ on $D \setminus \overline{D_a}$ with real part $w$ (cf. [2], Theorem 15.1.3). Evidently, $e^F$ is holomorphic on $D \setminus \overline{D_a}$, with $|e^F| = e^w$.

Since $|z| \in (a, 1)$ and $\|v\|_{L^\infty} < +\infty$, it follows from the definition of $w$ that

$$w(z) \leq u(z) + C(a, \lambda, \|v\|_{L^\infty}).$$

Then

$$\left| \frac{1}{2\pi i} \int_{\gamma_k} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta \right| < C \int_{\gamma_k} \left| e^{F} \right| < C \int_{\gamma_k} e^u \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let $\gamma_{\epsilon_0} = (a + \epsilon_0)(\cos \theta, \sin \theta)$ for some fixed $\epsilon_0 > 0$. The Deformation Invariance Theorem for holomorphic functions then asserts

$$\frac{1}{2\pi i} \int_{\gamma_{\epsilon_0}} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \lim_{k \rightarrow \infty} \int_{\gamma_k} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta = 0.$$

Cauchy’s integral formula implies, for $z \in D_R \setminus \overline{D_{\epsilon_0}}$, $R \in (a + \epsilon_0, 1)$,

$$e^{F(z)} = \frac{1}{2\pi i} \int_{\partial D_R} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{\epsilon_0}} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\partial D_R} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta$$

where all loops are positively oriented. However, the Cauchy integral on the right hand side defines a holomorphic function on entire $D_R$, hence, $e^{F}$ can be extended to a holomorphic $F$ on $D_R$. By Cauchy’s integral formula, for any $z \in D_{a/2}$,

$$|F(z)| = \left| \frac{1}{2\pi i} \int_{\gamma_k} \frac{F(\zeta)}{\zeta - z} \, d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma_k} \frac{e^{F(\zeta)}}{\zeta - z} \, d\zeta \right|$$

$$\leq \frac{1}{a\pi} \int_{\gamma_k} |e^{F}|$$

where in the second equality we used the fact that $F$ equals $e^{F}$ on $\gamma_k$. But (3.2) then implies $F(z) = 0$ by letting $k \rightarrow \infty$, which in turn asserts $F(z) = 0$. Since $z$ is arbitrary in $D_{a/2}$, we conclude that $F$ vanishes identically on $D_{a/2}$, hence on $D_R$. This further implies that $e^{F}$ must vanish identically. But this is impossible for it could only happen if $w = -\infty$ everywhere in $D \setminus \overline{D_a}$.

**Theorem 3.5.** Let $f : D \setminus \{0\} \rightarrow \mathbb{R}^n$ be a smooth immersion that satisfies

1. $\|\nabla f\|_{L^\infty(D \setminus \{0\})} < +\infty$,
2. $\int_D |A|^2 < +\infty$, where $A$ is the second fundamental form of $f$ on $D \setminus \{0\}$.

Set $g = df \otimes df$. Then for any $\delta \in (0, 1)$, $(D_\delta \setminus \{0\}, g)$ is conformal to $(D \setminus \{0\}, g_0)$.

**Proof.** In light of the proof of Theorem 2.1, the punctured Riemannian disk $(D \setminus \{0\}, g)$ has only three distinct conformal types, and to establish the theorem, it suffices to prove that $(D \setminus \{0\}, g)$ is not conformal to $(D \setminus \overline{D_a}, g_0)$ for any $a \in (0, 1)$.

Suppose there is a conformal diffeomorphism

$$\varphi : D \setminus \overline{D_a} \rightarrow D \setminus \{0\}$$
Proposition 3.4. Let $f: D \setminus \{0\} \to \mathbb{R}^n$ be a smooth immersion such that $\|\nabla f\|_{L^\infty(D)} < +\infty$, and let $\gamma_\epsilon(\theta) = (\epsilon \cos \theta, \epsilon \sin \theta)$ on $D \setminus \{0\}$. Since $\|\nabla f\|_{L^\infty(D)} < +\infty$, the length of $\gamma_\epsilon$ in the induced metric $g$ by $f$ satisfies

$$L_g(\gamma_\epsilon) = \frac{2\pi \sqrt{g(\gamma'_\epsilon(\theta), \gamma'_\epsilon(\theta))}}{\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0.$$ 

Therefore, $L_g(\varphi^{-1}(\gamma_\epsilon)) = L_g(\gamma_\epsilon) \to 0$ as $\epsilon \to 0$, which contradicts Proposition 3.4. \qed

Remark. Theorem 3.5 is not true if we replace $\|\nabla f\|_{L^\infty} < +\infty$ with $\mu(f) < +\infty$. For example, $f(z) = f(re^{i\theta}) = (e^{i\theta}, r)$ is a smooth immersion of $D \setminus \{0\}$ in $\mathbb{R}^3$ with $\mu(f) + \|A\|_{L^2} < +\infty$. However, $f(D \setminus \{0\}) = S^1 \times (0, 1)$ is not conformal to $D \setminus \{0\}$.

3.2. Existence of conformal structures of non-smooth immersions away from the puncture. We first define the function space that we will work with. For $0 < \lambda < \Lambda$, we denote

$$BL^n(D, \lambda, \Lambda) = \{f \in W^{2,2}(D, \mathbb{R}^n) : \lambda g_0 < df \otimes df < \Lambda g_0, \quad \text{a.e.} \ x \in D \}.$$ 

Every $f \in BL^n(D, \lambda, \Lambda)$ is a local embedding in the sense of the following lemma.

Lemma 3.6. Let $f \in BL^n(D, \lambda, \Lambda)$. Then there exist $\delta$ and $\lambda_1 > 0$ which only depend on $\lambda$ and $\Lambda$, such that

$$\frac{|f(x) - f(x')|}{|x - x'|} \geq \lambda_1, \quad \forall \ |x|, |x'| < \delta.$$ 

Proof. Assume this is not true. Then we can find $x_k, x'_k \in D$, such that $x_k, x'_k \to 0$ and

$$\frac{|f(x_k) - f(x'_k)|}{|x_k - x'_k|} \to 0, \quad \text{as} \ k \to \infty.$$ 

Write $x'_k = x_k + r_k e_k$ for some unit vector $e_k$ for $r_k = |x_k - x'_k|$ and define

$$f_k(x) = \frac{f(x_k + r_k e_k) - f(x_k)}{r_k}.$$ 

Without loss of generality, we assume $e_k \to e := (0, 1) \in \mathbb{R}^2$. Then $f_k(e) \to 0 \in \mathbb{R}^2$ as

$$|f(x_k + r_k e) - f(x_k)| \leq |f(x_k + r_k e_k) - f(x_k)| + |f(x_k + r_k e) - f(x_k + r_k e_k)| \leq o(r_k) + \lambda r_k |e_k - e|.$$ 

It is easy to check that

$$\int_{D_2} |\nabla^2 f_k|^2 \to 0, \quad \|\nabla f_k\|_{L^\infty(D_2)} \leq \Lambda.$$ 

Then we may assume $f_k \to f_0 = ax^1 + bx^2$ for some $a, b \in \mathbb{R}^n$ strongly in $W^{2,2}(D) \cap C^0(D)$. Thus $f_0(0, 1) = 0$ which implies $b = 0$.

However, since $df_k \otimes df_k(x) = df \otimes df(x_k + r_k x)$, we have $df_k \otimes df_k \geq \lambda((dx^1)^2 + (dx^2)^2)$ for a.e. $x$, and $df_k \otimes df_k$ converges strongly in $L^p$ for any $p > 0$. Thus $b \neq 0$ as $\lambda > 0$. A contradiction. \qed
Next, we show that every \( f \) in \( BL^n(D, \lambda, \Lambda) \), which is an embedding in the sense of Lemma 3.6, can be approximated by smooth ones with similar geometric properties.

**Lemma 3.7.** Let \( f \in BL^n(D, \lambda, \Lambda) \) and assume \( f \) is an embedding. Then we can find \( f_k \in C^\infty(D, \mathbb{R}^n) \) such that for any fixed \( r \in (0, 1) \), \( f_k \) is an embedding on \( D_r \) and \( f_k \) converges to \( f \) strongly in \( W^{2,2}(D_r) \). Moreover, we have for large \( k \)

\[
\lambda \frac{1}{2} g_0 < df_k \otimes df_k < 2\Lambda g_0 \quad \text{on} \quad D_r, \quad \lim_{k \to +\infty} \int_{D_r} |A_{f_k}|^2 = \int_{D_r} |A_f|^2 d\mu_f.
\]

**Proof.** Take a nonnegative cut-off function \( \eta \) supported in \( D \) with \( \int_{\mathbb{R}^2} \eta = 1 \). Define

\[
f_k(x) = \int_{\mathbb{R}^2} f(x - t_k y) \eta(y) d\sigma_y, \quad \text{where} \quad x \in D_r \quad \text{and} \quad t_k \to 0.
\]

Then \( f_k \) is smooth and converges to \( f \) strongly in \( W^{2,2}(D_r) \).

First, we prove that

\[df_k \otimes df_k(x) \geq \frac{\lambda}{2} g_0\]

when \( x \in D_r \) and \( k \) is sufficiently large. Assume this is not true. Then we can find \( x_k \in D_r \), such that

\[df_k \otimes df_k(x_k) < \frac{\lambda}{2} g_0\]

and we may assume \( x_k \to x_0 \in D_r \). By (3.4), we have

\[
\nabla f_k(x_k) = \int_{\mathbb{R}^2} (\nabla_x f(x_k - t_k y)) \eta(y) d\sigma_y = -\int_D \left( \nabla_y \left( f(x_k - t_k y) - \frac{f(x_k)}{t_k} \right) \right) \eta(y) d\sigma_y.
\]

Let

\[f'_k(y) = \frac{f(x_k - t_k y) - f(x_k)}{t_k}.
\]

Since

\[|\nabla_y f'_k| < \sqrt{2\Lambda}\]

and

\[\int_D |\nabla^2 f'_k|^2 d\sigma_y = \int_{D_{t_k}(x_k)} |\nabla^2 f|^2 d\sigma \to 0,
\]

we may assume that \( f'_k(y) \) converges to \( ay^1 + by^2 \) strongly in \( W^{2,2}(D) \) for some \( a, b \in \mathbb{R}^n \). Then we get

\[\nabla f_k(x_k) \to \int_D (a, b) \eta = (a, b).
\]

However, since \( df'_k \otimes df'_k \geq \lambda g_0 \), we have \(|a|^2, |b|^2 \geq \lambda \). It contradicts the choice of \( x_k \).

Thus we can conclude \( df_k \otimes df_k \geq \frac{\lambda}{2} g_0 \). The upper estimate \( df_k \otimes df_k \leq 2\Lambda g_0 \) is obvious.

Next, we prove that \( f_k \) is an embedding on \( D_r \) for large \( k \). Since \( f_k \) is smooth with \( df_k \otimes df_k \geq \frac{\lambda}{2} g_0 \), we can find \( \delta > 0 \), such that \( f_k \) is an embedding on \( D_\delta(x) \) for any \( x \in D_r \). Assume \( f_k \) is not an embedding on \( D_r \). Then we can find \( x_k \) and \( x'_k \) in \( D_r \) with \(|x_k - x'_k| \geq \delta > 0 \) and \( f_k(x_k) = f_k(x'_k) \). Up to a subsequence, assume \( x_k \to x_0 \) and \( x'_k \to x'_0 \). Since \( f_k \) converges to \( f \) in \( C^0(D_r) \), we get \( f(x_0) = f(x'_0) \) which contradicts that \( f \) is an embedding. Therefore, \( f_k \) is an embedding for all large \( k \).
Lastly, we prove the equality in (3.3). Since $f_k$ converges strongly to $f$ in $W^{2,2}(D_r)$ for any $r$, $\nabla f_k$ converges strongly in $L^2$. Then a subsequence of $\nabla f_k$ converges almost everywhere. By Egorov’s Theorem, for any $\delta > 0$, we can choose $E$, s.t. $\mu(D_r \setminus E) < \delta$, and $\nabla f_k$ converges uniformly on $E$. The second fundamental form $A_k$ is the normal component of the Hessian of $f_k$: for $1 \leq i, j, p, q, m \leq 2$

\begin{equation}
A_{k,ij} = \frac{\partial^2 f_k}{\partial x^i \partial x^j} - \frac{\partial^2 f_k}{\partial x^i \partial x^j} \cdot \sum_{m,p} \frac{\partial f_k}{\partial x^m} g_{mp} \frac{\partial f_k}{\partial x^p},
\end{equation}

where $(g_k^{pq}) = \left( \frac{\partial f_k}{\partial x^p} \cdot \frac{\partial f_k}{\partial x^q} \right)^{-1}$ is the inverse matrix of $g_k$. Since

$$g_k^{-1} = \frac{1}{\det(g_k)} \begin{pmatrix} g_{k,22} & -g_{k,12} \\ -g_{k,12} & g_{k,11} \end{pmatrix}$$

and

$$\frac{\lambda^2}{4} \leq \det(g_k)$$

because $\frac{\lambda}{2} g_0 \leq df_k \otimes df_k$, the inverse matrix $g_k^{-1}$ is bounded: $|g_k^{pq}| < C$. Then the uniform convergence of $\nabla f_k$ on $E$ and the strong convergence of $f_k$ in $W^{2,2}(D_r)$ imply

$$\int_E |A_f|^2 d\mu_f = \lim_{k \to +\infty} \int_E |A_{f_k}|^2 d\mu_{f_k}.$$ 

Moreover, using $|\nabla f_k| \leq \sqrt{2} \Lambda$ on $D_r \setminus E$, we have

$$\left( \int_{D_r \setminus E} |A_{f_k}|^2 d\mu_{f_k} \right)^{\frac{1}{2}} < C \|\nabla^2 f_k\|_{L^2(D_r \setminus E)} \to C \|\nabla^2 f\|_{L^2(D_r \setminus E)} \text{, as } k \to \infty.$$ 

Then

$$\limsup_{k \to +\infty} \left| \int_{D_r} |A_{f_k}|^2 d\mu_{f_k} - \int_{D_r} |A_f|^2 d\mu_f \right| \leq C \int_{D_r \setminus E} |\nabla^2 f|^2 dx + \int_{D_r \setminus E} |A_f|^2.$$ 

Recall $|A_f| \in L^2(D)$ by assumption. Letting $\mu(D_r \setminus E) \to 0$, we finish the proof. \hfill \Box

Next, we prove that a bi-Lipschitz immersion of $D$ in $\mathbb{R}^n$ with small $\|A\|_{L^2}$ must be a $W^{2,2}$ conformal immersion. We begin with

Lemma 3.8. Let $f_k$ be a smooth conformal immersion of $D$ in $\mathbb{R}^n$ with

\begin{equation}
\liminf_{r \to 1} d_{g_k}(0, \partial D_r) \geq \delta > 0
\end{equation}

where $d_{g_k}$ is the distance function w.r.t. the metric $g_k = df_k \otimes df_k$, $\delta$ is a constant and

$$\int_D |A_{f_k}|^2 d\mu_{f_k} < 4\pi - \gamma.$$ 

Assume $g_k = e^{2u_k} g_0$, $\mu_{g_k}(D) \leq \Lambda$ and $f_k(0) = 0$. Then $f_k$ converges weakly in $W^{2,2}(D_r)$ to a nonconstant $W^{2,2}$-conformal map, for any $r < 1$. Moreover, we have

$$\|u_k\|_{L^\infty(D_r)} + \|\nabla u_k\|_{L^2(D_r)} < C(\gamma, \Lambda, r), \quad \forall r \in (0, 1).$$
Proof. By Theorem 5.1.1 in [4], it suffices to show $f_k$ does not converge to a point. Assume, on the contrary, that $f_k$ converges to a point. Then $u_k$ converges to $-\infty$ uniformly on $D_{\frac{1}{2}}$ (cf. [4], [6]). By (3.6), for any fixed $\theta$, we have
\[
\int_0^1 e^{u_k(te^{i\theta})} dt \geq \liminf_{r \to 1} d_{g_k}(0, \partial D_r) \geq \delta.
\]
Then we can find $r_k \in (\frac{1}{2}, 1)$ such that $u_k(r_ke^{i\theta}) > c(\delta)$ for all large $k$.

By (3.1) and Corollary 3.2, there is a function $v_k$ with $\|v_k\|_{L^\infty(D)} < C(\gamma)$ solving
\[
-\Delta v_k = K_{f_k}e^{2u_k}.
\]

Then we have
\begin{enumerate}
\item $(u_k - v_k) \to -\infty$ uniformly on $D_{\frac{1}{2}}$;
\item for any fixed $\theta$, there exists $r_k \in (\frac{1}{2}, 1)$, such that $(u_k - v_k)(r_ke^{i\theta}) \geq c'(\delta)$. Here $c'(\delta)$ may be negative.
\end{enumerate}

It follows that $u_k - v_k$ has a minimum in $D$. The strong maximum principle implies the harmonic function $u_k - v_k$ must be constant. However,
\[
c'(\delta) \leq (u_k - v_k)(r_ke^{i\theta}) = \min_D (u_k - v_k) \to -\infty \text{ as } k \to \infty
\]
yields a contradiction. \hfill \Box

Now we are ready to show the existence of conformal structures for the weak immersions with small total curvature.

**Proposition 3.9.** Let $f \in BL^n(D_2, \lambda, \Lambda)$ and assume $f$ is an embedding with
\[
\int_{D_2} |A_f|^2 < 4\pi - \gamma.
\]

Then there exists a bijective map $\varphi : D \to D$, such that $\varphi, \varphi^{-1} \in W_{loc}^{2,2}(D) \cap W_{loc}^{1,\infty}(D)$ and $f \circ \varphi \in W_{con,f,loc}^{2,2}(D, \mathbb{R}^n)$.

**Proof.** By Lemma 3.7, there exists a sequence of smooth embeddings $\{f_k\}$ converging to $f$ strongly in $W^{2,2}(D_r)$ for any $r \in (0, 2)$ with $\|A_{f_k}\|_{L^2(D_r)} \to \|A_f\|_{L^2(D_r)}$ and $\mu(D_r, g_k) \to \mu(D_r, g_f) < \infty$ as $k \to \infty$, where $g_k, g_f$ are the induced metrics by $f_k, f$, respectively.

Since $f_k$ is smooth, the Riemannian disk $(D_2, g_k)$ is either conformal to $\mathbb{C}$ or $(D, g_0)$ by the uniformization theorem. So the simply connected proper subdomain $(D, g_k)$ is conformal to $(D, g_0)$. There is a diffeomorphism $\varphi_k : (D, g_0) \to (D, g_k)$ with
\[
\tilde{g}_k := \varphi_k^*(g_k) = e^{2u_k}((dy^1)^2 + (dy^2)^2)
\]
for some smooth function $u_k$. Further, we can assume $f_k \circ \varphi_k(0) = 0$. The areas $\mu(D, \tilde{g}_k) = \mu(D, g_k)$ are uniformly bounded by some $\Lambda$ and for large $k$
\[
\|A_{f_k \circ \varphi_k}\|_{L^2(D, \tilde{g}_k)} = \|A_{f_k}\|_{L^2(D, g_k)} \leq 4\pi - \frac{\gamma}{2}.
\]

Note that $f_k$ converges to $f$ in $C^0(D_2)$ and $f$ is an embedding. For large $k$
\[
d_{\mathbb{R}^n}(0, f_k(\partial D)) > \frac{1}{2} d_{\mathbb{R}^n}(0, f(\partial D)) := \delta > 0.
\]
Since $f_k \circ \varphi_k$ is an isometry from $(D, \bar{g}_k)$ to $(f_k(D), g_k)$, for every $r \in (0, 1)$ it holds
\[
d_{\bar{g}_k}(0, f_k \circ \varphi_k(\partial D_r)) = d_{g_k}(0, f_k(\partial D_r)).\]
As the extrinsic distance is no larger than the intrinsic distance, we have for large $k$
\[
\liminf_{k \to 1} d_{\bar{g}_k}(0, f_k \circ \varphi_k(\partial D_r)) = \liminf_{k \to 1} d_{g_k}(0, f_k(\partial D_r))
\geq \liminf_{k \to 1} d_{g^n}(0, f_k(\partial D_r))
> \frac{1}{2} d_{g^n}(0, f(\partial D))
= \delta.
\]
Applying Lemma 3.8 to the conformal maps $f_k \circ \varphi_k$,
\[
\|u_k\|_{L^\infty(D_r)} + \|\nabla u_k\|_{L^2(D_r)} < C(\gamma, r, \Lambda) \quad \text{for any } r \in (0, 1).
\]
Let $G_k(y) = (g_{ij}(\varphi_k(y)))$ and let $J(\varphi_k)$ be the Jacobi of $\varphi_k$. Then
\[
e^{2u_k} I = J^T(\varphi_k)G_kJ(\varphi_k).
\]
Hence
\[
J(\varphi_k)J(\varphi_k)^T = e^{2u_k} G_k^{-1}.
\]
It follows
\[
\lambda(r)g_0 < d_{\varphi_k} \otimes d_{\varphi_k} \leq \Lambda(r)g_0,
\]
where $0 < \lambda(r) < \Lambda(r)$. Noting that $|\varphi_k| < 1$ as the image of $\varphi_k$ lies in $D$, we may assume $\varphi_k$ converges in $C^0(D_r)$ to a map $\varphi$ for any $r \in (0, 1)$.
Next, we bound the Hessian of $\varphi_k$ uniformly in $k$ for any $r \in (0, 1)$:
\[
\|\nabla^2 \varphi_k\|_{L^2(D_r)} < C(r).
\]
To see (3.9), we first let $h_k(y)$ be the $2 \times 2$-matrix whose entries are given by
\[
h_{k,ij}(y) = \frac{\partial \varphi_k(y)}{\partial y^i} \frac{\partial \varphi_k(y)}{\partial y^j} = e^{2u_k(y)} g_{k}^{ij}(\varphi_k(y)).
\]
Then
\[
h_k^{-1}(y) = (h_{k}^{ij}(y)) = e^{-2u_k(y)} G(y).
\]
By (3.7), (3.8) and the fact that $\frac{1}{2} g_0 \leq d_{\varphi_k} \otimes d_{\varphi_k} \leq 2\Lambda g_0$, we have
\[
\int_{D_r} \left| \nabla h_k^{-1} \right|^2 \leq C \int_{D_r} e^{-4u_k} \left( |\nabla f_k|^4 |\nabla u_k|^2 + |\nabla \varphi_k|^2 |\nabla^2 f_k|^2 \right) < C(r).
\]
Since $\nabla h_k = -h_k(\nabla h_k^{-1}) h_k$ and $h_k$ is bounded, we have
\[
\|\nabla h_k\|_{L^2(D_r)} + \|\nabla h_k^{-1}\|_{L^2(D_r)} < C(r).
\]
We now estimate $\nabla^2 \varphi_k$ as follows, for simplicity, write $\partial_i \varphi_k, \partial_{ij} \varphi_k$ for the first and second order derivatives respectively. Since $\partial_i \varphi_k(y), i = 1, 2$ form a basis of $T_{\varphi_k(y)} \varphi_k(D)$ and
\( h_k = \langle \nabla \varphi_k, \nabla \varphi_k \rangle \), we have
\[
\begin{align*}
\partial_{11} \varphi_k &= \langle \partial_{11} \varphi_k, \partial_i \varphi_k \rangle h_{ij} \partial_j \varphi_k \\
&= \langle \partial_{11} \varphi_k, \partial_i \varphi_k \rangle h_{ij} \partial_j \varphi_k + \langle \partial_{11} \varphi_k, \partial_2 \varphi_k \rangle h_{2j} \partial_j \varphi_k \\
&= \frac{1}{2} \left( \partial_1 |\partial_1 \varphi_k|^2 \right) h_{1j} \partial_j \varphi_k + \left( \partial_1 \langle \partial_1 \varphi_k, \partial_2 \varphi_k \rangle - \langle \partial_1 \varphi_k, \partial_2 \varphi_k \rangle \right) h_{2j} \partial_j \varphi_k \\
&= \frac{1}{2} \left( \partial_1 h_{1,11} \right) h_{1j} \partial_j \varphi_k + \frac{1}{2} \left( \partial_1 h_{k,11} \right) h_{2j} \partial_j \varphi_k
\end{align*}
\]

and similar for \( \partial_{22} \varphi_k \);
\[
\begin{align*}
\partial_{12} \varphi_k &= \langle \partial_{12} \varphi_k, \partial_i \varphi_k \rangle h_{ij} \partial_j \varphi_k \\
&= \langle \partial_{12} \varphi_k, \partial_1 \varphi_k \rangle h_{1j} \partial_j \varphi_k + \langle \partial_{12} \varphi_k, \partial_2 \varphi_k \rangle h_{2j} \partial_j \varphi_k \\
&= \frac{1}{2} \left( \partial_2 h_{k,11} \right) h_{1j} \partial_j \varphi_k + \frac{1}{2} \left( \partial_2 h_{k,22} \right) h_{2j} \partial_j \varphi_k
\end{align*}
\]

Hence (3.9) holds.

Now, we may assume \( \varphi_k \) converges to \( \varphi \) weakly in \( W^{2,2}_\text{loc}(D) \). Thus \( \varphi \in W^{2,2}_\text{loc}(D) \cap W^{1,\infty}_\text{loc}(D) \) and satisfies
\[
\begin{align*}
e^{2u(y)} \delta_{ij} &= \frac{\partial \psi^p}{\partial y^i}(y) \frac{\partial \psi^q}{\partial y^j}(y) g_{pq}(\varphi(y)), \\
J(\varphi)J(\varphi)^T &= e^{2u} G^{-1}, \\
\lambda' g_0 < d\varphi \otimes d\varphi < \Lambda' g_0,
\end{align*}
\]

where \( 0 < \lambda' < \Lambda \).

Let \( \psi_k(x) = \varphi_k^{-1}(x) \). We have
\[
ge_{k,ij}(x) = \frac{\partial \psi_k(x)}{\partial x^i} \frac{\partial \psi_k(x)}{\partial x^j} e^{2u_k(\psi_k(x))}.
\]

Hence \( \| \nabla \psi_k \|_{L^\infty(D_r)} < C \). Thus we may assume \( \psi_k \) converges in \( C^0 \) to a map \( \psi \). Since \( \varphi_k(\psi_k(x)) = x \), we see \( \varphi(\psi(x)) = x \). Then \( \varphi \) is a bijective.

Using similar arguments, we can prove \( \| \nabla^2 \psi_k \|_{L^2(D_r)} < C(r) \). In fact, we just need to replace \( h_k \) by \( h_k^{-1} \) and \( \varphi_k \) by \( \psi_k \) in (3.10) and (3.11). Then \( \psi \in W^{2,2}_\text{loc}(D) \cap W^{1,\infty}_\text{loc}(D) \). \( \square \)

3.3. Extending conformal structures at the puncture. With the preparation in sections 3.1 and 3.2, we now state and prove the main result in section 3.

**Theorem 3.10.** Let \( f \in W^{2,2}_\text{loc}(D_2 \setminus \{0\}, \mathbb{R}^n) \) be a \( C^0 \) immersion. Assume that

1. For any \( r \in (0,1) \) there is \( \lambda(r) > 0 \), s.t. \( g := df \otimes df > \lambda(r) g_0 \) almost everywhere in \( D_2 \setminus D_r \), where \( g_0 \) is the Euclidean metric.
2. \( \| \nabla f \|_{L^\infty(D_r)} < +\infty \).
3. \( \int_{f(D_2 \setminus \{0\})} |A|^2 < +\infty \), where \( A \) is the second fundamental form of \( f \) on \( D_2 \setminus \{0\} \).

Then there exists a homeomorphism \( \phi : D \to D \) with \( \phi(0) = 0 \) that satisfies \( \phi, \phi^{-1} \in W^{2,2}_\text{loc}(D \setminus \{0\}) \cap W^{1,\infty}_\text{loc}(D \setminus \{0\}) \) and \( (\phi^{-1})^* g = e^{2u} g_0 \) on \( D \setminus \{0\} \), where \( u \in W^{1,2}_\text{loc}(D \setminus \{0\}) \cap L^\infty(D \setminus \{0\}) \). Moreover, \( f \circ \phi^{-1} \) is branched \( W^{2,2} \)-conformal immersion of the Euclidean unit disk \( D \) in \( \mathbb{R}^n \), with 0 as its only possible branch point.
Theorem 3.9, there exists $\varphi_{D_\delta(p)} : D_\delta(p) \to D$, $\varphi_{D_\delta(p)}, \varphi_{D_\delta(p)}^{-1} \in W^{1,\infty} \cap W^{2,2}$ and
\begin{equation}
(\varphi_{D_\delta(p)})^* g = e^{2u_{D_\delta(p)}} g_0.
\end{equation}

Thus, we can find a countable open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $D_2 \setminus \{0\}$, where $U_i$ is a disk $D_\delta(p_i)$ as above, such that there is a homeomorphism $\varphi_i$ from $U_i$ to an open set of $\mathbb{R}^2$, such that $\varphi_i, \varphi_i^{-1} \in W^{1,\infty} \cap W^{2,2}$. Then
\[
\phi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)
\]
is in $W^{2,2} \cap W^{1,\infty}$ with
\[
e^{2u_i} g_0 = \phi_{ij}^*(e^{2u_j} g_0).
\]
Thus, $\phi_{ij}$ is conformal. Furthermore, we can assume that for each pair of $i, j$ the orientation induced by $\varphi_{ij}$ on $U_i \cup U_j$ is the same as that of the oriented surface $D \setminus \{0\}$. Then $\phi_{ij}$ is holomorphic. Then $\mathcal{A} = \{\{U_i, \varphi_i\} : i \in I\}$ is an atlas on $D_2 \setminus \{0\}$ which defines a smooth structure and a complex structure $\mathcal{C}$ on $D_2 \setminus \{0\}$. The identity map $\text{Id} : (D_2 \setminus \{0\}, \mathcal{C}) \to (D \setminus \{0\}, g_0)$, in local coordinates, is $\text{Id} \circ \varphi_i^{-1}(y)$ which is in $W^{2,2} \cap W^{1,\infty}$.

Next, we show $\lim_{\mathcal{A}} \phi(z) = 0$. We will use the following:

Fact: if $\varphi : D \setminus \{0\} \to D \setminus \{0\}$ is a continuous homeomorphism, then for any $z_k \to 0$, $|\varphi(z_k)| \to 0$ or 1, i.e., $\varphi(z_k)$ converges to $\partial D \cup \{0\}$.

This is because $\varphi(z_k)$ cannot have any accumulation points in $D \setminus \{0\}$: if this were not true, then there would exist a small disk $D(r_0)(q) \subset D \setminus \{0\}$ centered at an accumulation point $q \in D \setminus \{0\}$ and $\varphi^{-1}(D(r_0)(q))$ contains infinitely many $z_k$; but this contradicts $x_k \to 0$. Moreover, for a given $\varphi$, only one of the limiting behavior of $|\varphi(z_k)|$ can occur: if there were $z_k \to 0$ and $z_k' \to 0$ with $|\varphi(z_k)| \to 0$ and $|\varphi(z_k')| \to 1$ then the intermediate value theorem would imply existence of $z_k'' \to 0$ with $|\varphi(z_k'')| = \frac{1}{2}$ hence $z_k''$ would have at least one accumulation point on $\partial D_{\frac{1}{2}}$.

By this Fact, $\lim_{z \to 0} \phi(z) = 0$ is equivalent to $\lim_{w \to 0} \phi^{-1}(w) = 0$. Now, we prove $\lim_{w \to 0} \phi^{-1}(w) = 0$ by contradiction. Assume that there exists $w_k \to 0$, such that $\phi^{-1}(w_k) \to p \in \partial D$. We claim that there exists a small $r > 0$ with $\phi^{-1}(D_r \setminus \{0\}) \subset D \setminus D_{\frac{1}{2}}$. Therefore, $\lim_{w \to 0} \phi^{-1}(w) = 0$
Otherwise, we can find $w'_k \to 0$ with $|\phi^{-1}(w'_k)| \leq \frac{1}{2}$. By the intermediate value theorem, there exists $w''_k$ in $D[w_k] \setminus D[w'_k]$ such that $|\phi^{-1}(w''_k)| = \frac{1}{2}$. But this is impossible for $w''_k \to 0$. The claim asserts $f \circ \phi \in W^{2,2}_{\text{conf,loc}}(D \setminus D_{\frac{1}{2}}, \mathbb{R}^n)$, and $\phi^{-1}(\gamma_\epsilon) \subset D \setminus D_{\frac{1}{2}}$, where $\gamma_\epsilon = (\epsilon \cos \theta, \epsilon \sin \theta)$. Then

$$
\lim_{\epsilon \to 0} L_{g_{f(\phi)}}(\phi^{-1}(\gamma_\epsilon)) = \lim_{\epsilon \to 0} L_{g_f}(\gamma_\epsilon) = 0.
$$

But this contradicts Proposition 3.4. \hfill \Box

For general surfaces with punctures, we have:

**Corollary 3.11.** Let $(\Sigma, h)$ be an oriented surface (may not compact) and let $S \subset \Sigma$ be a finite set. Suppose $f \in W^{2,2}_{\text{loc}}(\Sigma \setminus S, \mathbb{R}^n)$ and $f$ is a $C^0$ immersion on $\Sigma \setminus S$. Assume that

1) For any $r \in (0, 1)$, there exists $\lambda(r)$, such that it holds a.e. in $\Sigma \setminus \bigcup_{p \in S} B_r(p)$

$$
g = df \otimes df > \lambda(r) h.
$$

2) $\|\nabla f\|_{L^\infty(\Sigma, h)} < \infty$.

3) $\int_{\Sigma} |A|^2 < \infty$, where $A$ is the second fundamental form of $f$ on $\Sigma \setminus S$.

Then there is a complex structure $c$ on $\Sigma$ such that $f : (\Sigma, c) \to \mathbb{R}^n$ is a branched $W^{2,2}$-conformal immersion with its branch locus contained in $S$.

**Proof.** Applying Lemma 3.6 and Proposition 3.9, for any $p \notin S$, we choose a neighborhood $U$ of $p$ and a homeomorphism $\varphi_U$ from $U$ to $D$, such that $\varphi_U \in W^{2,2}(U, \mathbb{R}^2) \cap W^{1,\infty}(U, \mathbb{R}^2)$ and

$$
(\varphi_U^{-1})^*(g) = e^{2w} g_0.
$$

For $p \in S$, by Theorem 3.10, there exist a neighborhood $U$ of $p$ with $U \cap S = p$ and a homeomorphism $\varphi_U$ from $U$ to $D$ that is in $W^{2,2}_{\text{loc}}(U \setminus \{0\}, \mathbb{R}^2) \cap W^{1,\infty}_{\text{loc}}(U \setminus \{0\}, \mathbb{R}^2)$ and $\varphi_U(p) = 0$ and (3.14) holds. As in the proof of Theorem 3.10, we have an atlas $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ of $\Sigma$, such that for any $i, j \in I$ the transition function $\phi_{ij} = \varphi_j \circ \varphi_i^{-1}$ is holomorphic, using the given orientation on $\Sigma$ to adjust if necessary, and in $W^{2,2}_{\text{loc}} \cap W^{1,\infty}_{\text{loc}}$. Thus $\mathcal{A}$ determines a smooth structure and a complex structure on $\Sigma$. Take a smooth metric $h'$ that is compatible with the new complex structure $c$. The identity map from $(\Sigma, h')$ to $(\Sigma, h)$ is homeomorphic and is in $W^{2,2}_{\text{loc}}(\Sigma \setminus S) \cap W^{1,\infty}_{\text{loc}}(\Sigma \setminus S)$. Therefore, $f : (\Sigma, c) \to \mathbb{R}^n$ is a branched $W^{2,2}$-conformal immersion with $S$ as its possible branch locus. \hfill \Box

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