The large genus limit of the infimum of the Willmore energy

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Abstract: The Willmore energy of a surface immersed into $\mathbb{R}^n$ is the integral of its squared mean curvature. It is known that the infimum $\beta^n_p$ of the Willmore energy among all closed oriented surfaces of genus $p$ in $\mathbb{R}^n$ is attained by a smooth embedded surface, and that $4\pi < \beta^n_p < 8\pi$ for $p \geq 1$. We show that $\beta^n_p$ converges to $8\pi$ as $p$ goes to infinity.

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1 Introduction

The Willmore energy of an immersed surface $\Sigma \hookrightarrow \mathbb{R}^n$ with mean curvature vector $\vec{H}$ and induced area measure $\mu$ is given by

$$\mathcal{W}(\Sigma) = \frac{1}{4} \int_{\Sigma} |\vec{H}|^2 \, d\mu.$$ 

Let $\mathcal{C}(n, p)$ be the class of oriented, closed (i.e. compact without boundary), smoothly immersed surfaces $\Sigma$ with genus($\Sigma$) = $p$, and put

$$\beta^n_p = \inf \{ \mathcal{W}(\Sigma) | f \in \mathcal{C}(n, p) \}.$$  \hspace{1cm} (1.1)
It is well-known that $W(\Sigma) \geq 4\pi$ for any closed immersed surface, with equality only for round spheres \cite{W}. In \cite{Sim93} L. Simon proved the existence of smooth minimizers in $C(n, p)$ under the Douglas-type condition

$$\beta^n_p < 4\pi + \min \left\{ \sum_{i=1}^r (\beta^n_{p_i} - 4\pi) : 1 \leq p_i < p, \sum_{i=1}^r p_i = p \right\} =: \tilde{\beta}_p^n. \quad (1.2)$$

In particular he obtained the existence for $p = 1$. The inequality (1.2) was proved later in \cite{BaKu03}, so $\beta^n_p$ is attained for all $n, p$ and $\beta^n_p > 4\pi$ for $p \geq 1$. By conformal invariance the area of a minimal surface in $S^3$ equals the Willmore energy of the surface in $\mathbb{R}^3$ obtained by stereographic projection \cite{Wei}, which leads to an upper bound for $\beta^n_p$. Namely, Pinkall \cite{KP86} and independently Kusner \cite{Kus87, Kus89} observed that the minimal surfaces $\xi_{p,1}$ in $S^3$ described by Lawson in \cite{Lw70} have area less than $8\pi$. In summary we know that

$$4\pi < \beta^n_p < 8\pi \quad \text{for} \quad p \geq 1. \quad (1.3)$$

An important consequence of the upper bound is that minimizers are automatically embedded, due to an inequality of Li and Yau \cite{LY82}. It was conjectured that the $\beta^n_p$ might be monotonically increasing in $p$, see \cite{KP86, p. 446}, and that the projected $\xi_{p,1}$ could in fact be minimizers for their genus \cite{Kus89, p. 318 and p. 344}. For large $p$ these surfaces look like two spheres connected by minimal handles, see \cite{Kus93, p. 293} for $p = 5$, in particular their Willmore energy converges to $8\pi$ as $p \rightarrow \infty$ \cite{Kus87}. Here we prove the following.

**Theorem 1.1** Let $\beta^n_p$ be the infimum of the Willmore energy among oriented, closed surfaces of genus $p$ immersed into $\mathbb{R}^n$. Then

$$\lim_{p \rightarrow \infty} \beta^n_p = 8\pi. \quad (1.4)$$

The actual indirect argument to prove the theorem is presented in the final section 4. It involves the construction of a limit surface, whose main properties are obtained in Lemma 3.1. An important point is to rule out that the surface is minimal, and for this we need the classification in Theorem 2.1.

We would like to thank Tom Ilmanen for asking the question addressed in this paper when one of us gave a talk in Zürich.

2 Stationary currents with small density at infinity

In this section we prove that stationary integral currents without boundary and density smaller than two at infinity are planes. Here we call an integral current $M$ stationary if the underlying varifold $\mu_M$ is stationary, that is $\bar{H}_{\mu_M} = 0$. In the following we always use $\omega_2 := \frac{\pi}{2}$.

**Theorem 2.1** Let $M$ be a stationary integral 2-current in $\mathbb{R}^n$ with $\partial M = 0$ and density at infinity satisfying

$$\theta^2(\mu_M, \infty) := \lim_{r \rightarrow \infty} \frac{\mu_M(B_r(0))}{\omega_2 r^2} < 2. \quad (2.1)$$

Then $M = 0$ or $M$ is a unit density plane. \hfill \Box
We prove this theorem in several steps. First we collect for $\mu_M$ as in the theorem and more generally for any stationary integral 2–varifold $\mu$ some consequences of the monotonicity formula, see [Sim] (17.5). We have for all $x \in \mathbb{R}^n$

$$\theta^2(\mu, x) \leq \frac{\mu(B_\rho(x))}{\omega_2 \rho^2} \leq \lim_{r \to \infty} \frac{\mu(B_r(x))}{\omega_2 r^2} = \theta^2(\mu, \infty) < 2. \quad (2.2)$$

Next $\theta^2(\mu, \cdot) \geq 1$ almost everywhere with respect to $\mu$ since $\mu$ is integral, hence by upper semicontinuity of $\theta^2(\mu, \cdot)$, see [Sim] Corollary 17.8, $\theta^2(\mu, x) \geq 1$ for all $x \in \text{spt } \mu. \quad (2.3)$

We start with the case when $\mu$ has unit density at infinity.

**Proposition 2.1** Let $\mu$ be stationary integral 2–varifold. If

$$\theta^2(\mu, \infty) = 1, \quad (2.4)$$

then $\mu = \mathcal{H}^2 \llcorner P$ is a unit density plane.

**Proof:**

As $\mu \neq 0$, we have $\text{spt } \mu \neq \emptyset$. Combining (2.2) - (2.4), we see

$$1 \leq \theta^2(\mu, x) \leq \frac{\mu(B_\rho(x))}{\omega_2 \rho^2} \leq \theta^2(\mu, \infty) = 1 \quad \text{for all } x \in \text{spt } \mu.$$

Exploiting the non-negative term in the monotonicity formula as in the proof of [Sim] Theorem 19.3, we see that $\mu$ is a cone about any $x \in \text{spt } \mu$. Taking a point $x \in \text{spt } \mu$ where $T_x \mu$ exists with multiplicity $\theta(x) = 1$, which is true $\mu$-almost-everywhere, we see that $\mu = \mathcal{H}^2 \llcorner (x + T_x \mu)$, hence the proposition is proved. 

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**Remark:**

Let $M$ be as in Theorem 2.1 with $\theta^2(\mu_M, \infty) = 1$, then $\mu_M$ is a unit density plane by the theorem above. Moreover, the condition $\partial M = 0$ implies by means of the constancy theorem, see [Sim] Theorem 26.27, that the orientation function of $M$ is constant, and hence that $M$ is a standard, unit density plane in the sense of currents. \(\square\)

We want to apply the abstract dimension reduction argument of Federer, see [Sim] §A, to obtain smoothness for any $M$ as in Theorem 2.1. To this end, we need the following convergence result.

**Proposition 2.2** Let $M_j$ be a sequence of stationary integral 2–currents in $\mathbb{R}^n$ with $\partial M_j = 0$ and

$$\theta^2(\mu_{M_j}, \infty) \leq 2 - \delta \quad (2.5)$$

for some $\delta > 0$. Then for a subsequence

$$M_j \to M \quad \text{weakly as currents,} \quad (2.6)$$

$$\mu_{M_j} \to \mu_M \quad \text{weakly as varifolds,}$$

where $M$ is a stationary integral 2–current in $\mathbb{R}^n$ with $\partial M = 0$ and

$$\theta^2(\mu_M, \infty) \leq \liminf_{j \to \infty} \theta^2(\mu_{M_j}, \infty) \leq 2 - \delta. \quad (2.7)$$
Proof:
By (2.2), we get for a subsequence

$$M_j \to M$$ weakly as currents,

$$\mu_{M_j} \to \mu_\infty$$ weakly as varifolds.

The 2-current $M$ is integral by the compactness theorem for integral currents, see [Sim] 27.3, and has $\partial M = \mu_\infty$. On the other hand, the 2-varifold $\mu_\infty$ is stationary and integral by Allard’s integral compactness theorem, see [All72] Theorem 6.4 or [Sim] Remark 42.8. Using weak convergence and (2.2), we get

$$\liminf_{j \to \infty} \mu_{M_j}(B_\rho(x)) \leq \mu_\infty(\Omega),$$

hence again by (2.2) we deduce for the stationary varifold $\mu_\infty$ that

$$\theta^2(\mu_\infty, x) \leq \theta^2(\mu_\infty),$$

which gives (2.7) for $\mu_\infty$. It remains to prove

$$\mu_\infty = \mu_M. \quad (2.8)$$

By lower semicontinuity of the mass under convergence of currents and by weak convergence of the varifolds, we get for all open sets $\Omega \subseteq \mathbb{R}^n$ with $\mu_\infty(\partial \Omega) = 0$ that

$$\mu_M(\Omega) \leq \liminf_{j \to \infty} \mu_{M_j}(\Omega) = \mu_\infty(\Omega),$$

which yields $\mu_M \leq \mu_\infty$. Since $\theta^2(\mu_\infty, x) < 2$ and $\mu_\infty$ is integral, we conclude $\theta^2(\mu_\infty, x) = 1$ almost everywhere with respect to $\mu_\infty$. We select $x \in \mathbb{R}^n$ with $\theta^2(\mu_\infty, x) = 1$. By Allard’s regularity theorem, see [All72] Theorem 8.19 or [Sim] Theorem 23.1, there exists a neighbourhood $U(x)$ of $x$ in which $\mu_\infty$ is $C^{1,\alpha}$, actually smooth as $\mu_\infty$ is stationary, and moreover the convergence $\mu_{M_j}U(x) \to \mu_\infty U(x)$ is in $C^{1,\alpha}$. This yields

$$\mu_\infty U(x) \leftarrow \mu_{M_j}U(x) \to \mu_m U(x),$$

hence for $\Omega := \bigcup_{\theta^2(\mu_\infty, x) = 1} U(x)$ that $\mu_\infty \Omega = \mu_\infty \Omega$. As we have already seen that $\mu_M(\mathbb{R}^n - \Omega) \leq \mu_\infty(\mathbb{R}^n - [\theta^2(\mu_\infty, x) = 1]) = 0$, we get (2.8), and the proposition is proved.

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Proposition 2.3 Let $M$ be as in Theorem 2.1. Then $M$ is a smooth.

Proof:
The set of all $M$ as in Theorem 2.1 with $\theta^2(\mu_M, x) \leq 2 - \delta$ for some fixed $\delta > 0$ is closed under weak convergence of currents by Proposition 2.2.

By Allard’s regularity theorem we see that $x \in \mathbb{R}^n$ with $\theta^2(\mu_M, x) < 1 + \varepsilon$ for appropriate $\varepsilon = \varepsilon(n) > 0$ is a regular point of $\mu_M$. As regular points always have integral density and as $\theta^2(\mu_M) \leq 2 - \delta$ by (2.2), we get

$$\text{reg } M = \text{reg } \mu_M = [1 \leq \theta^2(\mu_M) < 1 + \varepsilon] = [\theta^2(\mu_M) = 1].$$
Combining with (2.3), we see
\[ \text{sing } M = \text{sing } \mu_M = [\theta^2(\mu_M) \geq 1 + \varepsilon]. \] (2.9)

To apply the abstract dimension reduction argument of Federer, see [Sim] §A, we consider \( M_j \) as above with \( M_j \to M \) weakly as currents and \( x_j \in \text{sing } M_j, x_j \to x_0 \). By Proposition 2.2, we get \( \mu_{M_j} \to \mu_M \) weakly as varifolds and by the monotonicity formula

\[ \frac{\mu_M(B_\varepsilon(x_0))}{\omega_2 \varepsilon^2} \geq \limsup_{j \to \infty} \frac{\mu_{M_j}(B_\varepsilon(x_j))}{\omega_2 \varepsilon^2} \geq \limsup_{j \to \infty} \theta^2(\mu_{M_j}, x_j), \]

hence

\[ \theta^2(\mu_M, x_0) = \lim_{\varepsilon \to 0} \frac{\mu_M(B_\varepsilon(x_0))}{\omega_2 \varepsilon^2} \geq \limsup_{j \to \infty} \theta^2(\mu_{M_j}, x_j) \geq 1 + \varepsilon, \]

and we conclude \( x_0 \in \text{sing } M \). Therefore the dimension reduction argument yields the alternative that either

\[ \text{sing } M = \emptyset \quad \text{for all above } M, \] (2.10)

or we have \( \dim \text{sing } M \leq d \) for a \( d \in \{0,1\} \), and there exists an \( M \) as above and a subspace \( L \subseteq \mathbb{R}^n \) of dimension \( d \) with

\[ \text{sing } M = L \quad \text{and} \quad (x \mapsto y + \varrho(x-y))_\varrho M = M \]

for all \( y \in L, \varrho > 0 \).

For \( d = 0 \), we see that \( M \) is regular in \( \mathbb{R}^n - \{0\} \), and \( \gamma := M \cap S^{n-1} \) is a smooth stationary curve on the sphere \( S^{n-1} \), hence consists of finitely many great circles which do not intersect. Then \( M \) is a union of finitely many planes, hence of one plane, as the number of planes cannot exceed \( \theta^2(\mu_M, \infty) < 2 \). We conclude that \( \text{sing } M = \emptyset \), which contradicts \( \text{sing } L \neq \emptyset \).

For \( d = 1 \), we put \( \Gamma := M \cap L^\perp \) and see \( M = \Gamma \times L \) where \( \Gamma \) is a one-dimensional cone at the origin which is smooth and stationary in \( L^\perp - \{0\} \). Therefore \( \Gamma \) consists of finitely many half lines. The number of these lines is at least two, as \( \theta^2(\mu_{L^\perp}, 0) \geq 1 \), and at most three since \( \theta^2(\mu_{L^\perp \times L}, \infty) = \theta^2(\mu_M, \infty) < 2 \). On the other hand since \( 0 = \partial M = (\partial \Gamma) \times L \), we conclude \( \partial \Gamma = 0 \), and the number of half lines must be even, hence must be two. Therefore \( M \) is the union of two half planes which contain \( L \). Since \( M \) is stationary, these half planes are contained in one plane, and \( M \) is indeed a plane. We conclude that \( \text{sing } M = \emptyset \), which contradicts \( \text{sing } L \neq \emptyset \).

Therefore the only possibility left is (2.10), and all \( M \) are smooth as claimed.

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Proposition 2.4 Let \( C \) be as in Theorem 2.1, and assume that \( C \) is a cone. Then either \( C = 0 \) or \( C \) is a unit density plane.

Proof:
Since \( C \) is cone we have, assuming that the tip of \( C \) is at the origin,

\[ \theta^2(\mu_C, 0) = \frac{\mu_C(B_\varepsilon(0))}{\omega_2 \varepsilon^2} = \theta^2(\mu_C, \infty). \]

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Now \( C \) is smooth by Proposition 2.3, hence \( \theta^2(\mu_C, 0) \in \mathbb{N}_0 \). Assuming \( C \neq 0 \), we get \( 0 \in \text{spt } \mu_C \) and
\[
1 = \theta^2(\mu_C, 0) = \theta^2(\mu_C, \infty),
\]
when recalling (2.3) and \( \theta^2(\mu, \infty) < 2 \) by (2.1). Then the remark following Proposition 2.1 yields that \( C \) is a unit density plane.

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**Proof of Theorem 2.1:**
Let \( M \neq 0 \) be as in the theorem. As the currents \( r^{-1}M \) satisfy the same assumptions, Proposition 2.2 yields for any sequence \( r_j \to \infty \), after passing to a subsequence,
\[
\begin{align*}
  & r_j^{-1}M \to C \quad \text{weakly as currents,} \\
  & \mu_{r_j^{-1}M} \to \mu_C \quad \text{weakly as varifolds.}
\end{align*}
\]

Then by weak convergence
\[
\frac{\mu_C(B_\rho(0))}{\omega_2 \rho^2} \geq \lim_{j \to \infty} \frac{\mu_{r_j^{-1}M}(B_\rho(0))}{\omega_2 \rho^2} = \lim_{j \to \infty} \frac{\mu_M(B_{gr_j}(0))}{\omega_2 (gr_j)^2} = \theta^2(\mu_M, \infty) \geq \frac{\mu_C(B_\rho(0))}{\omega_2 \rho^2},
\]
hence
\[
\theta^2(\mu_C, 0) = \frac{\mu_C(B_\rho(0))}{\omega_2 \rho^2} = \theta^2(\mu_C, \infty) = \theta^2(\mu_M, \infty) < 2. \tag{2.11}
\]

By the monotonicity formula and the argument as in [Sim] §19 and §42, we see that \( C \) is a cone at the origin. Since by (2.2) for any \( x \in \text{spt } M \neq \emptyset \)
\[
1 \leq \theta^2(\mu_M, x) \leq \theta^2(\mu_M, \infty),
\]
we get \( C \neq 0 \), and \( C \) is a unit density plane by Proposition 2.4, hence
\[
\theta^2(\mu_M, \infty) = \theta^2(\mu_C, \infty) = 1 \tag{2.12}
\]
by (2.11). Then \( M \) is a unit density plane by the remark following Proposition 2.1, and the theorem follows.

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### 3 An inductive argument

**Lemma 3.1** Let \( \Sigma_j \subseteq \mathbb{R}^n \) be a sequence of closed oriented Willmore surfaces and assume that for some \( \delta > 0, \Lambda < \infty \) and \( R_j \to \infty \) we have
\[
\begin{align*}
  & W(\Sigma_j) \leq 8\pi - \delta, \\
  & \| A_{\Sigma_j} \|_{L^\infty(\Sigma_j \cap B_{R_j}(0))} \leq \Lambda, \\
  & |A_{\Sigma_j}(0)| = 1,
\end{align*}
\]

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where \( A_{\Sigma_j} \) is the second fundamental form. Then for a subsequence

\[ \Sigma_j \to \Sigma \text{ smoothly in compact subsets of } \mathbb{R}^n \]  

(3.2)

to a smooth embedded Willmore surface \( \Sigma \) with

\[ \theta^2(\mathcal{H}^2 \Sigma, \cdot) \leq \theta^2(\mathcal{H}^2 \Sigma, \infty) + \frac{1}{4\pi} \mathcal{W}(\Sigma) \leq 2 - \frac{\delta}{4\pi}, \]

\[ \mathcal{W}(\Sigma) > 0, \]  

(3.3)

and

\[ \sup_{x \in \Sigma} |x| |A_{\Sigma}(x)| < \infty. \]  

(3.4)

Moreover if \( \Sigma \) is non-compact then for \( r_j \to \infty \) we get after passing to a subsequence

\[ r_j^{-1} \Sigma \to P \text{ smoothly in compact subsets of } \mathbb{R}^n - \{0\}, \]  

(3.5)

where \( P \) is a plane containing the origin which possibly depends on the subsequence. \( \square \)

The proof of the lemma is divided into three steps.

**Proposition 3.2** Let \( \Sigma_j \subseteq \mathbb{R}^n \) be a sequence of closed oriented surfaces, such that for some \( \delta > 0 \), \( \Lambda < \infty \) and \( R_j \to \infty \) there holds

\[ \mathcal{W}(\Sigma_j) \leq 8\pi - \delta, \]

\[ \| A_{\Sigma_j}, \nabla A_{\Sigma_j} \|_{L^\infty(\Sigma_j \cap B_{R_j}(0))} \leq \Lambda, \]

\[ |A_{\Sigma_j}(0)| = 1. \]  

(3.6)

Then

\[ \liminf_{j \to \infty} \mathcal{W}(\Sigma_j \cap B_{R_j}(0)) \geq \varepsilon_0 \]  

(3.7)

for some \( \varepsilon_0 = \varepsilon_0(n, \delta, \Lambda) > 0 \).

**Proof:**

First, let \( \mu_j \) be any sequence of integral 2-varifolds in \( \mathbb{R}^n \) with \( \tilde{H}_{\mu_j} \in L^2(\mu_j) \) and

\[ \theta^2(\mu_j, \infty) + \frac{1}{4\pi} \mathcal{W}(\mu_j) \leq 2 - \delta. \]  

(3.8)

Since \( \varphi^{-2}\mu_j(B_\varphi) \leq C(\theta^2(\mu_j, \infty) + \mathcal{W}(\mu_j)) \leq C \) by [KuSch04] (A.6), we get after passing to a subsequence

\[ \mu_j \to \mu_\infty \text{ weakly as varifolds}. \]

The limit \( \mu_\infty \) is an integral 2-varifold by Allard’s integral compactness theorem, and has \( \tilde{H}_{\mu_\infty} \in L^2(\mu_\infty) \) by lower semicontinuity. Moreover combining [KuSch04] (A.8), (A.14) and the inequality before (A.23), we get

\[ \theta^2(\mu_\infty, \cdot) \leq \theta^2(\mu_\infty, \infty) + \frac{1}{4\pi} \mathcal{W}(\mu_\infty) \leq \limsup_{j \to \infty} \left( \theta^2(\mu_j, \infty) + \frac{1}{4\pi} \mathcal{W}(\mu_j) \right) \leq 2 - \delta. \]  

(3.9)
Now if there is no $\varepsilon_0 > 0$ as claimed in (3.7), then there exist $\Sigma_j, R_j$ as in (3.6) with
\[ W(\Sigma_j \cap B_{R_j}(0)) \to 0. \]  
(3.10)

Putting $\mu_j := \mathcal{H}^2_\nu \Sigma_j$, we have $\theta^2(\mu_j, \infty) = 0$ since the $\Sigma_j$ are compact, and get
\[ \mathcal{H}^2_\nu \Sigma_j \to \mu_\infty \quad \text{weakly as varifolds}. \]

The integral 2-varifold $\mu_\infty$ is stationary, i.e., $\bar{H}_{\mu_\infty} \equiv 0$, by lower semicontinuity. Moreover from (3.9) and $\theta^2(\mathcal{H}^2_\nu \Sigma_j, \infty) = 0$ we see\[ \theta^2(\mu_\infty, \cdot) \leq \theta^2(\mu_\infty, \infty) + \limsup_{j \to \infty} \frac{1}{4\pi} W(\Sigma_j) \leq 2 - \frac{\delta}{4\pi}. \]  
(3.11)

Using the bounds on the second fundamental form in (3.6), we see that $\Sigma_j \to \Sigma$ in $C^{2,\alpha}$ in compact subsets of $\mathbb{R}^n$ to a $C^{2,\alpha}$-surface $\Sigma$, which is embedded by (3.11), and $|A_{\Sigma_j}(0)| \to |A_{\Sigma_j}(0)| = 1$. Moreover $C^{2,\alpha}$-convergence and the embeddedness of $\Sigma$ yield $\mathcal{H}^2_\nu \Sigma = \mu_\infty$. Therefore $\bar{H}_\Sigma = \bar{H}_{\mu_\infty} \equiv 0$, and $\Sigma$ is a minimal surface and in particular smooth. Considering $\Sigma_j$ with the given orientation as an integral 2-current, we see $\partial \Sigma_j = 0$ and $\Sigma_j \to \Sigma$ weakly as currents, hence $\Sigma$ has an orientation and is a stationary integral 2-current with $\partial \Sigma = 0$. Then Theorem 2.1 implies by (3.11) that $\Sigma$ is a plane, which contradicts $A_{\Sigma}(0) \neq 0$, and the proposition follows.

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In the above proposition the $\Sigma_j$ were general. Now we consider Willmore surfaces.

**Proposition 3.3** Let $\Sigma_j \subseteq \mathbb{R}^n$ be a sequence of closed oriented Willmore surfaces, such that for some $\delta > 0$, $\Lambda < \infty$ and $R_j \to \infty$ we have the assumptions (3.1), that is
\[ W(\Sigma_j) \leq 8\pi - \delta, \]
\[ \| A_{\Sigma_j} \|_{L^\infty(\Sigma_j \cap B_{R_j}(0))} \leq \Lambda, \]
\[ |A_{\Sigma_j}(0)| = 1. \]

If
\[ \lim_{j \to \infty} \sup_{x \in \Sigma_j \cap B_{R_j}(0)} |x| |A_{\Sigma_j}(x)| = \infty, \]  
(3.12)

then there exist $\tilde{R}_j \leq R_j, \tilde{R}_j \to \infty$ with
\[ \liminf_{j \to \infty} W(\Sigma_j \cap B_{\tilde{R}_j}(0)) \leq \liminf_{j \to \infty} W(\Sigma_j \cap B_{R_j}(0)) - \varepsilon_0 \]  
(3.13)

for some $\varepsilon_0 = \varepsilon_0(n, \delta) > 0$.
Proof:
We assume (3.1), (3.12), put
\[
2\varrho_j := \sup_{x \in \Sigma_j \cap B_{R_j}(0)} |x|(1 - |x|/R_j)|A_{\Sigma_j}(x)| \geq \sup_{x \in \Sigma_j \cap B_{R_j/2}(0)} |x||A_{\Sigma_j}(x)|/2 \to \infty \tag{3.14}
\]
by (3.12), and select \( x_j \in \Sigma_j \cap B_{R_j}(0) \) for \( \varrho_j > 0 \) with
\[
2\varrho_j = |x_j|(1 - |x_j|/R_j)|A_{\Sigma_j}(x_j)|.
\]
Clearly putting
\[
\tilde{R}_j := \varrho_j/|A_{\Sigma_j}(x_j)| \geq \varrho_j/\Lambda \to \infty
\]
by (3.1), we get
\[
2\tilde{R}_j = |x_j|(1 - |x_j|/R_j) \leq \min(|x_j|, R_j - |x_j|), \tag{3.15}
\]
hence for any \( x \in B_{\tilde{R}_j}(x_j) \)
\[
|x| \geq |x_j| - \tilde{R}_j \geq |x_j|/2,
R_j - |x| \geq R_j - |x_j| - \tilde{R}_j \geq (R_j - |x_j|)/2.
\]
This yields
\[
|x|(1 - |x|/R_j) \geq |x_j|(1 - |x_j|/R_j)/4,
\]
and
\[
|A_{\Sigma_j}(x)| \leq \frac{|A_{\Sigma_j}(x_j)|}{|x|(1 - |x|/R_j)} \leq 4|A_{\Sigma_j}(x_j)| \quad \text{for all } x \in \Sigma_j \cap B_{\tilde{R}_j}(x_j).
\]
Putting \( \tilde{\Sigma}_j := |A_{\Sigma_j}(x_j)|(|\Sigma_j - x_j|) \), we see \( |A_{\tilde{\Sigma}_j}(0)| = 1, \| A_{\tilde{\Sigma}_j} \|_{L^\infty(\tilde{\Sigma}_j \cap B_{\varrho_j/2}(0))} \leq 4 \) and, since \( \Sigma_j \) and \( \tilde{\Sigma}_j \) are Willmore surfaces, we get from the interior estimates in \([\text{KuSch01}]\) Theorem 3.5 for large \( j \) with \( \varrho_j \geq 1 \) that
\[
\| \nabla^k A_{\tilde{\Sigma}_j} \|_{L^\infty(\tilde{\Sigma}_j \cap B_{\varrho_j/2}(0))} \leq C_k(n) \quad \forall k \in \mathbb{N}.
\]
Applying Proposition 3.2 with \( R_j \) replaced by \( \varrho_j/2 \to \infty \), we get
\[
\liminf_{j \to \infty} W(\Sigma_j \cap B_{R_j}(x_j)) \geq \liminf_{j \to \infty} W(\tilde{\Sigma}_j \cap B_{\varrho_j/2}(0)) \geq \varepsilon_0(n, \delta). \tag{3.16}
\]
Since \( B_{\tilde{R}_j}(0) \subseteq B_{R_j}(0) - B_{\tilde{R}_j}(x_j) \) by (3.15), we calculate
\[
\liminf_{j \to \infty} W(\Sigma_j \cap B_{\tilde{R}_j}(0)) \leq \liminf_{j \to \infty} \left( W(\Sigma_j \cap B_{R_j}(0)) - W(\Sigma_j \cap B_{\tilde{R}_j}(x_j)) \right) \leq \liminf_{j \to \infty} W(\Sigma_j \cap B_{R_j}(0)) - \varepsilon_0,
\]
which is (3.13).

///
Proposition 3.4 Let $\Sigma_j \subseteq \mathbb{R}^n$ be a sequence of closed oriented Willmore surfaces, and assume that for some $\delta > 0$, $\Lambda < \infty$ and $R_j \to \infty$ we have the assumptions (3.1), that is

\[
W(\Sigma_j) \leq 8\pi - \delta, \\
\| A_{\Sigma_j} \|_{L^\infty(\Sigma_j \cap B_{R_j}(0))} \leq \Lambda, \\
|A_{\Sigma_j}(0)| = 1.
\]

Then there exist $\hat{R}_j \leq R_j$, $\hat{R}_j \to \infty$ with

\[
\liminf_{j \to \infty} \sup_{x \in \Sigma_j \cap B_{\hat{R}_j}(0)} |x||A_{\Sigma_j}(x)| < \infty.
\]

Proof:

If $\liminf_{j \to \infty} \sup_{x \in \Sigma_j \cap B_{R_j/2}(0)} |x||A_{\Sigma_j}(x)| < \infty$ we are done by putting $\hat{R}_j := R_j/2$. Otherwise we apply Proposition 3.3 and replace $R_j$ by $\tilde{R}_j$ at most $N < 8\pi/\delta_0(n, \delta)$ times to get the desired conclusion.

Proof of Lemma 3.1:

First we apply Proposition 3.4, replace $R_j$ by $\hat{R}_j$ and pass to a subsequence to obtain

\[
\limsup_{j \to \infty} \sup_{x \in \Sigma_j \cap B_{R_j}(0)} |x||A_{\Sigma_j}(x)| < \infty.
\]

Since $\Sigma_j$ are Willmore surfaces, we get from the interior estimates in [KuSch01] Theorem 3.5 and the uniform bound on the second fundamental form in $B_{R_j}(0)$ by (3.1) for large $j$ with $R_j \geq 1$ that

\[
\| \nabla^k A_{\Sigma_j} \|_{L^\infty(\Sigma_j \cap B_{R_j/2}(0))} \leq C_k(n, \Lambda) \quad \forall k \in \mathbb{N}.
\]

Recalling that $\theta^2(\mathcal{H}^2_{\Sigma_j}, \infty) = 0$, since $\Sigma_j$ are compact, we can pass to the limit as in Proposition 3.2 with the Remark after Proposition 2.2 and obtain for a subsequence $\Sigma_j \to \Sigma$ smoothly in compact subsets of $\mathbb{R}^n$, $\mathcal{H}^2_{\Sigma_j} \to \mu_{\infty}$ weakly as varifolds
to a smooth embedded Willmore surface $\Sigma$ and an integral 2-varifold $\mu_{\infty}$ satisfying

\[
\theta^2(\mu_{\infty}, \cdot) \leq \theta^2(\mu_{\infty}, \infty) + \frac{1}{4\pi} W(\mu_{\infty}) \leq \limsup_{j \to \infty} \frac{1}{4\pi} W(\Sigma_j) \leq 2 - \frac{\delta}{4\pi}.
\]

We see $\partial \Sigma_j = 0$ and $\Sigma_j \to \Sigma$ weakly as currents, hence $\Sigma$ has an orientation and is an integral 2-current with $\partial \Sigma = 0$. Moreover smooth convergence and the embeddedness of $\Sigma$ yield $\mu_{\Sigma} = \mathcal{H}^2_{\Sigma} = \mu_{\infty}$, and we obtain the first part of (3.3) from (3.19).

Further $|A_{\Sigma}(0)| = |A_{\Sigma_j}(0)| = 1$. If $W(\Sigma) = 0$, then $\Sigma$ would be stationary, hence $\Sigma$ would be a plane by Theorem 2.1 and (3.19), which contradicts $A_{\Sigma}(0) \neq 0$. Therefore
\( W(\Sigma) > 0 \), which is the second part of (3.3). The bound (3.4) follows from (3.18) and from the smooth convergence. Since (3.3) and (3.4) are scale invariant, we get

\[
\theta^2(\mathcal{H}^2, r^{-1} \Sigma, \infty) + \frac{1}{4\pi} W(r^{-1} \Sigma) \leq 2 - \frac{\delta}{4\pi},
\]

\[
\|A_{r^{-1} \Sigma}\|_{L^\infty(r^{-1} \Sigma - B_\delta(0))} \leq \Gamma \delta^{-1}
\]

for all \( r > 0 \) and the given \( \delta > 0 \), and some \( \Gamma < \infty \). As above by the interior estimates in [KuSch01] Theorem 3.5 and the Remark after Proposition 2.2, we can pass to the limit for a subsequence \( r_j \to \infty \) and get

\[
r_j^{-1} \Sigma \to M \quad \text{smoothly in compact subsets of } \mathbb{R}^n - \{0\},
\]

\[
\mathcal{H}^2, r_j^{-1} \Sigma \to \mu \quad \text{weakly as varifolds},
\]

\[
\theta^2(\mu) \leq \theta^2(\mu, \infty) + \frac{1}{4\pi} W(\mu) \leq 2 - \frac{\delta}{4\pi},
\]

where \( M \) is a smooth embedded Willmore surface in \( \mathbb{R}^n - \{0\} \) and an integral 2–current with \( \partial M = 0 \), and \( \mu \) is an integral 2–varifold. Again smooth convergence and the embeddednes of \( M \) yield \( \mu_M(\mathbb{R}^n - \{0\}) = \mathcal{H}^2(\mathbb{R}^n - \{0\}) = \mu_\Sigma(\mathbb{R}^n - \{0\}) \), hence \( \mu_M = \mathcal{H}^2, M = \mu \), since \( \mu_M(\{0\}) = \mathcal{H}^2(\{0\}) = \mu(\{0\}) = 0 \). Then by weak convergence

\[
\frac{\mu_M(B_{\delta}(0))}{\omega_2 \delta^2} \geq \lim_{j \to \infty} \frac{\mu_j^{-1} \Sigma(B_{\delta}(0))}{\omega_2 \delta^2} = \lim_{j \to \infty} \frac{\mu_\Sigma(B_{\varphi_j}(0))}{\omega_2 (\varphi_j)^2} = \theta^2(\mu_\Sigma, \infty) \geq \frac{\mu_M(B_{\delta}(0))}{\omega_2 \delta^2},
\]

hence the inequalities are all equalities, and

\[
\theta^2(\mu_M, 0) = \frac{\mu_M(B_{\delta}(0))}{\omega_2 \delta^2} = \theta^2(\mu_M, \infty) = \theta^2(\mu_\Sigma, \infty).
\]

If \( \Sigma \) is non-compact, we get \( \theta^2(\mu_\Sigma, \infty) \geq 1 \) by [KuSch04] (A.22) and \( M \neq 0 \). Next

\[
W(M \setminus B_{\delta}(0)) \leq \lim_{j \to \infty} W(r_j^{-1} \Sigma \setminus B_{\delta}(0)) = \lim_{j \to \infty} W(\Sigma \setminus B_{\varphi_j}(0)) = 0,
\]

as \( W(\Sigma) < \infty \) by (3.3), hence \( W(M \setminus \{0\}) = 0 \) and \( M \) is stationary. Recalling \( \theta^2(\mu_M, \infty) < 2 \) by (3.20) and \( M \neq 0 \), Theorem 2.1 implies that \( M \) is a plane, and the lemma follows.

///

4 Proof of Theorem 1.1

We assume by contradiction that for some \( \delta > 0 \) we have

\[
\beta^n_{\infty} := \liminf_{p \to \infty} \beta^p_n \leq 8\pi - \delta.
\]

Select a sequence \( p_k \to \infty \) with \( \beta^p_{n_k} \to \beta^n_{\infty} \), and choose oriented, closed, smoothly embeded Willmore surfaces \( \Sigma_k \subseteq \mathbb{R}^n \) with \( W(\Sigma_k) = \beta^p_{n_k} \) according to [Sim93] and [BaKu03]. We normalize by a translation and a dilation such that

\[
\|A_{\Sigma_k}\|_{L^\infty(\Sigma_k)} \leq 1 = |A_{\Sigma_k}(0)|.
\]
Applying Lemma 3.1, we get after passing to a subsequence that the $\Sigma_k$ converge smoothly in compact subsets to an embedded Willmore surface $\Sigma$ satisfying (3.3), in particular $W(\Sigma) > 0$. Clearly $\Sigma$ is non-compact as genus($\Sigma_k$) $\to \infty$, hence by (3.5) there is a sequence $r_j \to \infty$ such that $r_j^{-1}\Sigma \to P$ locally smoothly in $\mathbb{R}^n - \{0\}$, where $P$ is a plane containing the origin. By a rotation we can assume that $P = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^n$. We fix $\varepsilon > 0$ and have for all $j \geq j(\varepsilon)
$

$$\{p = (x, y) \in r_j^{-1}\Sigma : 1 \leq |x| \leq 2, |y| \leq 2\} = \text{graph } u_j,$$

(4.1)

where the smooth function $u_j : \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 2\} \to \mathbb{R}^{n-2}$ satisfies

$$\|u_j\|_{C^0(B_{\varepsilon}(0) - B_{\varepsilon}^2(0))} < \varepsilon.$$  

(4.2)

As $k \to \infty$ the surfaces $r_j^{-1}\Sigma_k$ converge locally smoothly to $r_j^{-1}\Sigma$, hence for fixed $j$ and $k$ sufficiently large depending on $j$ we have an analogous graph representation for

$$\{p = (x, y) \in r_j^{-1}\Sigma_k : 1 \leq |x| \leq 2, |y| \leq 2\},$$

by functions $u_{j,k} : \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 2\} \to \mathbb{R}^{n-2}$, which also satisfy the inequality (4.2). Fix a smooth cutoff function $\eta \in C^\infty(\mathbb{R})$, such that $\eta' \geq 0$ and

$$\eta(r) = \begin{cases} 0 & \text{for } r \leq 1, \\ 1 & \text{for } r \geq 2. \end{cases}$$

Putting $C = \{(x, y) : |x|, |y| < 2\}$ we now define an oriented, closed embedded surface $\tilde{\Sigma}_k$ as the union of the sets $\{(x, 0) : |x| \leq 1\}$, $r_j^{-1}\Sigma_k - C$ and the graph of $x \mapsto \eta(|x|)u_{j,k}(x)$ on $1 < |x| < 2$. We compute the total Gauß curvature

$$\left| \int_{\tilde{\Sigma}_k} K_{\tilde{\Sigma}_k} d\mu_{\tilde{\Sigma}_k} - \int_{r_j^{-1}\Sigma_k} K_{r_j^{-1}\Sigma_k} d\mu_{r_j^{-1}\Sigma_k} \right| \leq \int_{\Sigma_k \cap C} |K_{\tilde{\Sigma}_k}| d\mu_{\tilde{\Sigma}_k} + \int_{r_j^{-1}\Sigma_k \cap C} |K_{r_j^{-1}\Sigma_k}| d\mu_{r_j^{-1}\Sigma_k}.$$

Letting $k \to \infty$ we obtain, using (4.2) for $u_{j,k}$ and scaling the second integral on the right,

$$\limsup_{k \to \infty} \left| \int_{\tilde{\Sigma}_k} K_{\tilde{\Sigma}_k} d\mu_{\tilde{\Sigma}_k} - \int_{r_j^{-1}\Sigma_k} K_{r_j^{-1}\Sigma_k} d\mu_{r_j^{-1}\Sigma_k} \right| \leq C\varepsilon^2 + \int_{\Sigma \cap r_j C} |K| d\mu < \infty.$$

By Gauß-Bonnet we see that also genus($\tilde{\Sigma}_k$) $\to \infty$. But we can estimate

$$W(\tilde{\Sigma}_k) \leq W(r_j^{-1}\Sigma_k) - W(r_j^{-1}\Sigma_k; C) + C\varepsilon^2 = W(\Sigma_k) - W(\Sigma_k; r_j C) + C\varepsilon^2,$$

and we conclude from smooth convergence

$$\beta^n \leq \liminf_{k \to \infty} W(\tilde{\Sigma}_k) \leq \beta^n_0 - W(\Sigma; r_j C) + C\varepsilon^2.$$

Letting $j \to \infty$ and then $\varepsilon \to 0$ we arrive at $W(\Sigma) = 0$, which contradicts (3.3).
References


