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Solving the KdV hierarchy with self-consistent sources by inverse scattering method

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Abstract

The evolution of the eigenfunctions in the Lax representation of the KdV hierarchy with self-consistent sources possesses singularity. By proposing a method to treat the singularity to determine the evolution of scattering data, the KdV hierarchy with self-consistent sources is integrated by the inverse scattering method. The soliton solutions of these equations are obtained. It is shown that the insertion of a source may cause the variation of the speed of soliton. This approach can be applied to other (1+1)-dimensional soliton hierarchies. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Nonlinear evolution equations with self-consistent sources have important physical applications [1–10], for example, the KdV equation with source describes the interaction of long and short capillary-gravity waves [4]. There are some ways to derive the integrable nonlinear evolution equations with self-consistent sources. Mel'nikov

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constructed some of this kind of nonlinear integrable systems by adding a new consistent operator to the original Lax representation [11]. Leon et al. [4,12,13] relate the sources to the singular part of the dispersion law. In recent years, soliton equations with self-consistent sources (SESCS) were studied based on the constrained flows of soliton equations which are just the stationary equations of SESCSs [14–17]. Since the Lax representation for the constrained flows of soliton equations can always be deduced from the adjoint representation for soliton equations, this approach provides a simple and natural way to derive both the SESCSs and their Lax representation [15–17]. The SESCSs possess t-type Hamiltonian or bi-Hamiltonian formulation [18] and can be used to deduce sinh-Gorden type of equations [19].

Some methods for solving the SESCSs were presented. The integration of some SESCSs was proposed by means of inverse scattering method without use of explicit evolution equations of eigenfunctions in Refs. [20,21] and by means of matrix theory in Refs. [1,2]. The $\bar{\partial}$-method and gauge transformation were applied to give the soliton solution for the modified *NLS* equation with a source and the modified Manakov system with self-consistent source [9,10]. The Darboux transformation for the *Kaup–Newell* and *AKNS* hierarchy with self-consistent sources were constructed in Refs. [16,17].

Since the evolution equation of eigenfunction in the Lax representation for the SESCS was not obtained explicitly in Refs. [20,21], the determination of the evolution of the scattering data was quite complicated and required special skill in Refs. [20,21]. In contrast with the soliton equations, the evolution of eigenfunctions for the SESCS possesses singularity in spectral parameter. The aim of this paper is to propose a way to treat the singularity to determine the evolution of the scattering data so that we could integrate SESCSs through the inverse scattering method by directly using an explicit expression for evolution of eigenfunction and obtain the explicit N soliton solution for SESCSs. Our method for determining the evolution of scattering data is completely different from that in Refs. [20,21]. This approach seems more natural and simple and enables us to solve whole soliton hierarchy with self-consistent sources directly and systematically by the inverse scattering method. The result shows that the evolution of the reflection coefficient is the same as that for the soliton equations without source, however, the evolution of each normalization constant has an extra term related to the square norm of the eigenfunctions. It is found that the insertion of a source into the soliton equation may cause the variation of the speed of soliton. This phenomenon may result in a great variety of dynamics of soliton solutions [20,21]. We use KdV hierarchy with self-consistent sources to illustrate the approach and present N-soliton solution for them. In fact, this approach can be used to solve other $(1 + 1)$-dimensional soliton hierarchies with self-consistent sources by the inverse scattering method, for instance, the AKNS hierarchy with self-consistent sources, the MKdV hierarchy with self-consistent sources, the nonlinear Schrödinger equation hierarchy with self-consistent sources, the Kaup–Newell hierarchy with self-consistent sources, the derivative nonlinear Schrödinger equation hierarchy with self-consistent sources.

2. The KdV hierarchy with self-consistent sources

To make the paper self-contained, we first recall the high-order constrained flows of KdV hierarchy and briefly describe how to derive the Lax representation for the KdV hierarchy with self-consistent sources.

Consider the Schrödinger equation

$$\phi_{xx} + (\lambda + u)\phi = 0 , \tag{2.1}$$

where ϕ and u are functions of x and t, λ is a spectral parameter. Eq. (2.1) can be written in the matrix form as

$$\begin{pmatrix} \phi \\ \phi_x \end{pmatrix}_x = U \begin{pmatrix} \phi \\ \phi_x \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -\lambda - u & 0 \end{pmatrix} . \tag{2.2}$$

The adjoint representation of (2.2) reads [22] as

$$V_x = [U, V] \equiv UV - VU . \tag{2.3}$$

Set

$$V = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} . \tag{2.4}$$

Eq. (2.3) yields

$$a_0 = b_0 = 0, \quad c_0 = -1, \quad a_1 = 0, \quad b_1 = 1, \quad c_1 = -\tfrac{1}{2}u ,$$

$$a_2 = \tfrac{1}{4}u_x, \quad b_2 = -\tfrac{1}{2}u, \quad c_2 = \tfrac{1}{8}(u_{xx} + u^2), \ldots ,$$

and, in general, for $k = 1, 2, \ldots,$

$$a_k = -\tfrac{1}{2}b_{k,x}, \quad b_{k+1} = Lb_k = -\tfrac{1}{2}L^{k-1}u, \quad c_k = -\tfrac{1}{2}b_{k,xx} - b_{k+1} - b_k u , \tag{2.5}$$

where

$$L = -\tfrac{1}{4}D^2 - u + \tfrac{1}{2}D^{-1}u_x, \quad D = \tfrac{\partial}{\partial x}, \quad DD^{-1} = D^{-1}D = 1 .$$

Set

$$V^{(n)} = \sum_{i=0}^{n+1} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n+1-i} + \begin{pmatrix} 0 & 0 \\ b_{n+2} & 0 \end{pmatrix} , \tag{2.6}$$

and take

$$\begin{pmatrix} \phi \\ \phi_x \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda) \begin{pmatrix} \phi \\ \phi_x \end{pmatrix} . \tag{2.7}$$

Then the compatibility condition of Eqs. (2.2) and (2.7) gives rise to the KdV hierarchy

$$u_{t_n} = D\frac{\delta H_n}{\delta u} \equiv -2b_{n+2,x}, \quad n = 0, 1, \ldots , \tag{2.8}$$

where $H_n = (4b_{n+3})/(2n+3)$. We have

$$\frac{\delta\lambda}{\delta u} = \phi^2, \quad L\phi^2 = \lambda\phi^2 . \tag{2.9}$$

The high-order constrained flows of the KdV hierarchy consist of the equations obtained from the spectral problem (2.1) for N distinct λ_j and the restriction of the variational derivatives for conserved quantities H_n and λ_j [23]

$$\frac{\delta H_n}{\delta u} - \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} \equiv -2b_{n+2} - \frac{1}{2}\sum_{j=1}^{N}\phi_j^2 = 0\,, \tag{2.10a}$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad j = 1,\ldots,N\,, \tag{2.10b}$$

where $n = 0,1,\ldots$. Set

$$\Phi = (\phi_1,\ldots,\phi_N)^{\mathrm{T}}\,.$$

According to Eqs. (2.5), (2.9) and (2.10), we denote

$$\tilde{a}_i = a_i, \quad \tilde{b}_i = b_i, \quad \tilde{c}_i = c_i, \quad i = 0,1,\ldots,n+1\,,$$

$$\tilde{b}_{n+2+i} = -\tfrac{1}{4}\langle \Lambda^i\Phi,\Phi\rangle, \quad i = 0,1,2,\ldots\,,$$

$$\tilde{a}_{n+2+i} = -\tfrac{1}{2}\tilde{b}_{n+2+i,x} = \tfrac{1}{4}\langle \Lambda^i\Phi,\Phi_x\rangle\,,$$

$$\tilde{c}_{n+2+i} = -\tfrac{1}{2}\tilde{b}_{n+2+i,xx} - \tilde{b}_{n+3+i} - \tilde{b}_{n+2+i}u = \tfrac{1}{4}\langle \Lambda^i\Phi_x,\Phi_x\rangle\,.$$

Then

$$N^{(n)} \equiv \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} = \lambda^{n+1}\sum_{k=0}^{\infty}\begin{pmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{c}_k & -\tilde{a}_k \end{pmatrix}\lambda^{-k} + \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}$$

$$= \sum_{k=0}^{n+1}\begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix}\lambda^{n+1-k} + \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} + \frac{1}{4}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix} \phi_j\phi_{j,x} & -\phi_j^2 \\ \phi_{j,x}^2 & -\phi_j\phi_{j,x} \end{pmatrix},$$

where η is independent of x, also satisfies the adjoint representation (2.3), i.e.,

$$N_x^{(n)} = [U, N^{(n)}]\,. \tag{2.11}$$

In fact, Eq. (2.11) gives rise to the Lax representation of the constrained flow (2.10).

The KdV hierarchy with self-consistent sources is given by [16,17]

$$u_{t_n} = D\left[\frac{\delta H_n}{\delta u} - \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] \equiv D\left[-2b_{n+2} - \frac{1}{2}\sum_{j=1}^{N}\phi_j^2\right]\,, \tag{2.12a}$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad j = 1,\ldots,N\,, \tag{2.12b}$$

where λ_j are distinct and in the present paper we concentrate on the following case:

$$\lambda_j = -k_j^2 < 0, \quad k_j > 0, \;\; j = 1,\ldots,N\,. \tag{2.12c}$$

Since the high-order constrained flows (2.10) are just the stationary equations of the KdV hierarchy with self-consistent sources (2.12), it is obvious that the zero-curvature representation for the KdV hierarchy with self-consistent sources (2.12) is given by

$$U_{t_n} - N_x^{(n)} + [U, N^{(n)}] = 0 \tag{2.13}$$

with the auxiliary linear problems

$$\begin{pmatrix} \psi \\ \psi_x \end{pmatrix}_x = U \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \quad \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}_{t_n} = N^{(n)} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \tag{2.14}$$

or equivalently

$$\psi_{xx} + (\lambda + u)\psi = 0 , \tag{2.15a}$$

$$\begin{aligned} \psi_{t_n} &= A^{(n)}\psi + B^{(n)}\psi_x \\ &= \sum_{l=0}^{n+1} (a_l\psi + b_l\psi_x)\lambda^{n+1-l} + \eta\psi + \frac{1}{4}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j}\phi_j(\phi_{j,x}\psi - \phi_j\psi_x) . \end{aligned} \tag{2.15b}$$

When $n = 1$, Eq. (2.12) gives the KdV equation with self-consistent sources

$$u_{t_1} = -\frac{1}{4}(6uu_x + u_{xxx}) - \frac{1}{2}D\sum_{j=1}^{N} \phi_j^2 , \tag{2.16a}$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad j = 1, \ldots, N , \tag{2.16b}$$

and the auxiliary linear problem reads as

$$\psi_{xx} + (\lambda + u)\psi = 0 ,$$

$$\psi_{t_1} = \left(\frac{1}{4}u_x + \eta\right)\psi + \frac{1}{4}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j}\phi_j(\phi_{j,x}\psi - \phi_j\psi_x) + \left(\lambda - \frac{1}{2}u\right)\psi_x .$$

3. Solving the KdV hierarchy with self-consistent sources

In this section, we will use the inverse scattering method to solve the initial-value problem for the KdV hierarchy with self-consistent sources (2.12) with $\lambda_j < 0$ in the class of real functions rapidly decreasing with x, namely under the assumption that $u(x,t), \phi_j(x,t),\ j = 1, \ldots, N$, vanish rapidly as $|x| \to \infty$. More exactly, we want to find a solution of system (2.12) which satisfies the following requirements. Let $u_0 = u_0(x)$ be an arbitrary function with the following properties [20]:

(a) $u_0(x)$ and its derivatives decay sufficiently rapidly as $|x| \to \infty$ such that

$$\int_{-\infty}^{\infty} \left(|xu_0(x)| + \sum_{j=0}^{2n+1} |u_0^{(j)}(x)| \right) \mathrm{d}x < \infty ; \tag{3.1a}$$

(b) the Schrödinger equation

$$\psi_{xx} + (\lambda + u_0(x))\psi = 0 , \tag{3.1b}$$

has exactly N distinct discrete eigenvalues as that given by (2.12c) (it can be easily treated in the same way for the case where there are more than N discrete spectrums for $u_0(x)$)

$$\lambda_j = (\mathrm{i}k_j)^2 = -k_j^2 \quad \text{where } k_j > 0, \; j = 1, \ldots, N \,. \tag{3.1c}$$

Let $\beta_j(t)$, $j=1,\ldots,N$, be an arbitrary continuous function of t. Using the inverse scattering method, we shall point out the way of constructing the solution $u=u(x,t)$, $\phi_j = \phi_j(x,t)$, $j=1,\ldots,N$, of system (2.12), such that

$$u(x,0) = u_0(x), \qquad \frac{1}{8}\int_{-\infty}^{\infty} \phi_j^2(x,t)\,\mathrm{d}x = \beta_j(t), \quad j = 1, \ldots, N \,. \tag{3.1d}$$

The procedure of finding the above solution of system (2.12) is very similar to that given in [24] for obtaining a solution rapidly decreasing with x of the KdV hierarchy. The main difference is the way for determining the evolution of scattering data. Denote $\lambda = k^2$, $\mathrm{Im}\, k \geq 0$. Under the assumption for $u(x,t), \phi_j(x,t)$, in the same way as in [24], we define the eigenfunction f^+, f^- for Schrödinger equation (2.15a) and (2.15b) with the following boundary condition:

$$f^-(x,k,t) \sim \mathrm{e}^{-\mathrm{i}kx}, \quad x \to -\infty \,, \tag{3.2a}$$

$$f^+(x,k,t) \sim \mathrm{e}^{\mathrm{i}kx}, \quad x \to +\infty \,. \tag{3.2b}$$

Let η^- be the parameter η in Eq. (2.15b) corresponding to $f^-(x,k,t)$. Notice that for $k \in (-\infty,\infty)$, since $k \neq \mathrm{i}k_j$, we have

$$b_1 = 1, \quad \lim_{|x|\to\infty} a_j = 0, \; j = 0, 1, \ldots, \qquad \lim_{|x|\to\infty} b_j = 0, \; j \neq 1 \,,$$

$$\lim_{|x|\to\infty} \phi_j(x,t) = 0, \quad j = 1, \ldots, N \,,$$

then

$$\lim_{|x|\to\infty} A^{(n)} = \eta, \qquad \lim_{|x|\to\infty} B^{(n)} = \lambda^n = k^{2n} \,. \tag{3.3}$$

By substituting $f^-(x,k,t)$ and η^- into Eq. (2.15b) and letting $x \to -\infty$, according to property (3.2) and (3.3) , it is found that

$$\eta^- = \mathrm{i}k^{2n+1} \,. \tag{3.4}$$

For $k \in (-\infty,\infty)$ and $k \neq 0$, $f^+(x,k,t)$ and $f^+(x,-k,t)$ are linearly independent, we may write

$$f^-(x,k,t) = a(k,t) f^+(x,-k,t) + b(k,t) f^+(x,k,t) \,. \tag{3.5}$$

Substituting representation of $f^-(x,k,t)$ (3.5) and η^- into Eq. (2.15b) and using (3.2) and (3.3), letting $x \to +\infty$, we have for $k \in (-\infty,\infty)$

$$\frac{\partial a}{\partial t} = 0, \quad \frac{\partial b}{\partial t} = 2\mathrm{i}k^{2n+1} b \tag{3.6}$$

which implies that for the KdV hierarchy with self-consistent sources, the evolution of quantities $a(k,t)$ and $b(k,t)$ is the same as that of the KdV hierarchy without source.

It can be shown as in [24] that functions $a(k,t)$ and $f^-(x,k,t)$ admit an analytical continuation in k into the upper-plane and zeros of $a(k,t)$ in the upper half-plane correspond to the discrete eigenvalues of spectral problem (2.15a). Eq. (3.6) indicates that the discrete eigenvalues do not depend on t. So the zeros of $a(k,t)$ are just $\mathrm{i}k_j$ and at $k=\mathrm{i}k_j$ the following equality for discrete eigenfunctions holds

$$f^-(x,\mathrm{i}k_m,t)=\tilde{C}_m(t)f^+(x,\mathrm{i}k_m,t),\quad \phi_m(x,t)=\alpha_m(t)f^-(x,\mathrm{i}k_m,t),\quad m=1,\dots,N\,. \tag{3.7}$$

Combining Eq. (2.12b) for $\lambda_j=(\mathrm{i}k_j)^2$ with (2.15a) for $\lambda=k^2$ leads to the following non-local form:

$$\phi_{j,x}f^-(x,k,t)-\phi_j f^-_x(x,k,t)=(k^2+k_j^2)\int_{-\infty}^{x}\phi_j(z,t)f^-(z,k,t)\,\mathrm{d}z \tag{3.8}$$

which, letting $k\to \mathrm{i}k_m,\ x\to\infty$, leads to

$$\int_{-\infty}^{\infty}\phi_j(z,t)f^-(z,\mathrm{i}k_m,t)\,\mathrm{d}z=0\quad \text{when } j\neq m \tag{3.9}$$

and

$$\begin{aligned}&\lim_{k\to \mathrm{i}k_m}\sum_{j=1}^{N}\frac{1}{k^2+k_j^2}\phi_j[\phi_{j,x}f^-(x,k,t)-\phi_j f^-_x(x,k,t)]\\ &\quad\sim \alpha_m(t)\tilde{C}_m(t)f^+(x,\mathrm{i}k_m,t)\int_{-\infty}^{\infty}\phi_m(z,t)f^-(z,\mathrm{i}k_m,t)\,\mathrm{d}z\\ &\quad=\tilde{C}_m(t)f^+(x,\mathrm{i}k_m,t)\int_{-\infty}^{\infty}\phi_m^2(z,t)\,\mathrm{d}z,\quad x\to\infty\,.\end{aligned} \tag{3.10}$$

Eq. (3.9) is the orthogonal property of the discrete eigenfunctions. Denote the parameter η in Eq. (2.15b) corresponding to the discrete eigenfunction $f^-(x,\mathrm{i}k_m,t)$ by η_m^-, $m=1,\dots,N$, respectively. By substituting η_m^- and $f^-(x,\mathrm{i}k_m,t)$ into Eq. (2.15b) and letting $x\to-\infty$, it follows from (3.2) and (3.8) that

$$\eta_m^-=(-1)^{n+1}k_m^{2n+1},\quad m=1,\dots,N\,. \tag{3.11}$$

Inserting the representation of $f^-(x,\mathrm{i}k_m,t)$ (3.7) into Eq. (2.15b), letting $x\to+\infty$ and noting Eqs. (3.2) and (3.10), one obtains

$$\frac{\mathrm{d}\tilde{C}_m}{\mathrm{d}t}=\left[2(-1)^{n+1}k_m^{2n+1}+\frac{1}{4}\int_{-\infty}^{\infty}\phi_m^2(z,t)\,\mathrm{d}z\right]\tilde{C}_m,\quad m=1,\dots,N\,. \tag{3.12}$$

Since the normalization constants $\bar{c}_m^2(t)$ are defined by [24]

$$\bar{c}_m^2(t)\equiv\left[\int_{-\infty}^{\infty}f^{+^2}(x,\mathrm{i}k_m,t)\,\mathrm{d}x\right]^{-1}=-\mathrm{i}\tilde{C}_m(t)\left(\frac{\partial a(\mathrm{i}k_m)}{\partial k}\right)^{-1},\quad m=1,\dots,N\,. \tag{3.13}$$

Eq. (3.12) yields

$$\frac{\mathrm{d}\bar{c}_m^2(t)}{\mathrm{d}t}=2[(-1)^{n+1}k_m^{2n+1}+\beta_m(t)]\bar{c}_m^2(t)\quad m=1,\dots,N\,. \tag{3.14}$$

Thus, the evolution of normalization constant $\bar{c}_m^2(t)$ has an extra term $2\beta_m(t)\bar{c}_m^2(t)$ compared with that of the KdV hierarchy without source. This phenomenon was also found in Ref. [20]. However, our way of deriving (3.14) is completely different from that in [20] and is more direct and natural. It is found from (3.7) and (3.13) that

$$\begin{aligned}\beta_j(t) &= \frac{1}{8}\int_{-\infty}^{\infty} \phi_j^2(x,t)\,\mathrm{d}x = \alpha_j^2(t)\tilde{C}_j^2(t)\frac{1}{8}\int_{-\infty}^{\infty}(f^+(x,\mathrm{i}k_j,t))^2\,\mathrm{d}x\,,\\ &= \frac{1}{8}\alpha_j^2(t)\tilde{C}_j^2(t)(\bar{c}_j^2(t))^{-1}\,,\end{aligned}$$

$$\phi_j(x,t) = 2\sqrt{2\beta_j(t)}\bar{c}_j(t)f^+(x,\mathrm{i}k_j,t), \quad j=1,\ldots,N\,,$$

which is consistent with (3.1d) according to (3.13).

According to [24], by means of Eqs. (3.6), (3.14) and the Gel'fand–Levitan–Marchenko equation, we can obtain the solution of the nth KdV equation with self-consistent sources (2.12) under condition (3.1) in the following way:

$$u(x,t) = 2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x)\,, \tag{3.15a}$$

$$\phi_j(x,t) = 2\sqrt{2\beta_j(t)}\bar{c}_j(t)\left(\mathrm{e}^{-k_jx} + \int_x^{\infty} K(x,s)\mathrm{e}^{-k_js}\,\mathrm{d}s\right), \quad j=1,\ldots,N\,, \tag{3.15b}$$

where $K(x,y)$ satisfies

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,s)F(s+y)\,\mathrm{d}s = 0, \quad y > x \tag{3.16}$$

with

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(k)}{a(k)}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k + \sum_{j=1}^{N}\bar{c}_j^2(t)\mathrm{e}^{-k_jx}\,.$$

4. The soliton solution of the KdV hierarchy with self-consistent sources

Assume that $b(k)=0$ and there are N distinct eigenvalues $\mathrm{i}k_j$, $k_j>0$, $j=1,\ldots,N$, corresponding to the potential $u_0(x)$, the normalization constants for $t=0$ are $\bar{c}_j^2(0)$, $\bar{c}_j(0)>0$, $j=1,\ldots,N$. Then, $K(x,y)$ in Eq. (3.16) can be obtained as

$$K(x,y) = EA^{-1}B\,, \tag{4.1}$$

where

$$\begin{aligned}E &= (\bar{c}_1(t)\mathrm{e}^{-k_1y}\ \ \bar{c}_2(t)\mathrm{e}^{-k_2y}\ \ \cdots\ \ \bar{c}_N(t)\mathrm{e}^{-k_Ny})\,,\\ A &= (a_{jl})_{N\times N} = \left(\delta_{jl} + \frac{\bar{c}_j(t)\bar{c}_l(t)}{k_j+k_l}\mathrm{e}^{-(k_j+k_l)x}\right)_{N\times N}\,,\\ B &= (-\bar{c}_1(t)\mathrm{e}^{-k_1x}\ \ -\bar{c}_2(t)\mathrm{e}^{-k_2x}\ \ \cdots\ \ -\bar{c}_N(t)\mathrm{e}^{-k_Nx})^{\mathrm{T}},\end{aligned}$$

then it follows from (3.14) that

$$\bar{c}_j(t)=\bar{c}_j(0)\exp\left[(-1)^{n+1}k_j^{2n+1}t+\int_0^t\beta_j(z)\,\mathrm{d}z\right],\quad j=1,\ldots,N\,. \tag{4.2}$$

After some reduction, the solution of the KdV hierarchy with self-consistent sources under our assumption can be written in the form [24]

$$u(x,t)=2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det A\,, \tag{4.3a}$$

$$\phi_j(x,t)=2\sqrt{2\beta_j(t)}\bar{c}_j(t)\frac{1}{\det A}\sum_{m=1}^{N}\bar{c}_m(t)\mathrm{e}^{-k_m x}Q_{mn}\,, \tag{4.3b}$$

where Q_{mn} denotes the cofactors of the matrix A. Denote

$$\varepsilon_{jm}\equiv((-1)^{n+1}k_j^{2n}-(-1)^{n+1}k_m^{2n})t+\int_0^t\left(\frac{\beta_j(z)}{k_j}-\frac{\beta_m(z)}{k_m}\right)\mathrm{d}z,\quad j,m=1,\ldots,N\,, \tag{4.4}$$

if

$$\varepsilon_{jm}\to-\infty\quad\text{or}\quad\varepsilon_{jm}\to\infty,\quad\text{when } j\neq m \text{ and } t\to\pm\infty\,, \tag{4.5}$$

then Eqs. (4.3) present the N-soliton solution of the nth KdV equation with self-consitent sources (2.12). The speed of propagation for each soliton is

$$(-1)^{n+1}k_j^{2n}+\frac{\beta_j(t)}{k_j},\quad j=1,\ldots,N\,. \tag{4.6}$$

Thus, we see that the insertion of a source may cause variation in the speed of a soliton, the relation between the source and the change of the speed of soliton is given by Eq. (4.6). Some relevant phenomena were also observed in Refs. [1,2] by matrix theory.

Here we give two examples for the KdV equation with self-consistent sources (2.16).

If $b(k)=0$ and there is only one discrete eigenvalue $\mathrm{i}k_1$, $k_1>0$, for $u_0(x)$, the corresponding initial normalization constant is $\bar{c}_1^2(0)$, $\bar{c}_1(0)>0$, then the one-soliton solution is given by

$$u(x,t)=2k_1^2\,\mathrm{sech}^2\left(k_1x-k_1^3t-\int_0^t\beta_1(z)\,\mathrm{d}z+x_0\right)\,, \tag{4.7a}$$

$$\phi_1(x,t)=2\sqrt{k_1\beta_1(t)}\,\mathrm{sech}\left(k_1x-k_1^3t-\int_0^t\beta_1(z)\,\mathrm{d}z+x_0\right)\,, \tag{4.7b}$$

where

$$x_0=\log\frac{\sqrt{2k_1}}{\bar{c}_1(0)}\,.$$

Eq. (4.7a) implies that the speed of the soliton solution no longer has explicit relation with the amplitude as in the case of the KdV equation without source [24].

If $b(k)=0$ and there are two discrete eigenvalues 2i and i for $u_0(x)$, the corresponding initial normalization constants are $\bar{c}_1^2(0)=12$ and $\bar{c}_2^2(0)=6$, respectively, then we get

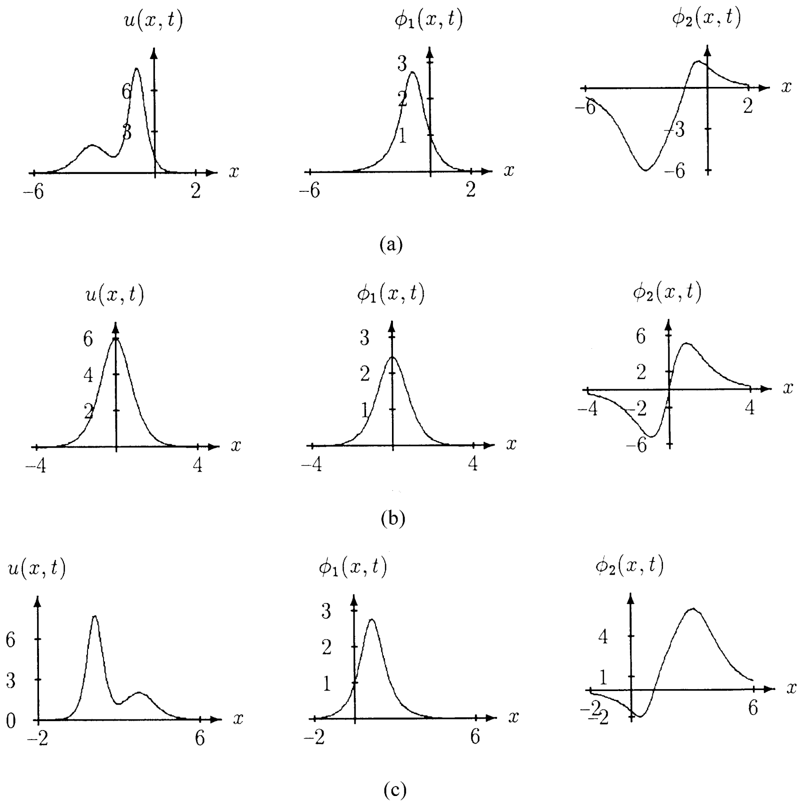

Fig. 1. The two-soliton solution (4.8) with $\beta_1(t)\equiv 1,\ \beta_2(t)\equiv 9$; (*a*) $t=-0.25$; (*b*) $t=0$; (*c*) $t=0.25$.

the two-soliton solution

$$u(x,t)=\frac{12\{3+4\cosh[2x-2t-2\int_0^t\beta_2(z)\,dz]+\cosh[4x-16t-2\int_0^t\beta_1(z)\,dz]\}}{\{\cosh[3x-9t-\int_0^t(\beta_1(z)+\beta_2(z))\,dz]+3\cosh[x-7t-\int_0^t(\beta_1(z)-\beta_2(z))\,dz]\}^2}\,, \tag{4.8a}$$

$$\phi_1(x,t)=4\sqrt{6\beta_1(t)}\,\frac{\cosh[x-t-\int_0^t\beta_2(z)\,dz]}{\cosh[3x-9t-\int_0^t(\beta_1(z)+\beta_2(z))\,dz]+3\cosh[x-7t-\int_0^t(\beta_1(z)-\beta_2(z))\,dz]}\,, \tag{4.8b}$$

$$\phi_2(x,t)=4\sqrt{3\beta_2(t)}\,\frac{\sinh[2x-8t-\int_0^t\beta_1(z)\,dz]}{\cosh[3x-9t-\int_0^t(\beta_1(z)+\beta_2(z))\,dz]+3\cosh[x-7t-\int_0^t(\beta_1(z)-\beta_2(z))\,dz]}\,. \tag{4.8c}$$

As plotted in Fig. 1, if we choose $\beta_1(t)$ and $\beta_2(t)$ appropriately, such as $\beta_1(t) = 1$ and $\beta_2(t) = 9$, the soliton with smaller amplitude may propagate faster than that with bigger amplitude. This phenomenon is completely different from that of solitons of KdV hierarchy without sources. Some other choices of $\beta_i(t)$ can give a great variety of dynamics of soliton solutions.

5. Conclusion

The Lax representation of the SESCS can always be deduced from the adjoint representation of the auxiliary linear problems for soliton equations. In contrast with the soliton equations, the evolution of eigenfunctions for the SESCS possessess singularity. We propose a method to treat the singularity to determine the evolution of scattering data. The evolution of each normalization constant has an extra term related to the square norm of the eigenfunction. We solve the whole soliton hierarchy with self-consistent sources by the inverse scattering method and obtain the soliton solution. The self-consistent sources may cause variation in the speed of soliton solution. Compared with the method in Refs. [20,21], our approach is quite different and seems more direct and simple. This approach can be used to solve other $(1 + 1)$-dimensional soliton hierarchies with self-consistent sources.

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References

[1] V.K. Mel'nikov, Capture and confinement of solitons in nonlinear integrable systems, Commun. Math. Phys. 120 (1989) 451–468.
[2] V.K. Mel'nikov, Interaction of solitary waves in the system described by the Kadomtsev–Petviashvili equation with a self-consistent source, Commun. Math. Phys. 126 (1989) 201–215.
[3] D.J. Kaup, Integrable Ponderomotive system: cavitons and solitons, Phys. Rev. Lett. 59 (1987) 2063–2066.
[4] J. Leon, A. Latifi, Solutions of an initial–boundary value problem for coupled nonlinear waves, J. Phys. A 23 (1990) 1385–1403.
[5] C. Claude, A. Latifi, J. Leon, Nonlinear resonant scattering and plasma instability: an integrable model, J. Math. Phys. 32 (1991) 3321–3330.
[6] R.A. Vlasov, E.V. Doktorov, Dokl. Akad. Nauk BSSR 26 (1991) 17.
[7] E.V. Doktorov, R.A. Vlasov, Optical solitons in media with resonant and non-resonant self-focusing nonlinearities, Opt. Acta 30 (1983) 223–232.
[8] M. Nakazawa, E. Yomada, H. Kubota, Coexistence of self-induced transparency soliton and nonlinear Schrödinger soliton, Phys. Rev. Lett. 66 (1991) 2625–2628.

[9] E.V. Doktorov, V.S. Shchesnovich, Nonlinear evolutions with singular dispersion laws associated with a quadratic bundle, Phys. Lett. A 207 (1995) 153–158.
[10] V.S. Shchesnovich, E.V. Doktorov, Modified Manakov system with self-consistent source, Phys. Lett. A 213 (1996) 23–31.
[11] V.K. Mel'nikov, New method for deriving nonlinear integrable systems, J. Math. Phys. 31 (1990) 1106–1113.
[12] J. Leon, Spectral transform and solitons for generalized coupled Bloch systems, J. Math. Phys. 29 (1988) 2012–2019.
[13] J. Leon, Nonlinear evolutions with singular dispersion laws and forced systems, Phys. Lett. A 144 (1990) 444–452.
[14] M. Antonowicz, Gelfand–Dikii hierarchies with the sources and Lax representation for restricted flows, Phys. Lett. A 165 (1992) 47–52.
[15] Yunbo Zeng, Yishen Li, The deduction of the Lax representation for constrained flows from the adjoint representation, J. Phys. A: Math. Gen. 26 (1993) L273–L278.
[16] Yunbo Zeng, New factorization of the Kaup–Newell hierarchy, Physica D 73 (1994) 171–188.
[17] Yunbo Zeng, Yishen Li, The Lax representation and Darboux transformation for constrained flows of AKNS hierarchy, Acta Math. Sinica, New Series 12 (1996) 217–224.
[18] Yunbo Zeng, Bi-hamiltonian structure of JM hierarchy with self-consistent sources, Physica A 262 (1999) 405–419.
[19] Yunbo Zeng, Constructing hierarchy of sinh-Gordon-type equations from soliton hierarchy with self-consistent source, Physica A 259 (1998) 278–290.
[20] V.K. Mel'nikov, Integration method of the Korteweg–de Vries equation with a self-consistent source, Phys. Lett. A 133 (1988) 493–496.
[21] V.K. Mel'nikov, Integration of the nonlinear Schrodinger equation with a source, Inverse Problems 8 (1992) 133–147.
[22] A.C. Newell, Solitons in Mathematics and Physics, SIAM, Philadelphia, 1985.
[23] Yunbo Zeng, An approach to the deduction of the finite-dimensional integrability from the infinite-dimensional integrability, Phys. Lett. A 160 (1991) 541–547.
[24] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.