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Physics Letters A 372 (2008) 3819–3823

PHYSICS LETTERS A

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A new extended KP hierarchy

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Received 25 September 2007; received in revised form 27 February 2008; accepted 28 February 2008

Available online 4 March 2008

Communicated by A.R. Bishop

Abstract

A method is proposed to construct a new extended KP hierarchy, which includes two types of KP equation with self-consistent sources and admits reductions to k-constrained KP hierarchy and to Gelfand–Dickey hierarchy with sources. It provides a general way to construct soliton equations with sources and their Lax representations.

PACS: 02.30.Ik

Keywords: Extended KP hierarchy; KP equation with self-consistent sources; Gelfand–Dickey hierarchy with self-consistent sources; k-constrained KP hierarchy; Lax representation

1. Introduction

Generalizations of KP hierarchy attract a lot of interests from both physical and mathematical points of view [1–10]. One kind of generalization is the so-called multi-component KP (mcKP) hierarchy [1]. The mcKP hierarchy contains many physically relevant nonlinear integrable systems, such as Davey–Stewartson equation, two-dimensional Toda lattice and three-wave resonant interaction ones. There exist several equivalent formulations of this mcKP hierarchy: matrix pseudo-differential operator (Sato) formulation, τ-function approach via matrix Hirota bilinear identities, multi-component free fermion formulation. A coupled KP hierarchy was generated through the procedure of so-called Pfaffianization [11]. It was shown in [12] that this coupled KP hierarchy can be reformulated as a reduced case of the 2-component KP hierarchy. Another kind of generalization of KP equation is the so-called KP equation with self-consistent sources, which was initiated by Mel'nikov [8–10]. For example, the first type of KP equation with self-consistent sources (KPSCS) reads [8,9,13]

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4\sum_{i=1}^{N}(q_i r_i)_{xx} = 0, \tag{1a}$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \quad i = 1, \dots, N, \tag{1b}$$

$$r_{i,y} = -r_{i,xx} - 2ur_i. \tag{1c}$$

The second type of KPSCS is [8,14]

$$4u_t - 12uu_x - u_{xxx} - 3D^{-1}u_{yy} = 3\sum_{i=1}^{N}\left[q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y\right], \tag{2a}$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i D^{-1}u_y + \frac{3}{2}q_i\sum_{j=1}^{N}q_j r_j + \frac{3}{2}u_x q_i, \tag{2b}$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i D^{-1}u_y - \frac{3}{2}r_i\sum_{j=1}^{N}q_j r_j + \frac{3}{2}u_x r_i, \tag{2c}$$

where D^{-1} stands for the inverse of $\frac{d}{dx}$.

The Lax equation of KP hierarchy is given by (see, e.g., [15])

$$L_{t_n} = [B_n, L], \tag{3}$$

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.02.070

where $L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\cdots$ is a pseudo-differential operator with potential functions u_i's, $B_n=L^n_+$ stands for the differential part of L^n, and $\partial=\frac{d}{dx}$. The notation $u_i'=u_{i,x}$ is used in this letter. The commutativity of ∂_{t_n} flows give rise to the zero-curvature equations of KP hierarchy

$$B_{n,t_k}-B_{k,t_n}+[B_n,B_k]=0.$$

In this letter, we first introduce a new vector field ∂_{τ_k} which is a linear combination of all vector fields ∂_{t_n}. Then we introduce a new Lax type equation which consist of the τ_k-flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of ∂_{τ_k}-flow and ∂_{t_k}-flows gives rise to a new extended KP hierarchy. This hierarchy enables us to obtain the first and second types of KPSCS (i.e., (1) and (2)) in a different way from those in [8–10,13,14,16] and to get their Lax representations directly. This implies that the new extended KP hierarchy obtained in this letter is different from the mcKP hierarchy given in [1]. Moreover, this new extended KP hierarchy can be reduced to two integrable hierarchies, i.e., the Gelfand–Dickey hierarchy with self-consistent source (GDHWS) [17] and the k-constrained KP hierarchy (k-KPH) [18,19]. The GDHWS includes the first type of KdV equation with self-consistent sources and the first type of Boussinesq equation with self-consistent sources. While, the k-KPH includes the second type of KdV equation with self-consistent sources (the Yajima–Oikawa equation) and the second type of Boussinesq equation with self-consistent sources. Thus, the method proposed in this letter to construct the new extended KP hierarchy provides a general way to find soliton equation with self-consistent sources as well as their Lax representations. Our letter will be organized as follows. In Section 2, we construct the new extended KP hierarchy and show that it contains the first and second types of KPSCS. In Section 3, the new extended KP hierarchy is reduced to the Gelfand–Dickey hierarchy with self-consistent source and the k-constrained KP hierarchy. In Section 4, some conclusions are given.

2. New extended KP hierarchy

With the help of formal dressing method, the L operator for KP hierarchy can be written in the following form [15]

$$L=\phi\partial\phi^{-1},\quad \phi=1+w_1\partial^{-1}+w_2\partial^{-2}+\cdots. \tag{4}$$

And the evolution of the dressing operator ϕ is given by

$$\phi_{t_n}=-L^n_-\phi. \tag{5}$$

The wave function and the adjoint one are then given by

$$\omega(t,z)=\phi\exp\bigl(\xi(t,z)\bigr),$$
$$\omega^*(t,z)=(\phi^*)^{-1}\exp\bigl(-\xi(t,z)\bigr),$$

where $\xi(t,z)=\sum_{i>0}t_iz^i$. They satisfy the following equations

$$Lw(t,z)=zw(t,z),\qquad \frac{\partial}{\partial t_n}w(t,z)=B_n\bigl(w(t,z)\bigr), \tag{6a}$$

$$L^*w^*(t,z)=zw^*(t,z),$$

$$\frac{\partial}{\partial t_n}w^*(t,z)=-B_n^*\bigl(w^*(t,z)\bigr). \tag{6b}$$

It was proved in [15] that the principle part of the resolvent defined by

$$T(z)_-=\sum_{i\in\mathbb{Z}}L^i_-z^{-i-1}, \tag{7a}$$

can be written as

$$T(z)_-=w(t,z)\partial^{-1}w^*(t,z). \tag{7b}$$

For any fixed $k\in\mathbb{N}$, we define a new variable τ_k whose vector field is

$$\partial_{\tau_k}=\partial_{t_k}-\sum_{i=1}^N\sum_{s\geqslant 0}\zeta_i^{-s-1}\partial_{t_s},$$

where ζ_i's are arbitrary distinct non-zero parameters. The τ_k-flow is given by

$$\begin{aligned}L_{\tau_k}&=\partial_{t_k}L-\sum_{i=1}^N\sum_{s\geqslant 1}\zeta_i^{-s-1}\partial_{t_s}L\\&=[B_k,L]-\sum_{i=1}^N\sum_{s\geqslant 0}\zeta_i^{-s-1}[B_s,L]\\&=[B_k,L]+\sum_{i=1}^N\sum_{s\in\mathbb{N}}\zeta_i^{-s-1}[L^s_-,L].\end{aligned}$$

Define $\tilde{B}_k$ by

$$\tilde{B}_k=B_k+\sum_{i=1}^N\sum_{s\in\mathbb{Z}}\zeta_i^{-s-1}L^s_-,$$

which, according to (7), can be written as

$$\tilde{B}_k=B_k+\sum_{i=1}^N w(t,\zeta_i)\partial^{-1}w^*(t,\zeta_i).$$

By setting $q_i=w(t,\zeta_i)$, $r_i=w^*(t,\zeta_i)$, we have

$$\tilde{B}_k=B_k+\sum_{i=1}^N q_i\partial^{-1}r_i, \tag{8a}$$

where q_i and r_i satisfy the following equations

$$q_{i,t_n}=B_n(q_i),\qquad r_{i,t_n}=-B_n^*(r_i),\quad i=1,\dots,N. \tag{8b}$$

Now we introduce a new Lax type equation given by

$$L_{\tau_k}=\left[B_k+\sum_{i=1}^N q_i\partial^{-1}r_i,L\right], \tag{9a}$$

with

$$q_{i,t_n}=B_n(q_i),\qquad r_{i,t_n}=-B_n^*(r_i),\quad i=1,\dots,N. \tag{9b}$$

We have the following lemma [20].

Lemma 1. $[B_n,q\partial^{-1}r]_-=B_n(q)\partial^{-1}r-q\partial^{-1}B_n^*(r)$.

Proof. Without loss of generality, we consider a monomial: $P = a\partial^n$ ($n \geqslant 0$). Then

$$[P, q\partial^{-1}r]_- = aq^{(n)}\partial^{-1}r - (q\partial^{-1}ra\partial^n)_-.$$

Notice that the second term can be rewritten in the following way

$$\begin{aligned}(q\partial^{-1}ra\partial^n)_- &= (q\partial^{-1}\partial(ra)\partial^{n-1} - q\partial^{-1}(ra)'\partial^{n-1})_- \\ &= (-q\partial^{-1}(ra)'\partial^{n-1})_- = \cdots \\ &= (-1)^n q\partial^{-1}(ar)^{(n)} = q\partial^{-1}P^*(r),\end{aligned}$$

then the lemma is proved. □

Proposition 1. *(3) and (9) give rise to the following new extended KP hierarchy*

$$B_{n,\tau_k} - \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right)_{t_n} + \left[B_n, B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right] = 0, \tag{10a}$$

$$q_{i,t_n} = B_n(q_i), \tag{10b}$$

$$r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \ldots, N. \tag{10c}$$

Proof. We will show that under (9b), (3) and (9a) give rise to (10a). For convenience, we assume $N = 1$, and denote q_1 and r_1 by q and r, respectively. By (3), (9) and Lemma 1, we have

$$\begin{aligned}B_{n,\tau_k} &= (L^n_{\tau_k})_+ \\ &= [B_k + q\partial^{-1}r, L^n]_+ \\ &= [B_k + q\partial^{-1}r, L^n_+]_+ + [B_k + q\partial^{-1}r, L^n_-]_+ \\ &= [B_k + q\partial^{-1}r, L^n_+] - [B_k + q\partial^{-1}r, L^n_+]_- + [B_k, L^n_-]_+ \\ &= [B_k + q\partial^{-1}r, B_n] - [q\partial^{-1}r, B_n]_- + [B_n, L^k]_+ \\ &= [B_k + q\partial^{-1}r, B_n] + B_n(q)\partial^{-1}r - q\partial^{-1}B_n^*(r) + B_{k,t_n} \\ &= [B_k + q\partial^{-1}r, B_n] + (B_k + q\partial^{-1}r)_{t_n}. \qquad \square\end{aligned}$$

Under (10a) and (10c), the Lax representation for (10a) is given by

$$\psi_{\tau_k} = \left(B_k + \sum_{i=1}^N q_i\partial^{-1}r_i\right)(\psi), \tag{11a}$$

$$\psi_{t_n} = B_n(\psi). \tag{11b}$$

Now, we list some examples in the new extended KP hierarchy (10).

Example 1 *(The first type of KPSCS).* For $n = 2$ and $k = 3$, (10) yields

$$u_{1,y} - u_1'' - 2u_2' = 0, \tag{12a}$$

$$2u_{1,t} - 3(u_2 + u_1')_{t_2} + 3u_2'' + u_1''' - 6u_1u_1' + 2\sum_{i=1}^N (q_ir_i)' = 0, \tag{12b}$$

$$q_{i,t_2} = q_i'' + 2u_1q_i, \qquad r_{i,t_2} = -r_i'' - 2u_1r_i. \tag{12c}$$

Set $y := t_2$, $t := \tau_3$, $u := u_1$, and eliminate u_2 by differentiating the second equation with respect to x, we get the first type of KP equation with self-consistent sources (1). The Lax representation of (1a) is

$$\psi_y = (\partial^2 + 2u)(\psi),$$

$$\psi_t = \left(\partial^3 + 3u\partial + \left(\frac{3}{2}D^{-1}u_y + \frac{3}{2}u_x\right) + \sum_{i=1}^N q_i\partial^{-1}r_i\right)(\psi).$$

Example 2 *(The second type of KPSCS).* For $n = 3$ and $k = 2$, (10) yields

$$u_{1,y} + \sum_{i=1}^N (q_ir_i)' - u_1'' - 2u_2' = 0, \tag{13a}$$

$$3(u_2 + u_1')_y - 2u_{1,t} - u_1''' + 3\sum_{i=1}^N (q_i'r_i)' + 6u_1u_1' - 3u_2'' = 0, \tag{13b}$$

$$q_{i,t_3} = q_{i,xxx} + 3u_1q_{i,x} + (3u_2 + 3u_1')q_i, \tag{13c}$$

$$r_{i,t_3} = r_{i,xxx} + 3u_1r_{i,x} - 3u_2r_i. \tag{13d}$$

Let $y := \tau_2$, $t = t_3$, $u := u_1$, and eliminate u_2 by integrating the first equation with respect to x, we get the second type of KPSCS (2). This equation was introduced in [8], and rediscovered by source generating method [14]. Under (13c) and (13d), the Lax representation for (2a) is

$$\psi_y = \left(\partial^2 + 2u + \sum_{i=1}^N q_i\partial^{-1}r_i\right)(\psi),$$

$$\psi_t = \left(\partial^3 + 3u\partial + \left(\frac{3}{2}D^{-1}u_y + \frac{3}{2}u_x + \frac{3}{2}\sum_{i=1}^N q_ir_i\right)\right)(\psi).$$

Example 3. For $n = 4$, $k = 2$, and $N = 1$, (10) gives higher order equations

$$u_{1,y} - u_{1,xx} - 2u_{2,x} + (qr)' = 0,$$

$$2u_{2,y} + 3u_{1,xy} + 3(qr)'' - 2(qr')' + 8u_1u_{1,x} - 3u_{1,xxx} - 4u_{3,x} - 8u_{2,xx} = 0,$$

$$\begin{aligned}&2u_{3,y} + 3u_{2,xy} + 2u_{1,xxy} - u_{1,t} + 2(qr)''' \\ &\quad - 3(qr')'' + 2(qr'')' + 4u_1u_{1,xx} \\ &\quad + 4u_1(qr)' + 4u_{1,x}u_2 + 6(u_{1,x})^2 \\ &\quad - 2u_{3,xx} - 3u_{2,xxx} - u_{1,xxxx} = 0,\end{aligned}$$

$$q_t = (\partial^4 + 4u_1\partial^2 + (4u_2 + 6u_1')\partial + (4u_3 + 6u_2' + 4u_1'' + 6u_1^2))q,$$

$$r_t = -(\partial^4 + 4u_1\partial^2 + (4u_2 + 6u_1')\partial + (4u_3 + 6u_2' + 4u_1''))^*r.$$

Here $y := \tau_2$, $t := t_4$. By using

$$2u_2 = D^{-1}u_{1,y} - u_{1,x} + qr,$$
$$4u_{3,x} = D^{-1}u_{1,yy} - 2u_{1,xy} + (qr)_y + 4(qr)_{xx} - 2(qr_x)_x + 8u_1u_{1,x} + u_{1,xxx}$$

the equations yield

$$\frac{1}{2}D^{-1}u_{1,yyy} + 2\big(u_1^2\big)_{xy} + \frac{1}{2}u_{1,xxxy} + \frac{3}{2}u_{1,xy} - u_{1,xt} + 6\big(u_{1,x}^2\big)_x + 4u_1u_{1,xxx} + 2u_{1,x}u_{1,y} + 2u_{1,xx}D^{-1}u_{1,y} - \frac{3}{2}u_{1,xyy} - 2(u_1^2)_{xxx} + \frac{1}{2}(qr)_{yy} + 3(qr)_{xxy} - (qr_x)_{xy} - 2\big(q_xr_x\big)_{xx} + 6u_{1,x}(qr)_x + 4u_1(qr)_{xx} + 2u_{1,xx}qr - \frac{3}{2}(qr)_{xxx} = 0,$$
$$q_t = \big(\partial^4 + 4u_1\partial^2 + (4u_2 + 6u_1')\partial + (4u_3 + 6u_2' + 4u_1'')\big)q,$$
$$r_t = -\big(\partial^4 + 4u_1\partial^2 + (4u_2 + 6u_1')\partial + (4u_3 + 6u_2' + 4u_1'')\big)^*r.$$

3. Reductions

The new extended KP hierarchy (10) admits reductions to several well-known $(1+1)$-dimensional systems.

3.1. The n-reduction of (10)

The n-reduction is given by

$$L^n = B_n \quad \text{or} \quad L^n_- = 0, \tag{14}$$

then (6) implies that

$$B_n(q_i) = L^nq_i = \zeta_i^nq_i, \tag{15a}$$
$$-B_n^*(r_i) = -L^{n*}r_i = -\zeta_i^nr_i. \tag{15b}$$

By using Lemma 1 and (15), we can see that the constraint (14) is invariant under the τ_k flow

$$\big(L^n_-\big)_{\tau_k} = \big[B_k, L^n\big]_- + \sum_{i=1}^N\big[q_i\partial^{-1}r_i, L^n\big]_- = \big[B_k, L^n_-\big]_- + \sum_{i=1}^N\big[q_i\partial^{-1}r_i, L^n_+\big]_- + \sum_{i=1}^N\big[q_i\partial^{-1}r_i, L^n_-\big]_- = \sum_{i=1}^N\big[q_i\partial^{-1}r_i, B_n\big]_-$$
$$= -\sum_{i=1}^N\big(q_{i,t_n}\partial^{-1}r_i + q_i\partial^{-1}r_{i,t_n}\big) \tag{16}$$
$$= -\sum_{i=1}^N\big(\zeta_i^nq_i\partial^{-1}r_i - \zeta_i^nq_i\partial^{-1}r_i\big) = 0. \tag{17}$$

Eqs. (14) and (5) imply that $\phi_{t_n} = 0$, so $(L^k)_{t_n} = 0$, which together with (17) means that one can drop t_n dependency from (10) and obtain

$$B_{n,\tau_k} = \left[(B_n)_+^{\frac{k}{n}} + \sum_{i=1}^Nq_i\partial^{-1}r_i, B_n\right], \tag{18a}$$
$$B_n(q_i) = \zeta_i^nq_i, \tag{18b}$$
$$B_n^*(r_i) = -\zeta_i^nr_i, \quad i = 1, \ldots, N. \tag{18c}$$

The system (18) is the so-called Gelfand–Dickey hierarchy with self-consistent sources [17].

For $n = 2$ and $k = 3$, (18) presents the first type of KdV equation with self-consistent sources ($t := \tau_3$, $u := u_1$)

$$u_t - 3uu_x - \frac{1}{4}u_{xxx} + \sum_{i=1}^N(q_ir_i)_x = 0,$$
$$q_{i,xx} + 2uq_i = \zeta_i^2q_i,$$
$$r_{i,xx} + 2ur_i = \zeta_i^2r_i, \quad i = 1, \ldots, N,$$

with Lax representation

$$\big(\partial^2 + 2u\big)(\psi) = \lambda\psi,$$
$$\psi_t = \left(\partial^3 + 3u\partial + \frac{3}{2}u' + \sum_{i=1}^Nq_i\partial^{-1}r_i\right)(\psi).$$

The first type of KdV equation with self-consistent sources can be solved by the inverse scattering method [10,21] and by the Darboux transformation (see [22] and the references therein).

For $n = 3$ and $k = 2$, (18) presents the first type of Boussinesq equation with self-consistent sources ($t := \tau_2$, $u := u_1$)

$$u_{tt} + \frac{1}{3}u_{xxxx} + 2\big(u^2\big)_{xx} + \sum_{i=1}^N(q_{i,x}r_i - q_ir_{i,x})_{xx} + \sum_{i=1}^N(q_ir_i)_{xt} = 0,$$
$$q_{i,xxx} + 3uq_{i,x} + q_i\left(\frac{3}{2}D^{-1}u_y + \frac{3}{2}u_x + \frac{3}{2}\sum_{j=1}^Nq_jr_j\right) = \zeta_i^3q_i,$$
$$r_{i,xxx} + 3ur_{i,x} - r_i\left(\frac{3}{2}D^{-1}u_y - \frac{3}{2}u_x + \frac{3}{2}\sum_{j=1}^Nq_jr_j\right) = \zeta_i^3r_i,$$
$$i = 1, \ldots, N,$$

with Lax representation

$$\big(\partial^3 + 3u_1\partial + 3u_2 + 3u_{1,x}\big)(\psi) = \lambda\psi,$$
$$\psi_t = \left(\partial^2 + 2u_1 + \sum_{i=1}^Nq_i\partial^{-1}r_i\right)(\psi).$$

3.2. The k-constrained hierarchy of (10)

The k-constraint is given by [18–20].

$$L^k = B_k + \sum_{i=1}^Nq_i\partial^{-1}r_i.$$

By dropping τ_k dependency from (10), we get

$$\left(B_k+\sum_{i=1}^{N}q_i\partial^{-1}r_i\right)_{t_n} = \left[\left(B_k+\sum_{i=1}^{N}q_i\partial^{-1}r_i\right)_+^{\frac{n}{k}}, B_k+\sum_{i=1}^{N}q_i\partial^{-1}r_i\right], \tag{19a}$$

$$q_{i,t_n}=\left(B_k+\sum_{j=1}^{N}q_j\partial^{-1}r_j\right)_+^{\frac{n}{k}}(q_i), \tag{19b}$$

$$r_{i,t_n}=-\left(B_k+\sum_{j=1}^{N}q_j\partial^{-1}r_j\right)_+^{\frac{n}{k}*}(r_i), \quad i=1,\dots,N, \tag{19c}$$

which is the so-called k-constrained KP hierarchy [18–20].

For $k=2$ and $n=3$, (19) gives rise to the second type of KdV equation with self-consistent sources.

$$u_t=\frac{1}{4}u_{xxx}+3uu_x+\frac{3}{4}\sum_{i=1}^{N}(q_{i,xx}r_i-q_ir_{i,xx}),$$

$$q_{i,t}=q_{i,xxx}+3uq_{i,x}+\frac{3}{2}q_i\sum_{j=1}^{N}q_jr_j+\frac{3}{2}u_xq_i,$$

$$r_{i,t}=r_{i,xxx}+3ur_{i,x}-\frac{3}{2}r_i\sum_{j=1}^{N}q_jr_j+\frac{3}{2}u_xr_i, \quad i=1,\dots,N.$$

For $k=3$ and $n=2$, (19) gives rise to the second type of Boussinesq equation with self-consistent sources.

$$u_{tt}+\frac{1}{3}u_{xxxx}+2\left(u^2\right)_{xx}-\frac{4}{3}\sum_{i=1}^{N}(q_ir_i)_{xx}=0,$$

$$q_{i,t}=q_{i,xx}+2uq_i,$$

$$r_{i,t}=-r_{i,xx}-2ur_i, \quad i=1,\dots,N.$$

4. Conclusions

A method is proposed in this letter to construct a new extended KP hierarchy, which enables us to find the first and second types of KPSCS (i.e., (1) and (2)) in a different way from those in [8–10,13,14] and to get their Lax representations directly. The new extended KP hierarchy offers natural reductions to the well-known Gelfand–Dickey hierarchy with self-consistent sources and to the k-constrained KP hierarchy. The k-constrained KP hierarchy includes the second type of KdV equation with self-consistent sources (the Yajima–Oikawa equation) and the second type of Boussinesq equation with self-consistent sources. The method proposed here provides a general way to construct the soliton equations with self-consistent sources and their Lax representations. This approach for constructing extended hierarchies can be applied to other $(2+1)$-dimensional systems (such as mKP, BKP, CKP, etc.) and semi-discrete systems (e.g., two-dimensional Toda lattice hierarchy). We will present some other new extended hierarchies in the forthcoming paper.

Acknowledgements

This work was supported by National Basic Research Program of China (973 Program) (2007CB814800) and National Natural Science Foundation of China (grand No. 10601028).

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