A generalized dressing approach for solving the extended KP and the extended mKP hierarchy

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A combination of dressing method and variation of constants as well as a formula for constructing the eigenfunction is used to solve the extended KP hierarchy, which is a hierarchy with one more series of time flow and based on the symmetry constraint of KP hierarchy. Similarly, extended mKP hierarchy is formulated and its zero-curvature form, Lax representation, and reductions are presented. Via gauge transformation, it is easy to transform dressing solutions of extended KP hierarchy to the solutions of extended mKP hierarchy. Wronskian solutions of extended KP and extended mKP hierarchies are constructed explicitly. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3126494]

I. INTRODUCTION

KP hierarchy is of fundamental importance not only in the theory of integrable systems but also in mathematical physics.[1–4] Besides interest of its own, many extensions and generalizations to KP hierarchy are also of great interest. One of these extensions is the multicomponent generalization.[5,6] Another kind of generalization is the so-called KP equation with self-consistent sources (KPSCS), which was discovered by Mel'nikov[7–9] and appeared later in Ref. 10 as a multiscale expansion of matrix analogy of the KP equation. Recently, we proposed an approach to construct an *extended KP hierarchy* (exKPH).[11] Inspired by the squared eigenfunction symmetry constraint of KP hierarchy, we introduced the new τ_k-flow by "extending" a specific t_k-flow of KP hierarchy. Then we find the exKPH consisting of t_n-flow of KP hierarchy, τ_k-flow, and the t_n-evolutions of eigenfunctions and adjoint eigenfunctions. The commutativity of t_n-flow and τ_k-flow gives rise to zero-curvature representation for exKPH. During to the introduction of τ_k-flow, the exKPH contains two time series $\{t_n\}$ and $\{\tau_k\}$ and more components by adding eigenfunctions and adjoint eigenfunctions.

The exKPH contains the first type and second type of KPSCS presented in Refs. 7–9. By t_n-reduction and τ_k-reduction, the exKPH reduces to the Gelfand–Dickey (GD) hierarchy with self-consistent sources and constrained KP hierarchy, respectively. For the τ_0 case, the exKPH with $k=0$ gives rise to the system considered in Refs. 12–14.

Since symmetry constraints are common objects for integrable systems, this idea for building exKPH was applied to many other KP-like 2+1 dimensional hierarchies, such as BKP hierarchy,[15] CKP hierarchy,[16] dispersionless KP hierarchy,[17] and even q-deformed KP hierarchy[18] and semi-discrete system such as two dimensional Toda lattice hierarchy.[19] The corresponding extended (2+1)-dimensional integrable systems and their reductions give both well-known and new integrable models.

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The *dressing* method is an important tool for solving GD and KP hierarchies.[20] However, this method cannot be applied directly for solving the "extended" hierarchy. For such reason, modifications to the traditional dressing method are needed. In this paper, with the combination of variation of constant method, a generalization to the dressing method is proposed, which is based on the dressing method for GD and KP hierarchy[20] and the approach for finding Wronskian solutions to constrained KP hierarchy.[21] In this way, we can solve the entire hierarchy of extended KP in a unified and simple manner. As the special cases, the *both* types of KPSCS, i.e., (1) and (5), are solved simultaneously.

As another part of our paper, we propose the extended modified KP hierarchy (exmKPH). Similar with the extended KP hierarchy, we introduce the τ_k-flow by the inspiration of squared eigenfunction symmetries constraints $(q_i\partial^{-1}r_i\partial)$ (Ref. 13) of mKPH. A special case of exmKPH (called nonstandard exmKPH) is obtained by choosing specific q_i and r_i. Two types of reductions, say t_n and τ_k reductions, of exmKPH are discussed. With t_n reduction, we obtain $1+1$ hierarchies, including mKdV equation with self-consistent sources. With τ_k reduction, we obtain constrained mKP hierarchies discussed in Refs. 22 and 23. A gauge transformation between exKPH and exmKPH is presented, based on the gauge transformation constructed by Refs. 22 and 24. Since solutions for exKPH are obtained, this transformation helps us to show the explicit formulation of Wronskian solutions for the exmKPH.

Our paper will be organized as follows. In Sec. II, we briefly recall the extended KP hierarchy. In Sec. III, we present the generalized dressing approach. In Sec. IV, we present the new extended mKP hierarchy and its reductions. In Sec. V, we give gauge transformation between the exKPH and exmKPH. In Sec. VI, we construct some solutions of exKPH and exmKPH. In the last section, we give conclusion.

II. THE EXTENDED KP HIERARCHY

In order to make our paper self-contained, we introduce the extended KP hierarchy[11] briefly. As well known, the Lax equation of KP hierarchy is given by

$$\partial_{t_n}L=[B_n,L] \quad \text{for } n\geq 1, \tag{1}$$

where $L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\cdots$ is a pseudodifferential operator (PDO). $B_n=L^n_{\geq 0}$ stands for the projection of PDO to its non-negative power part. The commutativity of ∂_{t_n} and ∂_{t_k} flows gives the zero-curvature equations of KP hierarchy,

$$B_{n,t_k}-B_{k,t_n}+[B_n,B_k]=0, \quad n,k>0.$$

It is known[25,26] that the squared eigenfunction symmetry constraint given by

$$L^k=B_k+\sum_{i=1}^{N}q_i\partial^{-1}r_i,$$

$$q_{i,t_n}=B_n(q_i),$$

$$r_{i,t_n}=-B_n^*(r_i), \quad i=1,\ldots,N,$$

is compatible with the KP hierarchy and reduces the KP hierarchy to the k-constrained KP hierarchy. Here the * denotes the adjoint operator. In this paper, we use $P(f)$ to denote an action of differential operator P on the function f, while Pf means the multiplication of differential operator P and zero order differential operator f.

Inspired by squared eigenfunction symmetry constraints, we presented the following integrable extended KP hierarchy (exKPH) in Ref. 11,

$$\partial_{\tau_k} L = \left[B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i, L \right], \quad N \geq 0, \tag{2a}$$

$$\partial_{t_n} L = [B_n, L], \quad \forall\, n \neq k, \tag{2b}$$

$$\partial_{t_n} q_i = B_n(q_i), \tag{2c}$$

$$\partial_{t_n} r_i = - B_n^*(r_i), \quad i = 1, \ldots, N. \tag{2d}$$

By showing the commutativity of (2a) and (2b) under (2c) and (2d), the zero-curvature equation for exKPH (2) is

$$B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] - \sum_{i=1}^{N} [q_i \partial^{-1} r_i, B_n]_{\geq 0} = 0, \tag{3a}$$

$$q_{i,t_n} = B_n(q_i), \tag{3b}$$

$$r_{i,t_n} = - B_n^*(r_i), \quad i = 1, \ldots, N. \tag{3c}$$

It is easy to see when $N=0$ Eqs. (2) and (3) return back to the ordinary KP hierarchy. Under (3b) and (3c) the Lax pair for (3a) is

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = \left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \right)(\Psi). \tag{4}$$

The extended KP is closely related to the KPSCS, for instance.

Example 1: *When $n=2$ and $k=3$, (2) provides the first type of KPSCS,*

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4\sum_{i=1}^{N} (q_i r_i)_{xx} = 0, \tag{5a}$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \quad i = 1, \ldots, N, \tag{5b}$$

$$r_{i,y} = - r_{i,xx} - 2ur_i. \tag{5c}$$

While for $n=3$ and $k=2$, it gives the second type of KPSCS,

$$4u_t - 12uu_x - u_{xxx} - 3D^{-1}u_{yy} = 3\sum_{i=1}^{N} [q_{i,xx} r_i - q_i r_{i,xx} + (q_i r_i)_y], \tag{6a}$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2} q_i D^{-1} u_y + \frac{3}{2} q_i \sum_{j=1}^{N} q_j r_j + \frac{3}{2} u_x q_i, \tag{6b}$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2} r_i D^{-1} u_y - \frac{3}{2} r_i \sum_{j=1}^{N} q_j r_j + \frac{3}{2} u_x r_i, \tag{6c}$$

respectively, where D^{-1} stands for the inverse of d/dx.

Remark 1: *In Ref. 13 the author considered the following flow generated by*

$$L_\tau = [q\partial^{-1}r, L],$$

$$q_{t_n} = B_n(q),$$

$$r_{t_n} = -B_n^*(r).$$

They proved that the ∂_τ-flow commutes with all the t_n-flow of (1). So the whole system is compatible. This system can be regarded as a special case of (2), if we consider $k=0$ and $\tau_0=\tau$.

III. GENERALIZED DRESSING APPROACH AND VARIATION OF CONSTANTS FOR exKPH

Inspired by Refs. 20 and 21, we consider the dressing formulation of exKPH. We assume that L operator of exKPH can be written in the dressing form,

$$L = W\,\partial\,W^{-1},$$

where $W=1+w_1\partial^{-1}+w_2\partial^{-2}+\cdots$ is called a *dressing operator*. Then in terms of W, exKPH can be written as

$$\partial_{t_n}W = -L_-^n W, \quad n \neq k, \tag{7a}$$

$$\partial_{\tau_k}W = -L_-^k W + \sum_{i=1}^{N} q_i\partial^{-1}r_i W, \tag{7b}$$

where q_i and r_i satisfy (2c) and (2d), respectively. The simplest W has finite many terms, which is $W=1+w_1\partial^{-1}+w_2\partial^{-2}+\cdots+w_N\partial^{-N}$. It is equivalent to assume that W is a pure differential operator of order N (since it is equivalent to multiplying ∂^N from right side to the above expression).

Let $h_1,\ldots,h_N$ be linearly independent functions satisfying

$$\partial_{t_n}h_i = \partial^n(h_i), \quad i=1,\ldots,N. \tag{8}$$

It is known[20] that the *Wronskian determinant*

$$\mathrm{Wr}(h_1,\ldots,h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ h_1' & h_2' & \cdots & h_N' \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)} \end{vmatrix}$$

is a τ-function of KP hierarchy and the Nth order differential operator given by

$$W = \frac{1}{\mathrm{Wr}(h_1,\cdots,h_N)}\begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1' & h_2' & \cdots & h_N' & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix} \tag{9}$$

provides a dressing operator for KP hierarchy satisfying (7a). The numerator of (9) is a formal determinant, which is denoted by $\mathrm{Wr}(h_1,\cdots,h_N,\partial)$. It is understood as an expansion with respect to its last column, in which all subdeterminants are collected on the left of the differential symbols.

Unfortunately, the dressing operator W defined by (8) and (9) neither satisfy (7b) nor provide formula for q_i and r_i. Since (7b) can be regarded as $\partial_{\tau_k}W=-L_-^kW$ with nonhomogeneous term, we can provide a new dressing operator by combining the usual dressing approach and the method of variation of constants. We also present formulas for q_i and r_i, which is motivated by. Ref. 21.

Let f_i, g_i satisfy

$$\partial_{t_n} f_i = \partial^n(f_i), \quad \partial_{\tau_k} f_i = \partial^k(f_i)(i=1,\ldots,N), \tag{10a}$$

$$\partial_{t_n} g_i = \partial^n(g_i), \quad \partial_{\tau_k} g_i = \partial^k(g_i). \tag{10b}$$

Now let h_i be the linear combination of f_i and g_i as

$$h_i = f_i + \alpha_i(\tau_k) g_i, \quad i=1,\ldots,N, \tag{11}$$

with the coefficient α_i being a differentiable function of τ_k. (Suppose $h_1,\ldots,h_N$ are still linearly independent.) We call this the variation of constants because if α_i is a constant independent of τ_k, the formulation returns back to (8) and (9) for KP case. Then clearly W defined by (9) and (11) still satisfy (7a). To claim that W satisfies (7b), we present

$$q_i = -\dot{\alpha}_i W(g_i), \quad r_i = (-1)^{N-i} \frac{\mathrm{Wr}(h_1,\ldots,\hat{h}_i,\ldots,h_N)}{\mathrm{Wr}(h_1,\ldots,h_N)}, \quad i=1,\ldots,N, \tag{12}$$

where the hat ˆ means rule out this term from the Wronskian determinant, $\dot{\alpha}_i = d\alpha_i/d\tau_k$, and here is a proposition.

Proposition 1: *Let W defined by (9) and (11), $L = W\partial W^{-1}$, then W, q_i, r_i satisfy (7).*

Remark 2: *Note that $r_1,\ldots,r_N$ defined in (12) satisfy the following linear equation:*

$$\sum_{i=1}^{N} h_i^{(j)} r_i = \delta_{j,N-1}, \quad j=0,1,\ldots,N-1, \tag{13}$$

where $\delta_{j,N-1}$ is the Kronecker's delta symbol.

To prove Proposition 1, we need several lemmas. The first one is given by Oevel and Strampp.[21]

Lemma 1: *(Oevel and Strampp) $W^{-1} = \sum_{i=1}^{N} h_i \partial^{-1} r_i$.*

Lemma 2: *For a pure differential operator P and a function f,*

$$\mathrm{Res}_\partial\, \partial^{-1} f P = P^*(f).$$

Lemma 3: *W^* annihilates each r_i, i.e., $W^*(r_i)=0$ for $i=1,\ldots,N$.*

Proof: Expanding the identity $W^*(W^{-1})^*\partial^j = \partial^j$ by using Lemma 1, and taking the residue, we have

$$0 = \mathrm{Res}_\partial\, W^*\left(\sum_{i=1}^{N} h_i \partial^{-1} r_i\right)^* \partial^j = -\mathrm{Res}_\partial\, W^* \sum_{i=1}^{N} r_i \partial^{-1} h_i \partial^j = (-1)^{j+1} \sum_{i=1}^{N} h_i^{(j)} W^*(r_i).$$

According to the last equality, $W^*(r_i)$ vanishes. ■

Lemma 4: (Key lemma) *The operator $\partial^{-1} r_i W$ is a pure differential operator for each i. Furthermore, for $1 \le i,j \le N$,*

$$(\partial^{-1} r_i W)(h_j) = \delta_{ij}. \tag{14}$$

Proof: Because $(\partial^{-1} r_i W)_- = \partial^{-1}[W^*(r_i)] = 0$ (by Lemma 3), $\partial^{-1} r_i W$ is a pure differential operator. We can define functions $c_{ij} = (\partial^{-1} r_i W)(h_j)$ and the x-derivative of c_{ij} shows $\partial(c_{ij}) = r_i W(h_j) = 0$, which means c_{ij} does not depend on x. Then

$$\sum_{i=1}^{N} h_i^{(k)} c_{ij} = \partial^k\left(\sum_i h_i c_{ij}\right) = \partial^k\left(\sum_i (h_i \partial^{-1} r_i W)(h_j)\right) = \partial^k(W^{-1}W)(h_j) = h_j^{(k)},$$

so $c_{ij} = \delta_{ij}$. □

Remark 3: *Lemma 4 is crucial to the proof of Proposition 1 and upcoming results. We believe* *it is important also because it shows the inherent connection between W, the function h_i($\in \ker W$), and the function r_i($\in \ker W^*$).*

Proof of Proposition 1: The proof of (7a) was given in Ref. 20 (p. 80). In a same way, taking ∂_{τ_k} to the identity $W(h_i)=0$, using (10a) and (11), the definition of q_i, and the *key lemma*, we have

$$0=(\partial_{\tau_k}W)(h_i)+(W\partial^k)(h_i)+\dot{\alpha}_i W(g_i)$$
$$=(\partial_{\tau_k}W)(h_i)+(L^kW)(h_i)-\sum_{j=1}^N q_j\delta_{ji}$$
$$=\left(\partial_{\tau_k}W+L_-^kW-\sum_{j=1}^N q_j\partial^{-1}r_jW\right)(h_i).$$

Since the pure differential operator acting on h_i in the last expression has degree $<N$, it cannot annihilate N independent functions unless the operator itself vanishes. Hence (7) is proven. □

Theorem 1: *W defined by (9) and (11), q_i and r_i defined by (12), $L=W\partial W^{-1}$ satisfy the extended KP hierarchy (2).*

Proof: The proof (2b) and (2c) can be found in Ref. 20 The proof of (2a) is straightforward by using Proposition 1,

$$L_{\tau_k}=W_{\tau_k}\partial W^{-1}-W\partial W^{-1}W_{\tau_k}W^{-1}=\left(-L_-^k+\sum_i q_i\partial^{-1}r_i\right)L+L\left(L_-^k-\sum_i q_i\partial^{-1}r_i\right)$$
$$=\left[-L_-^k+\sum_i q_i\partial^{-1}r_i,L\right]=\left[B_k+\sum_{i=1}^N q_i\partial^{-1}r_i,L\right].$$

So it remains to prove (2d). First, we see that

$$(W^{-1})_{t_n}=-W^{-1}W_{t_n}W^{-1}=-W^{-1}(L^n-B_n)=\partial^nW^{-1}-W^{-1}B_n.$$

Then we make substitutions to this equality by applying $W^{-1}=\Sigma h_i\partial^{-1}r_i$ at both ends, we have

$$(W^{-1})_{t_n}=\sum h_i^{(n)}\partial^{-1}r_i+\sum h_i\partial^{-1}r_{i,t_n}=\partial^nW^{-1}-W^{-1}B_n=\sum h_i^{(n)}\partial^{-1}r_i-\sum h_i\partial^{-1}B_n^*(r_i).$$

Then $\Sigma h_i\partial^{-1}r_{i,t_n}=-\Sigma h_i\partial^{-1}B_n^*(r_i)$ implies (2d). □

IV. THE EXTENDED mKP HIERARCHY

In the same way as in Sec. II, the extended mKP hierarchy is formulated. The L operator of mKP hierarchy is defined by

$$L=\partial+v_0+v_1\partial^{-1}+v_2\partial^{-2}+\cdots.$$

The Lax equations of mKP hierarchy are given by

$$L_{t_n}=[B_n,L],\quad n\geq 1,\quad B_n=L_{\geq 1}^n. \tag{15}$$

The commutativity of ∂_{t_n} and ∂_{t_k} flows gives the zero-curvature equation,

$$B_{n,t_k}-B_{k,t_n}+[B_n,B_k]=0.$$

When $n=2$ and $k=3$, we get *mKP equation*,

$$4v_t-v_{xxx}+6v^2v_x-3(D^{-1}v_{yy})-6v_x(D^{-1}v_y)=0,$$

where $t:=t_3$, $y:=t_2$, $v:=v_0$. Dropping y dependency, we obtain the mKdV equation.

Since the squared eigenfunction symmetry constraint given by

$$L^k = B_k + \sum q_i \partial^{-1} r_i \partial$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \ldots, N,$$

$$r_{i,t_n} = -(\partial B_n \partial^{-1})^*(r_i)$$

is consistent with the mKP hierarchy and reduces mKP hierarchy to k-constraint mKP hierarchy (see Refs. 13 and 22 and references therein), we can define extended mKP hierarchy by introduce a new time evolution of L,

$$L_{\tau_k} = \left[B_k + \sum q_i \partial^{-1} r_i \partial, L\right],$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \ldots, N,$$

$$r_{i,t_n} = -(\partial B_n \partial^{-1})^*(r_i).$$

Definition 1: The *exmKPH is defined by*

$$L_{\tau_k} = \left[B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial, L\right], \tag{16a}$$

$$L_{t_n} = [B_n, L], \quad n \neq k, \tag{16b}$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \ldots, N \tag{16c}$$

$$r_{i,t_n} = -(\partial B_n \partial^{-1})^*(r_i). \tag{16d}$$

We can verify the compatibility of (16a) and (16b) in the same way as in Ref. 11.

By using the identity $(q_i \partial^{-1} r_i \partial)_{t_n} = [B_n, q_i \partial^{-1} r_i \partial]_{\leq 0}$, the zero-curvature equation for exmKPH can be written as

$$B_{k,t_n} - B_{n,\tau_k} + [B_k, B_n] + \sum_{i=1}^{N} [q_i \partial^{-1} r_i \, \partial, B_n]_{\geq 1} = 0, \tag{17a}$$

$$q_{i,t_n} = B_n(q_i), \quad i = 1, \ldots, N, \tag{17b}$$

$$r_{i,t_n} = -(\partial^{-1} B_n^* \partial)(r_i). \tag{17c}$$

Under the conditions (17b) and (17c), the Lax pair for (17) is given by

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = \left(B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial\right)(\Psi).$$

Example 2: *When* $n=2$ *and* $k=3$, *we get the first type of mKP with self-consistent sources,*

$$4v_t - v_{xxx} + 6v^2 v_x - 3D^{-1} v_{yy} - 6v_x D^{-1} v_y + 4\sum_{i=1}^{N} (q_i r_i)_x = 0,$$

$$q_{i,y}=q_{i,xx}+2vq_{i,x},$$

$$r_{i,y}=-r_{i,xx}+2vr_{i,x},$$

where $t:=\tau_3$, $y:=t_2$, $v:=v_0$. *When* $n=3$ *and* $k=2$, *we get the second type of mKPSCS*

$$4v_t-v_{xxx}+6v^2v_x-3D^{-1}v_{yy}-6v_xD^{-1}v_y+\sum_{i=1}^{N}[3(q_ir_{i,xx}-q_{i,xx}r_i)-3(q_ir_i)_y-6(vq_ir_i)_x]=0, \tag{18a}$$

$$q_{i,t}=q_{i,xxx}+3vq_{i,xx}+\frac{3}{2}(D^{-1}v_y)q_{i,x}+\frac{3}{2}v_xq_{i,x}+\frac{3}{2}v^2q_{i,x}+\frac{3}{2}\sum_{j=1}^{N}(q_jr_j)q_{i,x}, \tag{18b}$$

$$r_{i,t}=r_{i,xxx}-3vr_{i,xx}+\frac{3}{2}(D^{-1}v_y)r_{i,x}-\frac{3}{2}v_xr_{i,x}+\frac{3}{2}v^2r_{i,x}+\frac{3}{2}\sum_{j=1}^{N}(q_jr_j)r_{i,x}, \tag{18c}$$

where $y:=\tau_2$, $t:=t_3$, $v:=v_0$.

Proposition 2: *It is easy to see that any constant is an (adj-)eigenfunction for (17c). Let* $N=2$, $q_1:=q$, $r_1=1$, $q_2=-1$, *and* $r_2=r$, *then we have a special case of exmKPH, called the* nonstandard exmKPH,

$$B_{k,t_n}-B_{n,\tau_k}+[B_k,B_n]+[q-r+\partial^{-1}r_x,B_n]_{\geq 1}=0, \tag{19a}$$

$$q_{t_n}=B_n(q), \tag{19b}$$

$$r_{t_n}=-(\partial^{-1}B_n^*\partial)(r). \tag{19c}$$

Example 3: *For* $n=2$ *and* $k=3$, *let* $\tau_3=t$, $t_2=y$, $v_0=v$, *we get the first non-standard extended mKP equation,*

$$4v_t-v_{xxx}+6v^2v_x-3D^{-1}v_{yy}-6v_xD^{-1}v_y+4(q-r)_x=0,$$

$$q_y=q_{xx}+2vq_x,$$

$$r_y=-r_{xx}+2vr_x.$$

Example 4: *For* $n=3$ *and* $k=2$, *let* $\tau_2=y$, $t_3=t$, $v_0=v$, *we get the second nonstandard extended mKP equation,*

$$4v_t-v_{xxx}+6v^2v_x-3D^{-1}v_{yy}-6v_xD^{-1}v_y-3(q-r)_y-3(q-r)_{xx}-6(vq-vr)_x-6r=0,$$

$$q_t=q_{xxx}+3vq_{xx}+\tfrac{3}{2}(D^{-1}v_y)q_x+\tfrac{3}{2}v_xq_x+\tfrac{3}{2}v^2q_x+\tfrac{3}{2}(q-r)q_x,$$

$$r_t=r_{xxx}-3vr_{xx}+\tfrac{3}{2}(D^{-1}v_y)r_x-\tfrac{3}{2}v_xr_x+\tfrac{3}{2}v^2r_x+\tfrac{3}{2}(q-r)r_x.$$

For the extended mKP hierarchies, there are two time series t_n and τ_k. So we can consider the reduction with respect to these time series, namely, the t_n-reduction and τ_k-reduction for the exmKP hierarchies, see the Appendix.

V. GAUGE TRANSFORMATION BETWEEN exKPH AND exmKPH

In Refs. 22 and 23, the authors give a gauge transformation between KP and mKP hierarchies with self-consistent sources. This transformation can be put parallelly to the exKPH and exmKPH cases. In this section, we use L and q_i, r_i for exKP hierarchy, and $\widetilde{L}$ and $\widetilde{q}_i$, $\widetilde{r}_i$ for exmKP hierarchy. The main result can be established as following.

Theorem 2: *Suppose L, q_i, r_i satisfy (1), f is a particular eigenfunction for Lax pair (4), then*

$$\widetilde{L} := f^{-1}Lf, \quad \widetilde{q}_i := f^{-1}q_i, \quad \widetilde{r}_i := -D^{-1}(fr_i)$$

satisfies the exmKP hierarchy (16).

We omit the proof since it is done by straightforward calculation.

Then we choose

$$f = W(1) = (-1)^N \frac{\mathrm{Wr}(h_1', \ldots, h_N')}{\mathrm{Wr}(h_1, \ldots, h_N)}$$

as the particular eigenfunction for Lax pair (4), where W is the dressing operator defined by (9), and (1) is clearly the *seed solution* for (4) when $L=\partial$, $q_i=r_i=0$. Then starting from the Lq_i and r_i given by Theorem 1, the Wronskian solution for exmKP hierarchy is

$$\widetilde{L} = \frac{\mathrm{Wr}(h_1, \ldots, h_N, \partial)}{\mathrm{Wr}(h_1', \ldots, h_N')} \partial \left[\frac{\mathrm{Wr}(h_1, \ldots, h_N, \partial)}{\mathrm{Wr}(h_1', \ldots, h_N')} \right]^{-1}, \tag{20a}$$

$$\widetilde{q}_i = -\dot{\alpha}_i \frac{\mathrm{Wr}(h_1, \ldots, h_N, g_i)}{\mathrm{Wr}(h_1', \ldots, h_N')}, \tag{20b}$$

$$\widetilde{r}_i = \frac{\mathrm{Wr}(h_1', \ldots, \widehat{h_i'}, \ldots, h_N')}{\mathrm{Wr}(h_1, \ldots, h_N)}, \tag{20c}$$

Equation (20c) can be proved by the trick of Laplace expansion formula.

VI. SOLUTIONS FOR THE exKP AND exmKP HIERARCHIES

Dressing approach with variation of constants and gauge transformation give us a simple way to construct explicit solutions to exKP and exmKP hierarchies. We will use the second type of KPSCS and the second type of mKPSCS as the example. The solution of the second type of KPSCS is constructed by source generating method,[27] the solutions of the second type of mKPSCS is not known yet.

Example 5: [Solutions for the second type of KPSCS (6)] *Let $k=2$, $n=3$ in (2) or (3), $y := \tau_2$, $t := t_3$, then (2) leads to second type of KPSCS (6). Then (10a) has following solutions:*

$$f_i = \exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) := e^{\xi_i}, \quad g_i = \exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) := e^{\eta_i},$$

where λ_i and μ_i (for $i=1,\ldots,N$) are different parameters. We use a linear combinations of f_i and g_i with coefficients α_i explicitly depending on y,

$$h_i = f_i + \alpha_i(y) g_i = 2\sqrt{\alpha_i} e^{(\xi_i+\eta_i)/2} \cosh(\Omega_i),$$

where $\Omega_i = (\xi_i - \eta_i)/2 - 1/2 \ln(\alpha_i)$.

(1) *One soliton solution. Let $N=1$, then we have*

$$L = W\partial W^{-1} = \partial + \frac{(\lambda_1 - \mu_1)^2}{4} \mathrm{sech}^2(\Omega_1)\partial^{-1} + \cdots, \quad W = \partial - h_1'/h_1.$$

The one soliton solution of (6) with $N=1$ is

$$u=\frac{(\lambda_1-\mu_1)^2}{4}\operatorname{sech}^2(\Omega),$$

$$q_1=\sqrt{\alpha_{1y}}(\lambda_1-\mu_1)e^{(\xi_1+\eta_1)/2}\operatorname{sech}(\Omega_1),$$

$$r_1=\frac{1}{2\sqrt{\alpha_1}}e^{-(\xi_1+\eta_1)/2}\operatorname{sech}(\Omega_1).$$

(2) *Two soliton solution. Let $N=2$, then we have*

$$L=\partial+\partial_x^2\ln\tau\partial^{-1}+\cdots.$$

Two soliton solution is

$$u=\partial^2\ln\tau,$$

$$q_1=\alpha_{1,y}\frac{(\lambda_1-\mu_1)(\lambda_2-\mu_1)}{\tau}\left(1+\alpha_2\frac{(\lambda_1-\mu_2)(\mu_2-\mu_1)}{(\lambda_1-\lambda_2)(\lambda_2-\mu_1)}e^{\chi_2}\right)e^{\eta_1},$$

$$q_2=\alpha_{2,y}\frac{(\lambda_2-\mu_2)(\lambda_1-\mu_2)}{\tau}\left(1+\alpha_1\frac{(\lambda_2-\mu_1)(\mu_1-\mu_2)}{(\lambda_2-\lambda_1)(\lambda_1-\mu_2)}e^{\chi_1}\right)e^{\eta_2},$$

$$r_1=\frac{1+\alpha_2e^{\chi_2}}{(\lambda_1-\lambda_2)\tau}e^{-\xi_1},\quad r_2=\frac{1+\alpha_1e^{\chi_1}}{(\lambda_2-\lambda_1)\tau}e^{-\xi_2},$$

where $\chi_i=\eta_i-\xi_i$ $(i=1,2)$ and

$$\tau=1+\alpha_1\frac{\lambda_2-\mu_1}{\lambda_2-\lambda_1}e^{\chi_1}+\alpha_2\frac{\mu_2-\lambda_1}{\lambda_2-\lambda_1}e^{\chi_2}+\alpha_1\alpha_2\frac{\mu_2-\mu_1}{\lambda_2-\lambda_1}e^{\chi_1+\chi_2}.$$

Example 6: [Solutions for the second type of mKPSCS (18)] *Since $\tilde{L}=f^{-1}Lf$*. Then it is easy to see

$$v_0=\partial_x\ln f,$$

where $f=W(1)=(-1)^N\mathrm{Wr}(h_1',\cdots,h_N')/\mathrm{Wr}(h_1,\cdots,h_N)$.

(1) *For $N=1$ we have*

$$\tilde{L}=\partial+\frac{\lambda_1-\mu_1}{2}[\tanh(\Omega_1+\theta_1)-\tanh\Omega_1]+\cdots,$$

where $\theta_1=\ln\sqrt{\lambda_1/\mu_1}$. One soliton solution for (2) with $N=1$ is

$$v=\frac{\lambda_1-\mu_1}{2}[\tanh(\Omega_1+\theta_1)-\tanh(\Omega_1)],$$

$$\tilde{q}_1=\partial_y(\sqrt{\alpha_1/(\lambda_1\mu_1)})(\mu_1-\lambda_1)e^{(\xi_1+\eta_1)/2}\operatorname{sech}(\Omega_1+\theta_1),$$

$$\tilde{r}_1=-\frac{1}{2\sqrt{\alpha_1}}e^{-(\xi_1+\eta_1)/2}\operatorname{sech}\Omega_1.$$

(2) *For $N=2$, we have*

$$\widetilde{L}=\partial+\partial_x \ln f+\cdots,$$

where $f=\lambda_1\lambda_2\widetilde{\tau}/\tau$, τ is defined in the previous example,

$$\widetilde{\tau}=1+\frac{\mu_1}{\lambda_1}\alpha_1\frac{\lambda_2-\mu_1}{\lambda_2-\lambda_1}e^{\chi_1}+\frac{\mu_2}{\lambda_2}\alpha_2\frac{\mu_2-\lambda_1}{\lambda_2-\lambda_1}e^{\chi_2}+\frac{\mu_1\mu_2}{\lambda_1\lambda_2}\alpha_1\alpha_2\frac{\mu_2-\mu_1}{\lambda_2-\lambda_1}e^{\chi_1+\chi_2}.$$

The two soliton solution for (18) with $N=2$ is

$$v=\partial \ln f,$$

$$\widetilde{q}_1=\alpha_{1,y}\frac{(\lambda_1-\mu_1)(\lambda_2-\mu_1)}{\lambda_1\lambda_2\widetilde{\tau}}\left(1+\alpha_2\frac{(\lambda_1-\mu_2)(\mu_2-\mu_1)}{(\lambda_1-\lambda_2)(\lambda_2-\mu_1)}e^{\chi_2}\right)e^{\eta_1},$$

$$\widetilde{q}_2=\alpha_{2,y}\frac{(\lambda_2-\mu_2)(\lambda_1-\mu_2)}{\lambda_1\lambda_2\widetilde{\tau}}\left(1+\alpha_1\frac{(\lambda_2-\mu_1)(\mu_1-\mu_2)}{(\lambda_2-\lambda_1)(\lambda_1-\mu_2)}e^{\chi_1}\right)e^{\eta_2},$$

$$\widetilde{r}_1=\frac{\lambda_2+\alpha_2\mu_2e^{\chi_2}}{(\lambda_2-\lambda_1)\tau}e^{-\xi_1},\quad \widetilde{r}_2=\frac{\lambda_1+\alpha_1\mu_1e^{\chi_1}}{(\lambda_2-\lambda_1)\tau}e^{-\xi_2}.$$

VII. CONCLUSION

In this paper, we generalize the dressing approach for KP hierarchy to exKPH by combining dressing approach and variation of constants, we also present formulas for q_i and r_i. The traditional dressing method cannot provide the evolution of extended τ_k variable. By introducing some varying constants, say $\alpha_i(\tau_k)$, we can obtain the evolution of τ_k correctly. In this way, we constructed Wronskian solutions to the whole exKP hierarchy.

In the same way as extended KP hierarchy, we considered the extended mKP hierarchy by introducing two series of time variables, say t_n and τ_k. The first and second types of mKPSCS are obtained as the first two nontrivial equations of exmKPH. We also considered a special case of exmKPH, called the nonstandard exmKP hierarchy, by choosing special q_i and r_i in exmKPH. Two types of reductions for exmKPH and nonstandard exmKPH, by dropping down t_n or τ_k dependencies, respectively, are discussed. Some $1+1$ dimensional systems, such as mKdV with self-consistent sources and k-constrained mKP hierarchy, are obtained as reductions of exmKPH.

A gauge transformation between exKPH and exmKPH is given. It helps us to obtain explicit solutions to the exmKP hierarchy. By using this transformation, we find soliton solutions for the exmKPH, especially for the second type of mKPSCS, which is unknown before.

It is known that besides KP and mKP, there is another $2+1$ dimensional hierarchy associated with the PDO $L:=w\partial+w_0+w_1\partial^{-1}+\cdots$, which has the Lax equation

$$L_{t_n}=[L^n_{\geq 2},L].$$

It is called Dym hierarchy. Note that the symmetry constraint for this hierarchy is known and a reciprocal transformation between mKP hierarchy and Dym hierarchy is discussed;[13] it is not difficult to find the extended Dym hierarchy and give its solutions.

Another interesting question is: Can we apply the dressing method with variation of constants to other kinds of $2+1$ dimensional systems? For example, the extended BKP and extended CKP hierarchy, the discrete extended KP hierarchy, the q-deformed extended KP hierarchy, and the KP-like hierarchy defined by using *star product* instead of PDO. The last one is worth considering because it admits a deformation to an extended version of dispersionless KP hierarchy. So, if we can apply dressing approach to extended version of such hierarchy, is it possible to deform the solutions of extended version of dispersionless KP hierarchy? If possible, it is another way to give explicit solutions for dispersionless hierarchy, compared with implicit solution given by hydrograph method. We will consider such problems in the future.

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APPENDIX: REDUCTIONS OF exmKP HIERARCHIES

1. The t_n-reduction

The t_n-reduction is given by

$$L^n_{\leq 0} = 0 \quad \text{or} \quad L^n = B_n. \tag{A1}$$

Then we have

$$(L^n)_{t_n} = [B_n, L^n] = 0, \quad B_{n,t_n} = 0,$$

L is independent of t_n and $B_n(q_i) = L^n(q_i) = \lambda_i^n(q_i)$, $-(\partial^{-1}B_n^*\partial)(r_i) = \lambda_i^n(r_i)$. In such case, the submanifold $L^n_{\leq 0}=0$ is invariant under τ_k-flow, namely,

$$(L^n_{\leq 0})_{\tau_k} = \left[B_k + \sum_i q_i\partial^{-1}r_i\partial, L^n\right]_{\leq 0} = [B_k, B_n]_{\leq 0} + \left[\sum_i q_i\partial^{-1}r\partial, B_n\right]_{\leq 0} = \left(\sum_i q_i\partial^{-1}r_i\,\partial\, B_n\right)_{\leq 0} - \sum_i (B_n q_i\partial^{-1}r_i\partial)_{\leq 0}.$$

Note that for a pure differential operator B without ∂^0 term,

$$(\partial^{-1}rB)_{\leq 0} = \partial^{-1}(\partial^{-1}B^*(r))\,\partial\,.$$

So the last express equals to zero,

$$-\sum_i q_i\partial^{-1}(\partial^{-1}B_n^*\,\partial\,(r_i))\,\partial - \sum_i B_n(q_i)\partial^{-1}r_i\partial = \sum_i \lambda_i^n q_i\partial^{-1}r_i\partial - \sum_i \lambda_i^n q_i\partial^{-1}r_i\partial = 0.$$

Therefore the exmKP hierarchy (1) can be reduced to this invariant submanifold, the t_n dependency is dropped. We call the following reduction the *t_n-reduction* for the exmKP hierarchy:

$$\mathcal{L}_{\tau_k} = \left[\mathcal{L}^{k/n}_{\geq 1} + \sum_{i=1}^N q_i\partial^{-1}r_i\partial, \mathcal{L}\right], \tag{A2a}$$

$$\lambda_i^n q_i = \mathcal{L}(q_i), \tag{A2b}$$

$$\lambda_i^n r_i = -\,\partial^{-1}\mathcal{L}^*\,\partial\,(r_i), \quad i = 1, \ldots, N, \tag{A2c}$$

where $\mathcal{L} = L^n = \partial^n + V_{n-2}\partial^{n-1} + \cdots + V_0\partial$.

Remark 4: *(A2) can be regarded as* $(1+1)$*-dimensional integrable soliton hierarchy with self-consistent sources.*

Example 7: *For* $n=2$, $k=3$, $\mathcal{L} = \partial^2 + V_0\partial$, *(A2) gives* the first type of mKdV equation with self-consistent sources,

$$V_t = \frac{1}{4}V_{xxx} - \frac{3}{4}V^2V_x - 2\sum_i (q_ir_i)_x,$$

$$q_{i,xx} + Vq_{i,x} = \lambda_i^2 q_i,$$

$$-r_{i,xx} + Vr_{i,x} = \lambda_i^2 r_i,$$

where $V := V_0$, $t := \tau_3$.

For $n=3$, $k=2$, $\mathcal{L} = \partial^3 + V_1\partial^2 + V_0\partial$, *(A1) gives*

$$V_{tt} = -\frac{1}{3}V_{xxxx} - \frac{2}{3}V_{xx}D^{-1}V_t - \frac{2}{3}V_xV_t + \frac{2}{3}(V^3)_{xxx} + \sum_i \left(3(q_ir_{i,x} - q_{i,x}r_i)_{xx} - (Vq_ir_i)_{xx} - 3(q_ir_i)_{xt}\right),$$

$$q_i''' + Vq_i'' + Uq_i' = \lambda_i^3 q_i,$$

$$r_i''' - Vr_i'' + (V - U')r_i' = \lambda_i^3 r_i,$$

where $t = \tau_2$, $V = V_1$, $U = \frac{1}{2}D^{-1}V_t + \frac{1}{2}V_x + \frac{1}{6}V^2 + \Sigma_i \frac{3}{2}(q_ir_i)$.

The t_n-reduction for the nonstandard exmKP hierarchy is formulated below. Since the nonstandard case is obtained by setting $N=2$, $r_1 = -q_2 = 1$, then λ_1 and λ_2 must be zero. Let $\eta := q - r$, $\psi := r_x$, then the t_n-reduction of nonstandard exmKP hierarchy is

$$\mathcal{L}_{\tau_k} = [\mathcal{L}^{k/n}_{\geq 1} + (q - r) + \partial^{-1}r_x, \mathcal{L}], \tag{A3a}$$

$$\mathcal{L}(q) = 0, \tag{A3b}$$

$$(\partial^{-1}\mathcal{L}^*)(r_x) = 0. \tag{A3c}$$

Example 8: *For* $n=2$ and $k=3$, $\mathcal{L} := \partial^2 + V\partial$,

$$V_t = \tfrac{1}{4}V_{xxx} - \tfrac{3}{8}V^2V_x - 2(q - r)_x,$$

$$q_{xx} + Vq_x = 0,$$

$$r_{xx} - Vr_x = 0.$$

For $n=3$ *and* $k=2$, $\mathcal{L} := \partial^3 + V\partial^2 + U\partial$,

$$V_{tt} = -\tfrac{1}{3}V_{xxxx} + \tfrac{2}{27}(V^3)_{xx} - \tfrac{4}{3}V_{xx}D^{-1}V_t - \tfrac{4}{3}V_xV_t - 3(q - r)_{xt} - 2(V(q - r))_{xx} - 6r_{xxx},$$

$$q_{xxx} + Vq_{xx} + Uq_x = 0,$$

$$r_{xxx} - (Vr_x)_x + Ur_x = 0,$$

where $U = \frac{1}{2}\left(V_x + \frac{1}{3}V^2 + 3\eta + D^{-1}V_t\right)$.

2. The τ_k-reduction

The τ_k-reduction is given by following symmetry constraint:

$$L^k_{\leq 0} = \sum q_i\partial^{-1}r_i\partial,$$

$$q_{i,t_n} = B_n(q_i),$$

$$r_{i,t_n} = -(\partial^{-1}B_n^*\partial)(r_i).$$

It is shown by Ref. 23 that the constraint is invariant under t_n-flow, therefore we reduce the standard exmKP hierarchy (1) to the following well-known *k-constrained mKP hierarchy*:[13,22]

$$(L_k)_{t_n} = [(L_k)^{n/k}_{\geq 1}, L_k], \tag{A4a}$$

$$q_{i,t_n} = B_n(q_i), \tag{A4b}$$

$$r_{i,t_n} = -(\partial^{-1}B_n^*\partial)(r_i), \tag{A4c}$$

where $L_k = L^k_{\geq 1} + \Sigma_i q_i \partial^{-1} r_i \partial$, $B_n = (L_k)^{n/k}_{\geq 1}$.

In Ref. 22 the author discussed the mKP constrained by $\eta + \partial^{-1}\psi$. Clearly this constraint is just the τ_k-reduction of nonstandard exmKP (2), by letting $\eta = q - r$, $\psi = r_x$.

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