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The relation between the Toda hierarchy and the KdV hierarchy

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Abstract

Under three relations connecting the field variables of Toda flows and that of KdV flows, we present three new sequences of combinations of the equations in the Toda hierarchy which have the KdV hierarchy as a continuous limit. The relation between the Poisson structures of the KdV hierarchy and the Toda hierarchy in the continuous limit is also studied. © 1999 Elsevier Science B.V.

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It is well known that the KdV-type equations can be obtained as continuous limits of suitably chosen discrete integrable systems. The relation between the KdV equation and the continuous limit of the Toda lattice was first described by Toda and Wadati [1]. The limit process in the realm of inverse scattering was discussed in Ref. [2]. Kupershmidt [3] presented a general setting for the integrable discrete system and studied the limit case in this framework.

In recent years the discussion on the relation between a hierarchy of soliton equations and the continuous limit of a hierarchy of integrable discrete systems has attracted some attention [4–9], since the relation provides a method of deforming the soliton hierarchy. One of the authors and Rauch-Wojciechowski [4] first proposed a recombination method to study the relation between a hierarchy of KdV flows and the continuous limit of a hierarchy of Kac–Van Moerbeke flows. This recombination method was also studied in Refs. [6,7]. An asymptotic series (in the lattice spacing) for field variables of the Toda hierarchy in terms of the field variable of the KdV hierarchy was proposed in Ref. [5] in order that the series be an approximate solution to the Toda flows to high accuracy and by choosing the initial data of the Toda flows in a canonical way the behavior of a certain Toda flow can mimic the KdV flows. Then a conjecture on the relation between the KdV hierarchy and the limit of the Toda hierarchy was suggested. The main purpose of this Letter is to present a clear and conclusive statement for the relation between the KdV hierarchy and the continuous limit of the Toda hierarchy by using three relations connecting the field variables of Toda flows and those of KdV flows proposed in Ref. [5]. We construct three new sequences of recombination equations in the Toda hierarchy which have the KdV hierarchy as continuous limit. Also we present the relation between the Poisson structures of the KdV hierarchy and the Toda hierarchy in the continuous limit. The method can be applied to all the relations connecting the field variables of Toda flows and those of KdV flows.

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First, we briefly describe the Toda hierarchy and the KdV hierarchy as presented in Refs. [10,11]. Consider the following discrete isospectral problem,

$$(E + w + vE^{(-1)})y = \lambda y, \tag{1}$$

where $w = w(n,t)$ and $v = v(n,t)$ depend on integer $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, λ is the spectral parameter, the shift operator E is defined as

$$(Ef)(n) = f(n+1),$$
$$f^{(k)}(n) = E^{(k)} f(n) = f(n+k), \quad n \in \mathbb{Z}.$$

The equation in the Toda hierarchy associated with (1) can be written as the following Hamiltonian equation [10],

$$\begin{pmatrix} w \\ v \end{pmatrix}_{t_m} = JK_{m+1} = J\frac{\delta H_{m+1}}{\delta u}, \quad m = 1,2,\ldots, \tag{2}$$

where $\delta/\delta u = (\delta/\delta w, \delta/\delta v)^{\mathrm{T}}$, Poisson tensor J, Hamiltonian H_i and K_i are defined as

$$J \equiv \begin{pmatrix} 0 & J_{12} \\ J_{21} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & (1-E)v \\ v(E^{(-1)}-1) & 0 \end{pmatrix},$$
$$K_i = \frac{\delta H_i}{\delta u} = \begin{pmatrix} -b_i^{(1)} \\ a_i/v \end{pmatrix}, \quad H_i = -\frac{b_{i+1}}{i}, \quad i = 0,1,\ldots, \tag{3}$$

with $a_0 = \frac{1}{2}$, $b_0 = 0$, and

$$b_{i+1}^{(1)} = wb_i^{(1)} - (a_i^{(1)} + a_i),$$
$$a_{i+1}^{(1)} - a_{i+1} = w(a_i^{(1)} - a_i) + vb_i - v^{(1)}b_i^{(2)},$$
$$c_i = -vb_i^{(1)}, \quad i = 0,1,\ldots. \tag{4}$$

Eqs. (2) have the bi-Hamiltonian formulation

$$GK_{i-1} = JK_i, \quad i = 1,2,\ldots,$$
$$G \equiv \begin{pmatrix} vE^{(-1)} - v^{(1)}E & w(1-E)v \\ v(E^{(-1)}-1)w & v(E^{(-1)}-E)v \end{pmatrix}, \tag{5}$$

where G is the second Poisson tensor. We take $a_1 = 0$, then the first four K_i's are

$$K_0 = \begin{pmatrix} 0 \\ 1/2v \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} w \\ 1 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} v + v^{(1)} + w^2 \\ w + w^{(-1)} \end{pmatrix}. \tag{6}$$

The Schrödinger spectral problem is of the form

$$(\partial_x^2 + q - \bar{\lambda})\bar{y} = 0, \tag{7}$$

which is associated with the following KdV hierarchy [11],

$$q_{t_m} = B_0 P_m = B_0 \frac{\delta \bar{H}_m}{\delta q}, \quad m = 1,2,\ldots, \tag{8}$$

where the vector field possesses the bi-Hamiltonian formulation with two Poisson tensors B_0 and B_1,

$$B_0 P_{k+1} = B_1 P_k, \quad k = 0,1,\ldots,$$
$$B_0 = \partial \equiv \partial_x, \quad B_1 = \tfrac{1}{4}\partial^3 + q\partial + \tfrac{1}{2}q_x,$$
$$\bar{H}_i = \frac{4\bar{b}_{i+2}}{2i+1}, \quad i = 0,1,\ldots,$$

with $\bar{b}_0 = 0$, $\bar{b}_1 = 1$, and

$$\bar{b}_{i+1} = (\tfrac{1}{4}\partial^2 + q - \tfrac{1}{2}\partial^{-1}q_x)\bar{b}_i, \quad i = 0,1,\ldots,$$

where $\partial^{-1}\partial = \partial\partial^{-1} = 1$. The first three P_k's are

$$P_0 = 2, \quad P_1 = q, \quad P_2 = \tfrac{1}{4}(3q^2 + q_{xx}). \tag{9}$$

The well-known KdV equations read

$$q_{t_2} = \tfrac{1}{4}(3q^2 + q_{xx})_x. \tag{10}$$

Consider the Toda hierarchy on a lattice with a small step h. We interpolate the sequences $(w(n))$ and $(v(n))$ with two smooth functions of a continuous variable x, and relate $w(n)$ and $v(n)$ to $q(x)$ and $g(x)$ by

$$w(n) = -2 + \tfrac{1}{2}q(x)h^2 + g(x)h^3,$$
$$v(n) = 1 + \tfrac{1}{2}q(x)h^2 - g(x)h^3, \tag{11a}$$

$$E^{(k)}w(n) = -2 + \tfrac{1}{2}q(x+kh)h^2 + g(x+kh)h^3,$$
$$E^{(k)}v(n) = 1 + \tfrac{1}{2}q(x+kh)h^2 - g(x+kh)h^3, \tag{11b}$$

where $g(x)$ is given by (14). Also we define

$$\lambda = \bar{\lambda}h^2, \quad y(n) = \bar{y}(x). \tag{12}$$

Then it is easy to see that the spectral problem operators in (1) have the expansion

$$(E + w + vE^{(-1)} - \lambda)y(n) = h^2(\partial^2 + q - \bar{\lambda})\bar{y}(x) + O(h^3), \qquad (13)$$

which implies that the Toda spectral problem goes to the Schrödinger spectral problem in a continuous limit. The expansion (13) does not depend on the choice of $g(x)$. Gieseker proposed a way in Ref. [5] to choose $g(x)$ by requirement that $w(n)$ and $v(n)$ given by (11) be approximate solutions to the Toda flows to different accuracy when $q(x)$ satisfies the KdV equation. For example, the first three choices of $g(x) = g_1(x) + g_2(x)h$ are as follows, respectively

$$g_1(x) = g_2(x) = 0, \qquad (14a)$$

$$g_1(x) = \tfrac{1}{8}q_x(x), \quad g_2(x) = 0, \qquad (14b)$$

$$g_1(x) = \tfrac{1}{8}q_x(x), \quad g_2(x) = -\tfrac{1}{32}q^2(x)h. \qquad (14c)$$

It was examined in Ref. [5] that a suitable chosen combination of first K_i's goes to first P_i's in the continuous limit. These results suggest that there exists some relation between the KdV hierarchy and the continuous limit of the Toda hierarchy. We will show that under (11) with (14a), (14b) and (14c), respectively, certain defined combinations of the equations in the Toda hierarchy have the KdV hierarchy as a continuous limit. In order to do so, we need following lemmas.

Lemma 1. Under the definition (11a), we have

$$K_i \equiv \begin{pmatrix} -b_i^{(1)} \\ a_i/v \end{pmatrix} = \begin{pmatrix} \alpha_i \\ \gamma_i \end{pmatrix} + O(h), \quad i = 0, 1, \dots, \qquad (15)$$

where the constants α_i and γ_i are given by

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \gamma_0 = \tfrac{1}{2},$$
$$\gamma_1 = 0, \quad \alpha_i = (-1)^{(i-1)}C_{2i-2}^{i-1}, \quad \gamma_i = (-1)^i C_{2i-2}^{i}, \quad i = 2, 3, \dots. \qquad (16)$$

Proof. Under the definition (11a), it is easy to see from K_0 that $\alpha_0 = 0$, $\gamma_0 = \frac{1}{2}$. Notice the first equation in (4), we have

$$\alpha_k = -2\alpha_{k-1} + 2\gamma_{k-1}, \quad k = 1, 2, \dots. \qquad (17)$$

The identity [10]

$$\sum_{i=0}^{k}(a_i a_{k-i} + b_i c_{k-i}) = 0, \quad k = 1, 2, \dots,$$

leads to the equation

$$\gamma_k = \sum_{i=1}^{k-1}(-\gamma_i\gamma_{k-i} + \alpha_i\alpha_{k-i}), \quad k = 1, 2, \dots. \qquad (18)$$

Using Eqs. (17), (18) and the combinational identity

$$\frac{1}{k}C_{2k-2}^{k-1} = \sum_{i=1}^{k-1}\frac{1}{i(k-i)}C_{2i-2}^{i-1}C_{2k-2i-2}^{k-i-1}, \quad k = 1, 2, \dots,$$

we can complete the proof of Lemma 1 by induction.

Under the definitions (11) and (14) we have the expansions

$$J_{12} = h\sum_{i=0}^{\infty} d_i h^i = -h\partial - \tfrac{1}{2}h^2\partial^2 - (\tfrac{1}{6}\partial^3 + \tfrac{1}{2}q\partial + \tfrac{1}{2}q_x)h^3 + O(h^4), \qquad (19)$$

$$d_4 = -\frac{1}{4!}\partial^4 - \tfrac{1}{4}\partial^2 q + \partial g_1,$$

$$d_i = -\frac{1}{i}\partial^i - \frac{1}{2(i-2)!}\partial^{i-2}q + \frac{1}{(i-3)!}\partial^{i-3}g_1 + \frac{1}{(i-4)!}\partial^{i-4}g_2, \quad i \geqslant 5,$$

$$J_{21} = h\sum_{i=0}^{\infty} e_i h^i = -h\partial + \tfrac{1}{2}h^2\partial^2 - (\tfrac{1}{6}\partial^3 + \tfrac{1}{2}q\partial)h^3 + O(h^4), \qquad (20)$$

$$e_4 = \frac{1}{4!}\partial^4 + \tfrac{1}{4}q\partial^2 + g_1\partial,$$

$$e_i = \frac{(-1)^i}{i}\partial^i + \frac{(-1)^{i-2}}{2(i-2)!}q\partial^{i-2} - \frac{(-1)^{i-3}}{(i-3)!}g_1\partial^{i-3} - \frac{(-1)^{i-4}}{(i-4)!}g_2\partial^{i-4}, \quad i \geqslant 5.$$

We define

$$\tilde{J} = \begin{pmatrix} 0 & \tilde{J}_{21} \\ \tilde{J}_{12} & 0 \end{pmatrix}$$

by requiring that

$$J\tilde{J} = I\,. \tag{21}$$

Then it is found that

$$\tilde{J}_{12} = h^{-1}\sum_{i=0}^{\infty}\tilde{d}_i h^i = -h^{-1}\partial^{-1} + \tfrac{1}{2} + (\tfrac{1}{2}q\partial^{-1} - \tfrac{1}{12}\partial)h + O(h^2)\,,$$

$$\tilde{J}_{21} = h^{-1}\sum_{i=0}^{\infty}\tilde{e}_i h^i = -h^{-1}\partial^{-1} - \tfrac{1}{2} + (\tfrac{1}{2}\partial^{-1}q - \tfrac{1}{12}\partial)h + O(h^2)\,, \tag{22}$$

where $\tilde{d}_i$, $\tilde{e}_i$ are determined by recurrence formulas

$$\tilde{d}_k = \partial^{-1}\sum_{i=1}^{k} d_i\tilde{d}_{k-i}\,,\quad \tilde{e}_k = \partial^{-1}\sum_{i=1}^{k} e_i\tilde{e}_{k-i}\,.$$

Then it is found that

$$\tilde{J}Jf = f + \eta\,, \tag{23}$$

where function vector η comes from the integration. Eqs. (21) and (23) imply that $J\eta = 0$. Since the kernel of J is $\{K_0, K_1\}$, we have the following lemma.

Lemma 2. The $\tilde{J}$ defined by Eqs. (21) and (22) satisfies

$$J\tilde{J} = I\,,\quad \tilde{J}Jf = f + \xi K_1 + \delta K_0\,, \tag{24}$$

where ξ, δ are constants.

Lemma 3. Under the definition (11a), we have the expansion

$$S \equiv \tfrac{1}{4}(\tilde{J}G)^2 + \tilde{J}G = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \tag{25}$$

$$S_{11} = S_{22} = (\tfrac{1}{4}\partial^2 + \tfrac{1}{2}\partial^{-1}q\partial + \tfrac{1}{4}\partial^{-1}q_x)h^2 + O(h^3)\,,$$

$$S_{12} = S_{21} = (\tfrac{1}{2}\partial^{-1}q\partial + \tfrac{1}{4}\partial^{-1}\partial^{-1}q_x)h^2 + O(h^3)\,,$$

and

$$S_{ij} + S_{kl} = B_0^{-1}B_1h^2 + O(h^3)\,,$$

where $(i,j,k,l) \in \{(1,1,1,2), (1,1,2,1), (1,2,2,2), (2,1,2,2)\}$.

Proof. Under the definition (11a), we have the following expansions,

$$G_{11} = -2h\partial - (\tfrac{1}{3}\partial^3 + q\partial + \tfrac{1}{2}q_x)h^3 + O(h^4)\,,$$

$$G_{12} = 2h\partial + h^2\partial^2 + (\tfrac{1}{3}\partial^3 + \tfrac{1}{2}q\partial + q_x)h^3 + O(h^4)\,,$$

$$G_{21} = 2h\partial - h^2\partial^2 + (\tfrac{1}{3}\partial^3 + \tfrac{1}{2}q\partial - \tfrac{1}{2}q_x)h^3 + O(h^4)\,,$$

$$G_{22} = -2h\partial - (\tfrac{1}{3}\partial^3 + 2q\partial + q_x)h^3 + O(h^4)\,.$$

Set

$$T \equiv \tilde{J}G = \begin{pmatrix} 0 & \tilde{J}_{21} \\ \tilde{J}_{12} & 0 \end{pmatrix}\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{26}$$

then the operator has the expansion

$$T_{11} = -2 + \tfrac{1}{2}h^2q + O(h^3)\,,$$

$$T_{12} = 2 + h\partial + (\tfrac{1}{2}\partial^2 + q)h^2 + O(h^3)\,,$$

$$T_{21} = 2 - h\partial + (\tfrac{1}{2}\partial^2 - \tfrac{1}{2}\partial^{-1}q_x)h^2 + O(h^3)\,,$$

$$T_{22} = -2 + \tfrac{1}{2}h^2\partial^{-1}q\partial + O(h^3)\,.$$

Denote

$$T^2 = \tilde{J}G\tilde{J}G = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{27}$$

then we get

$$M_{11} = 8 + (\partial^2 - \partial^{-1}q_x)h^2 + O(h^3)\,,$$

$$M_{12} = -8 - 4h\partial + (-2\partial^2 - 3q + \partial^{-1}q\partial)h^2 + O(h^3)\,,$$

$$M_{21} = -8 + 4h\partial + (-2\partial^2 + 2q + \partial^{-1}q_x)h^2 + O(h^3)\,,$$

$$M_{22} = 8 + (\partial^2 + \partial^{-1}q_x)h^2 + O(h^3)\,.$$

The expansions of (26) and (27) lead to Lemma 2.

Lemma 4. We have

$$TK_i = \tilde{J}GK_i = K_{i+1} + \delta_{i+1}K_0\,,\quad i = 0, 1, \ldots\,, \tag{28}$$

where

$$\delta_i = -2(\alpha_i + \gamma_i) = (-1)^i\frac{2}{i}C_{2i-2}^{i-1}\,,\quad i = 1, 2, \ldots\,. \tag{29}$$

Proof. Eqs. (5) and (24) give rise to

$$TK_i = \tilde{J}GK_i = K_{i+1} + \xi_{i+1}K_1 + \delta_{i+1}K_0 .$$

It follows from (15), (17) and (26) that

$$\xi_{i+1} = 0 , \quad \delta_{i+1} = -2(\alpha_{i+1} + \gamma_{i+1}) ,$$

which together with (16) leads to (29).

If we define

$$W \equiv \tfrac{1}{4}G\tilde{J}G + G = (W_{ij}) , \quad 1 \leqslant i, j \leqslant 2 ,$$

it follows from the proof of Lemma 2 that

$$W_{11} = W_{22} = (-\tfrac{1}{2}q\partial - \tfrac{1}{4}q_x)h^3 + O(h^4) ,$$
$$W_{12} = W_{21} = (-\tfrac{1}{4}\partial^3 - \tfrac{1}{2}q\partial - \tfrac{1}{4}q_x)h^3 + O(h^4) .$$

Then we arrive at the following propositions.

Proposition 1. The relation between the Poisson tensors of the Toda hierarchy and those of the KdV hierarchy is as follows,

$$J = -B_0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}h + O(h^2) ,$$
$$W_{ij} + W_{kl} = -B_1h^3 + O(h^4) , \qquad (30)$$

where $(i,j,k,l) \in \{(1,1,1,2), (1,1,2,1), (1,2,2,2), (2,1,2,2)\}$.

Proposition 2. Let

$$w(n) = -2 + \tfrac{1}{2}q(x)h^2 ,$$
$$v(n) = 1 + \tfrac{1}{2}q(x)h^2 , \qquad (31)$$

then

$$\tilde{P}_k \equiv \sum_{i=0}^{2k} \beta_{k,i}K_i = \tfrac{1}{2}P_kh^{2k}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(h^{2k+1}) , \quad k \geqslant 1 , \qquad (32)$$

and

$$\begin{pmatrix} w \\ v \end{pmatrix}_{t_k} + \frac{1}{h^{2k-1}}J\tilde{P}_k = \tfrac{1}{2}(q_{t_k} - B_0P_k)h^2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(h^3) , \quad k \geqslant 1 , \qquad (33)$$

where

$$\beta_{k,2k} = (\tfrac{1}{4})^{k-1} , \quad \beta_{k,i} = \tfrac{1}{4}\beta_{k-1,i-2} + \beta_{k-1,i-1} , \quad 2 \leqslant i \leqslant 2k-1 ,$$
$$\beta_{k,1} = \beta_{k-1,0} + \tfrac{1}{4}\sum_{i=0}^{2k-2}\beta_{k-1,i}\delta_{i+1} ,$$
$$\beta_{k,0} = \sum_{i=0}^{2k-2}\beta_{k-1,i}(\tfrac{1}{2}\delta_{i+1} + \tfrac{1}{4}\delta_{i+2}) ,$$

with

$$\beta_{1,0} = -2 , \quad \beta_{1,1} = 2 , \quad \beta_{1,2} = 1 ,$$
$$\delta_i = (-1)^i\frac{2}{i}C_{2i-2}^{i-1} , \quad i = 1, 2, \dots .$$

Proof. Under the definition (31), it is easy to verify Proposition 2 for $k = 1$. By induction we have

$$(\tfrac{1}{4}(\tilde{J}G)^2 + \tilde{J}G)\tilde{P}_{k-1} = (\tfrac{1}{4}\tilde{J}G + 1) \times \sum_{i=0}^{2k-2}\beta_{k-1,i}(k_{i+1} + \delta_{i+1}K_0)$$
$$= \tfrac{1}{4}\sum_{i=0}^{2k-2}\beta_{k-1,i}(K_{i+2} + \delta_{i+2}K_0 + \delta_{i+1}K_1 + \delta_{i+1}\delta_1K_0) + \sum_{i=0}^{2k-2}\beta_{k-1,i}(K_{i+1} + \delta_{i+1}K_0)$$
$$= \tfrac{1}{4}\beta_{k-1,2k-2}k_{2k} + \sum_{i=2}^{2k-1}(\tfrac{1}{4}\beta_{k-1,i-2} + \beta_{k-1,i-1})K_i + \left(\beta_{k-1,0} + \tfrac{1}{4}\sum_{i=0}^{2k-2}\beta_{k-1,i}\delta_{i+1}\right)K_1$$
$$+ \sum_{i=0}^{2k-2}\beta_{k-1,i}(\tfrac{1}{2}\delta_{i+1} + \tfrac{1}{4}\delta_{i+2})K_0 = \tilde{P}_k \equiv \sum_{i=0}^{2k}\beta_{k,i}K_i . \qquad (34)$$

Using S in (25), the following approximation is deduced,

$$(\tfrac{1}{4}(\tilde{J}G)^2+\tilde{J}G)\tilde{P}_{k-1}=\begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}$$
$$\times\left(\tfrac{1}{2}P_{k-1}h^{2k-2}\begin{pmatrix}1\\1\end{pmatrix}+\mathrm{O}(h^{2k-1})\right)$$
$$=\begin{pmatrix}S_{11}+S_{12}\\ S_{21}+S_{22}\end{pmatrix}\tfrac{1}{2}P_{k-1}h^{2k-2}+\mathrm{O}(h^{2k-1})$$
$$=\tfrac{1}{2}B_0^{-1}B_1P_{k-1}h^{2k}\begin{pmatrix}1\\1\end{pmatrix}+\mathrm{O}(h^{2k+1})$$
$$=\tfrac{1}{2}P_kh^{2k}\begin{pmatrix}1\\1\end{pmatrix}+\mathrm{O}(h^{2k+1}), \tag{35}$$

which together with Eq. (34) gives rise to Eq. (32). Eq. (33) can be obtained by combining Eqs. (30), (31) and (32).

For example, the first three $\tilde{P}_k$'s are

$$\tilde{P}_1=-2K_0+2K_1+K_2=\begin{pmatrix}2+w\\ 1-1/v\end{pmatrix},$$
$$\tilde{P}_2=\tfrac{3}{2}K_0-K_1+\tfrac{3}{2}K_2+\tfrac{3}{2}K_3\tfrac{1}{4}K_4,$$
$$\tilde{P}_3=-\tfrac{5}{4}K_0+\tfrac{3}{4}K_1-\tfrac{5}{8}K_2+\tfrac{5}{4}K_3+\tfrac{15}{8}K_4+\tfrac{5}{8}K_5+\tfrac{1}{16}K_6. \tag{36}$$

Proposition 3. Let

$$w(n)=-2+\tfrac{1}{2}q(x)h^2+\tfrac{1}{8}q_x(x)h^3,$$
$$v(n)=1+\tfrac{1}{2}q(x)h^2-\tfrac{1}{8}q_x(x)h^3, \tag{37}$$

then we have

$$Q_k\equiv\begin{pmatrix}Q_{k,1}\\ Q_{k,2}\end{pmatrix}\equiv\sum_{i=0}^{2k-1}\tilde{\beta}_{k,i}K_i, \tag{38}$$

$$Q_{k,1}+Q_{k,2}=P_kh^{2k}+\mathrm{O}(h^{2k+1}),\quad k\geqslant 2,$$

and

$$(w_{t_k}+v_{t_k})+\frac{1}{h^{2k-1}}(J_{12}Q_{k,2}+J_{21}Q_{k,1})$$
$$=(q_{t_k}-B_0P_k)h^2+\mathrm{O}(h^3),\quad k\geqslant 2, \tag{39}$$

where

$$\tilde{\beta}_{k,2k-1}=(\tfrac{1}{4})^{k-2},\quad \tilde{\beta}_{k,i}=\tfrac{1}{4}\tilde{\beta}_{k-1,i-2}+\tilde{\beta}_{k-1,i-1},$$
$$2\leqslant i\leqslant 2k-2,$$

$$\tilde{\beta}_{k,1}=\tilde{\beta}_{k-1,0}+\tfrac{1}{4}\sum_{i=0}^{2k-3}\tilde{\beta}_{k-1,i}\delta_{i+1},$$
$$\tilde{\beta}_{k,0}=\sum_{i=0}^{2k-3}\tilde{\beta}_{k-1,i}(\tfrac{1}{2}\delta_{i+1}+\tfrac{1}{4}\delta_{i+2}),$$

with δ_i given in Proposition 2, and

$$\tilde{\beta}_{2,0}=4,\quad \tilde{\beta}_{2,1}=-2,\quad \tilde{\beta}_{2,2}=2,\quad \tilde{\beta}_{2,3}=1.$$

Proof. It is also easy to check Proposition 3 for $k=2$. Applying the operator S on Q_{k-1}, we can get a similar equation as (34) ($\tilde{P}_{k-1}$ is substituted by Q_{k-1} and the superscripts of the summation are $2k-3$ and $2k-2$ instead of $2k-2$ and $2k-1$, respectively). Upon the assumption of Proposition 3 for $k-1$,

$$Q_{k-1,1}+Q_{k-1,2}=P_{k-1}h^{2k-2}+\mathrm{O}(h^{2k-1}),$$

we have

$$Q_{k,1}+Q_{k,2}=(S_{11}+S_{21})Q_{k-1,1}+(S_{12}+S_{22})Q_{k-1,2}$$
$$=B_0^{-1}B_1P_{k-1}h^{2k}+\mathrm{O}(h^{2k+1})$$
$$=P_kh^{2k}+\mathrm{O}(h^{2k+1}).$$

The above formulation and (30) lead to (39).

The first two Q_k's read

$$Q_2=4K_0-2K_1+2K_2+K_3,$$
$$Q_3=-3K_0+\tfrac{3}{2}K_1-K_2+\tfrac{3}{2}K_3+\tfrac{3}{2}K_4+\tfrac{1}{4}K_5. \tag{40}$$

Proposition 4. Let

$$w(n)=-2+\tfrac{1}{2}q(x)h^2+\tfrac{1}{8}q_x(x)h^3-\tfrac{1}{32}q^2(x)h^4,$$
$$v(n)=1+\tfrac{1}{2}q(x)h^2-\tfrac{1}{8}q_x(x)h^3+\tfrac{1}{32}q^2(x)h^4, \tag{41}$$

then

$$R_k\equiv\sum_{i=0}^{2k-1}\tilde{\beta}_{k,i}K_i=\tfrac{1}{2}P_kh^{2k}\begin{pmatrix}1\\1\end{pmatrix}+\mathrm{O}(h^{2k+1}),$$
$$k\geqslant 2, \tag{42}$$

and

$$\begin{pmatrix}w\\ v\end{pmatrix}_{t_k}+\frac{1}{h^{2k-1}}JR_k=\tfrac{1}{2}(q_{t_k}-B_0P_k)h^2\begin{pmatrix}1\\1\end{pmatrix}$$
$$+\mathrm{O}(h^3),\quad k\geqslant 2, \tag{43}$$

where $\tilde{\beta}_{k,i}$ are defined in Proposition 3.

Proposition 4 can be proved by a similar method as used in the proof of Proposition 2.

Remark 1. The $g(x)$ in (11a) is assumed to be a polynomial of $q(x)$ and the derivatives of q and is chosen in such a way in Ref. [5] that when $q(x)$ is a solution of KdV flows, the $w(n), v(n)$ are approximate solutions to the Toda flows to high accuracy. The geometric meaning of this operation is not clear. Comparing the above propositions, we can conclude that the higher accuracy is introduced in the definition (11a), the fewer components are needed in the recombination method to recover the KdV hierarchy through the limit process. For example, under (11) with (14a), we need a combination of $K_2, \ldots, K_6$ to have Eq. (10) as a continuous limit; however, for (11) with (14b), the combination of $K_2, \ldots, K_5$ leads to KdV equation (10) in the continuous limit.

Remark 2. Proposition 1 implies that the Poisson structure of the KdV hierarchy is recovered combining the entries of the matrices of the Toda Poisson tensors in the continuous limit. However, the geometric meaning of such combinations of different entries is not clear.

In addition, apart from the commuting vector fields, we can also show that the conserved functionals, the Lax pairs and restricted flows for the Toda hierarchy go to those for the KdV hierarchy in the continuous limit.

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