JOURNAL OF MATHEMATICAL PHYSICS VOLUME 39, NUMBER 11 NOVEMBER 1998

Families of dynamical r-matrices and Jacobi inversion problem for nonlinear evolution equations

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(Received 9 December 1997; accepted for publication 26 May 1998)

Three families of dynamical r-matrices are explicitly constructed by means of constrained flows of nonlinear evolution equations (NLEE) associated with polynomial second-order spectral problems. The dynamical classical Yang–Baxter equations for the three linear r-matrix algebras are presented with explicit expression for the extra term. The separation variables and the Jacobi inversion problems for the x- and t_n-constrained flow of the NLEEs are found. The factorization of NLEE allows us to obtain the Jacobi inversion problem for the NLEE by combining the above two Jacobi inversion problems. This provides a method of separation of variables to solve the NLEEs. *© 1998 American Institute of Physics.* [S0022-2488(98)01611-9]

I. INTRODUCTION

In recent years the study of completely integrable systems admitting a classical r-matrix Poisson structure with dynamical r-matrix has attracted some attention.[1–7] A family of systems with parabolic separable potentials and parabolic dynamical r-matrices has been presented by recurrence relation in Ref. 3. An approach of additive deformation to a quite general family of dynamical r-matrices including parabolic and elliptic r-matrices is described in Refs. 5 and 6. The stimulus for considering such recurrence relation and additive deformation was inspired by constrained flows of Korteweg–de Vries (KdV) and coupled KdV equations. However, explicit expressions for the family both of integrable systems and of parabolic and elliptic dynamical r-matrices have not been carried out. It was pointed out in Refs. 7 and 8 that various constrained flows of soliton equations can be directly used to construct some kinds of dynamical r-matrices, since their Lax representations can always be deduced from the adjoint representation of the spectral problem.[9] The first and primary aim of this paper is to construct explicit expressions for three families of integrable systems and their dynamical r-matrices directly from the constrained flows of nonlinear evolution equations (NLEEs) associated with two polynomial second-order spectral problems considered in Ref. 10. The first and third families of integrable systems possess elliptic but different dynamical r-matrices; the second one has the parabolic dynamical r-matrices. We also discuss the dynamical classical Yang–Baxter equations satisfied by the three dynamical r-matrix algebras and present a general way to find the explicit expressions for the extra term.[4]

The second aim of this paper is to establish the Jacobi inversion problem for the NLEEs. For the Lax matrices satisfying r-matrix algebra with the r-matrix of *XXX*, *XXZ*, and *XYZ* type, the problem of determining variables of separation has been essentially solved.[11–14] By using the Lax representation, the separation variables and Jacobi inversion problems for the x- and t_n-constrained flows are found. It was shown in Refs. 15–17 that the soliton equation can always be factorized by x- and t_n-constrained flow. The factorization of the NLEEs together with a combination of the above two Jacobi inversion problems gives rise to the Jacobi inversion problem for the NLEE. The standard Jacobi inversion techniques allow us to find the solution of the NLEE in terms of the Riemann theta function.[18] Therefore, this provides a method of separation of variables to solve the NLEE. The paper is organized as follows.

In Sec. II, for our purposes we reproduce two hierarchies of NLEEs associated with the polynomial second-order spectral problem and its modification, which were considered in Ref. 17, in the framework of zero-curvature representation. In Sec. III we study the first x-constrained flow and t_n-constrained flow and obtain x-finite-dimensional integrable Hamiltonian systems (FDIHS) and t_n-FDIHS. The NLEE is factorized by these two commuting x- and t_n-FDIHSs. In fact we obtain a family of x-FDIHSs depending on the order m of the polynomial. We deduce the Lax representation for the x- and t_n-FDIHSs from the adjoint representation of the spectral problem. Then we construct the classical Poisson structure for the family of x-FDIHSs and obtain explicitly a family of elliptic dynamical r-matrices. Finally we introduce the dynamical classical Yang–Baxter equations which are valid for the dynamical r-matrix algebra. We present a general way to obtain the explicit form for the extra term in the Yang–Baxter equations. In Sec. IV we determine canonically conjugated separation variables and the Jacobi inversion problem for the x- and t_n-FDIHSs by means of their Lax representation. By combining these two Jacobi inversion problems the factorization of the NLEE allows us to find the Jacobi inversion problem for the NLEE. In Secs. V and VI we consider the second x-constrained flow for the first hierarchy of the NLEEs and first x-constrained flow for the second hierarchy of the NLEEs, and present explicit expressions for the second and third families of x-FDIHSs and parabolic and elliptic dynamical r-matrices, respectively. Then we apply the same procedure as in Secs. III and IV to them. In Sec. VII a summary is presented.

II. TWO HIERARCHIES OF NONLINEAR EVOLUTION EQUATIONS

In this section we recall two hierarchies of nonlinear evolution equations (NLEEs) associated with the polynomial second-order spectral problem and its modification. These NLEEs have been described in Ref. 10. For our purpose we rederive these NLEEs in the framework of zero-curvature representation and present some formulas in different form.

A. The first hierarchy of NLEEs

Consider the following polynomial second-order spectral problem,[10]

$$\psi_x = U(u,\lambda)\psi, \quad U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ -\sum_{i=0}^{m} u_i\lambda^i & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.1}$$

where $u_m=-1$, $u=(u_{m-1},\ldots,u_0)^T$. The adjoint representation of (2.1) is[19]

$$V_x = [U,V] \equiv UV - VU. \tag{2.2}$$

Set

$$V = \sum_{i=0}^{\infty} V_i\lambda^{-i}, \quad V_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}. \tag{2.3}$$

Equation (2.2) yields

$$a_0 = \cdots = a_m = b_0 = \cdots = b_{m-1} = 0, \quad b_m = 1, \quad b_{m+1} = \tfrac{1}{2}u_{m-1},$$

$$a_{m+1} = -\tfrac{1}{4}u_{m-1,x}, \quad c_0 = 1, \quad c_1 = -\tfrac{1}{2}u_{m-1},\ldots,$$

and, in general,

$$\begin{pmatrix} b_{k+m} \\ \vdots \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} b_{k+m-1} \\ \vdots \\ b_k \end{pmatrix}, \tag{2.4a}$$

$$a_k = -\frac{1}{2}\, b_{k,x}, \quad c_k = -\frac{1}{2}\, b_{k,xx} - \sum_{i=0}^{m} u_i b_{k+i}, \quad k=1,2,..., \tag{2.4b}$$

where

$$L=\begin{pmatrix} L_{m-1} & L_{m-2} & \dots & L_1 & L_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$L_0=\tfrac{1}{4}D^2+u_0-\tfrac{1}{2}D^{-1}u_{0,x}, \quad L_i=u_i-\tfrac{1}{2}D^{-1}u_{i,x}, \quad i=1,...,m-1,$$

$$D=\frac{\partial}{\partial x}, \quad DD^{-1}=D^{-1}D=1. \tag{2.5}$$

Take

$$V^{(n)}(u,\lambda)=\sum_{i=0}^{n} V_i\lambda^{n-i}+\begin{pmatrix} 0 & 0 \\ -\sum_{k=1}^{m} \lambda^{m-k}\sum_{i=1}^{k} b_{n+i}u_{m-k+i} & 0 \end{pmatrix}, \tag{2.6}$$

and define

$$\psi_{t_n}=V^{(n)}(u,\lambda)\psi. \tag{2.7}$$

Then the compatibility condition of (2.1) and (2.7) gives rise to the zero-curvature representation[19]

$$U_{t_n}-V_x^{(n)}+[U,\ V^{(n)}]=0, \quad n=1,2,... . \tag{2.8}$$

Equation (2.8) leads to the following hierarchy of NLEEs which can be written as an infinite-dimensional Hamiltonian system,

$$u_{t_n}=\begin{pmatrix} u_{m-1} \\ \vdots \\ u_0 \end{pmatrix}_{t_n}=J\begin{pmatrix} b_{n+m} \\ \vdots \\ b_{n+1} \end{pmatrix}=J\ \frac{\delta H_n}{\delta u}, \quad n=1,2,..., \tag{2.9}$$

where the Hamiltonian H_n and Hamiltonian operator J are given by

$$J=\begin{pmatrix} 0 & 0 & \dots & 0 & 2D \\ 0 & 0 & \dots & 2D & J_{m-1} \\ 0 & 0 & \dots & J_{m-1} & J_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2D & J_{m-1} & \dots & J_1 & J_0 \end{pmatrix},$$

$$J_i=-u_{i,x}-2u_iD, \quad i=0,1,...,m-1, \quad H_n=\frac{2}{m-2n-2}\sum_{i=1}^{m} iu_ib_{n+i+1}.$$

Furthermore V satisfies the adjoint representation of (2.7):[19]

$$V_{t_n}=[V^{(n)},\ V], \quad n=1,2,... . \tag{2.10}$$

Under zero boundary condition we have

$$\frac{\delta\lambda}{\delta u}=(\lambda^{m-1}\psi_1^2,\dots,\psi_1^2)^T, \quad L\ \frac{\delta\lambda}{\delta u}=\lambda\ \frac{\delta\lambda}{\delta u}. \tag{2.11}$$

B. The second hierarchy of NLEEs

Consider the following modified polynomial second-order spectral problem:[10]

$$\psi_x = U(u,\lambda)\psi, \quad U(u,\lambda) = \begin{pmatrix} u_0 & \lambda \\ -\sum_{i=1}^{m} u_i \lambda^{i-1} & -u_0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.12}$$

where $u_m = -1$, $u = (u_0,\ldots,u_{m-1})^T$. Equations (2.2) and (2.3) yield

$$a_0 = \cdots = a_{m-2} = b_0 = \cdots = b_{m-3} = 0, \quad b_{m-2} = 1, \quad b_{m-1} = \tfrac{1}{2} u_{m-1},$$

$$a_{m-1} = u_0, \quad c_0 = 1, \quad c_1 = -\tfrac{1}{2} u_{m-1}, \ldots,$$

and, in general,

$$\begin{pmatrix} 2a_{k+1} \\ -b_{k+1} \\ \vdots \\ -b_{k+m-1} \end{pmatrix} = L \begin{pmatrix} 2a_k \\ -b_k \\ \vdots \\ -b_{k+m-2} \end{pmatrix}, \tag{2.13a}$$

$$c_{k+1} = a_{k,x} - \sum_{i=1}^{m} u_i b_{k+i-1}, \quad k = 1,2,\ldots, \tag{2.13b}$$

where

$$L = \begin{pmatrix} 0 & -2u_0 + D & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ L_0 & L_1 & L_2 & \ldots & L_{m-2} & L_{m-1} \end{pmatrix},$$

$$L_0 = \tfrac{1}{4} D + \tfrac{1}{2} D^{-1} u_0 D, \quad L_i = \tfrac{1}{2} u_i + \tfrac{1}{2} D^{-1} u_i D, \quad i = 1,\ldots,m-1.$$

Take

$$V^{(n)}(u,\lambda) = \sum_{i=0}^{n} V_i \lambda^{n-i} + \begin{pmatrix} 0 & -b_n \\ \sum_{j=1}^{m-1} \lambda^{j-1} \sum_{i=j+1}^{m} b_{n+i-j-1} u_i & 0 \end{pmatrix}, \tag{2.14}$$

and define

$$\psi_{t_n} = V^{(n)}(u,\lambda)\psi. \tag{2.15}$$

Then the zero-curvature representation (2.8) leads to the following hierarchy of NLEEs which can also be written as an infinite-dimensional Hamiltonian system

$$u_{t_n} = \begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix}_{t_n} = J \begin{pmatrix} 2a_n \\ -b_n \\ \vdots \\ -b_{n+m-2} \end{pmatrix} = J \frac{\delta H_n}{\delta u}, \quad n = 1,2,\ldots, \tag{2.16}$$

where the Hamiltonian H_n and Hamiltonian operator J are given by

$$J=\begin{pmatrix} \frac{1}{2}D & 0 & 0 & \dots & 0 & 0 \\ 0 & J_2 & J_3 & \dots & J_{m-1} & -2D \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & J_{m-1} & -2D & \dots & 0 & 0 \\ 0 & -2D & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$J_i=u_{i,x}+2u_iD, \quad i=0,1,...,m-1, \quad H_n=\frac{2}{m-2n-2}\left[a_{n,x}-\sum_{i=1}^{m} iu_ib_{n+i-1}\right].$$

Also we have

$$\frac{\delta\lambda}{\delta u}=(2\psi_1\psi_2,\psi_1^2,\dots,\lambda^{m-2}\psi_1^2)^T, \quad L\frac{\delta\lambda}{\delta u}=\lambda\frac{\delta\lambda}{\delta u}. \tag{2.17}$$

III. THE FIRST FAMILY OF DYNAMICAL *r*-MATRICES

A. The factorization of the NLEEs (2.9)

We first recall the constrained flows and factorization of the NLEEs. The x-constrained flow of NLEEs consists of the equations obtained from the spectral problem for N distinct λ_j and of restriction of the variational derivatives for conserved quantities H_k (for any fixed k) and λ_j:[15,16,20]

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_x=U(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \quad j=1,...,N, \tag{3.1a}$$

$$\frac{\delta H_k}{\delta u}-\frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}=0. \tag{3.1b}$$

The t_n-constrained flow of NLEEs consists of the equations obtained from the evolution equation of the eigenfunction for N distinct λ_j and of the NLEE itself:[15,16]

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_n}=V^{(n)}(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \quad j=1,...,N, \tag{3.2a}$$

$$u_{t_n}=J\frac{\delta H_n}{\delta u}. \tag{3.2b}$$

Hereafter we denote the inner product in $\mathbf{R}^N$ by $\langle .,. \rangle$ and $\Psi_i=(\psi_{i1},\dots,\psi_{iN})^T$, $i=1,2$, $\Lambda=\text{diag}(\lambda_1,...,\lambda_N)$. By means of (2.11), (3.1) for NLEE (2.9) can be written as

$$\Psi_{1,x}=\Psi_2, \quad \Psi_{2,x}=\Lambda^m\Psi_1-\sum_{i=0}^{m-1}u_i\Lambda^i\Psi_2, \tag{3.3a}$$

$$\begin{pmatrix} b_{m+k}\\ \vdots \\ b_{k+1}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\langle\Lambda^{m-1}\Psi_1,\Psi_1\rangle\\ \vdots \\ \langle\Psi_1,\Psi_1\rangle\end{pmatrix}. \tag{3.3b}$$

For $k=m$, (3.3b) reads

$$\begin{pmatrix} b_{2m} \\ \vdots \\ b_{m+k} \\ \vdots \\ b_{m+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} u_0 + \sum_{i=1}^{m-1} L_i b_{m+i} \\ \vdots \\ \frac{1}{2} u_{m-k} + \sum_{i=1}^{k-1} L_{m-k+i} b_{m+i} \\ \vdots \\ \frac{1}{2} u_{m-1} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \langle \Lambda^{m-1}\Psi_1, \Psi_1 \rangle \\ \vdots \\ \langle \Lambda^{k-1}\Psi_1, \Psi_1 \rangle \\ \vdots \\ \langle \Psi_1, \Psi_1 \rangle \end{pmatrix}, \tag{3.4a}$$

which leads to

$$u_{m-k} = \langle \Lambda^{k-1}\Psi_1, \Psi_1 \rangle - \sum_{i=1}^{k-1} L_{m-k+i} \langle \Lambda^{i-1}\Psi_1, \Psi_1 \rangle. \tag{3.4b}$$

Then, using the following kind of equality,

$$\sum_{j=l}^{k} A_{k-j} \sum_{i=l}^{j} B_i C_{j,i} = \sum_{i=l}^{k} B_i \sum_{j=0}^{k-i} A_j C_{k-j,i}, \quad l=0,1, \tag{3.5}$$

it is found from (3.4) and (3.3a) by a direct computation and induction that

$$u_{m-k} = \sum_{j=1}^{k} (-1)^{j-1} \frac{j+1}{2^j} \sum_{l_1+\cdots+l_j=k-j} \langle \Lambda^{l_1}\Psi_1, \Psi_1 \rangle \cdots \langle \Lambda^{l_j}\Psi_1, \Psi_1 \rangle,$$
$$k=1,\ldots,m, \tag{3.6}$$

where $l_1 \geqslant 0,\ldots,l_j \geqslant 0$. By substituting (3.6) into (3.3a), the first x-constrained flow of (2.9) can be written in canonical finite-dimensional Hamiltonian system (FDHS)

$$\Psi_{1,x} = \frac{\partial F_0}{\partial \Psi_2}, \quad \Psi_{2,x} = -\frac{\partial F_0}{\partial \Psi_1}, \tag{3.7}$$

with the Hamiltonian F_0 given by

$$F_0 = \frac{1}{2} \langle \Psi_2, \Psi_2 \rangle + \sum_{j=0}^{m} \frac{(-1)^{j-1}}{2^{j+1}} \sum_{l_1+\cdots+l_{j+1}=m-j} \langle \Lambda^{l_1}\Psi_1, \Psi_1 \rangle \cdots \langle \Lambda^{l_{j+1}}\Psi_1, \Psi_1 \rangle.$$

The integrals of motion for (3.7) are

$$F_k = \frac{1}{2} \langle \Lambda^k \Psi_2, \Psi_2 \rangle + \frac{1}{4} \sum_{j=0}^{k-1} [\langle \Lambda^j \Psi_2, \Psi_2 \rangle \langle \Lambda^{k-1-j}\Psi_1, \Psi_1 \rangle - \langle \Lambda^j \Psi_1, \Psi_2 \rangle \langle \Lambda^{k-1-j}\Psi_1, \Psi_2 \rangle]$$
$$+ \sum_{j=0}^{m} \frac{(-1)^{j-1}}{2^{j+1}} \sum_{l_1+\cdots+l_{j+1}=m-j} \langle \Lambda^{l_1}\Psi_1, \Psi_1 \rangle \cdots \langle \Lambda^{l_j}\Psi_1, \Psi_1 \rangle \langle \Lambda^{l_{j+1}+k}\Psi_1, \Psi_1 \rangle,$$
$$k=0,1,\ldots\,. \tag{3.8}$$

The FDHS (3.7) was first presented in Ref. 21. However, the Lax representation, r-matrix, and separability for (3.7) have not been studied.

Under (3.6) and (3.7), t_{m+1}-constrained flow (3.2) of NLEE (2.9) can be written as a FDHS,

$$\Psi_{1,t_{m+1}} = \frac{\partial F_1}{\partial \Psi_2}, \quad \Psi_{2,t_{m+1}} = -\frac{\partial F_1}{\partial \Psi_1}, \tag{3.9}$$

and t_{m+2}-constrained flow (3.2) of NLEE (2.9) becomes a FDHS:

$$\Psi_{1,t_{m+2}}=\frac{\partial F_2}{\partial \Psi_2}, \quad \Psi_{2,t_{m+2}}=-\frac{\partial F_2}{\partial \Psi_1}. \tag{3.10}$$

We will show in Sec. III C that FDHSs (3.7), (3.9), and (3.10) are a finite-dimensional integrable Hamiltonian system (FDIHS) with the integrals of motion given by (3.8). The construction of the x- and t_n-constrained flow ensures that if (Ψ_1,Ψ_2) satisfies two commuting FDIHSs (3.7) and (3.9) [(3.10)] simultaneously, then u given by (3.6) is a solution of the NLEE (2.9) with $n=m+1(n=m+2)$.

In general, the x-constrained flow (3.1) and t_n-constrained flow (3.2) under (3.1) can be transformed into an x-FDIHS and t_n-FDIHS. Then the NLEE is factorized into these two commuting x-FDIHS and t_n-FDIHS, namely if (Ψ_1,Ψ_2,u) satisfies these two commuting FDIHSs simultaneously, then u is a solution of the NLEE. Therefore, some kind of solution, such as finite-gap solution, for NLEE can be obtained through solving two commuting x- and t_n-FDIHS. We shall find the Jacobi inversion problem for these x- and t_n-FDIHSs later, and combine them to obtain the Jacobi inversion problem for the NLEE, which is solvable in terms of Riemann theta function.

B. Lax representation

By following the method in Ref. 9, the Lax representation for FDHS (3.7) can be deduced from the adjoint representation (2.2). First we have to find the expression of a_i, b_i, c_i under the constraint (3.6) and (2.11), which are denoted by $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$. According to (2.4), (2.11), (3.6), and (3.7), it is found by a direct calculation and using (3.5):

$$\tilde{a}_0=\cdots=\tilde{a}_m=\tilde{b}_0=\cdots=\tilde{b}_{m-1}=0, \quad \tilde{b}_m=1, \quad \tilde{c}_0=1,$$

$$\tilde{b}_{m+1+k}=\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_1\rangle, \quad k=0,1,..., \tag{3.11a}$$

$$\tilde{a}_{m+1+k}=-\tfrac{1}{2}\tilde{b}_{m+1+k,x}=-\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_2\rangle, \quad k=0,1,..., \tag{3.11b}$$

$$\tilde{c}_k=-\sum_{i=m-k}^{m} u_i\tilde{b}_{k+i}=\sum_{j=1}^{k}\left(-\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle,$$

$$k=1,...,m,$$

$$\tilde{c}_{m+1+k}=-\frac{1}{2}\,\tilde{b}_{m+1+k,xx}-\sum_{i=0}^{m}u_i\tilde{b}_{m+1+k+i}=-\frac{1}{2}\,\langle\Lambda^k\Psi_2,\Psi_2\rangle, \quad k=0,1,... . \tag{3.11c}$$

The construction of $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$ guarantees that under (3.7)

$$\tilde{V}=\sum_{m=0}^{\infty}\begin{pmatrix}\tilde{a}_i & \tilde{b}_i\\ \tilde{c}_i & -\tilde{a}_i\end{pmatrix}\lambda^{-i}$$

satisfies (2.2). Set

$$M\equiv\lambda^m\tilde{V}=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & -A(\lambda)\end{pmatrix}. \tag{3.12}$$

We have

$$A(\lambda)=-\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\psi_{2j}}{\lambda-\lambda_j}, \quad B(\lambda)=1+\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}, \tag{3.13a}$$

$$C(\lambda)=\lambda^m+\sum_{k=1}^{m}\lambda^{m-k}\sum_{j=1}^{k}\left(-\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle-\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}^2}{\lambda-\lambda_j}. \tag{3.13b}$$

The construction of M ensures the following proposition.

Proposition 1: The Lax representation for the FDHS (3.7) is given by

$$M_x=[\overline{U},\ M], \tag{3.14}$$

where $\overline{U}$ is obtained by inserting (3.6) into U in (2.1).

Similarly, $\tilde{V}$ under (3.9) [(3.10)] satisfies the adjoint representation (2.10). Thus the Lax representation for (3.9) $(n=m+1)$ [(3.10) $(n=m+2)$] is given by

$$M_{t_n}=[\overline{V^{(n)}},\ M], \tag{3.15}$$

where $\overline{V^{(n)}}$ is obtained by inserting (3.6) into $V^{(n)}$ in (2.6).

The equations (3.14) and (3.15) implies that Tr $M^2(\lambda)=A^2(\lambda)+B(\lambda)C(\lambda)$ is the generating function of the integrals of motion for (3.7), (3.9), and (3.10). We have

$$A(\lambda)^2+B(\lambda)C(\lambda)=\lambda^m+\sum_{i=1}^{N}\frac{F^{(i)}}{\lambda-\lambda_i}, \tag{3.16}$$

$$F^{(i)}=-\frac{1}{2}\psi_{2i}^2+\frac{1}{2}\left[\lambda_i^m+\sum_{k=1}^{m}\lambda_i^{m-k}\sum_{j=1}^{k}\left(-\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle\right]\psi_{1i}^2$$
$$-\frac{1}{4}\sum_{k\neq i}\frac{(\psi_{1i}\psi_{2k}-\psi_{1k}\psi_{2i})^2}{\lambda_k-\lambda_i},\quad i=1,...,N,$$

where $F^{(i)}$, $i=1,...,N$, are independent integrals of motion for (3.7), (3.9), and (3.10). In fact, we have

$$F_k=\sum_{i=0}^{N}\lambda_i^kF^{(i)},\quad k=0,1,...\ .$$

C. The first family of dynamical *r*-matrices

In this section we describe the classical Poisson structure associated with the Lax representation for (3.7), (3.9), and (3.10).

With respect to the standard Poisson bracket, it is found by a straightforward but lengthy calculation and using (3.5) that

$$\{A(\lambda),A(\mu)\}=\{B(\lambda),B(\mu)\}=0,\quad \{C(\lambda),C(\mu)\}=2g_1(\lambda,\mu)[A(\lambda)-A([\mu)], \tag{3.17a}$$

$$\{A(\lambda),B(\mu)\}=\frac{1}{\mu-\lambda}[B(\lambda)-B(\mu)],\quad \{B(\lambda),C(\mu)\}=\frac{2}{\mu-\lambda}[A(\lambda)-A(\mu)], \tag{3.17b}$$

$$\{A(\lambda),C(\mu)\}=\frac{1}{\mu-\lambda}[C(\mu)-C(\lambda)]-g_1(\lambda,\mu)B(\lambda), \tag{3.17c}$$

where

$$g_1(\lambda,\mu)=\sum_{k=0}^{m-1}\lambda^k\mu^{m-1-k}+\sum_{l=1}^{m-1}\sum_{k=0}^{m-1-l}\lambda^k\mu^{m-1-l-k}\sum_{j=1}^{l}\left(-\frac{1}{2}\right)^j \times(j+1)\sum_{l_1+\cdots+l_j=l-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle. \tag{3.18}$$

We use the standard notation σ_k, $k=0,1,2,3$, for the Pauli matrices and $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$. Denote

$$M_1(\lambda)=M(\lambda)\otimes I,\quad M_2(\mu)=I\otimes M(\mu),$$

where I is the 2×2 unit matrix. Let P be the permutation matrix. Then (3.17) gives rise to the the following proposition.

Proposition 2: The classical Poisson structure for (3.7), (3.9), and (3.10) is given in the following linear r-matrix form:[1]

$$\{M_1(\lambda),M_2(\mu)\}=[r_{12}(\lambda,\mu),M_1(\lambda)]-[r_{21}(\lambda,\mu),M_2(\mu)], \tag{3.19}$$

where the r-matrix is of the form

$$r_{12}(\lambda,\mu)=\frac{1}{\lambda-\mu}P+g_1(\lambda,\mu)S,\quad P=\frac{1}{2}\sum_{k=0}^{3}\sigma_k\otimes\sigma_k,\quad S=\sigma_-\otimes\sigma_-,$$
$$r_{21}(\lambda,\mu)=Pr_{12}(\lambda,\mu)P=r_{12}(\mu,\lambda). \tag{3.20}$$

Proof: The proof is a direct computation.

The family of r-matrices $r_{12}(\lambda,\mu)$ for $m=2,3,...$ is of the dynamical type which depends not only on the spectral parameter but also on the dynamical variables.

According to a remarkable theorem proved in Ref. 1, the involutivity of the integrals of motion

$$\{F^{(i)},F^{(j)}\}=0,\quad i,j=1,...,N, \tag{3.21}$$

is a consequence of (3.19). Therefore (3.7), (3.9), and (3.10) are FDIHSs.[22]

Let us denote

$$M_1(\lambda)=M(\lambda)\otimes I\otimes I,\quad M_2(\mu)=I\otimes M(\mu)\otimes I,\quad M_3(\nu)=I\otimes I\otimes M(\nu),$$

$$P_{12}=P\otimes I,\quad P_{23}=I\otimes P,\quad P_{13}=\frac{1}{2}\sum_{k=0}^{3}\sigma_k\otimes I\otimes\sigma_k,$$

$$S_{12}=S\otimes I,\quad S_{23}=I\otimes S,\quad S_{13}=\sigma_-\otimes I\otimes\sigma_-.$$

It is known that the r-bracket is a Poisson bracket if r-matrix obeys some classical dynamical Yang–Baxter equations. In the following we introduce dynamical Yang–Baxter equations for this family of dynamical r-matrices.

Proposition 3: For the linear r-matrix algebra (3.19) and (3.20), the following dynamical Yang–Baxter equations are valid:

$$[d_{12}(\lambda,\mu),d_{13}(\lambda,\nu)]+[d_{12}(\lambda,\mu),d_{23}(\mu,\nu)]+[d_{32}(\nu,\mu),d_{13}(\lambda,\nu)]+\{M_2(\mu),d_{13}(\lambda,\nu)\}$$
$$-\{M_3(\nu),d_{12}(\lambda,\mu)\}+f(\lambda,\mu,\nu)[S_0,M_2(\mu)-M_3(\nu)], \tag{3.22}$$

plus cyclic permutations, where

$$d_{ij}(\lambda,\mu)=\frac{1}{\lambda-\mu}P_{ij}+g_1(\lambda,\mu)S_{ij},$$

$$d_{ji}(\lambda,\mu)=d_{ij}(\lambda,\mu),\quad i<j,\quad i,j=1,2,3,\quad S_0=\sigma_-\otimes\sigma_-\otimes\sigma_-,$$

$$f(\lambda,\mu,\nu)=-2\sum_{k=0}^{m-2}\sum_{l=0}^{m-2-k}\lambda^k\mu^l\nu^{m-2-l-k}-2\sum_{l=1}^{m-2}\sum_{k=0}^{m-2-l}\sum_{i=0}^{m-2-l-k}\lambda^k\mu^i\nu^{m-2-l-k-i}$$
$$\times\sum_{j=1}^{l}\left(-\frac{1}{2}\right)^j(j+1)^2\sum_{l_1+\cdots+l_j=l-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle. \tag{3.23}$$

Proof: A direct calculation shows that the validity of (3.22) is due to the following equality

$$2\{A(\mu),g_1(\lambda,\nu)\}+\frac{2}{\lambda-\mu}[g_1(\mu,\nu)-g_1(\lambda,\nu)]-f(\lambda,\mu,\nu)B(\mu)=0, \tag{3.24}$$

where λ,μ are treated as spectral parameters. Notice that if (3.24) holds and $g_1(\lambda,\nu)$ is replaced by $g_1(\lambda,\nu)-\frac{1}{2}\Sigma_{j=1}^N\psi_{2j}^2/(\lambda-\lambda_j)$, then (3.24) still holds. Comparing (3.24) with (3.17c), we find that the relation between $g_1(\lambda,\mu)$ and $f(\lambda,\mu,\nu)$ is the same as the relation between $C(\mu)$ and $g_1(\lambda,\mu)$. Thus we may obtain $f(\lambda,\mu,\nu)$ from $-2g_1(\lambda,\mu)$ by replacing m by $m-1$ and μ^k by $\Sigma_{l=0}^k\mu^l\nu^{k-l}$. This completes the proof.

The proof provides a general way to obtain $f(\lambda,\mu,\nu)$ from $g(\lambda,\mu)$, namely, $f(\lambda,\mu,\nu)$ can be deduced from $-2g(\lambda,\mu)$ by the same relation as that between $g(\lambda,\mu)$ and $C(\mu)$. The dynamical Yang–Baxter equations (3.22) take the general form introduced in Ref. 4.

IV. THE JACOBI INVERSION PROBLEM FOR NLEE (2.9)

We introduce separation variables w_k,v_k, $k=1,...,N$ for (3.7), (3.9), and (3.10) by zeros of $B(\lambda)$ (Refs. 11–14):

$$B(\lambda)=1+\frac{1}{2}\sum_{j=1}^N\frac{\psi_{1j}^2}{\lambda-\lambda_j}=\frac{R(\lambda)}{K(\lambda)}, \tag{4.1a}$$

and

$$v_k=A(w_k)=\frac{1}{2}\sum_{j=1}^N\frac{\psi_{1j}\psi_{2j}}{w_k-\lambda_j},\quad k=1,...,N, \tag{4.1b}$$

where

$$K(\lambda)\equiv\prod_{j=1}^N(\lambda-\lambda_j)=\sum_{i=0}^N\alpha_i\lambda^{N-i},\quad R(\lambda)\equiv\prod_{j=1}^N(\lambda-w_j)=\sum_{i=0}^N\beta_i\lambda^{N-i},$$

$$\alpha_0=1,\quad \alpha_1=-\sum_{j=1}^N\lambda_j,\quad \alpha_2=\sum_{j=1}^N\sum_{k=j+1}^N\lambda_j\lambda_k,\ldots,\quad \alpha_N=(-1)^N\lambda_1\cdots\lambda_N,$$

$$\beta_0=1,\quad \beta_1=-\sum_{j=1}^N w_j,\quad \beta_2=\sum_{j=1}^N\sum_{k=j+1}^N w_jw_k,\ldots,\quad \beta_N=(-1)^Nw_1\cdots w_N. \tag{4.1c}$$

It follows from (4.1a) that the w_k, $k=1,...,N$, define elliptic coordinates given by the formula

$$\psi_{1j}^2=2\frac{R(\lambda_j)}{K'(\lambda_j)},\quad j=1,...,N, \tag{4.2}$$

where the prime denotes differentiation with respect to λ. Then we obtain from (4.2)

$$\sum_{j=1}^N\psi_{2j}d\psi_{1j}=\sum_{j=1}^N v_kdw_k,$$

which implies that the coordinates w_k, v_k are canonically conjugated.[22] From (4.1a) we find

$$\langle \Lambda^k \Psi_1, \Psi_1 \rangle = 2\beta_{k+1} + 2\sum_{i=0}^{N} \beta_i \sum_{l=1}^{k+1-i} (-1)^l \prod_{j_1+\cdots+j_l=k+1-i} \alpha_{j_1}\cdots\alpha_{j_l}, \tag{4.3}$$

where $j_i \geqslant 1$, $\alpha_k = \beta_k = 0$ for $k > N$. Combining (3.6) and (4.3) gives rise to the expression of u_k in terms of w_k, which are the symmetric functions of w_k, $k=1,...,N$. Let

$$A^2(\lambda) + B(\lambda)C(\lambda) = \frac{P(\lambda)}{K(\lambda)}, \quad P(\lambda) = \sum_{i=1}^{N+m+1} P_i \lambda^{i-1}, \tag{4.4}$$

where P_i, $i=1,...,N$, are also the integrals of motion for (3.7), (3.9), and (3.10). By substituting (3.13) we find

$$P_{N+m+1} = 1, \quad P_{N+m} = \alpha_1, \quad \dots, \quad P_{N+1} = \alpha_m, \tag{4.5}$$

$$F_0 = -P_N + \alpha_{m+1}, \quad F_1 = -P_{N-1} + \alpha_1 P_N - \alpha_1\alpha_{m+1} + \alpha_{m+2}, \tag{4.6a}$$

$$F_2 = -P_{N-2} + \alpha_1 P_{N-1} - (\alpha_1^2 - \alpha_2)P_N + \alpha_1^2\alpha_{m+1} - \alpha_1\alpha_{m+2} - \alpha_2\alpha_{m+1} + \alpha_{m+3}, \quad \dots . \tag{4.6b}$$

In order to write the Hamilton–Jacobi equation from (4.4), we must reinterpret the P_i as integration constants and replace v_k by the partial derivatives $\partial W/\partial w_k$ of the generating function W of canonical transformation.[22] Inserting $\lambda = w_k$, we find from (4.4)

$$v_k = \sqrt{\frac{P(w_k)}{K(w_k)}}, \quad k=1,...,N, \tag{4.7}$$

which implies that variables in the Hamilton–Jacobi equation are completely separable. Here W can be expressed in the separation form $W(w_1,\dots,w_N) = \Sigma_{k=1}^N W_k(w_k)$. By replacing $v_k = \partial W_k/\partial w_k$ and interpreting P_i as integration constants, (4.4) may be integrated to give the completely separated solution

$$W(w_1,\dots,w_N) = \sum_{k=1}^{N} \int^{w_k} \sqrt{\frac{P(\lambda)}{K(\lambda)}}\, d\lambda. \tag{4.8}$$

The linearizing coordinates are then

$$Q_i \equiv \frac{\partial W}{\partial P_i} = \frac{1}{2}\sum_{k=1}^{N} \int^{w_k} \frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\, d\lambda, \quad i=1,...,N. \tag{4.9}$$

This equality provides a map, called the Abel map, from the old coordinates w_k, $k=1,...,N$, which live on the Riemann surface, to new coordinates Q_k, $k=1,...,N$, which live on its Jacobi variety. The linear flow induced by (3.7) is then given by [using (4.6a)]

$$Q_i = \gamma_i + \frac{\partial F_0}{\partial P_i} x = \gamma_i - x\,\delta_{i,N}, \quad i=1,...,N, \tag{4.10}$$

where γ_i are constants.

The linear flow induced by (3.9) is of the form [using (4.6a)]

$$Q_i = \overline{\gamma_i} + \frac{\partial F_1}{\partial P_i} t_{m+1} = \overline{\gamma_i} - [\delta_{i,N-1} - \alpha_1 \delta_{i,N}] t_{m+1}, \quad i=1,...,N. \tag{4.11}$$

The factorization of NLEE (2.9) and combining of Eqs. (4.9), (4.10), and (4.11) give rise to the following proposition.

Proposition 4: The Jacobi inversion problem for FDHS (3.7) is given by

$$\frac{1}{2}\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i-\delta_{i,N}x,\quad i=1,\dots,N,\tag{4.12}$$

the Jacobi inversion problem for FDHS (3.9) is given by

$$\frac{1}{2}\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i+(\delta_{i,N}\alpha_1-\delta_{i,N-1})t_{m+1},\quad i=1,\dots,N,\tag{4.13}$$

and the Jacobi inversion problem for NLEE (2.9) with $n=m+1$ *is of the form*

$$\frac{1}{2}\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i-\delta_{i,N}(x-\alpha_1t_{m+1})-\delta_{i,N-1}t_{m+1},\quad i=1,\dots,N.\tag{4.14}$$

The linear flow induced by (3.10) is given by [using (4.6b)]

$$Q_i=\overline{\gamma_i}+\frac{\partial F_2}{\partial P_i}t_{m+2}=\overline{\gamma_i}-[\delta_{i,N-2}-\alpha_1\delta_{i,N-1}+(\alpha_1^2-\alpha_2)\delta_{i,N}]t_{m+2},\quad i=1,\dots,N.\tag{4.15}$$

The factorization of NLEE (2.9) and the combining of Eqs. (4.9), (4.10), and (4.15) lead to the Jacobi inversion problem for FDHS (3.10),

$$\frac{1}{2}\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i-[\delta_{i,N-2}-\alpha_1\delta_{i,N-1}+(\alpha_1^2-\alpha_2)\delta_{i,N}]t_{m+2},\quad i=1,\dots,N,\tag{4.16}$$

and the Jacobi inversion problem for NLEE (2.9) with $n=m+2$:

$$\frac{1}{2}\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i-[\delta_{i,N-2}-\alpha_1\delta_{i,N-1}]t_{m+2}-[x+(\alpha_1^2-\alpha_2)t_{m+2}]\delta_{i,N},\quad i=1,\dots,N.\tag{4.17}$$

By using standard Jacobi inversion techniques,[18] the solutions u of NLEE (2.9), which are the symmetric functions of w_k, $k=1,\dots,N$, determined by (3.6) and (4.3), can be given an explicit form in terms of Riemann theta functions. This provides a method of separation of variables to solve NLEE (2.9) for $n=m+1$, $m+2$. The above presented method can be applied to the whole hierarchy (2.9).

V. THE SECOND FAMILY OF DYNAMICAL *r*-MATRICES

We now consider the second x-constrained flow of NLEE (2.9) and its Lax representation, dynamical r-matrix, and Jacobi inversion problem.

A. The second *x*-constrained flow of the NLEE (2.9)

The second x-constrained flow of NLEE (2.9) is obtained from (3.3) for $k=m+1$. Equation (3.3b) with $k=m+1$ reads

$$\begin{pmatrix} b_{2m+1}\\ \vdots\\ b_{m+k}\\ \vdots\\ b_{m+2}\end{pmatrix}=\begin{pmatrix}\Sigma_{i=0}^{m-1}L_ib_{m+i+1}\\ \vdots\\ \frac{1}{2}u_{m-k}+\Sigma_{i=1}^{k-1}L_{m-k+i}b_{m+i}\\ \vdots\\ \frac{1}{2}u_{m-2}+\frac{1}{2}L_{m-1}u_{m-1}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\langle\Lambda^{m-1}\Psi_1,\Psi_1\rangle\\ \vdots\\ \langle\Lambda^{k-2}\Psi_1,\Psi_1\rangle\\ \vdots\\ \langle\Psi_1,\Psi_1\rangle\end{pmatrix},\tag{5.1a}$$

which leads to

$$u_{m-k}=\langle\Lambda^{k-2}\Psi_1,\Psi_1\rangle-L_{m-k+1}u_{m-1}-\sum_{i=2}^{k-1}L_{m-k+i}\langle\Lambda^{i-2}\Psi_1,\Psi_1\rangle,\quad k=2,...,m,\tag{5.1b}$$

$$L_0u_{m-1}=\langle\Lambda^{m-1}\Psi_1,\Psi_1\rangle-\sum_{i=1}^{m-1}L_i\langle\Lambda^{i-1}\Psi_1,\Psi_1\rangle.\tag{5.1c}$$

By using (3.5) and the following kind of equality,

$$\sum_{j=2l}^{k}A_{k-j}\sum_{i=l}^{[j/2]}B_iC_{j,2i}=\sum_{i=l}^{[k/2]}B_i\sum_{j=0}^{k-2i}A_jC_{k-j,2i},\quad l=0,1,\tag{5.2}$$

it is found from (5.1b) and (3.3a) by a direct but lengthy computation that

$$u_{m-k}=E_{k,0}u_{m-1}^k+\sum_{i=0}^{k-2}u_{m-1}^i\sum_{j=1}^{[(k-i)/2]}E_{i,j}\sum_{l_1+\cdots+l_j=k-i-2j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle,$$
$$k=2,...,m,\tag{5.3a}$$

where

$$E_{i,j}=-\left(-\frac{1}{2}\right)^{i+j}\frac{(i+j+1)!}{i!j!}=-(i+j+1)\beta_{i,j},\quad \beta_{i,j}=\left(-\frac{1}{2}\right)^{i+j}\frac{(i+j)!}{i!j!}.\tag{5.3b}$$

Denote

$$q=u_{m-1},\quad p=-\tfrac{1}{8}u_{m-1,x}.\tag{5.4}$$

By substituting (5.3) into (3.3a) and (5.1c), the second x-constrained flow of (2.9) can be written as a canonical FDHS with the extended phase space (Ψ_1,q,Ψ_2,p):

$$\Psi_{1,x}=\frac{\partial F_0}{\partial\Psi_2},\quad \Psi_{2,x}=-\frac{\partial F_0}{\partial\Psi_1},\quad q_x=\frac{\partial F_0}{\partial p},\quad p_x=-\frac{\partial F_0}{\partial q},\tag{5.5}$$

with the Hamiltonian F_0 given by

$$F_0=\frac{1}{2}\langle\Psi_2,\Psi_2\rangle+\beta_{m+2,0}q^{m+2}-4p^2$$
$$+\sum_{i=0}^{m}q^i\sum_{j=1}^{[(m+2-i)/2]}\beta_{i,j}\sum_{l_1+\cdots+l_j=m+2-i-2j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle.$$

B. Lax representation

We now derive the Lax representation for the family of FDHS (5.5) from the adjoint representation (2.2) in the same way as for (3.7). According to (2.4), (2.11), (5.1), and (5.5), it is found by a direct calculation

$$\tilde a_0=\cdots=\tilde a_m=\tilde b_0=\cdots=\tilde b_{m-1}=0,\quad \tilde b_m=1,$$
$$\tilde b_{m+1}=\tfrac{1}{2}q,\quad \tilde a_{m+1}=-\tfrac{1}{2}\tilde b_{m+1,x}=2p,\quad \tilde c_0=1\tag{5.6a}$$

$$\tilde b_{m+2+k}=\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_1\rangle,\quad k=0,1,...,$$
$$\tilde a_{m+2+k}=-\tfrac{1}{2}\tilde b_{m+2+k,x}=-\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_2\rangle,\quad k=0,1,...,\tag{5.6b}$$

$$\tilde{c}_k = -\frac{1}{2}\tilde{b}_{k,xx} - \sum_{i=m-k}^{m} u_i \tilde{b}_{k+i} = \beta_{k,0} q^k$$
$$+ \sum_{i=0}^{k-2} q^i \sum_{j=1}^{[(k-i)/2]} \beta_{i,j} \sum_{l_1+\cdots+l_j=k-i-2j} \langle \Lambda^{l_1}\Psi_1,\Psi_1\rangle \cdots \langle \Lambda^{l_j}\Psi_1,\Psi_1\rangle, \quad k=1,...,m+1, \tag{5.6c}$$
$$\tilde{c}_{m+2+k} = -\frac{1}{2}\tilde{b}_{m+2+k,xx} - \sum_{i=0}^{m} u_i \tilde{b}_{m+k+2+i} = -\frac{1}{2}\langle \Lambda^k \Psi_2,\Psi_2\rangle, \quad k=0,1,... .$$

Set

$$M \equiv \lambda^{m+1}\tilde{V} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}.$$

Then we have

$$A(\lambda) = 2p - \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}\psi_{2j}}{\lambda-\lambda_j}, \quad B(\lambda) = \lambda + \frac{1}{2}q + \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}, \tag{5.7a}$$

$$C(\lambda) = \lambda^{m+1} + \sum_{k=1}^{m+1} \lambda^{m+1-k}\tilde{c}_k - \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}^2}{\lambda-\lambda_j}. \tag{5.7b}$$

The construction of M gives rise to the following proposition.

Proposition 5: The Lax representation for the FDHS (5.5) is given by (3.14) with $\bar{U}$ obtained by inserting (5.3) and (5.4) into U in (2.1).

Similarly, the equality

$$A^2(\lambda) + B(\lambda)C(\lambda) = \lambda^{m+2} + F_0 + \sum_{i=1}^{N}\frac{F^{(i)}}{\lambda-\lambda_i} \tag{5.8}$$

determines $N+1$ independent integrals of motion $F_0, F^{(i)}$, $i=1,...,N$, for FDHS (5.5). We omit the expression for $F^{(i)}$.

C. The second family of dynamical *r*-matrices

By a straightforward but lengthy calculation and using (5.2), it is found that $A(\lambda), B(\lambda), C(\lambda)$ satisfies the same formulas as (3.17) with $g_1(\lambda,\mu)$ replaced by the following $g_2(\lambda,\mu)$:

$$g_2(\lambda,\mu) = -\sum_{l=0}^{m-1} E_{l,0} q^l \sum_{k=0}^{m-1-l} \mu^{m-1-l-k}\lambda^k - \sum_{n=0}^{m-3}\sum_{k=0}^{m-3-n} \lambda^k \mu^{m-3-n-k} \sum_{l=0}^{n} q^l \sum_{j=1}^{[(n+2-l)/2]} E_{l,j}$$
$$\times \sum_{l_1+\cdots+l_j=n+2-l-2j} \langle \Lambda^{l_1}\Psi_1,\Psi_1\rangle \cdots \langle \Lambda^{l_j}\Psi_1,\Psi_1\rangle. \tag{5.9}$$

Then we have the following proposition.

Proposition 6: The classical Poisson structure for FDHS (5.5) is given in the linear r-matrix form (3.19) with

$$r_{12}(\lambda,\mu) = \frac{1}{\lambda-\mu}P + g_2(\lambda,\mu)S, \quad r_{21}(\lambda,\mu) = Pr_{12}(\lambda,\mu)P = r_{12}(\mu,\lambda). \tag{5.10}$$

This gives the second family of dynamical r-matrices $r_{12}(\lambda,\mu)$ for $m=2,3,...$. The involutivity of the integrals of motion $F_0, F^{(i)}$, $i=1,...,N$, is a consequence of (3.9) and (5.10). Therefore (5.5) is a FDIHS.

As shown in the proof of Proposition 3, the following proposition is valid.

Proposition 7: For the linear r-matrix algebra (3.19) and (5.10) the dynamical Yang–Baxter equations (3.22) with $d_{i,j}(\lambda,\mu)$ *and f given by following formulas are valid:*

$$d_{ij}(\lambda,\mu)=\frac{1}{\lambda-\mu}P_{ij}+g_2(\lambda,\mu)S_{ij}, \quad d_{ji}(\lambda,\mu)=d_{ij}(\lambda,\mu), \quad i<j, \quad i,j=1,2,3,$$

$$f(\lambda,\mu,\nu)=2\sum_{l=0}^{m-3}\tilde{E}_{l,0}q^l\sum_{k=0}^{m-3-l}\sum_{i=0}^{m-3-l-k}\lambda^k\mu^i\nu^{m-3-l-k-i}$$
$$+2\sum_{n=0}^{m-5}\sum_{k=0}^{m-5-n}\sum_{i=0}^{m-5-n-k}\lambda^k\mu^i\nu^{m-5-n-k-i}$$
$$\times\sum_{l=0}^{n}q^l\sum_{j=2}^{[(n+2-l)/2]}\tilde{E}_{l,j}\sum_{l_1+\cdots+l_j=n+2-l-2j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle,$$

where

$$\tilde{E}_{i,j}=(i+j+1)^2\beta_{i,j}.$$

D. The Jacobi inversion problem for FDHS (5.5)

We introduce separation variables w_k, v_k, $k=1,\ldots,N+1$, for (5.5) as

$$B(\lambda)=\lambda+\frac{1}{2}q+\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}=\frac{R(\lambda)}{K(\lambda)}, \tag{5.11a}$$

and

$$v_k=A(w_k)=2p-\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\psi_{2j}}{w_k-\lambda_j}, \quad k=1,\ldots,N+1, \tag{5.11b}$$

where $K(\lambda)$ is given by (4.1c); $R(\lambda)$ and β_k are also given by (4.1c) but with N replaced by $N+1$. It follows from (5.11a) that w_k, $k=1,\ldots,N+1$, define parabolic coordinates given by the formula

$$\psi_{1j}^2=2\frac{R(\lambda_j)}{K'(\lambda_j)}, \quad j=1,\ldots,N, \quad q=2\beta_1-2\alpha_1. \tag{5.12}$$

The equation (5.12) leads to

$$\sum_{j=1}^{N}\psi_{2j}d\psi_{1j}+pdq=\sum_{j=1}^{N+1}v_k dw_k.$$

Therefore the coordinates w_k, v_k are canonically conjugated. From (4.1a) we find

$$\langle\Lambda^k\Psi_1,\Psi_1\rangle=2\beta_{k+1}+2\sum_{i=0}^{N+1}\beta_i\sum_{l=1}^{k+1-i}(-1)^l\prod_{j_1+\cdots+j_l=k+1-i}\alpha_{j_1}\ldots\alpha_{j_l}, \tag{5.13}$$

where $j_i\geqslant 1$, $\alpha_k=0$, for $k>N$, and $\beta_k=0$ for $k>N+1$. Combining (5.3), (5.12), and (5.13) gives rise to the expression of u_k in terms of w_k, which are the symmetric functions of w_k, $k=1,\ldots,N+1$. Let

$$A^2(\lambda)+B(\lambda)C(\lambda)=\frac{P(\lambda)}{K(\lambda)}, \quad P(\lambda)=\sum_{i=1}^{N+m+3}P_i\lambda^{i-1}, \tag{5.14}$$

where P_i, $i=1,...,N+1$, are the integrals of motion for FDHS (5.5). By substituting (5.7) into (5.14), one obtains

$$P_{N+m+3}=1, \quad P_{N+m+2}=\alpha_1, \quad \ldots, \quad P_{N+2}=\alpha_{m+1}, \quad P_{N+1}=\alpha_{m+2}-F_0, \quad \ldots . \tag{5.15}$$

In the exact same way as we did in Sec. IV, we can arrive at the following proposition.

Proposition 8: The Jacobi inversion problem for FDHS (5.5) is of the form

$$\frac{1}{2}\sum_{k=1}^{N+1}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}\,d\lambda=\gamma_i-\delta_{i,N+1}x, \quad i=1,...,N+1.$$

VI. THE THIRD FAMILY OF DYNAMICAL *r*-MATRIX

We now apply the same procedure in Sec. III and IV to the NLEEs (2.16).

A. The factorization of the NLEEs (2.16)

The first x-constrained flow (3.1) for $k=m-1$ of NLEE (2.16) reads

$$\Psi_{1,x}=u_0\Psi_1+\Lambda\Psi_2, \quad \Psi_{2,x}=\left(\Lambda^{m-1}-\sum_{i=1}^{m-1}u_i\Lambda^{i-1}\right)\Psi_1-u_0\Psi_2, \tag{6.1a}$$

$$\begin{pmatrix} 2a_{m-1}\\ -b_{m-1}\\ \vdots\\ -b_{2m-3}\end{pmatrix}=\frac{1}{2}\begin{pmatrix} 2\langle\Psi_1,\Psi_2\rangle\\ \langle\Psi_1,\Psi_1\rangle\\ \vdots\\ \langle\Lambda^{m-2}\Psi_1,\Psi_1\rangle\end{pmatrix}, \tag{6.1b}$$

which leads to

$$u_0=\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle, \tag{6.2a}$$

$$u_{m-k}=-\langle\Lambda^{k-1}\Psi_1,\Psi_1\rangle+\sum_{i=1}^{k-1}L_{m-i}\langle\Lambda^{k-i-1}\Psi_1,\Psi_1\rangle, \quad k=1,...,m-1. \tag{6.2b}$$

Then it is found by a direct but lengthy computation that

$$u_0=\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle, \tag{6.3a}$$

$$u_{m-k}=-\sum_{j=1}^{k}\frac{j+1}{2^j}\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle, \quad k=1,...,m-1. \tag{6.3b}$$

By substituting (6.3) into (6.1a), the first x-constrained flow of NLEE (2.16) can be written in canonical FDHS

$$\Psi_{1,x}=\frac{\partial F_0}{\partial\Psi_2}, \quad \Psi_{2,x}=-\frac{\partial F_0}{\partial\Psi_1}, \tag{6.4a}$$

with the Hamiltonian F_0 given by

$$F_0=\frac{1}{2}\langle\Lambda\Psi_2,\Psi_2\rangle+\frac{1}{4}\langle\Psi_1,\Psi_2\rangle^2-\sum_{j=1}^{m}\left(\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=m-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle. \tag{6.4b}$$

Under (6.3) and (6.4), the t_m-constrained flow (3.2) of NLEE (2.16) can be written as a FDHS

$$\Psi_{1,t_m}=\frac{\partial F_1}{\partial\Psi_2}, \quad \Psi_{2,t_m}=-\frac{\partial F_1}{\partial\Psi_1}, \tag{6.5a}$$

with the Hamiltonian F_1 given by

$$F_1=\frac{1}{2}\langle\Lambda^2\Psi_2,\Psi_2\rangle+\frac{1}{2}\langle\Psi_1,\Psi_2\rangle\langle\Lambda\Psi_1,\Psi_2\rangle-\frac{1}{4}\langle\Psi_1,\Psi_1\rangle\langle\Lambda\Psi_2,\Psi_2\rangle$$
$$-\sum_{j=1}^{m}\left(\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=m-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_{j-1}}\Psi_1,\Psi_1\rangle\langle\Lambda^{l_j+1}\Psi_1,\Psi_1\rangle. \tag{6.5b}$$

We will show later that both (6.4) and (6.5) are FDIHS. The construction of the x- and t_m-constrained flow ensures that if (Ψ_1,Ψ_2) satisfies two commuting FDIHSs (6.4) and (6.5) simultaneously, then u given by (6.3) is a solution of the NLEE (2.16) with $n=m$, namely the NLEE (2.16) with $n=m$ is factorized into two commuting x-FDIHS (6.4) and t_m-FDIHS (6.5).

B. Lax representation for (6.4) and (6.5)

It is found by a direct calculation

$$\tilde a_0=\cdots=\tilde a_{m-2}=\tilde b_0=\cdots=\tilde b_{m-3}=0,\quad \tilde b_{m-2}=1,\quad \tilde c_0=1,$$

$$\tilde b_{m-1+k}=-\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_1\rangle,\quad k=0,1,...,$$

$$\tilde a_{m-1+k}=(u_0-\tfrac{1}{2}D)\tilde b_{m-2+k}=\tfrac{1}{2}\langle\Lambda^k\Psi_1,\Psi_2\rangle,\quad k=0,1,...,$$

$$\tilde c_k=-\sum_{i=m-k}^{m}u_i\tilde b_{k+i-2}=\sum_{j=1}^{k}\left(\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle,\quad k=1,...,m-1,$$

$$\tilde c_{m-1+k}=\tilde a_{m-2+k,x}-\sum_{i=1}^{m}u_i\tilde b_{m+k+i-3}=\frac{1}{2}\langle\Lambda^k\Psi_2,\Psi_2\rangle,\quad k=1,\dots .$$

Set

$$M\equiv\lambda^{m-1}\tilde V=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & -A(\lambda)\end{pmatrix}, \tag{6.6a}$$

we have

$$A(\lambda)=\frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\psi_{1j}\psi_{2j}}{\lambda-\lambda_j},\quad B(\lambda)=\lambda-\frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}, \tag{6.6b}$$

$$C(\lambda)=\lambda^{m-1}+\sum_{k=1}^{m-1}\lambda^{m-1-k}\sum_{j=1}^{k}\left(\frac{1}{2}\right)^j\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle+\frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j\psi_{2j}^2}{\lambda-\lambda_j}. \tag{6.6c}$$

Proposition 9: The Lax representation for the FDHS (6.4) is given by (3.14) with $\bar U$ obtained by inserting (6.3) into U in (2.12) and M defined by (6.6). The Lax representation for the FDHS (6.5) is given by (3.15) with $\bar V^{(m)}$ obtained by inserting (6.3) into $V^{(m)}$ in (2.14).

C. The third family of dynamical *r*-matrices

With respect to the standard Poisson bracket one obtains

$$\{A(\lambda)A(\mu)\}=\{B(\lambda),B(\mu)\}=0,\quad \{C(\lambda),C(\mu)\}=2g_3(\lambda,\mu)[A(\mu)-A(\lambda)],$$

$$\{A(\lambda),B(\mu)\}=\frac{1}{\mu-\lambda}[\lambda B(\mu)-\mu B(\lambda)],\quad \{B(\lambda),C(\mu)\}=\frac{2\lambda}{\mu-\lambda}[A(\mu)-A(\lambda)],$$

$$\{A(\lambda),C(\mu)\}=\frac{\lambda}{\mu-\lambda}[C(\lambda)-C(\mu)]+g_3(\lambda,\mu)B(\lambda), \tag{6.7a}$$

where

$$g_3(\lambda,\mu)=\sum_{k=0}^{m-2}\mu^{m-2-k}\lambda^k+\sum_{l=1}^{m-2}\sum_{k=0}^{m-2-l}\lambda^k\mu^{m-2-l-k}\times\sum_{j=1}^{l}\left(\frac{1}{2}\right)^j(j+1)$$
$$\times\sum_{l_1+\cdots+l_j=l-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle. \tag{6.7b}$$

Then (6.7) gives rise to the following proposition.

Proposition 10: The classical Poisson structure for FDHS (6.4) and (6.5) is given by the linear r-matrix form (3.19) with the following r-matrix:

$$r_{12}(\lambda,\mu)=\frac{\mu}{\mu-\lambda}P-G-g_3(\lambda,\mu)S, \quad G=\sigma_+\otimes\sigma_-, \quad S=\sigma_-\otimes\sigma_-,$$
$$r_{21}(\lambda,\mu)=Pr_{12}(\lambda,\mu)P=r_{12}(\mu,\lambda). \tag{6.8}$$

The equation (6.8) presents the third family of dynamical r-matrices $r_{12}(\lambda,\mu)$ for $m=3,4,\ldots$. We have

$$A^2(\lambda)+B(\lambda)C(\lambda)=\lambda^m+\lambda\sum_{i=1}^{N}\frac{F^{(i)}}{\lambda-\lambda_i}, \tag{6.9}$$

where

$$F^{(i)}=\frac{1}{2}\lambda_i\psi_{2i}^2-\frac{1}{2}\lambda_i^{m-1}\psi_{1i}^2+\frac{1}{2}\psi_{1i}\psi_{2i}\langle\Psi_1,\Psi_2\rangle-\frac{1}{4}\psi_{2i}^2\langle\Psi_1,\Psi_1\rangle$$
$$-\sum_{k=1}^{m-1}\lambda_i^{m-k-1}\sum_{j=1}^{k}\left(\frac{1}{2}\right)^{j+1}\sum_{l_1+\cdots+l_j=k-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle\psi_{1i}^2$$
$$+\frac{1}{4}\sum_{k\neq i}\frac{\lambda_k(\psi_{1i}\psi_{2k}-\psi_{1k}\psi_{2i})^2}{\lambda_k-\lambda_i}, \quad i=1,\ldots,N.$$

The classical Poisson structure (3.19) and (6.8) guarantees that independent integrals of motion for (6.4) and (6.5) $F^{(i)}$, $i=1,\ldots,N$ are in involution. Therefore (6.4) and (6.5) are FDIHSs.

In the similar way as for Proposition 3, we can prove the following proposition.

Proposition 11: For the linear r-matrix algebra (3.19) and (6.8) the dynamical Yang–Baxter equations (3.22) with the $d_{i,j}$ and f given by following formulas are valid:

$$d_{ij}(\lambda,\mu)=\frac{\mu}{\mu-\lambda}P_{ij}-G_{ij}-g_3(\lambda,\mu)S_{ij}, \quad d_{ji}(\lambda,\mu)=d_{ij}(\lambda,\mu), \quad j>i, \quad i,j=1,2,3,$$

$$f(\lambda,\mu,\nu)=-2\sum_{k=0}^{m-3}\sum_{l=0}^{m-3-k}\lambda^k\mu^l\nu^{m-3-l-k}-2\sum_{l=1}^{m-3}\sum_{k=0}^{m-3-l}\sum_{i=0}^{m-3-l-k}\lambda^k\mu^i\nu^{m-3-l-k-i}$$
$$\times\sum_{j=1}^{l}\left(\frac{1}{2}\right)^j(j+1)^2\sum_{l_1+\cdots+l_j=l-j}\langle\Lambda^{l_1}\Psi_1,\Psi_1\rangle\cdots\langle\Lambda^{l_j}\Psi_1,\Psi_1\rangle.$$

D. The Jacobi inversion problem for NLEE (2.16)

We introduce canonically conjugated separation variables w_k, v_k, $k=1,...,N$ as

$$B(\lambda)=\lambda\left[1-\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}\right]=\lambda\frac{R(\lambda)}{K(\lambda)}, \tag{6.10a}$$

and

$$v_k=\frac{A(w_k)}{w_k}=\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\psi_{2j}}{w_k-\lambda_j}, \quad k=1,...,N, \tag{6.10b}$$

where $K(\lambda)$, $R(\lambda)$ are given by (4.1c). It follows from (6.10a) that w_k, $k=1,...,N$, define elliptic coordinates given by the formula

$$\psi_{1j}^2=-2\frac{R(\lambda_j)}{K'(\lambda_j)}, \quad j=1,...,N. \tag{6.11}$$

We have from (6.10) that

$$\langle\Lambda^k\Psi_1,\Psi_1\rangle=-2\beta_{k+1}-2\sum_{i=0}^{N}\beta_i\sum_{l=1}^{k+1-i}(-1)^l\prod_{j_1+\cdots+j_l=k+1-i}\alpha_{j_1}\cdots\alpha_{j_l}. \tag{6.12}$$

One obtains from (6.3a), (6.4), and (6.10a) that

$$u_0=\frac{\beta_{N,x}}{2\beta_N}. \tag{6.13}$$

Equations (6.3b), (6.12), and (6.13) imply that the expression of u_k, $k=0,...,m-1$, in terms of w_k are the symmetric functions of w_k, $k=1,...,N$.

Then in the same way as in Sec. IV, we obtain the following proposition.

Proposition 12: The Jacobi inversion problem for FDHS (6.4) is of the form

$$\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}d\lambda=\gamma_i+2\delta_{i,N}x, \quad i=1,...,N, \tag{6.14}$$

the Jacobi inversion problem for FDHS (6.5) is

$$\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}d\lambda=\gamma_i-2\delta_{i,N}\alpha_1 t_m+2\delta_{i,N-1}t_m, \quad i=1,...,N, \tag{6.15}$$

and the Jacobi inversion problem for NLEE (2.16) with $n=m$ is of the form

$$\sum_{k=1}^{N}\int^{w_k}\frac{\lambda^{i-1}}{\sqrt{P(\lambda)K(\lambda)}}d\lambda=\gamma_i+2\delta_{i,N}(x-\alpha_1 t_m)+2\delta_{i,N-1}t_m, \quad i=1,...,N. \tag{6.16}$$

The Jacobi inversion problem (4.14) and (6.16) are almost the same; however, the u of (2.9) and u of (2.16) are in terms of w_k, $k=1,...,N$, in different way.

VII. CONCLUSION

By directly using the constrained flows of soliton equations, we have constructed three families of FDIHSs and their dynamical r-matrices. In this way explicit expressions for the FDIHSs and for the dynamical r-matrices as well as for the dynamical Yang–Baxter equations arise naturally.

Since Lax representation for x- and t_n-constrained flow of soliton equation are naturally deduced from the adjoint representation of the spectral problem, the Jacobi inversion problem for

x- and t_n-constrained flow can always be found by means of the Lax representation. The factorization of soliton equation by x- and t_n-constrained flow allows us to obtain the Jacobi inversion problem for soliton equation by combining the Jacobi inversion problems for x- and t_n-constrained flow. This implies that some kind of solution for soliton equation can be found by the method of separation of variables.

ACKNOWLEDGMENTS

This work was supported by the Chinese Basic Research Project ‘‘Nonlinear Science.’’ The author wishes to express his gratitude to the Department of Mathematics, Sophia University, Professor Sunao Ouchi, and Professor Ken-ichi Shinoda for warm hospitality.

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