JOURNAL OF MATHEMATICAL PHYSICS VOLUME 41, NUMBER 8 AUGUST 2000

Integration of the soliton hierarchy with self-consistent sources

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(Received 29 March 2000; accepted for publication 12 May 2000)

In contrast with the soliton equations, the evolution of the eigenfunctions in the Lax representation of soliton equation with self-consistent sources (SESCS) possesses singularity. We present a general method to treat the singularity to determine the evolution of scattering data. The AKNS hierarchy with self-consistent sources, the MKdV hierarchy with self-consistent sources, the nonlinear Schrödinger equation hierarchy with self-consistent sources, the Kaup–Newell hierarchy with self-consistent sources and the derivative nonlinear Schrödinger equation hierarchy with self-consistent sources are integrated directly by using the inverse scattering method. The N soliton solutions for some SESCS are presented. It is shown that the insertion of a source may cause the variation of the velocity of soliton. This approach can be applied to all other $(1+1)$-dimensional soliton hierarchies. © *2000 American Institute of Physics.* [S0022-2488(00)05008-8]

I. INTRODUCTION

In recent years the nonlinear evolution equations with self-consistent sources have been studied through some different ways and have important physical applications,[1–11] for example, the nonlinear Schrödinger equation with self-consistent sources is relevant to some problems of plasma physics and solid state physics. Some of this kind of nonlinear integrable systems were constructed by adding a new operator to the original Lax representation in Ref. 10, by representing the source as the Fourier integral over the eigenfunctions of the so-called generating operator in Ref. 11, or by relating the sources to the singular part of the dispersion law in Refs. 3 and 12. Recently the soliton equations with self-consistent sources (SESCS) were studied based on the high-order constrained flows of soliton equations, namely the high-order constrained flows of soliton equations are consider as the stationary equations of the SESCS.[13–16] The ''self-consistent sources'' considered in Refs. 13–16 and in present paper are similar to those considered by Mel'nikov in Refs. 11 and 17 and differ fundamentally from the ones of Refs. 3 and 12 which are constructed with the scattering states (real k) instead of the the discrete eigenvalues. These SESCS do not have x-type Hamiltonian formulation, however possess t-type Hamiltonian or bi-Hamiltonian formulation.[18] They can be used to deduce sinh-Gordon type of equations.[19]

A few nonlinear evolution equations with self-consistent sources were solved. The integration of KdV equation with self-consistent sources, nonlinear Schrödinger equation hierarchy with self-consistent sources and some $(2+1)$-dimensional SESCS was proposed by means of inverse scattering method without use of explicit evolution equations of eigenfunctions in Refs. 11 and 17

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and by means of matrix theory in Ref. 1. The modified NLS (nonlinear Schrödinger) equation with a source and the modified Manakov system with a self-consistent sources was solved by the $\bar{\partial}$-method and gauge transformation in Refs. 8 and 9. The Darboux transformation for the Kaup-Newell hierarchy with self-consistent sources and AKNS hierarchy with self-consistent sources were presented in Refs. 15 and 16.

Since the evolution of the eigenfunction in the Lax representation for the KdV equation with sources and NLSE with source was not obtained explicitly in Refs. 11 and 17, the determination of the evolution of the scattering data was quite complicated and required special skill in Refs. 11 and 17. In contrast with the soliton equations, the evolution of eigenfunctions for the SESCS possesses singularity in spectral parameter. In present paper we systematically study the SESCS in the framework of the high-order constrained flows of soliton equations, since this approach provides a simple and natural way to derive both the SESCS and their Lax representation which can always be deduced from the adjoint representation of the eigenvalue problem for soliton equations.[14–16] By directly using explicit expression for evolution of eigenfunction, we propose a general way to treat the singularity in the evolution of eigenfunction to determine the evolution of the scattering data so that we could simply and naturally integrate the SESCS through the inverse scattering method and obtain the explicit soliton solution for some SESCS. The main point is to transform the singular part of the evolution of the eigenfunctions in the Lax representation into nonlocal form and introduce some arbitrary functions denoted by $\beta_j(t)$ related to the definition of the normalization constant so that we could obtain evolution of the normalization constant. Our method for determining the evolution of scattering data is quite different from that in Refs. 11 and 17. This approach seems more systematic and simple and enables us to solve whole soliton hierarchy with self-consistent sources directly and systematically by inverse scattering method. In particular we show how to integrate the AKNS hierarchy with self-consistent sources, the MKdV hierarchy with self-consistent sources, the nonlinear Schrödinger equation (NLSE) hierarchy with self-consistent sources, the Kaup–Newell (KN) hierarchy with self-consistent sources and the derivative nonlinear Schrödinger equation (DNLSE) hierarchy with self-consistent sources directly by using the inverse scattering method. The result shows that the evolution of the reflection coefficient is the same as that for the soliton equations without source, however, the evolution of each normalization constant has an extra term related to the $\beta_j(t)$. It is found that the insertion of a source into the soliton equation may cause the variation of the velocity of soliton.[11,17] The choices of $\beta_j(t)$ may result a great variety of dynamics of soliton solutions. Finally we would like to point out that the methods in Refs. 11 and 17 and in present paper can not solve the initial value problem of the SESCS in general, but only in the class of potential possessing the given set of discrete eigenvalues.

In Sec. II, we first recall the SESCS based on the high-order constrained flows and construction of their Lax representation, and illustrate the approach for solving SESCS through inverse scattering transformation by using AKNS hierarchy with self-consistent sources as a model. Then in the following sections we present the integration of the MKdV hierarchy with self-consistent sources, the NLSE hierarchy with self-consistent sources, the KN hierarchy with self-consistent sources and DNLSE hierarchy with self-consistent sources in a little different way, respectively. Also we present N soliton solution for some SESCS. In fact, this approach can be used to solve all other $(1+1)$-dimensional soliton hierarchy with self-consistent sources by inverse scattering method.

II. INTEGRATION OF THE AKNS HIERARCHY WITH SELF-CONSISTENT SOURCES

A. The AKNS hierarchy with self-consistent sources

To make the paper self-contained, we first recall the high-order constrained flows of AKNS hierarchy and briefly describe how to derive the Lax representation for the AKNS hierarchy with self-consistent sources.

Consider the AKNS spectral problem[20]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad U=\begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix}, \quad u=\begin{pmatrix}q\\ r\end{pmatrix}. \tag{2.1}$$

The adjoint representation of (2.1) reads[21]

$$V_x=[U,V]\equiv UV-VU. \tag{2.2}$$

Set

$$V=\sum_{i=0}^{\infty}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{-i}. \tag{2.3}$$

Equation (2.2) yields

$$a_0=-1, \quad b_0=c_0=a_1=0, \quad b_1=q, \quad c_1=r$$

$$a_2=\tfrac{1}{2}qr, \quad b_2=-\tfrac{1}{2}q_x, \quad c_2=\tfrac{1}{2}r_x, \ldots,$$

and in general

$$\begin{pmatrix}c_{m+1}\\ b_{m+1}\end{pmatrix}=L\begin{pmatrix}c_m\\ b_m\end{pmatrix}=L^m\begin{pmatrix}r\\ q\end{pmatrix}, \quad a_{m,x}=qc_m-rb_m, \tag{2.4}$$

where

$$L=\frac{1}{2}\begin{pmatrix}D-2rD^{-1}q & 2rD^{-1}r\\ -2qD^{-1}q & -D+2qD^{-1}r\end{pmatrix}, \quad D=\frac{\partial}{\partial x}, \quad DD^{-1}=D^{-1}D=1.$$

Set

$$V^{(n)}=\sum_{i=0}^{n}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{n-i}, \tag{2.5}$$

and take

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{t_n}=V^{(n)}(u,\lambda)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{2.6}$$

Then the compatibility condition of Eqs. (2.1) and (2.6) gives rise to the AKNS hierarchy[20]

$$u_{t_n}=\begin{pmatrix}q\\ r\end{pmatrix}_{t_n}=J\begin{pmatrix}c_{n+1}\\ b_{n+1}\end{pmatrix}=J\frac{\delta H_{n+1}}{\delta u}, \quad n=0,1,\ldots, \tag{2.7}$$

where

$$H_n=\frac{2a_{n+1}}{n}, \quad J=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}.$$

We have

$$\frac{\delta\lambda}{\delta q}=\phi_2^2, \quad \frac{\delta\lambda}{\delta r}=-\phi_1^2, \quad L\begin{pmatrix}\phi_2^2\\ -\phi_1^2\end{pmatrix}=\lambda\begin{pmatrix}\phi_2^2\\ -\phi_1^2\end{pmatrix}. \tag{2.8}$$

The high-order constrained flows of the AKNS hierarchy consist of the equations obtained from the spectral problem (2.1) for N distinct λ_j and the restriction of the variational derivatives for conserved quantities H_n and λ_j[22]

$$\frac{\delta H_{n+1}}{\delta u}-\frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}=\begin{pmatrix}c_{n+1}\\ b_{n+1}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\langle\Phi_2,\Phi_2\rangle\\ -\langle\Phi_1,\Phi_1\rangle\end{pmatrix}=0, \tag{2.9a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j},\quad \phi_{2j,x}=r\phi_{1j}+\lambda_j\phi_{2j},\quad j=1,\ldots,N, \tag{2.9b}$$

where $n=0,1,\ldots,\Phi_i=(\phi_{i1},\ldots,\phi_{iN})^T$, $i=1,2,\langle .,.\rangle$ denotes the inner product. According to Eqs. (2.4), (2.8), and (2.9), we define

$$\tilde{a}_i=a_i,\quad \tilde{b}_i=b_i,\quad \tilde{c}_i=c_i,\quad i=0,1,\ldots,n,$$

$$\tilde{b}_{n+1+i}=-\tfrac{1}{2}\langle\Lambda^i\Phi_1,\Phi_1\rangle,\quad \tilde{c}_{n+1+i}=\tfrac{1}{2}\langle\Lambda^i\Phi_2,\Phi_2\rangle,$$

$$\tilde{a}_{n+1+i}=D^{-1}(q\tilde{c}_{n+1+i}-r\tilde{b}_{n+1+i})=\tfrac{1}{2}\langle\Lambda^i\Phi_1,\Phi_2\rangle,\quad i=0,1,\ldots,$$

where $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$. Then

$$\begin{aligned}N^{(n)}&=\begin{pmatrix}A^{(n)} & B^{(n)}\\ C^{(n)} & D^{(n)}\end{pmatrix}\\ &\equiv\lambda^n\sum_{k=0}^{\infty}\begin{pmatrix}\tilde{a}_k & \tilde{b}_k\\ \tilde{c}_k & -\tilde{a}_k\end{pmatrix}\lambda^{-k}+\begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix}\\ &=\sum_{k=0}^{n}\begin{pmatrix}a_k & b_k\\ c_k & -a_k\end{pmatrix}\lambda^{n-k}+\begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix}+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2\\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j}\end{pmatrix},\end{aligned}$$

where η is some constant, also satisfies the adjoint representation (2.2), i.e.

$$N_x^{(n)}=[U,N^{(n)}], \tag{2.10}$$

which, in fact, gives rise to the Lax representation of the constrained flow (2.9).

The AKNS hierarchy with self-consistent sources is defined by Refs. 15 and 16.

$$\begin{pmatrix}q\\ r\end{pmatrix}_{t_n}=J\left[\frac{\delta H_{n+1}}{\delta u}-\frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right]=J\left[\begin{pmatrix}c_{n+1}\\ b_{n+1}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\langle\Phi_2,\Phi_2\rangle\\ -\langle\Phi_1,\Phi_1\rangle\end{pmatrix}\right], \tag{2.11a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j},\quad \phi_{2j,x}=r\phi_{1j}+\lambda_j\phi_{2j},\quad j=1,\ldots,N, \tag{2.11b}$$

for N distinct λ_j and assume that

$$\begin{aligned}&\lambda_j=i\zeta_j,\quad \mathrm{Im}\,\zeta_j>0,\quad \text{or}\quad \mathrm{Re}\,\lambda_j<0,\quad j=1,\ldots,N_1,\\ &\lambda_j=i\overline{\zeta_j},\quad \mathrm{Im}\,\overline{\zeta_j}<0,\quad \text{or}\quad \mathrm{Re}\,\lambda_j>0,\quad j=N_1+1,\ldots,N.\end{aligned} \tag{2.11c}$$

Since the high-order constrained flows (2.9) are just the stationary equations of the AKNS hierarchy with self-consistent sources (2.11), it is obvious that the zero-curvature representation for the AKNS hierarchy with self-consistent sources (2.11) is given by

$$U_{t_n}-N_x^{(n)}+[U,N^{(n)}]=0, \tag{2.12}$$

with the auxiliary linear problems

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-i\zeta & q\\ r & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2.13a}$$

where $\lambda=i\zeta$ and

$$\begin{aligned}\psi_{1,t_n}&=A^{(n)}\psi_1+B^{(n)}\psi_2\\ &=\sum_{k=0}^{n}(a_k\psi_1+b_k\psi_2)\lambda^{n-k}+\eta\psi_1+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2),\\ \psi_{2,t_n}&=C^{(n)}\psi_1+D^{(n)}\psi_2\\ &=\sum_{k=0}^{n}(c_k\psi_1-a_k\psi_2)\lambda^{n-k}+\eta\psi_2+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2).\end{aligned} \tag{2.13b}$$

When $n=1$, Eq. (2.11) gives the AKNS equation with self-consistent sources

$$q_{t_2}=-\tfrac{1}{2}q_{xx}+q^2r-\langle\Phi_1,\Phi_1\rangle,\quad r_{t_2}=\tfrac{1}{2}r_{xx}-qr^2-\langle\Phi_2,\Phi_2\rangle, \tag{2.14a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j},\quad \phi_{2j,x}=r\phi_{1j}+\lambda_j\phi_{2j},\quad j=1,\ldots,N, \tag{2.14b}$$

and the auxiliary linear problem (2.13b) reads

$$\begin{aligned}\psi_{1,t_2}&=\left(-\lambda^2+\frac{1}{2}qr+\eta\right)\psi_1+\left(q\lambda-\frac{1}{2}q_x\right)\psi_2+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2),\\ \psi_{2,t_2}&=\left(r\lambda+\frac{1}{2}r_x\right)\psi_1+\left(\lambda^2-\frac{1}{2}qr+\eta\right)\psi_2+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2).\end{aligned} \tag{2.15}$$

B. Integration of the AKNS hierarchy with self-consistent sources

We will use the inverse scattering method to solve the initial-value problem for the AKNS hierarchy with self-consistent sources (2.11) under assumption that $q(x,t),r(x,t),\phi_{1j}(x,t),\phi_{2j}(x,t),j=1,\ldots,N$, vanish rapidly as $|x|\to\infty$. Let $q_0(x),r_0(x)$ be arbitrary functions with the following properties:

(a) $q_0(x)$ and $r_0(x)$ vanish rapidly as $|x|\to\infty$, for convenience, we assume[20] that

$$\int_{-\infty}^{\infty}|x|^l|q_0(x)|dx<\infty,\quad \int_{-\infty}^{\infty}|x|^l|r_0(x)|dx<\infty, \tag{2.16a}$$

for all l;

(b) the AKNS spectral equation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q_0(x)\\ r_0(x) & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-i\zeta & q_0(x)\\ r_0(x) & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{2.16b}$$

has exactly N discrete eigenvalues as that given by (2.11c). Let $\beta_j(t),j=1,\ldots,N$, be arbitrary continuous functions of t. Using the inverse scattering method, we shall point out the way of constructing the solution $q=q(x,t)$, $r=r(x,t)$, $\phi_{1j}=\phi_{1j}(x,t)$, $\phi_{2j}=\phi_{2j}(x,t)$, $j=1,\ldots,N$, of the system (2.11), such that

$$q(x,0)=q_0(x), \quad r(x,0)=r_0(x),$$

$$\beta_j(t)=\int_{-\infty}^{\infty}\phi_{1j}(x,t)\phi_{2j}(x,t)dx, \quad j=1,\ldots,N_1,$$

$$\beta_j(t)=-\int_{-\infty}^{\infty}\phi_{1j}(x,t)\phi_{2j}(x,t)dx, \quad j=N_1+1,\ldots,N. \tag{2.16c}$$

The procedure of finding the above solution of the system (2.11) is very similar to that given in Ref. 20 for obtaining a solution rapidly decreasing with x of the AKNS hierarchy except the way for determining the evolution of scattering data. Denote $\lambda=i\zeta$, $\mathrm{Im}\,\zeta=\xi$. In the same way as in Ref. 20, we define the eigenfunction $\psi^-(x,t,\zeta),\overline{\psi}^-(x,t,\zeta),\psi^+(x,t,\zeta),\overline{\psi}^+(x,t,\zeta)$ for AKNS spectral equation (2.13a) and evolution of eigenfunctions (2.13b) with the following boundary condition on $\zeta=\xi$:

$$\psi^-(x,t,\zeta)\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x}, \quad \overline{\psi}^-(x,t,\zeta)\sim\begin{pmatrix}0\\-1\end{pmatrix}e^{i\zeta x}, \quad x\rightarrow-\infty, \tag{2.17a}$$

$$\psi^+(x,t,\zeta)\sim\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x}, \quad \overline{\psi}^+(x,t,\zeta)\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x}, \quad x\rightarrow\infty. \tag{2.17b}$$

Under the assumption for q,r,ϕ_{1j},ϕ_{2j}, we have

$$a_0=-1, \quad b_0=c_0=0, \quad \lim_{|x|\rightarrow\infty}a_j=\lim_{|x|\rightarrow\infty}b_j=\lim_{|x|\rightarrow\infty}c_j=0, \quad j=1,2,\ldots .$$

On $\zeta=\xi$, since $\zeta\neq\zeta_j$, we have

$$\lim_{|x|\rightarrow\infty}N^{(n)}=\begin{pmatrix}-\lambda^n+\eta & 0\\ 0 & \lambda^n+\eta\end{pmatrix}=\begin{pmatrix}-(i\zeta)^n+\eta & 0\\ 0 & (i\zeta)^n+\eta\end{pmatrix}. \tag{2.18}$$

Let $\eta^-,\overline{\eta}^-$ be the parameter η in the Eq. (2.13b) corresponding to $\psi^-(x,t,\zeta)$ and $\overline{\psi}^-(x,t,\zeta)$. By inserting Eq. (2.17a) into Eq. (2.13b), using (2.18) and let $x\rightarrow-\infty$, we get

$$\eta^-=(i\zeta)^n, \quad \overline{\eta}^-=-(i\zeta)^n.$$

On $\zeta=\xi$, $\psi^+(x,t,\zeta)$ and $\overline{\psi}^+(x,t,\zeta)$ are linearly independent, we may write

$$\psi^-(x,t,\zeta)=a(\zeta,t)\overline{\psi}^+(x,t,\zeta)+b(\zeta,t)\psi^+(x,t,\zeta), \tag{2.19a}$$

$$\overline{\psi}^-(x,t,\zeta)=-\overline{a}(\zeta,t)\psi^+(x,t,\zeta)+\overline{b}(\zeta,t)\overline{\psi}^+(x,t,\zeta). \tag{2.19b}$$

Then by substituting $\psi^-(x,t,\zeta)$ and $\eta^-(\overline{\psi}^-(x,t,\zeta),\overline{\eta}^-)$ into Eq. (2.13b), using (2.17), (2.18) and (2.19), let $x\rightarrow\infty$, it can be shown that on $\zeta=\xi$

$$\frac{\partial a}{\partial t}=0, \quad \frac{\partial b}{\partial t}=2(i\zeta)^n b, \quad \frac{\partial \rho}{\partial t}=2(i\zeta)^n\rho, \tag{2.20a}$$

$$\frac{\partial \overline{a}}{\partial t}=0, \quad \frac{\partial \overline{b}}{\partial t}=-2(i\zeta)^n\overline{b}, \quad \frac{\partial \overline{\rho}}{\partial t}=-2(i\zeta)^n\overline{\rho}, \tag{2.20b}$$

where $\rho=a/b$, $\overline{\rho}=\overline{a}/\overline{b}$. That is, for the AKNS hierarchy with self-consistent sources, the evolution of quantities $a(k,t),b(k,t),\overline{a}(k,t),\overline{b}(k,t)$ is the same as that of the AKNS hierarchy without source.

It can be shown as in Ref. 20 that function $a(\zeta,t)(\overline{a}(\zeta,t))$ admits an analytical continuation in ζ into the upper half (lower half) plane. The AKNS spectral equation (2.13a) can possess discrete eigenvalues which occur whenever $a(\zeta,t)$ has zeros in the upper half ζ plane and whenever $\overline{a}(\zeta,t)$ has zeros in the lower half η plane. Equation (2.20) indicate that the discrete eigenvalues do not depend on t. According to the assumption for q,r,ϕ_{1j},ϕ_{2j}, and (2.16b), the zeros of $a(\zeta,t)$ are $\lambda_j=i\zeta_j$, $j=1,\ldots,N_1$ and the zeros of $\overline{a}(\zeta,t)$ are $\lambda_j=i\overline{\zeta}_j$, $j=N_1+1,\ldots,N$, and at ζ_j and $\overline{\zeta}_j$ the following equalities for the discrete eigenfunctions hold

$$\psi^-(x,t,\zeta_m)=C_m(t)\psi^+(x,t,\zeta_m),\quad m=1,\ldots,N_1, \tag{2.21a}$$

$$\overline{\psi}^-(x,t,\overline{\zeta}_m)=\overline{C}_m(t)\overline{\psi}^+(x,t,\overline{\zeta}_m),\quad m=N_1+1,\ldots,N, \tag{2.21b}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\psi^-(x,t,\zeta_j),\quad j=1,\ldots,N_1, \tag{2.21c}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\overline{\alpha}_j(t)\overline{\psi}^-(x,t,\overline{\zeta}_j),\quad j=N_1+1,\ldots,N. \tag{2.21d}$$

It is found from Eqs. (2.11b) and (2.13a) that

$$\begin{aligned}&\phi_{2j}(x,t)\psi_1(x,t,\zeta)-\phi_{1j}(x,t)\psi_2(x,t,\zeta)\\ &\quad=(\lambda_j-i\zeta)\int_{-\infty}^{x}[\phi_{1j}(z,t)\psi_2(z,t,\zeta)+\phi_{2j}(z,t)\psi_1(z,t,\zeta)]dz,\end{aligned} \tag{2.22}$$

which gives rise to

$$\int_{-\infty}^{\infty}[\phi_{1j}(z,t)\psi_2^-(z,t,\zeta_m)+\phi_{2j}(z,t)\psi_1^-(z,t,\zeta_m)]dz=0,\quad j\neq m,\quad m=1,\ldots,N_1, \tag{2.23a}$$

$$\int_{-\infty}^{\infty}[\phi_{1j}(z,t)\overline{\psi}_2^-(z,t,\overline{\zeta}_m)+\phi_{2j}(z,t)\overline{\psi}_1^-(z,t,\overline{\zeta}_m)]dz=0,\quad j\neq m,\quad m=N_1+1,\ldots,N, \tag{2.23b}$$

and

$$\begin{aligned}&\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{i\zeta-\lambda_j}\phi_{2j}[\phi_{2j}\psi_1^-(x,t,\zeta)-\phi_{1j}\psi_2^-(x,t,\zeta)]\\ &\quad\sim-\alpha_m(t)C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}[\phi_{1m}(z,t)\psi_2^-(z,t,\zeta_m)+\phi_{2m}(z,t)\psi_1^-(z,t,\zeta_m)]\\ &\quad=-2C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz,\quad x\to\infty,\quad m=1,\ldots,N_1,\end{aligned} \tag{2.24a}$$

$$\begin{aligned}&\lim_{\zeta\to\overline{\zeta}_m}\sum_{j=1}^{N}\frac{1}{i\zeta-\lambda_j}\phi_{1j}[\phi_{2j}\overline{\psi}_1^-(x,t,\zeta)-\phi_{1j}\overline{\psi}_2^-(x,t,\zeta)]\\ &\quad\sim-2\overline{C}_m(t)\overline{\psi}_1^+(x,t,\overline{\zeta}_m)\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz,\quad x\to\infty,\quad m=N_1+1,\ldots,N.\end{aligned} \tag{2.24b}$$

Equation (2.23) are the orthogonal property of the discrete eigenfunctions. Denote the parameter η in Eq. (2.13b) corresponding to the eigenfunction $\psi^-(x,t,\zeta_m)$ by η_m^-, $m=1,\ldots,N_1$, and $\overline{\psi}^-(x,t,\zeta_m)$ by $\overline{\eta_m^-}$, $m=N_1+1,\ldots,N$, respectively. Substituting them into Eq. (2.13b) and let $x\to-\infty$, it can be seen that

$$\eta_m^-=(i\zeta_m)^n=\lambda_m^n,\quad m=1,\ldots,N_1,\quad \overline{\eta_m^-}=-(i\overline{\zeta_m})^n=-\lambda_m^n,\quad m=N_1+1,\ldots,N. \tag{2.25}$$

Inserting the representation of $\psi^-(x,t,\zeta_m)(\overline{\psi}^-(x,t,\zeta_m))$ (2.21) into Eq. (2.13b), let $x\to+\infty$ and using Eq. (2.24), we obtain

$$\frac{dC_m}{dt}=\left[2(i\zeta_m)^n-2\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz\right]C_m=2[(i\zeta_m)^n-\beta_m(t)]C_m,\quad m=1,\ldots,N_1, \tag{2.26a}$$

$$\frac{d\overline{C_m}}{dt}=2[-(i\overline{\zeta_m})^n+\beta_m(t)]\overline{C_m},\quad m=N_1+1,\ldots,N. \tag{2.26b}$$

Since the normalization constants $c_m,\overline{c_m}$ [the notation c_m here is not that given in (2.3)] are defined by

$$c_m(t)\equiv\left[2\int_{-\infty}^{\infty}\psi_1^+(x,t,\zeta_m)\psi_2^+(x,t,\zeta_m)dx\right]^{-1}=i\frac{C_m(t)}{a'(\zeta_m)},\quad m=1,\ldots,N_1, \tag{2.27a}$$

$$\overline{c_m}(t)\equiv\Bigg[2\int_{-\infty}^{\infty}\overline{\psi_1^+}(x,t,\overline{\zeta_m})\overline{\psi}$$

$$\beta_j(t)=\int_{-\infty}^{\infty}\phi_{1j}(x,t)\phi_{2j}(x,t)dx$$

$$=\alpha_j^2(t)C_j^2(t)\int_{-\infty}^{\infty}\psi_1^+(x,t,\zeta_j)\psi_2^+(x,t,\zeta_j)dx,$$

$$=\frac{1}{2}\alpha_j^2(t)C_j^2(t)(c_j(t))^{-1},\quad j=1,\ldots,N_1, \tag{2.30}$$

which lead to

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{2\beta_j(t)c_j(t)}\,\psi^+(x,t,\zeta_j),\quad j=1,\ldots,N_1, \tag{2.31a}$$

and similarly

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\,\overline{\psi}^+(x,t,\overline{\zeta_j}),\quad j=N_1+1,\ldots,N. \tag{2.31b}$$

The Eqs. (2.31) are consistent with (2.16c) according to (2.27).

According to Ref. 20, using Eqs. (2.20) and (2.28) and solving the Gel'fand–Levitan–Marchenko equation, we can get the solution of the nth AKNS equation with self-consistent sources (2.11) under the assumptions and the conditions (2.16) in the following way:

$$q(x,t)=-2K_1(x,x),\quad r(x,t)=-2\overline{K}_2(x,x), \tag{2.32a}$$

$$\phi_{1j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\int_x^{\infty}K_1(x,s)e^{\lambda_j s}ds, \tag{2.32b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\left(e^{\lambda_j x}+\int_x^{\infty}K_2(x,s)e^{\lambda_j s}ds\right),\quad j=1,\ldots,N_1, \tag{2.32c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\left(e^{-\lambda_j x}+\int_x^{\infty}\overline{K}_1(x,s)e^{-\lambda_j s}ds\right), \tag{2.32d}$$

$$\phi_{2j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\int_x^{\infty}\overline{K}_2(x,s)e^{-\lambda_j s}ds,\quad j=N_1+1,\ldots,N, \tag{2.32e}$$

where $K(x,y)=(K_1(x,y),\ K_2(x,y))^T,\overline{K}(x,y)=(\overline{K}_1(x,y),\overline{K}_2(x,y))^T$ satisfy

$$\overline{K}(x,y)+F(x+y)\begin{pmatrix}0\\1\end{pmatrix}+\int_x^{\infty}K(x,s)F(s+y)ds=0,\quad y>x, \tag{2.33a}$$

$$K(x,y)-\overline{F}(x+y)\begin{pmatrix}1\\0\end{pmatrix}-\int_x^{\infty}\overline{K}(x,s)\overline{F}(s+y)ds=0,\quad y>x, \tag{2.33b}$$

and

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\zeta)e^{i\zeta x}d\zeta-\sum_{j=1}^{N_1}c_j(t)e^{\lambda_j x},$$

$$\overline{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\rho}(\zeta)e^{-i\zeta x}d\zeta+\sum_{j=N_1+1}^{N}\overline{c_j}(t)e^{-\lambda_j x}.$$

We now present the exploding solitons for system (2.11) with $N_1=1,N=2$. Assume that $\rho(\zeta)=\overline{\rho(\zeta)}=0$ and there is one discrete eigenvalue ζ in the upper half plane and one discrete eigenvalue $\overline{\zeta}$ in the lower half plane. We have

$$F(x)=-c(t)e^{i\zeta x},\quad \overline{F}(x)=\overline{c}(t)e^{-i\overline{\zeta}x},$$

where $c(t),\overline{c}(t)$ are found from (2.29)

$$c(t)=c(0)\exp\left[2(i\zeta)^n t-2\int_0^t\beta_1(z)dz\right],\quad \overline{c}(t)=\overline{c}(0)\exp\left[-2(i\overline{\zeta})^n t+2\int_0^t\beta_2(z)dz\right]. \tag{2.34}$$

Solving Eq. (2.33) leads to

$$K_1(x,y)=\frac{1}{\Delta}\overline{c}(t)e^{-i\overline{\zeta}(x+y)},\quad K_2(x,y)=-\frac{1}{(\overline{\zeta}-\zeta)\Delta}ic(t)\overline{c}(t)e^{2i\zeta x-i\overline{\zeta}(x+y)}, \tag{2.35a}$$

$$\overline{K}_1(x,y)=-\frac{1}{(\overline{\zeta}-\zeta)\Delta}ic(t)\overline{c}(t)e^{-2i\overline{\zeta}x+i\zeta(x+y)},\quad \overline{K}_2(x,y)=\frac{1}{\Delta}c(t)e^{i\zeta(x+y)}, \tag{2.35b}$$

$$\Delta=1+\frac{1}{(\overline{\zeta}-\zeta)^2}c(t)\overline{c}(t)e^{2i(\zeta-\overline{\zeta})x}. \tag{2.35c}$$

Then we obtain the solution by means of Eq. (2.32)

$$q(x,t)=-\frac{2}{\Delta}\overline{c}(t)e^{-2i\overline{\zeta}x},\quad r(x,t)=-\frac{2}{\Delta}c(t)e^{2i\zeta x}, \tag{2.36a}$$

$$\phi_{11}(x,t)=i\sqrt{2\beta_1(t)c(t)}\frac{1}{(\zeta-\overline{\zeta})\Delta}\overline{c}(t)e^{i(\zeta-2\overline{\zeta})x}, \tag{2.36b}$$

$$\phi_{21}(x,t)=\sqrt{2\beta_1(t)c(t)}\left(e^{i\zeta x}-\frac{1}{(\overline{\zeta}-\zeta)^2\Delta}c(t)\overline{c}(t)e^{i(3\zeta-2\overline{\zeta})x}\right), \tag{2.36c}$$

$$\phi_{12}(x,t)=\sqrt{-2\beta_2(t)\overline{c}(t)}\left(e^{-i\overline{\zeta}x}-\frac{1}{(\overline{\zeta}-\zeta)^2\Delta}c(t)\overline{c}(t)e^{i(2\zeta-3\overline{\zeta})x}\right), \tag{2.36d}$$

$$\phi_{22}(x,t)=i\sqrt{-2\beta_2(t)\overline{c}(t)}\frac{1}{(\zeta-\overline{\zeta})\Delta}c(t)e^{i(2\zeta-\overline{\zeta})x}, \tag{2.36e}$$

where $c(t),\overline{c}(t)$ are given by (2.34). Equation (2.36) show that the velocity of the solution depends on $\mathrm{Re}[(i\zeta)^{n-1}-[\beta_1(t)/i\zeta]]$ or $\mathrm{Re}[-(i\overline{\zeta})^{n-1}-[\beta_2(t)/i\overline{\zeta}]]$. Therefore the insertion of sources may cause the variation of velocity of the soliton solution. This phenomenon is completely different from that of solitons of the AKNS hierarchy without sources. The choices of $\beta_i(t)$ can give a great variety of dynamics of soliton solutions.

III. INTEGRATION OF THE MKDV HIERARCHY WITH SELF-CONSISTENT SOURCES

A. The MKdV hierarchy with self-consistent sources

Consider the reduced case of the AKNS spectral problem for $r=-q$[20]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\quad U=\begin{pmatrix}-\lambda & q\\ -q & \lambda\end{pmatrix}. \tag{3.1}$$

The adjoint representation of (3.1), i.e., (2.2), and (2.3) yield

$$a_0=-1,\quad b_0=c_0=a_1=0,\quad b_1=-c_1=q,$$

$$a_2=-\tfrac{1}{2}q^2,\quad b_2=c_2=-\tfrac{1}{2}q_x,\ \dots,$$

and in general

$$b_{2m+1}=-c_{2m+1}=Lb_{2m-1},\quad L=\tfrac{1}{4}D^2+qD^{-1}qD,$$

$$b_{2m}=c_{2m}=-\tfrac{1}{2}Db_{2m-1},\quad a_{2m+1}=0,\quad a_{2m}=2D^{-1}qb_{2m}. \tag{3.2}$$

The MKdV hierarchy reads[20]

$$q_{t_{2n+1}}=-2b_{2n+2}=Db_{2n+1}=D\frac{\delta H_{2n+1}}{\delta q},\quad n=0,1,\dots, \tag{3.3}$$

where

$$H_{2n+1}=\frac{2a_{2n+2}}{2n+1}.$$

We have

$$\frac{\delta\lambda}{\delta q}=\phi_1^2+\phi_2^2,\quad L(\phi_1^2+\phi_2^2)=\lambda^2(\phi_1^2+\phi_2^2). \tag{3.4}$$

The MKdV hierarchy with self-consistent sources is given by

$$q_{t_{2n+1}}=D\left[\frac{\delta H_{2n+1}}{\delta q}+\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta q}\right]=D[b_{2n+1}+\langle\Phi_1,\Phi_1\rangle+\langle\Phi_2,\Phi_2\rangle], \tag{3.5a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j},\quad \phi_{2j,x}=-q\phi_{1j}+\lambda_j\phi_{2j},\quad j=1,\dots,N, \tag{3.5b}$$

for N distinct λ_j and assume that

$$\lambda_j<0,\quad j=1,\dots,N_1,\quad \lambda_j>0,\quad j=N_1+1,\dots,N. \tag{3.5c}$$

The zero-curvature representation for the MKdV hierarchy with self-consistent sources (3.5) is presented by (2.12) with the auxiliary linear problems

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q\\ -q & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-i\zeta & q\\ -q & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{3.6a}$$

where $\lambda=i\zeta$ and

$$\psi_{1,t_{2n+1}}=A^{(2n+1)}\psi_1+B^{(2n+1)}\psi_2\equiv\sum_{k=0}^{2n}(a_k\psi_1+b_k\psi_2)\lambda^{2n+1-k}+\eta\psi_1$$
$$+\lambda\sum_{j=1}^{N}\frac{1}{\lambda^2-\lambda_j^2}[2\lambda_j\phi_{1j}\phi_{2j}\psi_1-(\lambda+\lambda_j)\phi_{1j}^2\psi_2-(\lambda-\lambda_j)\phi_{2j}^2\psi_2],$$

$$\psi_{2,t_{2n+1}}=C^{(2n+1)}\psi_1+D^{(2n+1)}\psi_2\equiv\sum_{k=0}^{2n}(c_k\psi_1-a_k\psi_2)\lambda^{2n+1-k}+\eta\psi_2+\lambda\sum_{j=1}^{N}\frac{1}{\lambda^2-\lambda_j^2}[(\lambda$$
$$+\lambda_j)\phi_{2j}^2\psi_1+(\lambda-\lambda_j)\phi_{1j}^2\psi_1-2\lambda_j\phi_{1j}\phi_{2j}\psi_2]. \tag{3.6b}$$

When $n=1$, the system (3.5) gives the MKdV equation with self-consistent sources

$$q_{t_3}=D[\tfrac{1}{4}(2q^3+q_{xx})+\langle\Phi_1,\Phi_1\rangle+\langle\Phi_2,\Phi_2\rangle], \tag{3.7a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j},\quad \phi_{2j,x}=-q\phi_{1j}+\lambda_j\phi_{2j},\quad j=1,\ldots,N, \tag{3.7b}$$

and the auxiliary linear problem (3.6b) for $n=1$ reads

$$\psi_{1,t_3}=\left(-\lambda^3-\frac{1}{2}q^2\lambda+\eta\right)\psi_1+\left(q\lambda^2-\frac{1}{2}q_x\lambda\right)\psi_2$$
$$+\lambda\sum_{j=1}^{N}\frac{1}{\lambda^2-\lambda_j^2}[2\lambda_j\phi_{1j}\phi_{2j}\psi_1-(\lambda+\lambda_j)\phi_{1j}^2\psi_2-(\lambda-\lambda_j)\phi_{2j}^2\psi_2],$$

$$\psi_{2,t_3}=-\left(q\lambda^2+\frac{1}{2}q_x\lambda\right)\psi_1+\left(\lambda^3+\frac{1}{2}q^2\lambda+\eta\right)\psi_2$$
$$+\lambda\sum_{j=1}^{N}\frac{1}{\lambda^2-\lambda_j^2}[(\lambda+\lambda_j)\phi_{2j}^2\psi_1+(\lambda-\lambda_j)\phi_{1j}^2\psi_1-2\lambda_j\phi_{1j}\phi_{2j}\psi_2]. \tag{3.8}$$

B. Integration of the MKdV hierarchy with self-consistent sources

Based on the results of Sec. II, we now use the inverse scattering method to solve the initial-value problem for the MKdV hierarchy with self-consistent sources (3.5) under the assumptions described in Sec. II B and the assumption that the spectral equation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-i\zeta & q_0(x)\\ -q_0(x) & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{3.9a}$$

has exactly $2N$ discrete eigenvalues $\zeta_j,\overline{\zeta_j}$, $j=1,\ldots,N$, related to λ_j in (3.5c) in the following way:

$$\zeta_j=-i\lambda_j,\quad j=1,\ldots,N_1,\quad \zeta_j=i\lambda_j,\quad j=N_1+1,\ldots,N,$$
$$\overline{\zeta_j}=i\lambda_j,\quad j=1,\ldots,N_1,\quad \overline{\zeta_j}=-i\lambda_j,\quad j=N_1+1,\ldots,N, \tag{3.9b}$$

namely, $\overline{\zeta_j}=-\zeta_j$. Similarly we get

$$\eta^-=(i\zeta)^{2n+1},\quad \overline{\eta^-}=-(i\zeta)^{2n+1}.$$

For real function $r=-q$, it is known[20] that the eigenfunction $\psi^-(x,t,\zeta)$, $\overline{\psi^-}(x,t,\zeta)$, $\psi^+(x,t,\zeta)$, $\overline{\psi^+}(x,t,\zeta)$ defined by (2.17) for (3.6) have symmetry relations

$$\overline{\psi}^{-}(x,t,\zeta)=\begin{pmatrix}\psi_2^{-}(x,t,-\zeta)\\-\psi_1^{-}(x,t,-\zeta)\end{pmatrix},\quad \overline{\psi}^{+}(x,t,\zeta)=\begin{pmatrix}\psi_2^{+}(x,t,-\zeta)\\-\psi_1^{+}(x,t,-\zeta)\end{pmatrix},\tag{3.10a}$$

which imply that the $a,\overline{a},b,\overline{b}$ defined by (2.19) satisfy

$$\overline{a(}\zeta,t)=a(-\zeta,t),\quad \overline{b(}\zeta,t)=b(-\zeta,t).\tag{3.10b}$$

In the same way as in the previous section, one can show that

$$\frac{\partial a}{\partial t}=0,\quad \frac{\partial b}{\partial t}=2(i\zeta)^{2n+1}b,\quad \frac{\partial \rho}{\partial t}=2(i\zeta)^{2n+1}\rho.\tag{3.11}$$

So for the MKdV hierarchy with self-consistent sources, the evolution of quantities $a(k,t)$ and $b(k,t)$ is the same as that of the MKdV hierarchy without source. The discrete eigenvalues of spectral problem (3.6a) are given by the zeros of $a(\zeta,t)$ and $\overline{a(\zeta,t)}$ and independent of t according to (3.11). So the zeros of $a(\zeta,t)$ and $\overline{a(\zeta,t)}$ are just ζ_j and $\overline{\zeta_j}=-\zeta_j$, $j=1,\ldots,N$, given by (3.9b). At ζ_j and $\overline{\zeta_j}$ the following equalities for the discrete eigenfunctions hold:

$$\psi^{-}(x,t,\zeta_m)=C_m(t)\psi^{+}(x,t,\zeta_m),\quad \overline{\psi}^{-}(x,t,\overline{\zeta}_m)=\overline{C}_m(t)\overline{\psi}^{+}(x,t,\overline{\zeta}_m),\quad m=1,\ldots,N,\tag{3.12}$$

which together with (3.10a) yield

$$C_m=\overline{C}_m,\quad m=1,\ldots,N.\tag{3.13}$$

Also we have

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\psi^{-}(x,t,\zeta_j),\quad j=1,\ldots,N_1,\tag{3.14a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\overline{\psi}^{-}(x,t,\overline{\zeta}_j),\quad j=N_1+1,\ldots,N.\tag{3.14b}$$

As in the previous section, the main point for deriving the evolution of C_m is to treat the singularity in the evolution of eigenfunctions (3.6b). It is found from Eqs. (3.5b) and (3.6a) that

$$\begin{aligned}&(i\zeta+\lambda_j)\phi_{2j}(x,t)\psi_1^{-}(x,t,\zeta)-2\lambda_j\phi_{1j}(x,t)\psi_2^{-}(x,t,\zeta)\\&=(i\zeta-\lambda_j)\bigg[\frac{1}{2}\,(\phi_{1j}(x,t)\psi_2^{-}(x,t,\zeta)+\phi_{2j}(x,t)\psi_1^{-}(x,t,\zeta))\\&\quad-\left(\frac{1}{2}i\zeta+\frac{3}{2}\lambda_j\right)\int_{-\infty}^{x}(\phi_{1j}(z,t)\psi_2^{-}(z,t,\zeta)+\phi_{2j}(z,t)\psi_1^{-}(z,t,\zeta))dz\bigg],\end{aligned}\tag{3.15a}$$

$$\begin{aligned}&(i\zeta-\lambda_j)\phi_{1j}(x,t)\psi_1^{-}(x,t,\zeta)-2\lambda_j\phi_{2j}(x,t)\psi_2^{-}(x,t,\zeta)\\&=(i\zeta+\lambda_j)\bigg[\frac{1}{2}(\phi_{1j}(x,t)\psi_1^{-}(x,t,\zeta)-\phi_{2j}(x,t)\psi_2^{-}(x,t,\zeta))\\&\quad+\left(\frac{1}{2}i\zeta-\frac{3}{2}\lambda_j\right)\int_{-\infty}^{x}(\phi_{2j}(z,t)\psi_2^{-}(z,t,\zeta)-\phi_{1j}(z,t)\psi_1^{-}(z,t,\zeta))dz\bigg],\\&j=1,\ldots,N,\end{aligned}\tag{3.15b}$$

which together with (3.10a) and (2.17) gives rise to

$$\int_{-\infty}^{\infty}[\phi_{1j}(z,t)\psi_2^-(z,t,\zeta_m)+\phi_{2j}(z,t)\psi_1^-(z,t,\zeta_m)]dz=0, \quad j\neq m, \quad m=1,\ldots,N, \tag{3.16a}$$

$$\int_{-\infty}^{\infty}[\phi_{1j}(z,t)\overline{\psi_2^-}(z,t,\overline{\zeta_m})+\phi_{2j}(z,t)\overline{\psi_1^-}(z,t,\overline{\zeta_m})]dz=0, \quad j\neq m, \quad m=1,\ldots,N, \tag{3.16b}$$

and

$$\begin{aligned}
&\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{(i\zeta)^2-\lambda_j^2}\phi_{2j}[(i\zeta+\lambda_j)\phi_{2j}(x,t)\psi_1^-(x,t,\zeta)-2\lambda_j\phi_{1j}(x,t)\psi_2^-(x,t,\zeta)]\\
&\sim-\alpha_m(t)C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}[\phi_{1m}(z,t)\psi_2^-(z,t,\zeta_m)+\phi_{2m}(z,t)\psi_1^-(z,t,\zeta_m)]dz\\
&=-2C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz, \quad \text{for} \quad x\to\infty,\\
&\qquad m=1,\ldots,N_1,
\end{aligned} \tag{3.17a}$$

$$\begin{aligned}
&\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{(i\zeta)^2-\lambda_j^2}\phi_{1j}[(i\zeta-\lambda_j)\phi_{1j}(x,t)\psi_1^-(x,t,\zeta)-2\lambda_j\phi_{2j}(x,t)\psi_2^-(x,t,\zeta)]\\
&\sim\alpha_m(t)\overline{\psi_1^-}(x,t,\overline{\zeta_m})\int_{-\infty}^{\infty}[\phi_{1m}(z,t)\overline{\psi_2^-}(z,t,\overline{\zeta_m})+\phi_{2m}(z,t)\overline{\psi_1^-}(z,t,\overline{\zeta_m})]dz\\
&=-2C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz, \quad \text{for} \quad x\to\infty,\\
&\qquad m=N_1+1,\ldots,N.
\end{aligned} \tag{3.17b}$$

Equations (3.16) are the orthogonal property of the discrete eigenfunctions. Also we have

$$\eta_m^-=(i\zeta_m)^{2n+1}=\lambda_m^{2n+1}, \quad m=1,\ldots,N_1, \quad \eta_m^-=(i\zeta_m)^{2n+1}=-\lambda_m^{2n+1}, \quad m=N_1+1,\ldots,N. \tag{3.18}$$

Take the representation of $\psi^-(x,t,\zeta_m)$ (3.12) into Eq. (3.6b), then let $x\to+\infty$ and using Eq. (3.17), we have

$$\begin{aligned}
\frac{dC_m}{dt}&=\left[2(i\zeta_m)^{2n+1}-2i\zeta_m\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz\right]C_m\\
&=2\lambda_m[\lambda_m^{2n}-\beta_m(t)]C_m,\\
&\quad m=1,\ldots,N_1,
\end{aligned} \tag{3.19a}$$

$$\frac{dC_m}{dt}=-2\lambda_m[\lambda_m^{2n}-\beta_m(t)]C_m, \quad m=N_1+1,\ldots,N, \tag{3.19b}$$

with the arbitrary real functions $\beta_m(t)$. According to the definition of the normalization constants $c_m,\overline{c_m}$ given by (2.27), one obtains

$$\overline{c_m}(t)=-c_m(t), \quad m=1,\ldots,N, \tag{3.20}$$

and

$$\frac{dc_m}{dt}=2\lambda_m[\lambda_m^{2n}-\beta_m(t)]c_m, \quad m=1,\dots,N_1, \tag{3.21a}$$

$$\frac{dc_m}{dt}=-2\lambda_m[\lambda_m^{2n}-\beta_m(t)]c_m, \quad m=N_1+1,\dots,N, \tag{3.21b}$$

which gives rise to

$$c_m(t)=c_m(0)\exp\left[2\lambda_m\left(\lambda_m^{2n}t-\int_0^t\beta_m(z)dz\right)\right], \quad m=1,\dots,N_1, \tag{3.22a}$$

$$c_m(t)=c_m(0)\exp\left[-2\lambda_m\left(\lambda_m^{2n}t-\int_0^t\beta_m(z)dz\right)\right], \quad m=N_1+1,\dots,N. \tag{3.22b}$$

Thus, the evolution of $c_m(t)=-\overline{c_m}(t)$ has an extra term $-2\lambda_m\beta_m(t)c_m(t)$ comparing with that of the MKdV hierarchy without source.

Equations (2.30) and (3.14) lead to

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{2\beta_j(t)c_j(t)}\,\psi^+(x,t,\zeta_j), \quad j=1,\dots,N_1, \tag{3.23a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\,\overline{\psi}^+(x,t,\overline{\zeta}_j), \quad j=N_1+1,\dots,N, \tag{3.23b}$$

which are consistent with (2.16c) according to (2.27). It is known[20] that for $r=-q$, one has

$$\overline{F}(x)=F(x), \quad \overline{K}(x,y)=\begin{pmatrix}K_2(x,y)\\ -K_1(x,y)\end{pmatrix}. \tag{3.24}$$

Then using Eqs. (3.11) and (3.22), we can get the solution of the nth MKdV equation with self-consistent sources (3.5) under the assumption in the following way:[20]

$$q(x,t)=-2K_1(x,x), \tag{3.25a}$$

$$\phi_{1j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\int_x^\infty K_1(x,s)e^{\lambda_j s}ds, \quad j=1,\dots,N_1, \tag{3.25b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\left(e^{\lambda_j x}+\int_x^\infty K_2(x,s)e^{\lambda_j s}ds\right), \quad j=1,\dots,N_1, \tag{3.25c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\left(e^{-\lambda_j x}+\int_x^\infty K_2(x,s)e^{-\lambda_j s}ds\right), \quad j=N_1+1,\dots,N, \tag{3.25d}$$

$$\phi_{2j}(x,t)=-\sqrt{-2\beta_j(t)\overline{c_j}(t)}\int_x^\infty K_1(x,s)e^{-\lambda_j s}ds, \quad j=N_1+1,\dots,N, \tag{3.25e}$$

where $K(x,y)=(K_1(x,y),K_2(x,y))^T$, satisfy

$$K_2(x,y)+\int_x^\infty K_1(x,s)F(s+y)ds=0, \quad y>x, \tag{3.26a}$$

$$K_1(x,y)-F(x+y)-\int_x^\infty K_2(x,s)F(s+y)ds=0,\quad y>x, \tag{3.26b}$$

and

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\zeta)e^{i\zeta x}d\zeta-\sum_{j=1}^{N_1}c_j(t)e^{\lambda_j x}-\sum_{j=N_1+1}^{N}c_j(t)e^{-\lambda_j x}. \tag{3.27}$$

C. The *N* soliton solution of the MKdV hierarchy with self-consistent sources

For the nth MKdV equation with self-consistent sources (3.5), assume $\rho(\zeta)=0$ and there are $2N$ distinct eigenvalues $\zeta_j=i\tilde{\lambda}_j=-i\lambda_j$, $j=1,\dots,N_1$, $\zeta_j=i\tilde{\lambda}_j=i\lambda_j$, $j=N_1+1,\dots,N$, $\overline{\zeta_j}=-\zeta_j$, $j=1,\dots,N$. Denote

$$E(x,t)=(c_1(t)e^{-\tilde{\lambda}_1 x}\,c_2(t)e^{-\tilde{\lambda}_2 x}\cdots c_N(t)e^{-\tilde{\lambda}_N x}),$$

$$M(x,t)=(M_{jl})_{N\times N}=\left(\frac{c_l(t)}{\tilde{\lambda}_j+\tilde{\lambda}_l}e^{-(\tilde{\lambda}_j+\tilde{\lambda}_l)x}\right)_{N\times N},$$

$$B(y)=(e^{-\tilde{\lambda}_1 y}e^{-\tilde{\lambda}_2 y}\cdots e^{-\tilde{\lambda}_N y})^T,$$

$$A_j(x)=\left(\frac{1}{\tilde{\lambda}_1+\tilde{\lambda}_j}e^{-(\tilde{\lambda}_1+\tilde{\lambda}_j)x}\cdots\frac{1}{\tilde{\lambda}_N+\tilde{\lambda}_j}e^{-(\tilde{\lambda}_N+\tilde{\lambda}_j)x}\right)^T,$$

$$D(x,t)=I+M(x,t)M(x,t).$$

Then the $K_1(x,y),K_2(x,y)$ in Eq. (3.26) can be obtained as

$$K_1(x,y,t)=-E(x,t)D^{-1}(x,t)B(y),\quad K_2(x,y,t)=-E(x,t)M(x,t)D^{-1}(x,t)B(y). \tag{3.28}$$

After some reduction, the solution of the MKdV hierarchy with self-consistent sources (3.5) under our assumption can be written in the form

$$q(x,t)=2\frac{d}{dx}\mathrm{tg}^{-1}\frac{\mathrm{Im}\det(I-iM(x,t))}{\mathrm{Re}\det(I-iM(x,t))}, \tag{3.29a}$$

$$\phi_{1j}(x,t)=-\sqrt{2\beta_j(t)c_j(t)}E(x,t)D^{-1}(x,t)A_j(x),\quad j=1,\dots,N_1, \tag{3.29b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}[e^{-\tilde{\lambda}_j x}-E(x,t)M(x,t)D^{-1}(x,t)A_j(x)],\quad j=1,\dots,N_1, \tag{3.29c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}[e^{-\tilde{\lambda}_j x}-E(x,t)M(x,t)D^{-1}(x,t)A_j(x)],\quad j=N_1+1,\dots,N, \tag{3.29d}$$

$$\phi_{2j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}E(x,t)D^{-1}(x,t)A_j(x),\quad j=N_1+1,\dots,N. \tag{3.29e}$$

In particular, if $\tilde{\lambda}_j\neq\tilde{\lambda}_m$ when $j\neq m$ and properly choose β_j in such way that

$$\epsilon_{jm}=(\tilde{\lambda}_j^{2n}-\tilde{\lambda}_m^{2n})t-\int_0^t(\beta_j(z)-\beta_m(z))dz,\quad j,m=1,\dots,N,$$

satisfy

$$\epsilon_{jm}\to -\infty, \quad \text{or} \quad \epsilon_{jm}\to\infty, \quad \text{when} \quad j\neq m, \quad \text{and} \quad t\to\pm\infty, \tag{3.30}$$

then Eq. (3.29) present the N-soliton solution of the nth MKdV equation with self-consistent sources (3.5). The velocity of each soliton for propagating depends on

$$v_j = 2\tilde{\lambda}_j^{2n} - 2\beta_j(t), \quad j=1,\ldots,N. \tag{3.31}$$

For instance, if take $\beta_j(t)$ to be constant and let $\tilde{\lambda}_j$ and β_j to satisfy

$$0<\tilde{\lambda}_1^{2n}-\beta_1<\tilde{\lambda}_2^{2n}-\beta_2<\cdots<\tilde{\lambda}_N^{2n}-\beta_N,$$

then Eq. (3.29) present the N-soliton solution of the nth MKdV equation with self-consistent sources (3.5).

We see that the insertion of a source may cause the variation of the velocity of a soliton. Since there are many choices of β_j, the dynamics of soliton solutions is variety.

IV. INTEGRATION OF THE NLSE HIERARCHY WITH SELF-CONSISTENT SOURCES

A. The NLSE hierarchy with self-consistent sources

Consider the reduced case of the AKNS spectral problem for $r=-q^*$[23]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad U=\begin{pmatrix}-\lambda & q\\ -q^* & \lambda\end{pmatrix}=\begin{pmatrix}-i\zeta & q\\ -q^* & i\zeta\end{pmatrix}, \tag{4.1}$$

where the * denotes complex conjugation. Equations (2.2) and (2.3) yields

$$a_0=-2i, \quad b_0=c_0=a_1=0, \quad b_1=2iq, \quad c_1=-2iq^*$$

$$a_2=-iqq^*, \quad b_2=-iq_x, \quad c_2=-iq_x^*, \ldots,$$

and in general

$$\begin{pmatrix}c_{m+1}\\ b_{m+1}\end{pmatrix}=L\begin{pmatrix}c_m\\ b_m\end{pmatrix}, \quad a_{m,x}=qc_m+q^*b_m, \tag{4.2}$$

where

$$L=\frac{1}{2}\begin{pmatrix}D+2q^*D^{-1}q & 2q^*D^{-1}q^*\\ -2qD^{-1}q & -D-2qD^{-1}q^*\end{pmatrix}.$$

We have

$$c_{2m+1}=b_{2m+1}^*, \quad c_{2m}=-b_{2m}^*, \quad a_{2m+1}=a_{2m+1}^*, \quad a_{2m}=-a_{2m}^*. \tag{4.3}$$

The hierarchy of the nonlinear Schrödinger equations (NLSE) reads[23]

$$q_{t_{2n}}=-2b_{2n+1}=-2\frac{\delta H_{2n+1}}{\delta q}, \quad H_{2n+1}=\frac{2a_{2n+2}}{2n+1}, \quad n=0,1,\ldots . \tag{4.4}$$

It is easy to verify that

$$L^m\begin{pmatrix}-\phi_1^{*2}+\phi_2^2\\ -\phi_1^2+\phi_2^{*2}\end{pmatrix}=\begin{pmatrix}-(-\lambda^*)^m\phi_1^{*2}+\lambda^m\phi_2^2\\ -\lambda^m\phi_1^2+(-\lambda^*)^m\phi_2^{*2}\end{pmatrix}, \quad m=1,2,\ldots . \tag{4.5}$$

The NLSE hierarchy with self-consistent sources is given by

$$q_{t_{2n}}=-2[b_{2n+1}+\langle\Phi_1,\Phi_1\rangle-\langle\Phi_2^*,\Phi_2^*\rangle], \tag{4.6a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j}, \quad \phi_{2j,x}=-q^*\phi_{1j}+\lambda_j\phi_{2j}, \quad j=1,\ldots,N, \tag{4.6b}$$

for N distinct λ_j and assume that

$$\lambda_j=i\zeta_j, \quad \mathrm{Im}\,\zeta_j>0 \quad \text{or} \quad \mathrm{Re}\,\lambda_j<0, \quad j=1,\ldots,N_1,$$

$$\lambda_j=i\overline{\zeta_j}, \quad \mathrm{Im}\,\overline{\zeta_j}<0 \quad \text{or} \quad \mathrm{Re}\,\lambda_j>0, \quad j=N_1+1,\ldots,N. \tag{4.6c}$$

According to Eqs. (4.2), (4.5), and (4.6), we define

$$\tilde{a}_i=a_i, \quad \tilde{b}_i=b_i, \quad \tilde{c}_i=c_i, \quad i=0,1,\ldots,2n,$$

$$\tilde{b}_{2n+1+m}=-\langle\Lambda^m\Phi_1,\Phi_1\rangle+\langle(-\Lambda^*)^m\Phi_2^*,\Phi_2^*\rangle,$$

$$\tilde{c}_{2n+1+m}=-\langle(-\Lambda^*)^m\Phi_1^*,\Phi_1^*\rangle+\langle\Lambda^m\Phi_2,\Phi_2\rangle,$$

$$\tilde{a}_{2n+1+m}=D^{-1}(q\tilde{c}_{2n+1+m}+q^*\tilde{b}_{2n+1+m})=\langle\Lambda^m\Phi_1,\Phi_2\rangle+\langle(-\Lambda^*)^m\Phi_1^*,\Phi_2^*\rangle, \quad m=0,1,\ldots,$$

which are consistent with (4.3). Then in the same way as in the Sec. II, it is found that the zero-curvature representation for the NLSE hierarchy with self-consistent sources (4.6) is given by (2.12) with the auxiliary linear problems

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q\\ -q^* & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-i\zeta & q\\ -q^* & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{4.7a}$$

and

$$\psi_{1,t_{2n}}=A^{(2n)}\psi_1+B^{(2n)}\psi_2\equiv\sum_{k=0}^{2n}(a_k\psi_1+b_k\psi_2)\lambda^{2n-k}+\eta\psi_1+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2),$$
$$+\sum_{j=1}^{N}\frac{1}{\lambda+\lambda_j^*}\phi_{2j}^*(\phi_{1j}^*\psi_1+\phi_{2j}^*\psi_2)$$

$$\psi_{2,t_{2n}}=C^{(2n)}\psi_1+D^{(2n)}\psi_2\equiv\sum_{k=0}^{2n}(c_k\psi_1-a_k\psi_2)\lambda^{2n-k}+\eta\psi_2+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2)$$
$$-\sum_{j=1}^{N}\frac{1}{\lambda+\lambda_j^*}\phi_{1j}^*(\phi_{1j}^*\psi_1+\phi_{2j}^*\psi_2). \tag{4.7b}$$

When $n=1$, Eq. (4.6) gives the NLSE equation with self-consistent sources

$$iq_{t_2}=q_{xx}+2q^2q^*-2i\langle\Phi_1,\Phi_1\rangle+2i\langle\Phi_2^*,\Phi_2^*\rangle, \tag{4.8a}$$

$$\phi_{1j,x}=-\lambda_j\phi_{1j}+q\phi_{2j}, \quad \phi_{2j,x}=-q^*\phi_{1j}+\lambda_j\phi_{2j}, \quad j=1,\ldots,N, \tag{4.8b}$$

and the auxiliary linear problem (4.7b) reads

$$\psi_{1,t_3}=(-2i\lambda^2-iqq^*+\eta)\psi_1+(2iq\lambda-iq_x)\psi_2+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{1j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2),$$

$$+\sum_{j=1}^{N}\frac{1}{\lambda+\lambda_j^*}\phi_{2j}^*(\phi_{1j}^*\psi_1+\phi_{2j}^*\psi_2),$$

$$\psi_{2,t_3}=(-2iq^*\lambda-iq_x^*)\psi_1+(2i\lambda^2+iqq^*+\eta)\psi_2+\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1-\phi_{1j}\psi_2)$$

$$-\sum_{j=1}^{N}\frac{1}{\lambda+\lambda_j^*}\phi_{1j}^*(\phi_{1j}^*\psi_1+\phi_{2j}^*\psi_2). \tag{4.9}$$

B. Integration of the NLSE hierarchy with self-consistent sources

We will use the inverse scattering method to solve the initial-value problem for the NLSE hierarchy with self-consistent sources (4.6) under the same assumptions as in Sec. II B as well as the assumption that the spectral equation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q_0(x)\\ -q_0^*(x) & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}-i\zeta & q_0(x)\\ -q_0^*(x) & i\zeta\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{4.10a}$$

has exactly $2N$ discrete eigenvalues given by (4.6c)

$$\zeta_j=-i\lambda_j,\quad j=1,\dots,N_1,\quad \zeta_j=i\lambda_j^*,\quad j=N_1+1,\dots,N,$$

$$\overline{\zeta_j}=i\lambda_j^*=\zeta_j^*,\quad j=1,\dots,N_1,\quad \overline{\zeta_j}=-i\lambda_j=\zeta_j^*,\quad j=N_1+1,\dots,N. \tag{4.10b}$$

We have

$$\eta^-=2i(i\zeta)^{2n},\quad \overline{\eta^-}=-2i(i\zeta)^{2n}.$$

For $r=-q^*$, it is known[20] that

$$\overline{\psi^-}(x,t,\zeta)=\begin{pmatrix}\psi_2^{-*}(x,t,\zeta^*)\\ -\psi_1^{-*}(x,t,\zeta^*)\end{pmatrix},\quad \overline{\psi^+}(x,t,\zeta)=\begin{pmatrix}\psi_2^{+*}(x,t,\zeta^*)\\ -\psi_1^{+*}(x,t,\zeta^*)\end{pmatrix}, \tag{4.11a}$$

which imply that the $a,\overline{a},b,\overline{b}$ defined by (2.19) satisfy

$$\overline{a}(\zeta,t)=a^*(\zeta^*,t),\quad \overline{b}(\zeta,t)=b^*(\zeta^*,t). \tag{4.11b}$$

In the same way as in the previous section, one can show that

$$\frac{\partial a}{\partial t}=0,\quad \frac{\partial b}{\partial t}=4i(-1)^n\zeta^{2n}b,\quad \frac{\partial \rho}{\partial t}=4i(-1)^n\zeta^{2n}\rho. \tag{4.12}$$

So for the NLSE hierarchy with self-consistent sources, the evolution of quantities $a(k,t)$ and $b(k,t)$ is the same as that of the NLSE hierarchy without source. The discrete eigenvalues of spectral problem (4.7a) are given by the zeros of $a(\zeta,t)$ and $\overline{a}(\zeta,t)$ and independent of t according to (4.12). So the zeros of $a(\zeta,t)$ and $\overline{a}(\zeta,t)$ are just ζ_j and $\overline{\zeta_j}=\zeta_j^*$, $j=1,\dots,N$, given by (4.10b). At ζ_j and $\overline{\zeta_j}$, the following equalities for the discrete eigenfunctions hold

$$\psi^-(x,t,\zeta_m)=C_m(t)\psi^+(x,t,\zeta_m), \tag{4.13a}$$

$$\overline{\psi}^{-}(x,t,\overline{\zeta}_m)=\overline{C}_m(t)\overline{\psi}^{+}(x,t,\overline{\zeta}_m), \quad m=1,\dots,N, \tag{4.13b}$$

which together with (4.11a) yield

$$C_m=\overline{C}_m^*, \quad m=1,\dots,N. \tag{4.14}$$

Also we have

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\psi^{-}(x,t,\zeta_j), \quad j=1,\dots,N_1, \tag{4.15a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\overline{\psi}^{-}(x,t,\overline{\zeta}_j), \quad j=N_1+1,\dots,N. \tag{4.15b}$$

It is found from Eqs. (4.6b) and (4.7a) that

$$\begin{aligned}&\phi_{2j}(x,t)\psi_1^{-}(x,t,\zeta)-\phi_{1j}(x,t)\psi_2^{-}(x,t,\zeta)\\ &\quad=(-i\zeta+\lambda_j)\int_{-\infty}^{x}(\phi_{1j}(z,t)\psi_2^{-}(z,t,\zeta)+\phi_{2j}(z,t)\psi_1^{-}(z,t,\zeta))dz,\end{aligned} \tag{4.16a}$$

$$\begin{aligned}&\phi_{1j}^*(x,t)\psi_1^{-}(x,t,\zeta)+\phi_{2j}^*(x,t)\psi_2^{-}(x,t,\zeta)\\ &\quad=(i\zeta+\lambda_j^*)\int_{-\infty}^{x}(\phi_{2j}^*(z,t)\psi_2^{-}(z,t,\zeta)-\phi_{1j}^*(z,t)\psi_1^{-}(z,t,\zeta))dz], \quad j=1,\dots,N,\end{aligned} \tag{4.16b}$$

which together with (4.11a) and (2.17) gives rise to

$$\int_{-\infty}^{\infty}[\phi_{1j}(z,t)\psi_2^{-}(z,t,\zeta_m)+\phi_{2j}(z,t)\psi_1^{-}(z,t,\zeta_m)]dz=0, \quad j\neq m, \quad 1,\dots,N, \tag{4.17a}$$

$$\int_{-\infty}^{\infty}[\phi_{2j}^*(z,t)\psi_2^{-}(z,t,\zeta_m)-\phi_{1j}^*(z,t)\psi_1^{-}(z,t,\zeta_m)]dz=0, \quad j\neq m, \quad m=1,\dots,N, \tag{4.17b}$$

and

$$\begin{aligned}&\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{i\zeta-\lambda_j}\phi_{2j}[\phi_{2j}(x,t)\psi_1^{-}(x,t,\zeta)-\phi_{1j}(x,t)\psi_2^{-}(x,t,\zeta)]\\ &\sim-\alpha_m(t)C_m(t)\psi_2^{+}(x,t,\zeta_m)\int_{-\infty}^{\infty}[\phi_{1m}(z,t)\psi_2^{-}(z,t,\zeta_m)+\phi_{2m}(z,t)\psi_1^{-}(z,t,\zeta_m)]dz\\ &=-2C_m(t)\psi_2^{+}(x,t,\zeta_m)\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz, \quad \text{for} \quad x\to\infty,\\ &\qquad m=1,\dots,N_1,\end{aligned} \tag{4.18a}$$

$$\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{i\zeta+\lambda_j^*}\phi_{1j}^*[\phi_{1j}^*(x,t)\psi_1^-(x,t,\zeta)+\phi_{2j}^*(x,t)\psi_2^-(x,t,\zeta)]\sim\alpha_m^*(t)C_m(t)$$

$$\times(\overline{\psi}_1^+(x,t,\overline{\zeta}_m))^*\int_{-\infty}^{\infty}[\phi_{1m}^*(z,t)(\overline{\psi}_2^-(z,t,\overline{\zeta}_m))^*+\phi_{2m}^*(z,t)(\overline{\psi}_1^-(z,t,\overline{\zeta}_m))^*]dz$$

$$=2C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}\phi_{1m}^*(z,t)\phi_{2m}^*(z,t)dz,\quad \text{for}\quad x\to\infty,$$

$$m=N_1+1,\ldots,N. \tag{4.18b}$$

Equation (4.17) are the orthogonal property of the discrete eigenfunctions. Also one gets

$$\eta_m^-=2i(i\zeta_m)^{2n}=2i\lambda_m^{2n},\quad m=1,\ldots,N_1,\quad \eta_m^-=-2i\lambda_m^{2n},\quad m=N_1+1,\ldots,N. \tag{4.19}$$

Then we have

$$\frac{dC_m}{dt}=\left[4i(i\zeta_m)^{2n}-2\int_{-\infty}^{\infty}\phi_{1m}(z,t)\phi_{2m}(z,t)dz\right]C_m=[4i\lambda_m^{2n}-2\beta_m(t)]C_m,\quad m=1,\ldots,N_1, \tag{4.20a}$$

$$\frac{dC_m}{dt}=[-4i\lambda_m^{2n}+2\beta_m^*(t)]C_m,\quad m=N_1+1,\ldots,N, \tag{4.20b}$$

which leads to

$$\frac{dc_m}{dt}=[4i\lambda_m^{2n}-2\beta_m(t)]c_m,\quad m=1,\ldots,N_1, \tag{4.21a}$$

$$\frac{dc_m}{dt}=[-4i\lambda_m^{2n}+2\beta_m^*(t)]c_m,\quad m=N_1+1,\ldots N, \tag{4.21b}$$

or

$$c_m(t)=c_m(0)\exp\left[4i\lambda_m^{2n}t-2\int_0^t\beta_m(z)dz\right),\quad m=1,\ldots,N_1, \tag{4.22a}$$

$$c_m(t)=c_m(0)\exp\left[-4i\lambda_m^{2n}t+2\int_0^t\beta_m^*(z)dz\right),\quad m=N_1+1,\ldots,N. \tag{4.22b}$$

Thus, the evolution of $c_m(t)=-\overline{c}_m^*(t)$ has an extra term $-2\beta_m(t)c_m(t)$ or $2\beta_m^*(t)c_m(t)$ comparing with that of the NLSE hierarchy without source.

Equations (2.30) and (4.15) lead to

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{2\beta_j(t)c_j(t)}\,\psi^+(x,t,\zeta_j),\quad j=1,\ldots,N_1, \tag{4.23a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{-2\beta_j(t)\overline{c}_j(t)}\,\overline{\psi}^+(x,t,\overline{\zeta}_j),\quad j=N_1+1,\ldots,N, \tag{4.23b}$$

which are consistent with (2.16c). It is known[20] that for $r=-q^*$, one has

$$\overline{F}(x)=F^*(x),\quad \overline{K}(x,y)=\begin{pmatrix}K_2^*(x,y)\\ -K_1^*(x,y)\end{pmatrix}. \tag{4.24}$$

Then according to Eqs. (4.12) and (4.22), we can get the solution of the nth NLSE equation with self-consistent sources (4.6) in the following way:[20]

$$q(x,t)=-2K_1(x,x), \tag{4.25a}$$

$$\phi_{1j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\int_x^\infty K_1(x,s)e^{\lambda_j s}ds, \quad j=1,\ldots,N_1, \tag{4.25b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\left(e^{\lambda_j x}+\int_x^\infty K_2(x,s)e^{\lambda_j s}ds\right), \quad j=1,\ldots,N_1, \tag{4.25c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\left(e^{-\lambda_j x}+\int_x^\infty K_2^*(x,s)e^{-\lambda_j s}ds\right), \quad j=N_1+1,\ldots,N, \tag{4.25d}$$

$$\phi_{2j}(x,t)=-\sqrt{-2\beta_j(t)\overline{c_j}(t)}\int_x^\infty K_1^*(x,s)e^{-\lambda_j s}ds, \quad j=N_1+1,\ldots,N, \tag{4.25e}$$

where $K(x,y)=(K_1(x,y),K_2(x,y))^T$, satisfy

$$K_2^*(x,y)+\int_x^\infty K_1(x,s)F(s+y)ds=0, \quad y>x, \tag{4.26a}$$

$$K_1(x,y)-F^*(x+y)-\int_x^\infty K_2^*(x,s)F^*(s+y)ds=0, \quad y>x, \tag{4.26b}$$

and

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^\infty \rho(\zeta)e^{i\zeta x}d\zeta-\sum_{j=1}^{N_1} c_j(t)e^{\lambda_j x}-\sum_{j=N_1+1}^{N} c_j(t)e^{-\lambda_j^* x}. \tag{4.27}$$

C. The *N* soliton solution of the NLSE hierarchy with self-consistent sources

For the nth NLSE equation with self-consistent sources (4.5), assume $\rho(\zeta)=0$ and there are $2N$ distinct eigenvalues $\tilde\lambda_j=-\lambda_j$, $j=1,\ldots,N_1$, $\tilde\lambda_j=\lambda_j$, $j=N_1+1,\ldots,N$, $\tilde\lambda_j=(\tilde\lambda_j)^*$, $j=1,\ldots,N$. Denote

$$E(x,t)=(c_1(t)e^{-\tilde\lambda_1 x}c_2(t)e^{-\tilde\lambda_2 x}\cdots c_N(t)e^{-\tilde\lambda_N x}),$$

$$M(x,t)=(M_{jl})_{N\times N}=\left(\frac{c_l(t)}{\tilde\lambda_j^*+\tilde\lambda_l}e^{-(\tilde\lambda_j^*+\tilde\lambda_l)x}\right)_{N\times N},$$

$$B(y)=(e^{-\tilde\lambda_1 y}e^{-\tilde\lambda_2 y}\cdots e^{-\tilde\lambda_N y})^T,$$

$$A_j(x)=\left(\frac{1}{\tilde\lambda_1^*+\tilde\lambda_j}e^{-(\tilde\lambda_1^*+\tilde\lambda_j)x}\cdots\frac{1}{\tilde\lambda_N^*+\tilde\lambda_j}e^{-(\tilde\lambda_N^*+\tilde\lambda_j)x}\right)^T,$$

$$D(x,t)=I+M(x,t)M^*(x,t).$$

Then the $K_1(x,y),K_2(x,y)$ in Eq. (4.26) can be obtained as

$$K_1(x,y,t)=-E^*(x,t)D^{-1}(x,t)B^*(y), \quad K_2(x,y,t)=-E(x,t)(D^{-1})^*(x,t)M^*(x,t)B^*(y). \tag{4.28}$$

After some reduction, the solution of the NLSE hierarchy with self-consistent sources (3.5) under our assumption can be written in the form[20]

$$q(x,t)=-2K_1(x,x,t)=2E^*(x,t)D^{-1}(x,t)B^*(x), \tag{4.29a}$$

$$\phi_{1j}(x,t)=-\sqrt{2\beta_j(t)c_j(t)}E^*(x,t)D^{-1}(x,t)A_j(x), \quad j=1,\ldots,N_1, \tag{4.29b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}[e^{-\tilde{\lambda}_j x}-E(x,t)(D^{-1}(x,t))^*M^*(x,t)A_j(x)], \quad j=1,\ldots,N_1, \tag{4.29c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}[e^{-\tilde{\lambda}_j^* x}-E^*(x,t)D^{-1}(x,t)M(x,t)A_j^*(x)], \quad j=N_1+1,\ldots,N, \tag{4.29d}$$

$$\phi_{2j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}E^*(x,t)D^{-1}(x,t)A_j^*(x), \quad j=N_1+1,\ldots,N. \tag{4.29e}$$

In particular, if $\tilde{\lambda}_j\neq\tilde{\lambda}_m$ when $j\neq m$ and properly choose β_j in such way that

$$\epsilon_{jm}=\mathrm{Re}\left[2i(\tilde{\lambda}_j^{2n-1}-\tilde{\lambda}_m^{2n-1})t-\int_0^t\left(\frac{\beta_j(z)}{\tilde{\lambda}_j}-\frac{\beta_m(z)}{\tilde{\lambda}_m}\right)dz\right], \quad j,m=1,\ldots,N,$$

satisfy

$$\epsilon_{jm}\to-\infty, \quad \text{or} \quad \epsilon_{jm}\to\infty, \quad \text{when} \quad j\neq m, \quad \text{and} \quad t\to\pm\infty, \tag{4.30}$$

then Eq. (4.29) present the N-soliton solution of the nth NLSE equation with self-consistent sources (4.5). The velocity of each soliton for propagating depends on

$$v_j=\mathrm{Re}\left[2i\tilde{\lambda}_j^{2n-1}-\frac{\beta_j(t)}{\tilde{\lambda}_j}\right], \quad j=1,\ldots,N. \tag{4.31}$$

For instance, if take $\beta_j(t)$ to be constant and let $\tilde{\lambda}_j$ and β_j to satisfy

$$0<\mathrm{Re}\left[2i\tilde{\lambda}_1^{2n-1}-\frac{\beta_1}{\tilde{\lambda}_1}\right]<\mathrm{Re}\left[2i\tilde{\lambda}_2^{2n-1}-\frac{\beta_2}{\tilde{\lambda}_2}\right]<\cdots<\mathrm{Re}\left[2i\tilde{\lambda}_N^{2n-1}-\frac{\beta_N}{\tilde{\lambda}_N}\right],$$

then Eq. (4.29) present the N-soliton solution of the nth NLSE equation with self-consistent sources (4.5).

Therefore we see that the insertion of a source may cause the variation of the velocity of a soliton.

V. INTEGRATION OF THE KAUP-NEWELL HIERARCHY WITH SELF-CONSISTENT SOURCES

A. The Kaup–Newell hierarchy with self-consistent sources

Consider the Kaup–Newell (KN) spectral problem[24]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad U=\begin{pmatrix}-i\zeta^2 & \zeta q\\ \zeta r & i\zeta^2\end{pmatrix}, \quad u=\begin{pmatrix}q\\ r\end{pmatrix}. \tag{5.1}$$

Equations (2.2) and (2.3) yields

$$a_0=-i,\quad b_0=c_0=a_1=0,\quad b_1=q,\quad c_1=r,\quad a_2=-\tfrac{1}{2}iqr,\quad b_2=0,\quad c_2=0,\ldots,$$

$$a_{2k+1}=b_{2k}=c_{2k}=0,\quad k=0,1,\ldots,$$

and in general

$$\begin{pmatrix} c_{2k+1}\\ b_{2k+1}\end{pmatrix}=L\begin{pmatrix} c_{2k-1}\\ b_{2k-1}\end{pmatrix}=L^m\begin{pmatrix} r\\ q\end{pmatrix},\quad a_{2m,x}=qc_{2m+1}-rb_{2m+1},\tag{5.2}$$

where

$$L=\frac{1}{2}\begin{pmatrix} -iD+rD^{-1}qD & rD^{-1}rD\\ qD^{-1}qD & iD+qD^{-1}rD\end{pmatrix}.$$

The KN hierarchy is given by

$$u_{t_n}=\begin{pmatrix} q\\ r\end{pmatrix}_{t_n}=J\begin{pmatrix} c_{2n-1}\\ b_{2n-1}\end{pmatrix}=J\frac{\delta H_{2n-2}}{\delta u},\quad n=1,2,\ldots,\tag{5.3}$$

where

$$H_0=qr,\quad H_{2k}=\frac{1}{2k}(4ia_{2k+2}-rb_{2k+1}-qc_{2k+1}),\quad J=\begin{pmatrix} 0 & D\\ D & 0\end{pmatrix}.$$

We have

$$\frac{\delta\zeta}{\delta q}=\zeta\phi_2^2,\quad \frac{\delta\zeta}{\delta r}=-\zeta\phi_1^2,\quad L\begin{pmatrix}\phi_2^2\\ -\phi_1^2\end{pmatrix}=\zeta^2\begin{pmatrix}\phi_2^2\\ -\phi_1^2\end{pmatrix}.\tag{5.4}$$

For N distinct ζ_j, the KN hierarchy with self-consistent sources is given by[15]

$$\begin{pmatrix} q\\ r\end{pmatrix}_{t_n}=J\left[\frac{\delta H_{2n-2}}{\delta u}-\frac{1}{2}\sum_{j=1}^{N}\frac{\delta\zeta_j}{\delta u}\right]=J\left[\begin{pmatrix} c_{2n-1}\\ b_{2n-1}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\langle\Theta\Phi_2,\Phi_2\rangle\\ -\langle\Theta\Phi_1,\Phi_1\rangle\end{pmatrix}\right],\tag{5.5a}$$

$$\phi_{1j,x}=-i\zeta_j^2\phi_{1j}+\zeta_j q\phi_{2j},\quad \phi_{2j,x}=\zeta_j r\phi_{1j}+i\zeta_j^2\phi_{2j},\quad j=1,\ldots,N,\tag{5.5b}$$

where $\Theta=\text{diag}(\zeta_1,\ldots\ \zeta_N)$, and assume that

$$\lambda_j=\zeta_j^2,\quad \text{Im}\,\lambda_j=\text{Im}\,\zeta_j^2>0,\quad j=1,\ldots,N_1,$$

$$\overline{\lambda_j}=\zeta_j^2,\quad \text{Im}\,\overline{\lambda_j}=\text{Im}\,\zeta_j^2<0,\quad j=N_1+1,\ldots,N,$$

$$\zeta_j^2\neq\zeta_m^2,\quad j\neq m,\quad j,m=1,\ldots,N.\tag{5.5c}$$

The zero-curvature representation for the KN hierarchy with self-consistent sources (5.5) is given by (2.12) with the auxiliary linear problems

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix} -i\zeta^2 & \zeta q\\ \zeta r & i\zeta^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\tag{5.6a}$$

and

$$\psi_{1,t_n}=A^{(n)}\psi_{1j}+B^{(n)}\psi_{2j}\equiv\sum_{k=0}^{n-1}(a_{2k}\zeta^{2n-2k}\psi_1+b_{2k+1}\zeta^{2n-2k-1}\psi_2)+\eta\psi_1$$
$$+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{1j}(\zeta^2\zeta_j^2\phi_{2j}\psi_1-\zeta\zeta_j^3\phi_{1j}\psi_2),$$

$$\psi_{2,t_n}=C^{(n)}\psi_{1j}+D^{(n)}\psi_{2j}\equiv\sum_{k=0}^{n-1}(c_{2k+1}\zeta^{2n-2k-1}\psi_1-a_{2k}\zeta^{2n-2k}\psi_2)+\eta\psi_2$$
$$+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{2j}(\zeta\zeta_j^3\phi_{2j}\psi_1-\zeta^2\zeta_j^2\phi_{1j}\psi_2). \tag{5.6b}$$

B. Integration of the Kaup–Newell hierarchy with self-consistent sources

We now use the inverse scattering method to solve the initial-value problem for the KN hierarchy with self-consistent sources (5.5) in the sense described in Sec. II B. We assume that the KN spectral equation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-i\zeta^2 & \zeta q_0(x)\\ \zeta r_0(x) & i\zeta^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{5.7a}$$

has exactly N discrete eigenvalues which are same as that given by (5.5c). Let $\beta_j(t),j=1,\ldots,N$, be arbitrary continuous functions of t. Using the inverse scattering method, we construct the solution $q(x,t),r(x,t),\phi_{1j}(x,t),\phi_{2j}(x,t),j=1,\ldots,N$, of the system (5.5) in such way that

$$q(x,0)=q_0(x),\quad r(x,0)=r_0(x),$$

$$\beta_j(t)=\int_{-\infty}^{\infty}[i\phi_{1j,x}(x,t)\phi_{2j}(x,t)+\zeta_j^2\phi_{1j}(x,t)\phi_{2j}(x,t)]dx,\quad j=1,\ldots,N_1,$$

$$\beta_j(t)=-\int_{-\infty}^{\infty}[i\phi_{1j,x}(x,t)\phi_{2j}(x,t)+\zeta_j^2\phi_{1j}(x,t)\phi_{2j}(x,t)]dx,\quad j=N_1+1,\ldots,N. \tag{5.7b}$$

We define the eigenfunction $\psi^-(x,t,\zeta)$, $\overline{\psi}^-(x,t,\zeta)$, $\psi^+(x,t,\zeta)$, $\overline{\psi}^+(x,t,\zeta)$ for KN spectral equations (5.6a) and (5.6b) with the following boundary condition on Im $\zeta^2=0$[24]

$$\psi^-(x,t,\zeta)\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta^2x},\quad \overline{\psi}^-(x,t,\zeta)\sim\begin{pmatrix}0\\-1\end{pmatrix}e^{i\zeta^2x},\quad x\to-\infty, \tag{5.8a}$$

$$\psi^+(x,t,\zeta)\sim\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta^2x},\quad \overline{\psi}^+(x,t,\zeta)\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta^2x},\quad x\to\infty. \tag{5.8b}$$

Under the assumption for q,r,ϕ_{1j},ϕ_{2j}, for Im $\zeta^2=0$, since $\zeta^2\neq\zeta_j^2$, we have

$$\lim_{|x|\to\infty}N^{(n)}=\begin{pmatrix}-i\zeta^{2n}+\eta & 0\\ 0 & i\zeta^{2n}+\eta\end{pmatrix}. \tag{5.9}$$

Then we get

$$\eta^-=i\zeta^{2n},\quad \overline{\eta}^-=-i\zeta^{2n}.$$

Denote $\lambda=\zeta^2$, it was shown in Refs. 24 and 25 that $a(\zeta)$ and $\overline{a}(\zeta)$ are even functions of ζ and may be denoted by $a(\lambda)$ and $\overline{a}(\lambda)$, $b(\zeta)$ and $\overline{b}(\zeta)$ are odd functions of ζ. We define $\rho(\lambda)=b(\zeta)/\zeta a(\zeta), \overline{\rho}(\lambda)=\overline{b}(\zeta)/\zeta\overline{a}(\zeta)$. Then by substituting $\psi^-(x,t,\zeta), \eta^-(\overline{\psi}^-(x,t,\zeta),\overline{\eta}^-)$ into Eq. (5.6b), using (5.9) and (2.19a), let $x\to\infty$, it can be shown that on $\operatorname{Im}\lambda=\operatorname{Im}\zeta^2=0$,

$$\frac{\partial a}{\partial t}=0,\quad \frac{\partial b}{\partial t}=2i\zeta^{2n}b,\quad \frac{\partial \rho}{\partial t}=2i\lambda^n\rho, \tag{5.10a}$$

$$\frac{\partial \overline{a}}{\partial t}=0,\quad \frac{\partial \overline{b}}{\partial t}=-2i\zeta^{2n}\overline{b},\quad \frac{\partial \overline{\rho}}{\partial t}=-2i\lambda^n\overline{\rho}. \tag{5.10b}$$

That is, for KN hierarchy with self-consistent sources, the evolution of quantities $a(\zeta,t)$, $b(\zeta,t)$, $\overline{a}(\zeta,t)$, $\overline{b}(\zeta,t)$ is the same as that of the KN hierarchy without source.

It is known[24,25] that function $a(\lambda,t)(\overline{a}(\lambda,t))$ admits an analytical continuation into the upper half (lower half) plane of λ. The KN spectral equation (5.6a) can possess discrete eigenvalues which occur whenever $a(\lambda,t)$ has zeros in the upper half plane and whenever $\overline{a}(\lambda,t)$ has zeros in the lower half plane. Equation (5.10) indicate that the discrete eigenvalues don't depend on t. According to the assumption for q,r,ϕ_{1j},ϕ_{2j}, and (5.7a), the zeros of $a(\lambda,t)$ are $\lambda_j=\zeta_j^2$, $j=1,\ldots,N_1$ and the zeros of $\overline{a}(\lambda,t)$ are $\overline{\lambda}_j=\zeta_j^2$, $j=N_1+1,\ldots,N$. At λ_j and $\overline{\lambda}_j$ the following equalities for the discrete eigenfunctions hold:

$$\psi^-(x,t,\zeta_m)=C_m(t)\psi^+(x,t,\zeta_m),\quad m=1,\ldots,N_1, \tag{5.11a}$$

$$\overline{\psi}^-(x,t,\zeta_m)=\overline{C}_m(t)\overline{\psi}^+(x,t,\zeta_m),\quad m=N_1+1,\ldots,N. \tag{5.11b}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\psi^-(x,t,\zeta_j),\quad j=1,\ldots,N_1. \tag{5.11c}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\overline{\psi}^-(x,t,\zeta_j),\quad j=N_1+1,\ldots,N. \tag{5.11d}$$

It is found from Eqs. (5.5b) and (5.6a) that

$$\begin{aligned}&\zeta\zeta_j^3\phi_{2j}(x,t)\psi_1^-(x,t,\zeta)-\zeta^2\zeta_j^2\phi_{1j}(x,t)\psi_2^-(x,t,\zeta)\\ &\quad=(\zeta^2-\zeta_j^2)\zeta_j^2\int_{-\infty}^{x}[-\psi_{2,x}^-(z,t,\zeta)\phi_{1j}(z,t)-i\zeta\zeta_j\psi_1^-(z,t,\zeta)\phi_{2j}(z,t)]dz,\end{aligned} \tag{5.12a}$$

$$\begin{aligned}&\zeta^2\zeta_j^2\phi_{2j}(x,t)\overline{\psi_1^-}(x,t,\zeta)-\zeta\zeta_j^3\phi_{1j}(x,t)\overline{\psi_2^-}(x,t,\zeta)\\ &\quad=(\zeta^2-\zeta_j^2)\zeta_j^2\int_{-\infty}^{x}[\overline{\psi_{1,x}^-}(z,t,\zeta)\phi_{2j}(z,t)-i\zeta\zeta_j\overline{\psi_2^-}(z,t,\zeta)\phi_{1j}(z,t)]dz,\end{aligned} \tag{5.12b}$$

which give rise to

$$\int_{-\infty}^{\infty}[\phi_{1j}(x,t)\psi_{2,x}^-(x,t,\zeta_m)+i\zeta_j\zeta_m\phi_{2j}(x,t)\psi_1^-(x,t,\zeta_m)]dx=0,\quad j\neq m,\quad m=1,\ldots,N_1, \tag{5.13a}$$

$$\int_{-\infty}^{\infty}[\phi_{2j}(x,t)\overline{\psi_{1,x}^-}(x,t,\zeta_m)-i\zeta_j\zeta_m\phi_{1j}(x,t)\overline{\psi_2^-}(x,t,\zeta_m)]dx=0,\quad j\neq m,\quad m=N_1+1,\ldots,N. \tag{5.13b}$$

It is found that

$$0=\frac{1}{\zeta_j}\int_{-\infty}^{\infty}[\psi_1(x,t,\zeta_j)\psi_2(x,t,\zeta_j)]_x dx=\int_{-\infty}^{\infty}[q\psi_2^2(x,t,\zeta_j)+r\psi_1^2(x,t,\zeta_j)]dx,$$

$$\zeta[q\psi_2^2(x,t,\zeta)-r\psi_1^2(x,t,\zeta)]$$
$$=\psi_{1,x}(x,t,\zeta)\psi_2(x,t,\zeta)-\psi_1(x,t,\zeta)\psi_{2,x}(x,t,\zeta)+2i\zeta^2\psi_1(x,t,\zeta)\psi_2(x,t,\zeta),$$

and

$$\frac{da(\lambda_j)}{d\zeta}=\frac{2C_j}{\zeta_j}\int_{-\infty}^{\infty}[\psi_{1x}^{+}(x,t,\zeta_j)\psi_2^{+}(x,t,\zeta_j)-i\zeta_j^2\psi_1^{+}(x,t,\zeta_j)\psi_2^{+}(x,t,\zeta_j)]dx,$$

$$j=1,\ldots,N_1,$$

$$\frac{d\overline{a}(\overline{\lambda}_j)}{d\zeta}=\frac{2\overline{C}_j}{\zeta_j}\int_{-\infty}^{\infty}[\overline{\psi_{1x}^{+}}(x,t,\zeta_j)\overline{\psi_2^{+}}(x,t,\zeta_j)-i\zeta_j^2\overline{\psi_1^{+}}(x,t,\zeta_j)\overline{\psi_2^{+}}(x,t,\zeta_j)]dx,$$

$$j=N_1+1,\ldots,N.$$

One obtains from (5.12)

$$\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{2j}(x,t)[\zeta\zeta_j^3\phi_{2j}(x,t)\psi_1^{-}(x,t,\zeta)-\zeta^2\zeta_j^2\phi_{1j}(x,t)\psi_2^{-}(x,t,\zeta)]$$
$$\sim\zeta_m^2\alpha_m(t)C_m(t)\psi_2^{+}(x,t,\zeta_m)\int_{-\infty}^{\infty}[\psi_2^{-}(z,t,\zeta_m)\phi_{1m,z}(z,t)-i\zeta_m^2\psi_1^{-}(z,t,\zeta_m)\phi_{2m}(z,t)]dz$$
$$=\zeta_m^2C_m(t)\psi_2^{+}(x,t,\zeta_m)\left\{\int_{-\infty}^{\infty}[\phi_{1m,z}(z,t)\phi_{2m}(z,t)-i\zeta_m^2\phi_{1m}(z,t)\phi_{2m}(z,t)]dz\right\}$$
$$=-i\zeta_m^2\beta_m(t)C_m(t)\psi_2^{+}(x,t,\zeta_m),\quad x\to\infty,\quad m=1,\ldots,N_1, \tag{5.14a}$$

$$\lim_{\zeta\to\zeta_m}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{1j}[\zeta^2\zeta_j^2\phi_{2j}(x,t)\overline{\psi_1^{-}}(x,t,\zeta)-\zeta\zeta_j^3\phi_{1j}(x,t)\overline{\psi_2^{-}}(x,t,\zeta)]$$
$$\sim i\zeta_m^2\beta_m(t)\overline{C}_m(t)\overline{\psi_1^{+}}(x,t,\zeta_m),\quad x\to\infty,\quad m=N_1+1,\ldots,N. \tag{5.14b}$$

Also we have

$$\eta_m^{-}=i\zeta_m^{2n},\quad m=1,\ldots,N_1,\quad \overline{\eta_m^{-}}=-i\zeta_m^{2n},\quad m=N_1+1,\ldots,N. \tag{5.15}$$

Then one gets

$$\frac{dC_m}{dt}=\left[2i\zeta_m^{2n}-\frac{1}{2}i\zeta_m^2\beta_m(t)\right]C_m(t),\quad m=1,\ldots,N_1, \tag{5.16a}$$

$$\frac{d\overline{C}_m}{dt}=-\left[2i\zeta_m^{2n}-\frac{1}{2}i\zeta_m^2\beta_m(t)\right]\overline{C}_m(t),\quad m=N_1+1,\ldots,N. \tag{5.16b}$$

The normalization constants c_m, $\overline{c_m}$ are defined by

$$c_m(t)\equiv\left[2\int_{-\infty}^{\infty}[i\psi_{1,x}^{+}(x,t,\zeta_m)\psi_2^{+}(x,t,\zeta_m)+\zeta_j^2\psi_1^{+}(x,t,\zeta_m)\psi_2^{+}(x,t,\zeta_m)]dx\right]^{-1}$$

$$=-iC_m(t)\left(\zeta_m\frac{da(\lambda_m)}{d\zeta}\right)^{-1},\quad m=1,\ldots,N_1, \tag{5.17a}$$

$$\overline{c_m}(t)\equiv\left[2\int_{-\infty}^{\infty}[i\overline{\psi_{1,x}^{+}}(x,t,\zeta_m)\overline{\psi_2^{+}}(x,t,\zeta_m)+\zeta_j^2\overline{\psi_1^{+}}(x,t,\zeta_m)\overline{\psi_2^{+}}(x,t,\zeta_m)]dx\right]^{-1}$$

$$=-i\overline{C}_m(t)\left(\zeta_m\frac{d\overline{a}(\overline{\lambda_m})}{d\zeta}\right)^{-1},\quad m=N_1+1,\ldots,N, \tag{5.17b}$$

one obtains

$$\frac{dc_m}{dt}=\left[2i\zeta_m^{2n}-\frac{1}{2}i\zeta_m^2\beta_m(t)\right]c_m,\quad m=1,\ldots,N_1, \tag{5.18a}$$

$$\frac{d\overline{c_m}}{dt}=\left[-2i\zeta_m^{2n}+\frac{1}{2}i\zeta_m^2\beta_m(t)\right]\overline{c_m},\quad m=N_1+1,\ldots,N. \tag{5.18b}$$

Thus, the evolution of $c_m(t)$ and $\overline{c_m}(t)$ has an extra term $-i\zeta_m^2\beta_m(t)c_m(t)$ and $i\zeta_m^2\beta_m(t)\overline{c_m}(t)$ comparing with that of the KN hierarchy without source.

It is found from, (5.7b), (5.11), and (5.17) that

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{2\beta_j(t)c_j(t)}\,\psi^{+}(x,t,\zeta_j),\quad j=1,\ldots,N_1, \tag{5.19a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\,\overline{\psi}^{+}(x,t,\zeta_j),\quad j=N_1+1,\ldots,N, \tag{5.19b}$$

which is consistent with (5.7b).

According to Refs. 24 and 25, using Eqs. (5.10) and (5.18) and solving the Gel'fand–Levitan–Marchenko equation, we can get the solution of the nth KN equation with self-consistent sources (5.5) in the following way:

$$q(x,t)e^{2i\mu^{+}(x)}=-2K_1(x,x),\quad r(x,t)e^{-2i\mu^{+}(x)}=-2\overline{K_2}(x,x), \tag{5.20a}$$

$$\phi_{1j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\int_x^{\infty}e^{i\zeta_j^2 s}K_1(x,s)\zeta_j e^{-i\mu^{+}(x)}ds, \tag{5.20b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\left(e^{i[\zeta_j^2x+\mu^{+}(x)]}+\int_x^{\infty}e^{i\zeta_j^2 s}K_2(x,s)e^{i\mu^{+}(x)}ds\right),\quad j=1,\ldots,N_1, \tag{5.20c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\left(e^{i[-\zeta_j^2x-\mu^{+}(x)]}+\int_x^{\infty}e^{-i\zeta_j^2 s}\overline{K_1}(x,s)e^{-i\mu^{+}(x)}ds\right), \tag{5.20d}$$

$$\phi_{2j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\int_x^{\infty}e^{-i\zeta_j^2 s}\overline{K_2}(x,s)\zeta_j e^{i\mu^{+}(x)}ds,\quad j=N_1+1,\ldots,N, \tag{5.20e}$$

where $\mu^{+}(x)=\frac{1}{2}\int_x^{\infty}r(s)q(s)ds=2\int_x^{\infty}K_1(s,s)\overline{K_2}(s,s)ds$, $K_1(x,y)$, $K_2(x,y)$, and $\overline{K_1}(x,y)$, $\overline{K_2}(x,y)$ satisfy $(y>x)$

$$\overline{K_1}(x,y)-i\int_x^\infty K_1(x,s)F'(s+y)ds=0, \tag{5.21a}$$

$$\overline{K_2}(x,y)+F(x+y)+\int_x^\infty K_2(x,s)F(s+y)ds=0, \tag{5.21b}$$

$$-K_1(x,y)+\overline{F}(x+y)+\int_x^\infty \overline{K_1}(x,s)\overline{F}(s+y)ds=0, \tag{5.21c}$$

$$-K_2(x,y)+i\int_x^\infty \overline{K_2}(x,s)\overline{F'}(s+y)ds=0, \tag{5.21d}$$

and

$$F(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\lambda)e^{i\lambda z}d\lambda+\sum_{j=1}^{N_1}c_j(t)e^{i\lambda_j z},$$

$$\overline{F}(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\rho}(\lambda)e^{-i\lambda z}d\lambda+\sum_{j=N_1+1}^{N}\overline{c_j}(t)e^{-i\overline{\lambda}_j z}.$$

We now present the exploding soliton for system (5.5). Assume that $\rho(\lambda)=\overline{\rho}(\lambda)=0,N_1=1,N=2$, namely there is one discrete eigenvalue $\lambda=\zeta_1^2$ in the upper half plane and one discrete eigenvalue $\overline{\lambda}=\zeta_2^2$ in the lower half plane. We have

$$F(z)=c(t)e^{i\lambda x},\quad \overline{F}(z)=\overline{c}(t)e^{-i\overline{\lambda}x},$$

where $c(t),\overline{c}(t)$ are found from (5.18)

$$c(t)=c(0)\exp\left[2i\zeta_1^{2n}-\frac{1}{2}i\zeta_1^2\int_0^t\beta_1(z)dz\right],\quad \overline{c}(t)=\overline{c}(0)\exp\left[-2i\zeta_2^{2n}+\frac{1}{2}i\zeta_2^2\int_0^t\beta_2(z)dz\right]. \tag{5.22}$$

Solving Eq. (5.21) leads to

$$K_1(x,y)=\frac{1}{\Delta_1}\overline{c}(t)e^{-i\overline{\lambda}(x+y)},\quad K_2(x,y)=-\frac{1}{i(\lambda-\overline{\lambda})\Delta_2}c(t)\overline{c}(t)\overline{\lambda}e^{2i\lambda x-i(\overline{\lambda}x-\lambda y)}, \tag{5.23a}$$

$$\overline{K_1}(x,y)=\frac{1}{i(\lambda-\overline{\lambda})\Delta_1}c(t)\overline{c}(t)\lambda e^{-2i\overline{\lambda}x+i(\lambda x-\overline{\lambda}y)},\quad \overline{K_2}(x,y)=\frac{1}{\Delta_2}c(t)e^{i\lambda(x+y)}, \tag{5.23b}$$

$$\Delta_1=1-\frac{1}{(\lambda-\overline{\lambda})^2}c(t)\overline{c}(t)\lambda e^{2i(\lambda-\overline{\lambda})x},\quad \Delta_2=-1+\frac{1}{(\lambda-\overline{\lambda})^2}c(t)\overline{c}(t)\overline{\lambda}e^{2i(\lambda-\overline{\lambda})x}. \tag{5.23c}$$

Then we obtain the solution by means of Eq. (5.20)

$$q(x,t)e^{2i\mu^+(x)}=-\frac{2}{\Delta_1}\overline{c}(t)e^{-2i\overline{\lambda}x},\quad r(x,t)e^{-2i\mu^+(x)}=-\frac{2}{\Delta_2}c(t)e^{2i\lambda x}, \tag{5.24a}$$

$$\phi_{11}(x,t)=i\sqrt{2\beta_1(t)c(t)}\frac{1}{(\lambda-\bar{\lambda})\Delta_1}\zeta\overline{c(t)}e^{i(\lambda-2\bar{\lambda})x-i\mu^+(x)}, \tag{5.24b}$$

$$\phi_{21}(x,t)=\sqrt{2\beta_1(t)c(t)}\left(e^{i[\zeta^2x+\mu^+(x)]}-\frac{1}{2\lambda(\lambda-\bar{\lambda})\Delta_2}c(t)\overline{c(t)}\bar{\lambda}e^{i(4\lambda-\bar{\lambda})x+i\mu^+(x)}\right), \tag{5.24c}$$

$$\phi_{12}(x,t)=\sqrt{-2\beta_2(t)\overline{c(t)}}\left(e^{i[-\bar{\zeta}^2x-\mu^+(x)]}-\frac{1}{2\bar{\lambda}(\lambda-\bar{\lambda})\Delta_1}c(t)\overline{c(t)}\lambda e^{i(\lambda-4\bar{\lambda})x-i\mu^+(x)}\right), \tag{5.24d}$$

$$\phi_{22}(x,t)=i\sqrt{-2\beta_2(t)\overline{c(t)}}\;\frac{1}{(\lambda-\bar{\lambda})\Delta_2}\bar{\zeta}c(t)e^{i(2\lambda-\bar{\lambda})x+i\mu^+(x)}, \tag{5.24e}$$

where $c(t),\overline{c(t)}$ are given by (5.22) and

$$\mu^+(x)=i\ln\frac{\Delta_1}{-\Delta_2}.$$

VI. INTEGRATION OF THE DNLSE HIERARCHY WITH SELF-CONSISTENT SOURCES

A. The DNLSE hierarchy with self-consistent sources

Consider the reduced case of the Kaup–Newell spectral problem for $r=-q^*$[24]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}-i\zeta^2 & \zeta q\\ -\zeta q^* & i\zeta^2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{6.1}$$

Solving the adjoint representation of the above problem yields

$$a_0=-i,\quad b_0=c_0=a_1=0,\quad b_1=q,\quad c_1=-q^*,$$

$$a_2=\tfrac{1}{2}iqq^*,\quad b_2=c_2=a_3=0,\ldots$$

and in general

$$a_{2m+1}=b_{2m}=c_{2m}=0,\quad c_{2m+1}=-b^*_{2m+1},\ \ldots,$$

$$\begin{pmatrix}c_{2m+1}\\ b_{2m+1}\end{pmatrix}=L\begin{pmatrix}c_{2m-1}\\ b_{2m-1}\end{pmatrix},\quad a_{2m,x}=qc_{2m+1}+q^*b_{2m+1}, \tag{6.2}$$

where

$$L=\frac{1}{2}\begin{pmatrix}-iD-q^*D^{-1}qD & q^*D^{-1}q^*D\\ qD^{-1}qD & iD-qD^{-1}q^*D\end{pmatrix}.$$

The hierarchy of the derivative nonlinear Schrödinger hierarchy (DNLSE) reads

$$q_{t_n}=Db_{2n-1}=-D\left(\frac{\delta H_{2n-2}}{\delta q}\right)^*, \tag{6.3}$$

where

$$H_0=-qq^*,\quad H_{2n}=\frac{1}{2n}(4ia_{2n+2}+q^*b_{2n+1}-qc_{2n+1}),\quad n=1,\ldots .$$

It is easy to verify that

$$L^m\begin{pmatrix}\zeta\phi_2^2+\zeta^*\phi_1^{*2}\\ -\zeta\phi_1^2-\zeta^*\phi_2^{*2}\end{pmatrix}=\begin{pmatrix}\zeta^{2m+1}\phi_2^2+\zeta^{*2m+1}\phi_1^{*2}\\ -\zeta^{2m+1}\phi_1^2-\zeta^{*2m+1}\phi_2^{*2}\end{pmatrix},\quad m=1,2,\ldots . \tag{6.4}$$

The DNLSE hierarchy with self-consistent sources is defined by

$$q_{t_n}=D[b_{2n-1}+\tfrac{1}{2}(\langle\Theta\Phi_1,\Phi_1\rangle+\langle\Theta^*\Phi_2^*,\Phi_2^*\rangle)], \tag{6.5a}$$

$$\phi_{1j,x}=-i\zeta_j^2\phi_{1j}+\zeta_j q\phi_{2j},\quad \phi_{2j,x}=-\zeta_j q^*\phi_{1j}+i\zeta_j^2\phi_{2j},\quad j=1,\ldots,N, \tag{6.5b}$$

and assume that

$$\lambda_j=\zeta_j^2,\quad \mathrm{Im}\,\lambda_j=\mathrm{Im}\,\zeta_j^2>0,\quad j=1,\ldots,N_1,$$

$$\overline{\lambda_j}=\zeta_j^2,\quad \mathrm{Im}\,\overline{\lambda_j}=\mathrm{Im}\,\zeta_j^2<0,\quad j=N_1+1,\ldots,N. \tag{6.5c}$$

The zero-curvature representation for the DNLSE hierarchy with self-consistent sources (6.5) is given by (2.11) with the auxiliary linear problems

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-i\zeta^2 & \zeta q\\ -\zeta q^* & i\zeta^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{6.6a}$$

and

$$\begin{aligned}\psi_{1,t_n}=A^{(n)}\psi_1+B^{(n)}\psi_2\equiv&\sum_{k=0}^{n-1}(a_{2k}\zeta^{2n-2k}\psi_1+b_{2k+1}\zeta^{2n-2k-1}\psi_2)+\eta\psi_1\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{1j}(\zeta^2\zeta_j^2\phi_{2j}\psi_1-\zeta\zeta_j^3\phi_{1j}\psi_2)\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^{*2}}\phi_{2j}^*(-\zeta^2\zeta_j^{*2}\phi_{1j}^*\psi_1-\zeta\zeta_j^{*3}\phi_{2j}^*\psi_2),\\ \psi_{2,t_n}=C^{(n)}\psi_1+D^{(n)}\psi_2\equiv&\sum_{k=0}^{n-1}(c_{2k+1}\zeta^{2n-2k-1}\psi_1-a_{2k}\zeta^{2n-2m}\psi_2)+\eta\psi_2\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{2j}(\zeta\zeta_j^3\phi_{2j}\psi_1-\zeta^2\zeta_j^2\phi_{1j}\psi_2)\\ &+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^{*2}}\phi_{1j}^*(\zeta\zeta_j^{*3}\phi_{1j}^*\psi_1+\zeta^2\zeta_j^{*2}\phi_{2j}^*\psi_2).\end{aligned} \tag{6.6b}$$

When $n=2$, Eq. (6.5) gives the DNLSE equation with self-consistent sources

$$q_{t_2}=\tfrac{1}{2}iq_{xx}-\tfrac{1}{2}(q^*q^2)_x+\tfrac{1}{2}D(\langle\Theta\Phi_1,\Phi_1\rangle+\langle\Theta^*\Phi_2^*,\Phi_2^*\rangle), \tag{6.7a}$$

$$\phi_{1j,x}=-i\zeta_j^2\phi_{1j}+\zeta_j q\phi_{2j},\quad \phi_{2j,x}=-\zeta_j q^*\phi_{1j}+i\zeta_j^2\phi_{2j},\quad j=1,\ldots,N, \tag{6.7b}$$

and the auxiliary linear problem (6.6b) reads

$$\psi_{1,t_2}=\left(-i\zeta^4+\frac{1}{2}i\zeta^2qq^*+\eta\right)\psi_1+\left(q\zeta^3-\frac{1}{2}\zeta q^*q^2+\frac{1}{2}i\zeta q_x\right)\psi_2+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{1j}(\zeta^2\zeta_j^2\phi_{2j}\psi_1$$

$$-\zeta\zeta_j^3\phi_{1j}\psi_2)+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^{*2}}\phi_{2j}^*(-\zeta^2\zeta_j^{*2}\phi_{1j}^*\psi_1-\zeta\zeta_j^{*3}\phi_{2j}^*\psi_2),$$

$$\psi_{2,t_2}=\left(-q^*\zeta^3+\frac{1}{2}\zeta qq^{*2}+\frac{1}{2}i\zeta q_x^*\right)\psi_1+\left(i\zeta^4-\frac{1}{2}i\zeta^2qq^*+\eta\right)\psi_2+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^2}\phi_{2j}(\zeta\zeta_j^3\phi_{2j}\psi_1$$

$$-\zeta^2\zeta_j^2\phi_{1j}\psi_2)+\frac{1}{2}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_j^{*2}}\phi_{1j}^*(\zeta\zeta_j^{*3}\phi_{1j}^*\psi_1+\zeta^2\zeta_j^{*2}\phi_{2j}^*\psi_2). \tag{6.8}$$

B. Integration of the DNLSE hierarchy with self-consistent sources

We now integrate the initial-value problem for the DNLSE hierarchy with self-consistent sources (6.5) by the inverse scattering method under the assumptions in the last section and the assumption that the spectral equation

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-i\zeta^2 & \zeta q_0(x)\\ -\zeta q_0^*(x) & i\zeta^2\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{6.9a}$$

have $2N$ discrete eigenvalues given by (6.5c) in the following way

$$\lambda_j=\xi_j^2,\quad \overline{\lambda}_j=\overline{\xi}_j^2,\quad j=1,\dots,N, \tag{6.9b}$$

where

$$\xi_j=\zeta_j,\quad j=1,\dots,N_1,\quad \xi_j=\zeta_j^*,\quad j=N_1+1,\dots,N,$$

$$\overline{\xi}_j=\zeta_j^*,\quad j=1,\dots,N_1,\quad \overline{\xi}_j=\zeta_j,\quad j=N_1+1,\dots,N.$$

We have

$$\eta^-=i(\zeta)^{2n},\quad \overline{\eta}^-=-i(\zeta)^{2n}.$$

For $r=-q^*$, it is known[24] that

$$\overline{\psi}^-(x,t,\zeta)=\begin{pmatrix}\psi_2^{-*}(x,t,\zeta^*)\\ -\psi_1^{-*}(x,t,\zeta^*)\end{pmatrix},\quad \overline{\psi}^+(x,t,\zeta)=\begin{pmatrix}\psi_2^{+*}(x,t,\zeta^*)\\ -\psi_1^{+*}(x,t,\zeta^*)\end{pmatrix}, \tag{6.10a}$$

which imply that the a, $\overline{a}$, b, $\overline{b}$ satisfy

$$\overline{a}(\zeta,t)=a^*(\zeta^*,t),\quad \overline{b}(\zeta,t)=b^*(\zeta^*,t). \tag{6.10b}$$

In the same way as in the previous section, one can show that

$$\frac{\partial a}{\partial t}=0,\quad \frac{\partial b}{\partial t}=2i\zeta^{2n}b,\quad \frac{\partial \rho}{\partial t}=2i\zeta^{2n}\rho. \tag{6.11}$$

So for the DNLSE hierarchy with self-consistent sources, the evolution of quantities $a(\zeta,t)$ and $b(\zeta,t)$ is the same as that of the DNLSE hierarchy without source. The discrete eigenvalues of spectral problem (6.6a) are given by the zeros of $a(\zeta,t)$ and $\overline{a(\zeta,t)}$ and independent of t according to (6.11). So the zeros of $a(\zeta,t)$ and $\overline{a(\zeta,t)}$ are just ξ_j and $\overline{\xi_j}$, respectively. At ξ_j and $\overline{\xi_j}$ the following equalities for the discrete eigenfunctions hold

$$\psi^-(x,t,\xi_m)=C_m(t)\psi^+(x,t,\xi_m),\quad \overline{\psi}^-(x,t,\overline{\xi}_m)=\overline{C}_m(t)\overline{\psi}^+(x,t,\overline{\xi}_m),\quad m=1,\ldots,N, \tag{6.12}$$

which together with (6.10a) yield

$$C_m=\overline{C}_m^*,\quad m=1,\ldots,N. \tag{6.13}$$

Also we have

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\psi^-(x,t,\zeta_j),\quad j=1,\ldots,N_1, \tag{6.14a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\alpha_j(t)\overline{\psi}^-(x,t,\zeta_j),\quad j=N_1+1,\ldots,N. \tag{6.14b}$$

It is found from Eqs. (6.5b) and (6.6a) that

$$\zeta\zeta_j^3\phi_{2j}(x,t)\psi_1^-(x,t,\zeta)-\zeta^2\zeta_j^2\phi_{1j}(x,t)\psi_2^-(x,t,\zeta)$$
$$=(\zeta^2-\zeta_j^2)\zeta_j^2\int_{-\infty}^x[\phi_{1j,z}(z,t)\psi_2^-(z,t,\zeta)-i\zeta\zeta_j\phi_{2j}(z,t)\psi_1^-(z,t,\zeta)]dz, \tag{6.15a}$$

$$\zeta\zeta_j^{*3}\phi_{1j}^*(x,t)\psi_1^-(x,t,\zeta)+\zeta^2\zeta_j^{*2}\phi_{2j}^*(x,t)\psi_2^-(x,t,\zeta)$$
$$=-(\zeta^2-\zeta_j^{*2})\zeta\zeta_j^*\int_{-\infty}^x[\phi_{1j,x}^*(z,t)\psi_1^-(z,t,\zeta)-i\zeta\zeta_j^*\phi_{2j}^*(z,t)\psi_2^-(z,t,\zeta)]dz,$$
$$j=1,\ldots,N, \tag{6.15b}$$

which together with (6.10a) gives rise to

$$\int_{-\infty}^{\infty}[\phi_{1j,z}(z,t)\psi_2^-(z,t,\xi_m)-i\zeta_m\xi_j\phi_{2j}(z,t)\psi_1^-(z,t,\xi_m)]dz=0,\quad j\neq m,\quad m=1,\ldots,N, \tag{6.16a}$$

$$\int_{-\infty}^{\infty}[\phi_{1j,z}^*(z,t)\psi_1^-(z,t,\xi_m)-i\zeta_m\xi_j^*\phi_{2j}^*(z,t)\psi_2^-(z,t,\xi_m)]dz=0,\quad j\neq m,m=1,\ldots,N, \tag{6.16b}$$

and

$$\lim_{\zeta\to\zeta_m}\sum_{j=1}^N\frac{1}{\zeta^2-\zeta_j^2}\phi_{2j}[\zeta\zeta_j^3\phi_{2j}(x,t)\psi_1^-(x,t,\zeta)-\zeta^2\zeta_j^2\phi_{1j}(x,t)\psi_2^-(x,t,\zeta)]$$
$$\sim\alpha_m(t)C_m(t)\zeta_m^2\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}[\phi_{1m,z}(z,t)\psi_2^-(z,t,\zeta_m)-i\zeta_m^2\phi_{2m}(z,t)\psi_1^-(z,t,\zeta_m)]dz$$
$$=\zeta_m^2C_m(t)\psi_2^+(x,t,\zeta_m)\int_{-\infty}^{\infty}[\phi_{1m,z}(z,t)\phi_{2m}(z,t)-i\zeta_m^2\phi_{2m}(z,t)\phi_{1m}(z,t)]dz$$
$$\text{for }x\to\infty,\quad m=1,\ldots,N_1, \tag{6.17a}$$

$$\lim_{\zeta\to\zeta_m^*}\sum_{j=1}^{N}\frac{1}{\zeta^2-\zeta_m^{*2}}\phi_{1j}^*[\zeta\zeta_j^{*3}\phi_{1m}^*(x,t)\psi_1^-(x,t,\zeta)+\zeta^2\zeta_m^{*2}\phi_{2m}^*(x,t)\psi_2^-(x,t,\zeta)]$$

$$\sim-\zeta_m^{*2}\alpha_m^*(t)(\overline{\psi}_1^+(x,t,\zeta_m^*))^*\int_{-\infty}^{\infty}[\psi_{1,z}^-(z,t,\zeta_m^*)\phi_{1m}^*(z,t)-i\zeta_m^{*2}\psi_2^-(z,t,\zeta_m^*)\phi_{2m}^*(z,t)]dz$$

$$=\zeta_m^{*2}C_m(t)\psi_2^+(x,t,\zeta_m^*)\int_{-\infty}^{\infty}[\phi_{1m,z}^*(z,t)\phi_{2m}^*(z,t)+i\zeta_m^{*2}\phi_{1m}^*(z,t)\phi_{2m}^*(z,t)]dz,$$

$$\text{for }x\to\infty,\quad m=N_1+1,\dots,N, \tag{6.17b}$$

Eq. (6.16) are the orthogonal property of the discrete eigenfunctions. Also one gets

$$\eta_m^-=i\zeta_m^{2n},\quad m=1,\dots,N_1,\quad \overline{\eta_m^-}=i(\zeta_m^*)^{2n},\quad m=N_1+1,\dots,N. \tag{6.18}$$

Using the $\beta_j(t)$ defined by (5.7b), then we have

$$\frac{dC_m}{dt}=\left[2i\zeta_m^{2n}-\frac{1}{2}i\zeta_m^2\beta_m(t)\right]C_m,\quad m=1,\dots,N_1, \tag{6.19a}$$

$$\frac{dC_m}{dt}=\left[-2i\zeta_m^{*2n}-\frac{1}{2}i\zeta_m^{*2}\beta_m^*(t)\right]C_m,\quad m=N_1+1,\dots,N, \tag{6.19b}$$

which leads to

$$\frac{dc_m}{dt}=\left[2i\zeta_m^{2n}-\frac{1}{2}i\zeta_m^2\beta_m(t)\right]c_m,\quad m=1,\dots,N_1, \tag{6.20a}$$

$$\frac{dc_m}{dt}=\left[-2i\zeta_m^{*2n}-\frac{1}{2}i\zeta_m^{*2}\beta_m^*(t)\right]c_m,\quad m=N_1+1,\dots,N, \tag{6.20b}$$

or

$$c_m(t)=c_m(0)\exp\left[2i\zeta_m^{2n}-\tfrac{1}{2}i\zeta_m^2\int_0^t\beta_m(z)dz\right],\quad m=1,\dots,N_1, \tag{6.21a}$$

$$c_m(t)=c_m(0)\exp\left[-2i\zeta_m^{*2n}-\frac{1}{2}i\zeta_m^{*2}\int_0^t\beta_m^*(z)dz\right],\quad m=N_1+1,\dots,N, \tag{6.21b}$$

where $c_m(t)=-\overline{c_m^*}(t)$ are defined by (5.17). Thus, the evolution of $c_m(t)$ has an extra term $-i\zeta_m^2\beta_m(t)c_m$ or $-i\zeta_m^{*2}\beta_m^*(t)c_m$ comparing with that of the DNLSE hierarchy without source.

According to the definition of $\beta_j(t)$ and (6.14) lead to

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{2\beta_j(t)c_j(t)}\,\psi^+(x,t,\zeta_j),\quad j=1,\dots,N_1, \tag{6.22a}$$

$$\begin{pmatrix}\phi_{1j}(x,t)\\ \phi_{2j}(x,t)\end{pmatrix}=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\,\overline{\psi}^+(x,t,\zeta_j),\quad j=N_1+1,\dots,N, \tag{6.22b}$$

which are consistent with the definition of $\beta_j(t)$. It is known[24] that for $r=-q^*$, one has

$$\overline{F}(x)=F^*(x),\quad \overline{K}(x,y)=\begin{pmatrix}K_2^*(x,y)\\ -K_1^*(x,y)\end{pmatrix}. \tag{6.23}$$

Then according to Eqs. (6.11) and (6.21), we can get the solution of the nth DNLSE equation with self-consistent sources (6.5) in the following way:

$$q(x,t)e^{2i\mu^+(x)}=-2K_1(x,x), \quad \mu^+(x)=-\frac{1}{2}\int_x^\infty |q|^2 dx=-2\int_x^\infty K_1(x,x)K_1^*(x,x)dx, \tag{6.24a}$$

$$\phi_{1j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\int_x^\infty e^{i\zeta_j^2 s}\zeta_j K_1(x,s)e^{-i\mu^+(x)}ds, \quad j=1,\ldots,N_1, \tag{6.24b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}\left(e^{i[\zeta_j^2 x+\mu^+(x)]}+\int_x^\infty e^{i\zeta_j^2 s}K_2(x,s)e^{i\mu^+(x)}ds\right), \quad j=1,\ldots,N_1, \tag{6.24c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}\left(e^{i[-\zeta_j^2 x-\mu^+(x)]}+\int_x^\infty e^{-i\zeta_j^2 s}K_2^*(x,s)e^{-i\mu^+(x)}ds\right), \quad j=N_1+1,\ldots,N, \tag{6.24d}$$

$$\phi_{2j}(x,t)=-\sqrt{-2\beta_j(t)\overline{c_j}(t)}\int_x^\infty e^{-i\zeta_j^2 s}\zeta_j K_1^*(x,s)e^{i\mu^+(x)}ds, \quad j=N_1+1,\ldots,N, \tag{6.24e}$$

where $K(x,y)=(K_1(x,y),K_2(x,y))^T$, satisfy

$$K_2^*(x,y)-i\int_x^\infty K_1(x,s)F'(s+y)ds=0, \quad y>x, \tag{6.25a}$$

$$-K_1(x,y)+F^*(x+y)+\int_x^\infty K_2^*(x,s)F^*(s+y)ds=0, \quad y>x, \tag{6.25b}$$

and

$$F(z)=\frac{1}{2\pi}\int_{-\infty}^\infty \rho(\lambda)e^{i\lambda z}d\lambda-\sum_{j=1}^N c_j(t)e^{i\lambda_j z}. \tag{6.26}$$

C. The *N* soliton solution of the DNLSE hierarchy with self-consistent sources

For the nth DNLSE with self-consistent sources (6.4), assume $\rho(\zeta)=0$ and there are $2N$ distinct eigenvalues λ_j and $\overline{\lambda_j}$, $j=1,\ldots,N$, given by (6.9b). Denote

$$E(x,t)=(-c_1(t)e^{i\lambda_1 x}\ \ -c_2(t)e^{i\lambda_2 x}\cdots\ -c_N(t)e^{i\lambda_N x}),$$

$$M(x,t)=(M_{jl})_{N\times N}=\left(\frac{c_l(t)}{i(-\lambda_j^*+\lambda_l)}e^{i(-\lambda_j^*+\lambda_l)x}\right)_{N\times N},$$

$$B(y)=(e^{i\lambda_1 x}\ \ e^{i\lambda_2 x}\ \cdots\ e^{i\lambda_N x})^T,$$

$$A_j(x)=\left(\frac{i}{\lambda_j-\lambda_1^*}e^{i(\lambda_j-\lambda_1^*)x}\cdots\frac{i}{\lambda_j-\lambda_N^*}e^{i(\lambda_j-\lambda_N^*)x}\right)^T, \quad j=1,\ldots,N,$$

$$D(x,t)=I+M(x,t)\Lambda M^*(x,t).$$

Then the $K_1(x,y)$, $K_2(x,y)$ in Eq. (6.25) can be obtained as

$$K_1(x,y,t)=E^*(x,t)D^{-1}(x,t)B^*(y), \quad K_2(x,y,t)=-E(x,t)D^{-1*}(x,t)M^*(x,t)\Lambda^*B^*(y). \tag{6.27}$$

After some reduction, the solution of the nth DNLSE with self-consistent sources (6.4) under our assumption can be written in the form

$$q(x,t)e^{2i\mu^+(x)}=-2E^*(x,t)D^{-1}(x,t)B^*(x), \tag{6.28a}$$

$$\phi_{1j}(x,t)=\zeta_j\sqrt{2\beta_j(t)c_j(t)}E^*(x,t)D(x,t)^{-1}A_j(x)e^{-i\mu^+(x)}, \quad j=1,\ldots,N_1, \tag{6.28b}$$

$$\phi_{2j}(x,t)=\sqrt{2\beta_j(t)c_j(t)}(e^{i[\zeta_j^2x+\mu^+(x)]}-E(x,t)D^{-1*}(x,t)M^*(x,t)\Lambda^*A_j(x)e^{i\mu^+(x)}), \quad j=1,\ldots,N_1, \tag{6.28c}$$

$$\phi_{1j}(x,t)=\sqrt{-2\beta_j(t)\overline{c_j}(t)}(e^{i[-\zeta_j^2x-\mu^+(x)]}-E^*(x,t)D^{-1}(x,t)M(x,t)\Lambda A_j^*(x,t)e^{-i\mu^+(x)}), \quad j=N_1+1,\ldots,N, \tag{6.28d}$$

$$\phi_{2j}(x,t)=-\zeta_j\sqrt{-2\beta_j(t)\overline{c_j}(t)}E(x,t)D^{-1*}(x,t)A_j^*(x,t)e^{i\mu^+(x)}, \quad j=N_1+1,\ldots,N, \tag{6.28e}$$

and

$$\mu^+(x)=-2\int_x^\infty E^*(x,t)D^{-1}(x,t)B^*(x)E(x,t)(D^{-1})^*(x,t)B(x)dx.$$

Similarly, by properly choosing β_j, the formula (6.28) present the N-soliton solution for the nth DNLSE with self-consistent sources (6.4).

VII. CONCLUSION

We systematically study the soliton equation with self-consistent sources (SESCS) based on the high-order constrained flows of soliton equations. The Lax representation of the SESCS can always be deduced from the adjoint representation of the auxiliary linear problems for soliton equations. In contrast with the soliton equations, the evolution of eigenfunctions for the SESCS possess singularity. We propose a general method to treat the singularity to determine the evolution of scattering data. The evolution of each normalization constant has an extra term related to the eigenfunction. We directly integrate the AKNS hierarchy with self-consistent sources, the MKdV hierarchy with self-consistent sources, the NLSE hierarchy with self-consistent sources, the KN hierarchy with self-consistent sources and the DNLSE hierarchy with self-consistent sources by inverse scattering method and obtain the soliton solutions. The self-consistent sources may cause the variation of the velocity of soliton solutions. Compared with the method in Refs. 11 and 17, our approach seems more natural and simple. This approach can be used to solve all other $(1+1)$-dimensional soliton hierarchies with self-consistent sources.

ACKNOWLEDGMENTS

The work was supported by the City University of Hong Kong (SRG: 7000945), the Research Grants Council of Hong Kong (CERG: 9040395 and CERG: 9040466), the Chinese Basic Research Project ''Nonlinear Science'' and the Doctorate Dissertation Foundation of Tsinghua University.

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