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An extended two-dimensional Toda lattice hierarchy and two-dimensional Toda lattice with self-consistent sources

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We extend the two-dimensional Toda lattice hierarchy (2DTLH) by its squared eigenfunction symmetries. This extended 2DTLH (ex2DTLH) includes the two-dimensional Toda lattice equation with self-consistent sources (2DTLSCS) as its first nontrivial equation. Lax representation of ex2DTLH is also presented. With the help of the Lax representation, we construct a nonauto-Bäcklund Darboux transformation (DT) for 2DTLSCS by applying the variation of constants to 2DTLSCS auto-Bäcklund DT. This DT enables us to find many solutions to 2DTLSCS, including solitons, rational solutions, positons, negatons, and complexitons. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2976685]

I. INTRODUCTION

In this paper, we will discuss an extension to the well-known two-dimensional Toda lattice hierarchy[1] (2DTLH) by using its squared eigenfunction symmetry. This extended hierarchy will contain the two-dimensional Toda lattice equation with self-consistent sources.[2]

The 2DTLH is a very important integrable hierarchy studied by Ueno and Takasaki,[1] the construction of which was inspired by the studies of the KP hierarchy. It includes the two-dimensional Toda lattice equation[3] and involves infinite collection of Lax-type equations in two sets of independent variables. The role of pseudodifferential operators in the machinery and construction of the KP hierarchy is played by *shift operators* in the 2DTLH. The importance of 2DTLH is not only represented by its physical applications (e.g., in interface growth[4]), geometrical applications [in connection with Laplace–Darboux transformation (DT) for general second order partial differential equations[5] and classification of certain surfaces in space related as focal or caustic surfaces[6]], and theoretical values (connection with infinite dimensional Lie algebras[1,7]) but also by the links between it and other integrable hierarchies. For instance, 2DTLH can be embedded into the two-component KP hierarchy and can be reduced to the one-dimensional Toda lattice hierarchy, to sinh-Gordon equation, etc.[1]

Squared eigenfunction symmetry is a widely used object in integrable systems. "Squared" usually means a product of eigenfunction and its adjoint associated with the auxiliary linear problem of the nonlinear integrable equations. In integrable equations of one spatial (continuous or discrete) and one temporal (i.e., $1+1$) dimensions, squared eigenfunction arises from the variational derivative of the eigenvalue that is associated with the spectral problem of the Lax pair. It is related to symmetries of the integrable equation. Such an idea is crucial in nonlinearization procedure and (high order) restricted flows.[8,9] By its identification with an usual symmetry of the

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integrable equation, it leads to finite dimensional integrable systems[8–10] (or integrable symplectic maps[11]). Many of such systems reported can be derived from a general r-matrix structure.[12]

The same idea was generalized to the reduction in $(2+1)$-dimensional integrable systems. In the KP and 2DTLH cases, the x-derivative of squared eigenfunction $\psi\psi^*$ was shown to be symmetry generation functions.[13] The symmetry reduction in the KP hierarchy utilizes the squared eigenfunction to produce the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy, Yajima–Oikawa hierarchy, etc. In the Sato theory, $\psi\psi^*$ is naturally connected with pseudodifferential operators $\psi\partial^{-1}\psi^*$. So the above reductions can be described as k-constrained KP hierarchy[14,15] by using pseudodifferential operators.

The restricted flows and constrained KP hierarchy also play important roles in obtaining soliton equations with self-consistent sources (SESCSs). SESCSs have wide physical applications in fields such as plasma, hydrodynamics, and solid-state physics.[16–18] In the $(2+1)$-dimensional case such as the KP hierarchy, the KP hierarchy with self-consistent sources was generated by identifying its stationary equation with constrained KP hierarchies.[19] However, this approach is incomplete in the sense that although it is effective in obtaining the KP equation with self-consistent sources (called the first-type KPSCS), it cannot obtain the other (called the second) type of KP with self-consistent sources, which also appeared in Ref. 16 and rediscoverd with the source generating approach by Ref. 20.

Our extended KP hierarchy[21] solved this problem. The idea of extended KP was to replace one particular flow $L_{t_k}=[B_k,L]$ of the KP hierarchy by $L_{\tau_k}=[B_k+\Sigma q_i\partial^{-1}r_i,L]$, where the squared eigenfunction symmetries were used. This idea is quite close to the work of Oevel,[22] Oevel and Carillo,[23] and Aratyn *et al.*,[24] where they considered an extra flow $L_z=[\Sigma q_i\partial^{-1}r_i,L]$ for KP. However, this is the first time that the above extension is used to implement the KP with sources of both the first and second types.

In this paper, we focus on the same extension of 2DTLH based on Ref. 21. It will help us get the two-dimensional Toda lattice equation with self-consistent sources (2DTLSCS) and its Lax presentation simultaneously. Due to the close relationship between 2DTLH and interface growth,[4] it is reasonable to speculate on the potential application of 2DTLSCS in the same field. However, it is beyond the scope of the current paper. To show the extended 2DTLH (ex2DTLH), as the first step, the squared eigenfunction symmetry for 2DTLH is studied in the framework of difference operator via Sato's approach. Although the squared eigenfunction symmetry and symmetry generating function of KP are studied extensively for a long time, this is not the case for 2DTLH. So it is still an interesting topic to study.

First let us briefly recall Ueno and Takasaki's[1] construction of 2DTLH: Let $\mathbf{x}=(x_1,x_2,\ldots)$ and $\mathbf{y}=(y_1,y_2,\ldots)$ be two series of independent continuous variables and $n\in\mathbb{Z}$ be a discrete variable. Then 2DTLH is defined by four infinite collections of Lax-type equations:

$$L_{x_m}=[B_m,L], \tag{1a}$$

$$L_{y_m}=[C_m,L], \tag{1b}$$

$$M_{x_m}=[B_m,M], \tag{1c}$$

$$M_{y_m}=[C_m,M], \quad m=1,2,\ldots . \tag{1d}$$

The difference operators L and M are defined by

$$L=\Lambda+u_0+u_1\Lambda^{-1}+u_2\Lambda^{-2}+\cdots, \quad M=v_{-1}\Lambda^{-1}+v_0+v_1\Lambda+v_2\Lambda^2+\cdots,$$

where Λ stands for a *shift operator*, which satisfies $\Lambda f(n)=f(n+1)\Lambda$.[1] Potentials u_i and v_i are

[1]Here the notation is different from Ref. 1: instead of an infinite dimensional matrix we use a shift operator.

functions of $\mathbf{x}$, $\mathbf{y}$, and n, $B_m = L_+^m$ denotes the positive part (≥ 0) of L^m with respect to the powers of Λ and $C_m = M_-^m$ denotes the negative part (<0) of M^m, and the bracket stands for the usual commutator of operators.

The key point for the definition of (1) is that the four sets of flows can be proven to mutually commute, the compatibilities of which give a *zero curvature* form for the 2DTLH:

$$B_{k,x_m} - B_{m,x_k} + [B_k, B_m] = 0, \tag{2a}$$

$$C_{k,y_m} - C_{m,y_k} + [C_k, C_m] = 0, \tag{2b}$$

$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0. \tag{2c}$$

When $m = k = 1$, (2) lead to the equations

$$u_y = v - v^{(1)}, \tag{3a}$$

$$v_x = v(u - u^{(-1)}). \tag{3b}$$

The Lax pair for (3) is

$$\psi_x = B(\psi) = (\Lambda + u)(\psi), \tag{4a}$$

$$\psi_y = C(\psi) = (v\Lambda^{-1})(\psi), \tag{4b}$$

where $x := x_1$, $y := y_1$, $B := B_1$, $C := C_1$, $u := u_0$, and $v := v_{-1}$. Eliminating u from (3) and replacing v by

$$v = \exp(q - q^{(-1)}), \tag{5}$$

we will obtain the so-called two-dimensional Toda lattice equation:

$$q_{xy} = \exp(q - q^{(-1)}) - \exp(q^{(1)} - q). \tag{6}$$

The 2DTLSCS was first presented in Ref. 25 by the source generating method as follows:

$$q_{xy} = e^{q - q^{(-1)}} - e^{q^{(1)} - q} + \sum (w_i w_i^*)_y, \tag{7a}$$

$$w_{i,y} = e^{q - q^{(-1)}} w_i^{(-1)} \quad (i = 1, \ldots, N), \tag{7b}$$

$$w_{i,y}^* = -e^{q^{(1)} - q} w_i^{*(1)}. \tag{7c}$$

Our ex2DTLH can be presented briefly as follows: First we introduce a new vector field $\partial_{\bar{y}_k}$, which is a linear combination of all vector fields ∂_{y_m}. The corresponding flow is proved to be connected with the squared eigenfunction of 2DTLH. Then we derive a new Lax-type equation accompanied with the condition of time evolutions of eigenfunctions. Commutativities of $\partial_{\bar{y}_k}$, ∂_{y_m}, and ∂_{x_k} flows give rise to our ex2DTLH. This hierarchy helps us derive the 2DTLSCS in a different way from Refs. 2 and 25 and to obtain their Lax representations.

In the second part of our paper, we will solve 2DTLSCS by DT. Since the Lax representation for 2DTLSCS is obtained, it is not difficult to construct a DT for 2DTLSCS, which can transform solutions of 2DTLSCS to solutions with the same number of source terms. Unfortunately this auto-Bäcklund transformation cannot give a nontrivial solution from a trivial one. So we virtually need a nonauto-Bäcklund transformation for constructing solutions. The idea is to consider 2DTLSCS as 2DTL with *nonhomogeneous* terms (i.e., self-consistent source terms). By applying the method of variation of constant (VC) to the DT, we find a new nonauto-Bäcklund DT which transforms a solution of the two-dimensional Toda lattice with N self-consistent sources to a solution with $(N+1)$ self-consistent sources. Next, we obtain the m-time DT formula by induction. With the help of this formula, we can find a number of solutions to 2DTLSCS, such as solitons, rational solutions, positons, negatons, and complexitons.

This paper will be organized as follows: In Sec. II we present the new ex2DTLH, which includes 2DTLSCS. A crucial step in resolvent identities is proved. In Sec. III we first construct the usual DT for 2DTLSCS; then by applying VC to this DT, we find a DT which can increase the sources' number by 1. The m-time DT formula follows by induction. Its final form is expressed in terms of Casorati determinants. In Sec. IV, we show some smooth and nonsmooth solutions of 2DTLSCS by DT. Some properties of these solutions are also illustrated. In Sec. V, we give conclusion and comments.

II. NEW EXTENDED TWO-DIMENSIONAL TODA LATTICE HIERARCHY

A. Sato approach and resolvent identities

First we show some useful notations and definitions.

Definition 1 (Residue of shift operator): *Let* $P=\Sigma_{i\in\mathbb{Z}}P_i\Lambda^i$. Then the residue of P is

$$\mathrm{Res}_\Lambda\, P = P_0.$$

Definition 2 (Adjoint operator *): $P^*=\Sigma_{i\in\mathbb{Z}}\Lambda^{-i}P_i$.

Definition 3 [Shift operator's action $P(\lambda^n)$]:

$$P(\lambda^n) = \sum_{i\in\mathbb{Z}} P_i\Lambda^i(\lambda^n) = \left(\sum_{i\in\mathbb{Z}} P_i\lambda^i\right)\cdot\lambda^n.$$

Definition 4 (Formal inverses of difference operator Δ): *For difference operator* $\Delta=\Lambda-1$, the formal inverses are given by

$$\Delta_+^{-1} = -\sum_{i\ge 0}\Lambda^i \quad \text{or} \quad \Delta_-^{-1} = \sum_{i\le -1}\Lambda^i.$$

Define *wave operators*

$$\hat{W}^{(\infty)} = b_0 + b_1\Lambda^{-1} + b_2\Lambda^{-2} + \cdots, \quad \hat{W}^{(0)} = c_0 + c_1\Lambda + c_2\Lambda^2 + \cdots,$$

where $b_0=1$. Ueno and Takasaki[1] proved that if L and M are solutions to (1) then there are wave operators by which L and M can be written as

$$L = \hat{W}^{(\infty)}\Lambda\hat{W}^{(\infty)-1}, \quad M = \hat{W}^{(0)}\Lambda^{-1}\hat{W}^{(0)-1} \tag{8a}$$

and

$$\partial_{x_m}\hat{W}^{(\infty)} = -L^m_{<0}\hat{W}^{(\infty)}, \quad \partial_{x_m}\hat{W}^{(0)} = L^m_{\ge 0}\hat{W}^{(0)}, \tag{8b}$$

$$\partial_{y_m}\hat{W}^{(\infty)} = M^m_{<0}\hat{W}^{(\infty)}, \quad \partial_{y_m}\hat{W}^{(0)} = -M^m_{\ge 0}\hat{W}^{(0)}. \tag{8c}$$

Define *wave function*

$$w^{(\infty)} = \hat{W}^{(\infty)}(\lambda^n)e^{\xi(x,\lambda)} = \left(\sum_{i\ge 0} b_i\lambda^{-i}\right)\lambda^n e^{\xi(x,\lambda)} := \hat{w}^{(\infty)}\lambda^n e^{\xi(x,\lambda)},$$

$$w^{(0)} = \hat{W}^{(0)}(\lambda^n)e^{\xi(y,\lambda^{-1})} = \left(\sum_{i\ge 0} c_i\lambda^{i}\right)\lambda^n e^{\xi(y,\lambda^{-1})} := \hat{w}^{(0)}\lambda^n e^{\xi(y,\lambda^{-1})}$$

and *adjoint wave function*

$$w^{(\infty)*} = \hat{W}^{(\infty)*-1}(\lambda^{-n})e^{-\xi(x,\lambda)} := \hat{w}^{(\infty)*}\lambda^{-n}e^{-\xi(x,\lambda)},$$

$$w^{(0)*} = \hat{W}^{(0)*-1}(\lambda^{-n})e^{-\xi(y,\lambda^{-1})} := \hat{w}^{(0)*}\lambda^{-n}e^{-\xi(y,\lambda^{-1})},$$

where $\xi(x,\lambda)=\Sigma_{i\geq 1}x_i\lambda^i$, $\xi(y,\lambda^{-1})=\Sigma_{i\geq 1}y_i\lambda^{-i}$. Then (1) can also be written as the compatibilities of following eigenvalue problems:

$$Lw^{(\infty)} = \lambda w^{(\infty)}, \quad Mw^{(0)} = \lambda^{-1}w^{(0)}, \tag{9a}$$

$$\partial_{x_m}w^{(\infty)} = B_m w^{(\infty)}, \quad \partial_{x_m}w^{(0)} = B_m w^{(0)}, \tag{9b}$$

$$\partial_{y_m}w^{(\infty)} = C_m w^{(\infty)}, \quad \partial_{y_m}w^{(0)} = C_m w^{(0)}. \tag{9c}$$

Lemma 1 (Ueno and Takasaki[1]): *Suppose* $P=\Sigma P_i\Lambda^i$, $Q=\Sigma Q_j\Lambda^j$. *Then*

$$\mathrm{Res}_\Lambda\, P\cdot Q^* = \mathrm{Res}_\lambda\, \lambda^{-1}P(\lambda^n)Q\cdot(\lambda^{-n}).$$

Proof: It is sufficient to prove a special case, say, $P=P_i\Lambda^i$, $Q=Q_j\Lambda^j$: $\mathrm{Res}_\Lambda\, PQ^*=\mathrm{Res}_\Lambda\, P_i\Lambda^{i-j}Q_j=\delta_{i,j}P_iQ_j$, while $\mathrm{Res}_\lambda\, \lambda^{-1}P(\lambda^n)Q(\lambda^{-n})=\mathrm{Res}_\lambda\, \lambda^{-1}P_i\lambda^{n+i}Q_j\lambda^{-n-j}=\delta_{i,j}P_iQ_j$. □

Similar to the KP theory, in which the principal part of the resolvent can be expressed in terms of a quadratic form of the wave function and adjoint wave function,[26] we have the following resolvent identities for 2DTLH.

Proposition 1 (Resolvent identities):

$$\sum_{k\geq 0} L^k_{\geq 0}\lambda^{-k} = -w^{(\infty)}\Delta_+^{-1}w^{(\infty)*}, \quad \sum_{k\in\mathbb{Z}} M^k_{\geq 0}\lambda^k = -w^{(0)}\Delta_+^{-1}w^{(0)*},$$

$$\sum_{k\in\mathbb{Z}} L^k_{<0}\lambda^{-k} = w^{(\infty)}\Delta_-^{-1}w^{(\infty)*}, \quad \sum_{k>0} M^k_{<0}\lambda^k = w^{(0)}\Delta_-^{-1}w^{(0)*}.$$

Proof: We prove one of them. The others are similar. Since $L=\hat{W}^{(\infty)}\Lambda\hat{W}^{(\infty)-1}$,

$$\begin{aligned}
L^k_{<0} &= (\hat{W}^{(\infty)}\Lambda^k\hat{W}^{(\infty)-1})_{<0} \\
&= \sum_{m\geq 1}\mathrm{Res}_\Lambda(\hat{W}^{(\infty)}\Lambda^k\hat{W}^{(\infty)-1}\Lambda^m)\Lambda^{-m} \\
&= \sum_{m\geq 1}\mathrm{Res}_\lambda\, \lambda^{-1}(\hat{W}^{(\infty)}\Lambda^k(\lambda^n)e^{\xi(x,\lambda)})\cdot(\Lambda^{-m}\hat{W}^{(\infty)*-1}(\lambda^{-n})e^{-\xi(x,\lambda)})\Lambda^{-m} \\
&= \sum_{m\geq 1}\mathrm{Res}_\lambda\, \lambda^{k-1}w^{(\infty)}\Lambda^{-m}w^{(\infty)*} \\
&= \mathrm{Res}_\lambda\, \lambda^{k-1}w^{(\infty)}\Delta_-^{-1}w^{(\infty)*}.
\end{aligned}$$

So $\Sigma_{k\in\mathbb{Z}}L^k_{<0}\lambda^{-k}=w^{(\infty)}\Delta_-^{-1}w^{(\infty)*}$. □

B. New extended two-dimensional Toda lattice hierarchy

For fixed $k\geq 1$, $N>0$, we define a new time variable $\bar{y}_k$ such that the corresponding vector field is

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^{N}\sum_{j>1}\lambda_i^j\partial_{y_j}, \tag{10}$$

where λ_i are distinct nonzero parameters. Then the $\bar{y}_k$ flow is given by

$$\frac{\partial}{\partial_{\bar{y}_k}}L=[\bar{C}_k,L], \quad \frac{\partial}{\partial_{\bar{y}_k}}M=[\bar{C}_k,M], \tag{11}$$

with

$$\bar{C}_k=C_k+\sum_{i=1}^{N}\sum_{j\geq 1}\lambda_i^j C_j,$$

which, according to Proposition 1, can be rewritten as

$$\bar{C}_k=C_k+\sum_{i=1}^{N}w_i^{(0)}\Delta_-^{-1}w_i^{(0)^*}.$$

For simplicity, we denote $w_i^{(0)}$ and $w_i^{(0)^*}$ by w_i and w_i^*. Then the compatibility of (1) and (11) gives the following:

Proposition 2: *New ex2DTLH. For $m\neq k$:*

$$B_{m,x_k}-B_{k,x_m}+[B_m,B_k]=0, \tag{12a}$$

$$C_{m,\bar{y}_k}-\bar{C}_{k,y_m}+[C_m,\bar{C}_k]=0, \tag{12b}$$

$$B_{m,\bar{y}_k}-\bar{C}_{k,x_m}+[B_m,\bar{C}_k]=0, \tag{12c}$$

$$B_{k,y_m}-C_{m,x_k}+[B_k,C_m]=0, \tag{12d}$$

$$w_{i,x_m}=B_m(w_i), \quad w_{i,y_m}=C_m(w_i) \quad (i=1,\ldots,N), \tag{12e}$$

$$w^*_{i,x_m}=-B_m^*(w_i^*), \quad w^*_{i,y_m}=-C_m^*(w_i^*) \tag{12f}$$

and for $m=k$:

$$B_{k,\bar{y}_k}-\bar{C}_{k,x_k}+[B_k,\bar{C}_k]=0, \tag{13a}$$

$$\partial_{x_k}w_i=B_k(w_i), \quad \partial_{x_k}w_i^*=-B_k^*(w_i^*) \quad (i=1,\ldots,N), \tag{13b}$$

where $\bar{C}_k=C_k+\sum_{i=1}^N w_i\Delta_-^{-1}w_i^*$.

Note that hereafter w_i and w_i^* do not need to be precisely the wave function and adjoint wave function. So we call them eigenfunction and adjoint eigenfunction. In fact, Eqs. (12e) and (12f) [or (13b)] ensure the closeness of Eqs. (12a)–(12d) [or (13a)]. Furthermore, under conditions (12e) and (12f) [or (13b)], one can easily obtain the Lax representations of (12a)–(12d) as

$$\psi_{x_m}=B_m(\psi), \quad \psi_{x_k}=B_k(\psi), \tag{14a}$$

$$\psi_{y_m}=C_m(\psi), \quad \psi_{\bar{y}_k}=\bar{C}_k(\psi) \tag{14b}$$

or get the Lax representation of (13a) as

$$\psi_{x_k}=B_k(\psi), \quad \psi_{\bar{y}_k}=\bar{C}_k(\psi). \tag{15}$$

Example 1 (Two-dimensional Toda lattice equation with self-consistent sources): *When $m=k=1$, let $u=u_0$, $v=v_{-1}$, $x=x_1$, $y=\bar{y}_1$,*

$$B_1=\Lambda+u, \quad C_1=v\Lambda^{-1}.$$

Then (13) becomes

$$u_y=-\Delta\left(v+\sum_{i=1}^{N}w_i w_i^{*(-1)}\right), \quad v_x=v(u-u^{(-1)}), \tag{16a}$$

$$w_{i,x}=B_1(w_i), \quad w_{i,x}^*=-B_1^*(w_i^*), \quad i=1,\ldots,N. \tag{16b}$$

With substitutions $u=q_x$, $v=\exp(q-q^{(-1)})$, (16) yields

$$q_{xy}=e^{q-q^{(-1)}}-e^{q^{(1)}-q}+\sum_{i=1}^{N}(w_i w_i^*)_x, \tag{17a}$$

$$w_{i,x}=w_i^{(1)}+q_x w_i \quad (i=1,\ldots,N), \tag{17b}$$

$$w_{i,x}^*=-w_i^{*(-1)}-q_x w_i^*. \tag{17c}$$

This is the two-dimensional Toda lattice equation with N self-consistent sources.

Analogously, let us introduce $\bar{x}_k$ such that

$$\partial_{\bar{x}_k}=\partial_{x_k}+\sum_{i=1}^{N}\sum_{j\geq 1}\lambda_i^{-j}\partial_{x_j},$$

then we will get *another* new ex2DTLH as follows.

For $m\neq k$,

$$B_{m,\bar{x}_k}-\bar{B}_{k,x_m}+[B_m,\bar{B}_k]=0, \tag{18a}$$

$$C_{m,\bar{x}_k}-\bar{B}_{k,y_m}+[C_m,\bar{B}_k]=0, \tag{18b}$$

$$C_{m,y_k}-C_{k,y_m}+[C_m,C_k]=0, \tag{18c}$$

$$B_{m,y_k}-C_{k,x_m}+[B_m,C_k]=0, \tag{18d}$$

$$w_{i,y_m}=C_m(w_i), \quad w_{i,x_m}=B_m(w_i), \quad i=1,\ldots,N, \tag{18e}$$

$$w_{i,y_m}^*=-C_m^*(w_i^*), \quad w_{i,x_m}^*=-B_m^*(w_i^*), \tag{18f}$$

and for $m=k$,

$$C_{k,\bar{x}_k}-\bar{B}_{k,y_k}+[C_k,\bar{B}_k]=0, \tag{19a}$$

$$\partial_{y_k}w_i=C_k(w_i), \quad \partial_{y_k}w_i^*=-C_k^*(w_i^*), \quad i=1,\ldots,N, \tag{19b}$$

where $\bar{B}_k=B_k-\Sigma_{i=1}^{N}w_i\Delta_+^{-1}w_i^*$.

Example 2 [2DTLSCS (Refs. 2 and 25)]: *When $m=k=1$, (19) leads to (7). It is interesting to see that (7) is equivalent to (17) under the transformations*

$$x \to -y, \quad y \to -x, \quad q \to q,$$

$$w_i \to -e^{q} w_i^*, \quad w_i^* \to e^{-q} w_i.$$

So hereafter we may concentrate on 2DTLSCS (17). This transformation was discovered by Hu.

III. DARBOUX TRANSFORMATION FOR 2DTLSCS

In the second part of our paper, we discuss 2DTLSCS (16). First recall the Lax pair of 2DTL (4). For convenience, we denote $B=B_1$, $C=C_1$, $u=u_0$, $v=v_{-1}$, $x=x_1$, $y=y_1$.

A. Applying the method of variation of constant to Darboux transformation for 2DTLSCS

Let us first present the definition of the Casorati determinant: for m discrete functions $h_1, \ldots, h_m$, the Casorati determinant

$$\mathrm{Cas}(h_1, \ldots, h_m) := \begin{vmatrix} h_1 & \cdots & h_m \\ h_1^{(1)} & \cdots & h_m^{(1)} \\ \vdots & \vdots & \vdots \\ h_1^{(m-1)} & \cdots & h_m^{(m-1)} \end{vmatrix}.$$

DT for 2DTL (3) is given by Ref. 27. Let us give a slightly different proof in the following lemma.

Lemma 2: *Let h be a particular eigenfunction of (4), $\mathcal{D}=\Lambda+\sigma$, $\sigma:=-h^{(1)}/h$. Then DT*

$$\tilde{u} := u^{(1)} + \sigma - \sigma^{(1)}, \tag{20a}$$

$$\tilde{v} := v\sigma/\sigma^{(-1)}, \tag{20b}$$

$$\tilde{\psi} := \mathcal{D}(\psi) = \frac{\mathrm{Cas}(h,\psi)}{h} \tag{20c}$$

gives a new solution to (4). Hence $(\tilde{u},\tilde{v})$ is a new solution to (3).

Proof: Since $\tilde{B} := \Lambda + \tilde{u}$, $\tilde{C} := \tilde{v}\Lambda^{-1}$, $\tilde{\psi} = \mathcal{D}(\psi)$, a sufficient condition to ensure (4) is

$$\mathcal{D}_x + \mathcal{D}B - \tilde{B}\mathcal{D} = 0, \tag{21a}$$

$$\mathcal{D}_y + \mathcal{D}C - \tilde{C}\mathcal{D} = 0. \tag{21b}$$

Being aware of

$$\mathcal{D}(h) = 0 \tag{22}$$

and taking partial derivatives ∂_x, ∂_y to (22), one gets

$$\mathcal{D}_x(h) + \mathcal{D}(h_x) = (\mathcal{D}_x + \mathcal{D}B)(h) = \tilde{B}\mathcal{D}(h) = 0,$$

$$\mathcal{D}_y(h) + \mathcal{D}(h_y) = (\mathcal{D}_y + \mathcal{D}C)(h) = \tilde{C}\mathcal{D}(h) = 0.$$

This means

$$(\mathcal{D}_x + \mathcal{D}B - \widetilde{B}\mathcal{D})(h) = 0, \quad (\mathcal{D}_y + \mathcal{D}C - \widetilde{C}\mathcal{D})(h) = 0. \tag{23}$$

From (20a) and (20b) we realize that the operator's actions in (23) are simply some function multiplications. So (21) holds. □

Lax representation for 2DTL with N self-consistent sources is

$$\psi_x = B(\psi), \tag{24a}$$

$$\psi_y = \left(C + \sum_{i=1}^{N} w_i \Delta_-^{-1} w_i^* \right)(\psi). \tag{24b}$$

Notice that Lax representation (24) holds under following conditions:

$$w_{i,x} = B(w_i), \quad i = 1, \ldots, N, \tag{24c}$$

$$w_{i,x}^* = -B^*(w_i). \tag{24d}$$

Proposition 3 [DT for 2DTLSCS (16)]: *Let h be a particular eigenfunction to (24), $\mathcal{D}=\Lambda+\sigma$, $\sigma := -h^{(1)}/h$. Based on DT (20), we define*

$$\widetilde{w}_i := \mathcal{D}(w_i) = \frac{\mathrm{Cas}(h, w_i)}{h}, \tag{25a}$$

$$\widetilde{w}_i^* := \mathcal{D}^{*^{-1}}(w_i^*) = -\frac{\mathrm{S}(h w_i^*)}{h^{(1)}}, \tag{25b}$$

where $\mathrm{S} := \Lambda\Delta_-^{-1}$. Then (20) and (25) give a new solution to (24). So we get a new solution of (16).

Proof: From Lemma 2 it is easy to see that $\widetilde{w}_i$ defined by (25a) satisfies (24c) (B replaced by $\widetilde{B}$). It is necessary to prove that $\widetilde{w}_i^*$ satisfies (24d). From the proof of Lemma 2 we know

$$(\partial_x - \widetilde{B})\mathcal{D} = \mathcal{D}(\partial_x - B).$$

Taking the formal adjoint * to this equality and rewriting it as

$$(-\partial_x - \widetilde{B}^*)\mathcal{D}^{*^{-1}} = \mathcal{D}^{*^{-1}}(-\partial_x - B^*),$$

then we find a sufficient condition for $\mathcal{D}^{*^{-1}}$ to be the DT for (24d). As a result we have proved (25b). Lastly we need to prove that the DT given by (20) and (25) makes (24b) covariant. That is to say

$$\mathcal{D}_y + \mathcal{D}C + \sum_{i=1}^{N} \mathcal{D}w_i\Delta_-^{-1}w_i^* - \widetilde{C}\mathcal{D} - \sum_{i=1}^{N} \widetilde{w}_i\Delta_-^{-1}\widetilde{w}_i^*\mathcal{D} = 0. \tag{26}$$

Based on (21a), we have to prove that the extra terms in (26) with respect to w_i, w_i^* are equal. We find

$$
\begin{aligned}
&-\frac{w_i^{(1)}\mathrm{S}(hw_i^*)}{h}+\frac{h^{(1)}}{h^2}w_i\Delta_-^{-1}(hw_i^*)+w_i^{(1)}\Lambda\Delta_-^{-1}w_i^*-\frac{h^{(1)}}{h}w_i\Delta_-^{-1}w_i^*-\widetilde{w}_i\Delta_-^{-1}\widetilde{w}_i^*\Lambda+\widetilde{w}_i\Delta_-^{-1}\widetilde{w}_i^*\frac{h^{(1)}}{h}\\
&=-\frac{\widetilde{w}_i}{h}\mathrm{S}(hw_i^*)-\frac{h^{(1)}}{h}w_iw_i^*+w_i^{(1)}\Lambda\Delta_-^{-1}w_i^*-\frac{h^{(1)}}{h}w_i\Delta_-^{-1}w_i^*+\widetilde{w}_i\Delta_-^{-1}\frac{\mathrm{S}(hw_i^*)}{h^{(1)}}\Lambda-\widetilde{w}_i\Delta_-^{-1}\frac{\mathrm{S}(hw_i^*)}{h}\\
&=-\frac{\widetilde{w}_i}{h}\mathrm{S}(hw_i^*)+w_i^{(1)}\Lambda\Delta_-^{-1}w_i^*-\frac{h^{(1)}}{h}w_i\Lambda\Delta_-^{-1}w_i^*+\widetilde{w}_i\Delta_-^{-1}\Lambda\frac{\mathrm{S}(hw_i^*)}{h}-\widetilde{w}_i\Delta_-^{-1}\frac{\mathrm{S}(hw_i^*)}{h}-\widetilde{w}_i\Lambda\Delta_-^{-1}w_i^*\\
&=0
\end{aligned}
$$

for each i. This ends the proof. □

Theorem 1 [DT and method of VC for 2DTLSCS (16)]: *Let f and g be two linear independent eigenfunctions of (24). Suppose $a(y)$ is an arbitrary function of y. Let $h:=f+a(y)g$,*

$$\widetilde{w}_{N+1}=c(y)\mathcal{D}(f), \tag{27a}$$

$$\widetilde{w}^*_{N+1}=\frac{d(y)}{h^{(1)}}, \tag{27b}$$

then (20), (25), and (27) give a new solution for (16) and (24) with $(N+1)$ self-consistent sources, where $c(y)$, $d(y)$ satisfy the equality $c(y)d(y)=\partial_y \log a(y)$.

Proof: It is easy to see that $\widetilde{w}_{N+1}$ satisfies (24c). To prove that $\widetilde{w}^*_{N+1}$ satisfies (24d), we have

$$\widetilde{w}^*_{N+1,x}=-\frac{d(y)}{h^{(1)}{}_2}h_x^{(1)}=-\frac{d(y)}{h^{(1)}{}_2}(h^{(2)}+u^{(1)}h^{(1)}),$$

$$-\widetilde{w}^{*(-1)}_{N+1}-\widetilde{u}\widetilde{w}^*_{N+1}=-\frac{d(y)}{h}-\left(u^{(1)}-\frac{h^{(1)}}{h}+\frac{h^{(2)}}{h^{(1)}}\right)\frac{d(y)}{h^{(1)}}=-\frac{d(y)h^{(2)}}{h^{(1)}{}_2}-u^{(1)}\frac{d(y)}{h^{(1)}}=\widetilde{w}^*_{N+1,x}.$$

Based on Proposition 3, we need to show that extra terms coming from

$$\mathcal{D}_y+\mathcal{D}\left(C+\sum_{i=1}^{N}w_i\Delta_-^{-1}w_i^*\right)-\left(\widetilde{C}+\sum_{i=1}^{N+1}w_i\Delta_-^{-1}\widetilde{w}_i^*\right)\mathcal{D}$$

can be canceled out. This is true:

$$
\begin{aligned}
-\frac{a_yg^{(1)}}{h}+\frac{h^{(1)}a_yg}{h^2}-\widetilde{w}_{N+1}\Delta_-^{-1}\widetilde{w}^*_{N+1}(\Lambda-h^{(1)}/h)&=\frac{a_y}{h^2}\mathrm{Cas}(g,h)-c(y)d(y)\frac{\mathrm{Cas}(h,f)}{h}\Delta_-^{-1}\left(h^{(1)-1}\Lambda-\frac{1}{h}\right)\\
&=\frac{a_y}{h^2}\mathrm{Cas}(g,f)-c(y)d(y)a(y)\frac{\mathrm{Cas}(g,f)}{h^2}=0.
\end{aligned}
$$

□

B. *m*-time nonauto-Bäcklund Darboux transformations

Theorem 2: *Let f_j* and *g_j $(j=1,2,\ldots,m)$ be m pairs of independent eigenfunctions to (24). Suppose $a_j(y)$ are arbitrary functions of time. Let*

$$h_j:=f_j+a_j(y)g_j.$$

Then after m -time DTs (Theorem 1), we find a solution for (24) with $(N+m)$ self-consistent sources, which is

$$u[m] = u^{(m)} + \frac{\widetilde{\mathrm{Cas}}^{(1)}(h_1, \ldots, h_m)}{\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)} - \frac{\widetilde{\mathrm{Cas}}(h_1, \ldots, h_m)}{\mathrm{Cas}(h_1, \ldots, h_m)}, \tag{28a}$$

$$v[m] = v\frac{\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)\mathrm{Cas}^{(-1)}(h_1, \ldots, h_m)}{\mathrm{Cas}^2(h_1, \ldots, h_m)}, \tag{28b}$$

$$w_i[m] = \frac{\mathrm{Cas}(h_1, \ldots, h_m, w_i)}{\mathrm{Cas}(h_1, \ldots, h_m)}, \quad i = 1, \ldots, N, \tag{28c}$$

$$w_i^*[m] = (-1)^m \frac{\overline{\mathrm{Cas}}(h_1, \ldots, h_m, w_i^*)}{\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)}, \tag{28d}$$

$$w_{N+j}[m] = c_j(y) f_j[m] = c_j(y)\frac{\mathrm{Cas}(h_1, \ldots, h_m, f_j)}{\mathrm{Cas}(h_1, \ldots, h_m)}, \quad j = 1, \ldots, m \tag{28e}$$

$$w_{N+j}^*[m] = (-1)^{m-j} d_j(y)\frac{\mathrm{Cas}^{(1)}(h_1, \ldots, \hat{h}_j, \ldots, h_m)}{\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)}, \tag{28f}$$

where

$$\widetilde{\mathrm{Cas}}(h_1, \ldots, h_m) = \begin{vmatrix} h_1 & \cdots & h_m \\ \vdots & \vdots & \vdots \\ h_1^{(m-2)} & \cdots & h_m^{(m-2)} \\ h_1^{(m)} & \cdots & h_m^{(m)} \end{vmatrix}, \quad \overline{\mathrm{Cas}}(h_1, \ldots, h_m, f) = \begin{vmatrix} S(h_1 f) & \cdots & S(h_m f) \\ h_1^{(1)} & \cdots & h_m^{(1)} \\ \vdots & \vdots & \vdots \\ h_1^{(m-1)} & \cdots & h_m^{(m-1)} \end{vmatrix}$$

and $c_j(y) d_j(y) = \partial_y \log a_j(y)$.

Proof: Since each time DT has the form $\mathcal{D} = \Lambda + \sigma$, after m-time DTs, the corresponding DT operator has the form

$$\mathcal{D}(m) = \Lambda^m + \sigma_{m-1}\Lambda^{m-1} + \cdots + \sigma_0.$$

There are m undetermined coefficients σ_i, $i = 0, \ldots, m-1$. Being aware of

$$\mathcal{D}(m)(h_i) = 0, \quad i = 1, 2, \ldots, m,$$

we find the linear equation for undetermined coefficients,

$$\begin{vmatrix} h_1 & h_1^{(1)} & \cdots & h_1^{(m-1)} \\ h_2 & h_2^{(1)} & \cdots & h_2^{(m-1)} \\ \vdots & \vdots & \vdots & \vdots \\ h_m & h_m^{(1)} & \cdots & h_m^{(m-1)} \end{vmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{m-1} \end{bmatrix} = - \begin{bmatrix} h_1^{(m)} \\ h_2^{(m)} \\ \vdots \\ h_m^{(m)} \end{bmatrix}.$$

By Cramer's rule

$$\sigma_0 = (-1)^m \frac{\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)}{\mathrm{Cas}(h_1, \ldots, h_m)}, \quad \sigma_{m-1} = -\frac{\widetilde{\mathrm{Cas}}(h_1, \ldots, h_m)}{\mathrm{Cas}(h_1, \ldots, h_m)}.$$

In order to obtain (28a) and (28b), noticing that the transformations (20a) and (20b) are unchanged from 2DTL to 2DTLSCS, we may consider the m-time DTs for 2DTL temporary. So assuming that $\mathcal{D}(m)$ transforms ψ to $\tilde{\psi} = \psi[m]$ and satisfies $\tilde{\psi}_x = \tilde{B}\tilde{\psi}$ and $\tilde{\psi}_y = \tilde{C}\tilde{\psi}$, we have

$$\mathcal{D}(m)_x + \mathcal{D}(m)B - \widetilde{B}\mathcal{D}(m) = 0, \tag{29a}$$

$$\mathcal{D}(m)_y + \mathcal{D}(m)C - \widetilde{C}\mathcal{D}(m) = 0. \tag{29b}$$

Comparing the coefficient of Λ^m in (29a) and the coefficient of Λ^{-1} in (29b), respectively, we find (28a) and (28b).

For an arbitrary eigenfunction w, its DT $\widetilde{w}=\mathcal{D}(m)(w)$ can be expressed in form (28c) according to the Laplace expansion formula. For (28d) we need induction. Suppose that for any adjoint eigenfunction w^* the m-time DT formula is (28d), then by (25b), the $(m+1)$th DT is

$$\begin{aligned} w^*[m+1] &= -\frac{S(h_{m+1}[m]w^*[m])}{h_{m+1}[m]^{(1)}} \\ &= \frac{S\left[\Delta\left(\dfrac{h_{m+1}[m-1]}{h_m[m-1]}S(h_m[m-1]w^*[m-1])\right) - h_{m+1}[m-1]^{(1)}w^*[m-1]^{(1)}\right]}{h_{m+1}[m]^{(1)}} \\ &= \frac{h_{m+1}[m-1]^{(1)}S(h_m[m-1]w^*[m-1])}{h_{m+1}[m]^{(1)}h_m[m-1]^{(1)}} - \frac{S(h_{m+1}[m-1]w^*[m-1])}{h_{m+1}[m]^{(1)}}. \end{aligned}$$

By induction hypothesis

$$\frac{S(h_m[m-1]w^*[m-1])}{h_m[m-1]^{(1)}} = (-1)^{m+1}\frac{\overline{\mathrm{Cas}}(h_1,\ldots,h_m,w^*)}{\mathrm{Cas}^{(1)}(h_1,\ldots,h_m)},$$

$$\frac{S(h_{m+1}[m-1]w^*[m-1])}{h_{m+1}[m-1]^{(1)}} = (-1)^{m+1}\frac{\overline{\mathrm{Cas}}(h_1,\ldots,h_{m-1},h_{m+1},w^*)}{\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m-1},h_{m+1})},$$

we find

$$\begin{aligned} w^*[m+1] &= \frac{(-1)^{m+1}}{\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m-1})\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m+1})}[\overline{\mathrm{Cas}}(h_1,\ldots,h_m,w^*)\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m-1},h_{m+1}) \\ &\quad - \overline{\mathrm{Cas}}(h_1,\ldots,h_{m-1},h_{m+1},w^*)\mathrm{Cas}^{(1)}(h_1,\ldots,h_m)] \\ &= \frac{(-1)^{m+1}}{\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m-1})\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m+1})} \\ &\quad \times \begin{vmatrix} \mathrm{S}(h_1w^*) & \cdots & \mathrm{S}(h_{m-1}w^*) & \mathrm{S}(h_mw^*) & \mathrm{S}(h_{m+1}w^*) & 0 & \cdots & 0 \\ h_1^{(1)} & \cdots & h_{m-1}^{(1)} & h_m^{(1)} & h_{m+1}^{(1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ h_1^{(m-1)} & \cdots & h_{m-1}^{(m-1)} & h_m^{(m-1)} & h_{m+1}^{(m-1)} & 0 & \cdots & 0 \\ h_1^{(1)} & \cdots & h_{m-1}^{(1)} & h_m^{(1)} & h_{m+1}^{(1)} & h_1^{(1)} & \cdots & h_{m-1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ h_1^{(m)} & \cdots & h_{m-1}^{(m)} & h_m^{(m)} & h_{m+1}^{(m)} & h_1^{(m)} & \cdots & h_{m-1}^{(m)} \end{vmatrix} \\ &= (-1)^{m+1}\frac{\overline{\mathrm{Cas}}(h_1,\ldots,h_{m+1},w^*)}{\mathrm{Cas}^{(1)}(h_1,\ldots,h_{m+1})}. \end{aligned}$$

Next we want to prove (28e) and (28f). First, it is easy to see that $w_{N+j}[m]=c_j(y)\mathcal{D}(f_j)$ can be derived from (28c). For $w_{N+j}^*[m]$ we use induction. Suppose that when $j \le m$ the formula for $w_{N+j}^*[m]$ is given by (28f). Then

$$w^*_{N+j}[m+1] = -\frac{S(h_{m+1}[m]w^*_{N+j}[m])}{h_{m+1}[m]^{(1)}}.$$

Because $h_{m+1}[m]$ is obtained by m-time DTs by using eigenfunctions $h_1, \ldots, h_m$ sequentially, it is equivalent to m-time DTs by successively using $h_1, \ldots, h_{j-1}, h_{j+1}, \ldots, h_m$ and h_j:

$$h_{m+1}[m] = h_{m+1}[m-1]^{(1)} - \frac{h_j[m-1]^{(1)}}{h_j[m-1]}h_{m+1}[m-1].$$

Since $w^*_{N+j}[m]=d_j(y)/h_j[m-1]^{(1)}$,

$$w^*_{N+j}[m+1] = -\frac{d_j(y)}{h_{m+1}[m]^{(1)}}S\Delta\left(\frac{h_{m+1}[m-1]}{h_j[m-1]}\right) = (-1)^{m+1-j}\frac{d_j(y)\mathrm{Cas}^{(1)}(h_1, \ldots, \hat{h}_j, \ldots, h_{m+1})}{\mathrm{Cas}^{(1)}(h_1, \ldots, h_{m+1})}.$$

When $j=m+1$,

$$w^*_{N+m+1}[m+1] = \frac{d_{m+1}(y)}{h_{m+1}[m]^{(1)}} = \frac{d_{m+1}(y)\mathrm{Cas}^{(1)}(h_1, \ldots, h_m)}{\mathrm{Cas}^{(1)}(h_1, \ldots, h_{m+1})}.$$

Thus the proof ends. □

IV. SOLUTIONS FOR 2DTLSCS

Let us start from trivial solution $q=1$, $v=1$, $u=0$, $N=0$ of (24). The Lax pair reads

$$\psi_x = \psi^{(1)}, \quad \psi_y = \psi^{(-1)}. \tag{30}$$

A. Solitons

Equations (30) have linear independent eigenfunctions,

$$f(n,x,y) = \exp(n\omega + zx + z^{-1}y), \quad g(n,x,y) = \exp(-n\omega + z^{-1}x + zy), \quad (z = e^{\omega}).$$

Let $a(y)=e^{\alpha(y)}$; then

$$h = f + a(y)g = 2\exp\Omega\cosh Z,$$

where

$$\Omega = \cosh\omega \cdot x + \cosh\omega \cdot y + \alpha/2, \quad Z = n\omega + \sinh\omega \cdot x - \sinh\omega \cdot y - \alpha/2.$$

From (28), taking $m=1$ we have the following 1-soliton solution for (16):

$$u[1] = \frac{\cosh(Z+2\omega)}{\cosh(Z+\omega)} - \frac{\cosh(Z+\omega)}{\cosh Z},$$

$$v[1] = \frac{\cosh(Z+\omega)\cosh(Z-\omega)}{\cosh^2 Z},$$

$$w[1] = c(y)\frac{\sinh\omega \cdot e^{\Omega}}{\cosh Z},$$

$$w^*[1] = \frac{d(y)e^{-\Omega}}{2\cosh(Z+\omega)}, \quad (c(y)d(y) = \dot{\alpha}).$$

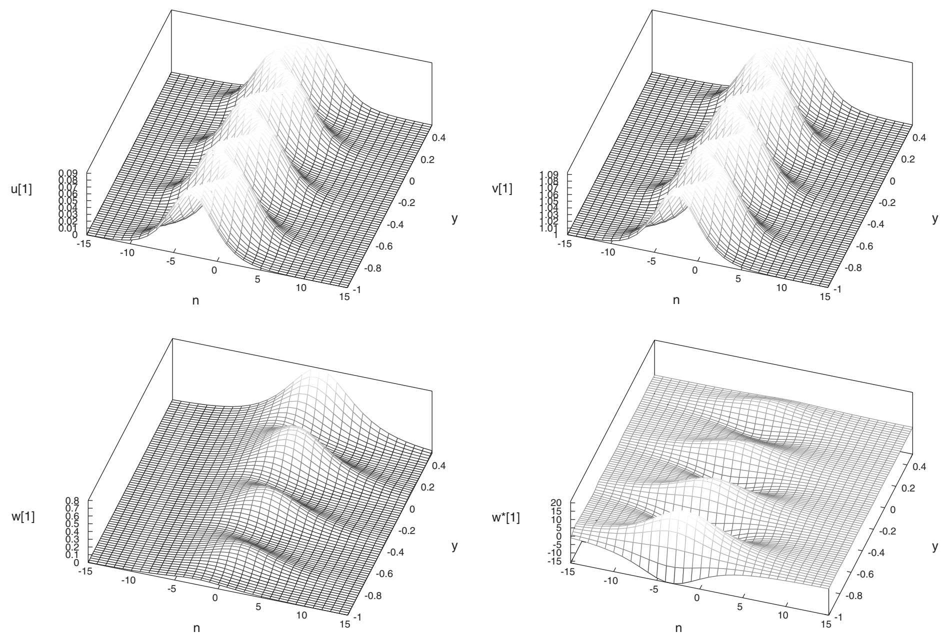


FIG. 1. Snapshot of 1-soliton at $x=0$ with $\omega=0.3$, $\alpha(y)=\sin((14\pi/3)y)$, $c(y)=1$, $d(y)=\dot{\alpha}$.

The 1-soliton solution for 2DTLSCS is different from usual 1-soliton for 2DTL since the soliton travel speed is determined by $\alpha(y)$. In particular, it provides a soliton with a nonconstant speed during its propagation. Furthermore the amplitude of $w[1]w^*[1]$ is controlled by $\dot{\alpha}$; therefore the change in solitary wave speed leads to the synchronous change in the amplitude and vice versa. Similar phenomena were observed in other systems with self-consistent sources, such as KP (Ref. 28) and Toda lattice.[29] Figure 1 shows a solitary wave with nonconstant speed.

If we take two pairs of independent eigenfunctions, with respect to $z_j=e^{\omega_j}$ $(j=1,2)$ respectively, i.e.,

$$f_j = \exp(n\omega_j + z_j x + z_j^{-1} y), \quad g_j = \exp(-n\omega_j + z_j^{-1} x + z_j y), \quad j = 1,2,$$

let $a_j(y)=(-1)^j e^{\alpha_j(y)}$, then

$$h_1 = f_1 + a_1 g_1 = 2 \exp \Omega_1 \cdot \sinh Z_1, \quad h_2 = f_2 + a_2 g_2 = 2 \exp \Omega_2 \cdot \cosh Z_2,$$

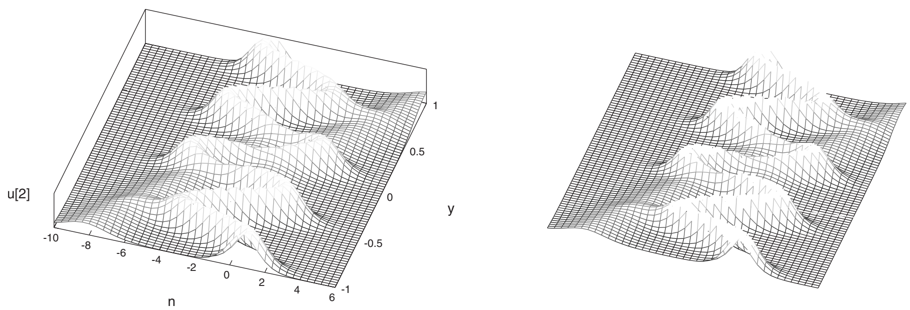
where

$$\Omega_j = \cosh \omega_j \cdot x + \cosh \omega_j \cdot y + \alpha_j/2, \quad Z_j = n\omega_j + \sinh \omega_j \cdot x - \sinh \omega_j \cdot y - \alpha_j/2.$$

To simplify expressions, we define

$$\begin{aligned} H_k &= \begin{vmatrix} \sinh Z_1 & \cosh Z_2 \\ \sinh(Z_1 + k\omega_1) & \cosh(Z_2 + k\omega_2) \end{vmatrix} \\ &= \sinh\frac{k(\omega_1 - \omega_2)}{2}\cosh\left(Z_1 + Z_2 + \frac{k}{2}(\omega_1 + \omega_2)\right) + \sinh\frac{k(\omega_1 + \omega_2)}{2} \\ &\quad \times\cosh\left(Z_1 - Z_2 + \frac{k}{2}(\omega_1 - \omega_2)\right) \quad (k = 1,2). \end{aligned}$$

Then the 2-soliton solution for (16) is



$$u[2]=\frac{H_2^{(1)}}{H_1^{(1)}}-\frac{H_2}{H_1},\quad v[2]=\frac{H_1^{(1)}H_1^{(-1)}}{H_1^2},$$

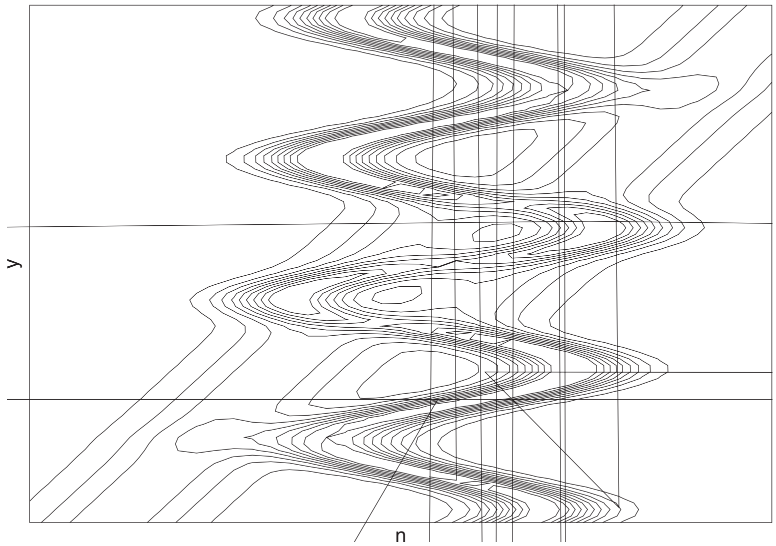
$$w_1[2]=c_1(y)a_1(y)\frac{2\sinh\omega_1(\cosh\omega_1-\cosh\omega_2)\exp\Omega_1\cosh(Z_2+\omega_2)}{H_1},$$

$$w_2[2]=c_2(y)a_2(y)\frac{2\sinh\omega_2(\cosh\omega_1-\cosh\omega_2)\exp\Omega_2\sinh(Z_1+\omega_1)}{H_1},$$

$$w_1^*[2]=-\frac{d_1(y)\exp(-\Omega_1)\cosh(Z_2+\omega_2)}{2H_1^{(1)}},\quad c_1d_1=\dot{\alpha}_1,$$

$$w_2^*[2]=\frac{d_2(y)\exp(-\Omega_2)\sinh(Z_1+\omega_1)}{2H_1^{(1)}},\quad c_2d_2=\dot{\alpha}_2.$$

The speed of two solitary waves are determined by α_1 and α_2, respectively. So it produces rich dynamic behaviors. For example, we can make the first solitary wave travel at a periodically changing speed. The 2-soliton interactions are shown by Fig. 2. The interaction



makes phase shifts, which can be shown by the contour graph of $u[2]$ in Fig. 3.

B. Rational solution

In Eq. (30), noticing that $\partial_z^k \psi$ is another eigenfunction, we find that $g(n,x,y)=z^n \exp(zx+z^{-1}y)$ and $f_k(n,x,y) := \partial_z^k g$ $(k \geq 1)$ are all independent eigenfunctions for (30). So let $\xi := zx+z^{-1}y$,

$$f_1(n,x,y) = \partial_z g = z^{n-1} e^{\xi}(n + z\xi_z),$$

$$f_2(n,x,y) = \partial_z^2 g = z^{n-2} e^{\xi}(n^2 + 2nz\xi_z - n + z^2\xi_z^2 + z^2\xi_{zz}),$$

$$f_3(n,x,y) = \cdots,$$

let $h_k = f_k + a(y)g$, and take $k=1$, $m=1$. We have a rational solution for (16):

$$u[1] = -\frac{z}{(\eta + za + 1/2)^2 - 1/4},$$

$$v[1] = 1 - \frac{1}{(\eta + za)^2},$$

$$w[1] = c(y)\frac{z^{n+1}e^{\xi}a}{\eta + za},$$

$$w^*[1] = d(y)\frac{z^{-n}e^{-\xi}}{\eta + za + 1},$$

where $\eta = n + z\xi_z$, $c(y)d(y) = (d/dy)\log a$. This is a nonsmooth weak solution which has singularities.

If take $k=2$, $m=1$, we find another rational solution:

$$u[1] := z\left(\frac{\eta^{(2)}+z^2a}{\eta^{(1)}+z^2a} - \frac{\eta^{(1)}+z^2a}{\eta+z^2a}\right),$$

$$v[1] := \frac{(\eta^{(1)}+z^2a)(\eta^{(-1)}+z^2a)}{(\eta+z^2a)^2},$$

$$w[1] := 2c(y)az^{n+1}e^{\xi}\frac{n+z\xi_z}{\eta+z^2a},$$

$$w^*[1] := d(y)\frac{z^{-n+1}e^{-\xi}}{\eta^{(1)}+z^2a},$$

where $\eta=n^2+2nz\xi_z-n+z^2\xi_z^2+z^2\xi_{zz}$, $c(y)d(y)=(d/dy)\log a$.

C. Other weak solutions

We can obtain other types of weak solutions, such as negatons, positons, complexitons. Let

$$f = z^n e^{zx+z^{-1}y} := z^n e^{F(x,y,z)}, \quad g = z^{-n}e^{z^{-1}x+zy} := z^{-n}e^{G(x,y,z)}$$

be pair of eigenfunctions to (30). Then f_z and g_z are another pair of eigenfunctions. Let $h=f+a(y)g=2\exp\Omega\cdot\cosh Z$, where Ω, $a(y)$, and Z are defined in Sec. IV A. Then

$$h_z = f_z + a(y)g_z = 2\Omega_z e^{\Omega}\cosh Z + 2e^{\Omega}Z_z\sinh Z.$$

Taking $m=2$, we obtain the following solution. For simplicity, we define

$$\begin{vmatrix} h & h_z \\ h^{(k)} & h_z^{(k)} \end{vmatrix} = 4e^{2\Omega}C_k, \quad \text{where} \quad C_k = \left(Z_z + \frac{k}{2z}\right)\sinh(k\omega) + \frac{k}{2z}\sinh(2Z+k\omega),$$

$$\mathrm{Cas}(h,h_z,f) = -8a(x)e^{3\Omega}\frac{\sinh^2\omega}{z}\cosh(Z+\omega),$$

$$\mathrm{Cas}(h,h_z,f_z) = 4a(x)e^{3\Omega}(D_1^{(1)}\cosh Z + D_1\cosh(Z+2\omega) - D_2\cosh(Z+\omega)),$$

where

$$D_k = -\left(\frac{n}{z}+\frac{k}{2z}+F_z\right)\left(\frac{n}{z}+\frac{k}{2z}-G_z\right)\sinh(k\omega) + \frac{k^2}{4z^2}\sinh(k\omega) + \frac{k\Omega_z}{z}\cosh(k\omega), \quad k=1,2.$$

Then we find the following solution for (16):

$$u[2] = \frac{C_2^{(1)}}{C_1^{(1)}} - \frac{C_2}{C_1}, \quad v[2] = \frac{C_1^{(1)}C_1^{(-1)}}{C_1^2}, \tag{31a}$$

$$w_1[2] = -2c_1(y)a(y)\frac{\sinh^2\omega e^{\Omega}}{zC_1}\cosh(Z+\omega), \tag{31b}$$

$$w_2[2] = c_2(y)a(y)\frac{e^{\Omega}(D_1^{(1)}\cosh Z + D_1\cosh(Z+2\omega) - D_2\cosh(Z+\omega))}{C_1}, \tag{31c}$$

$$w_1^*[2] = -d_1(y)\frac{\Omega_z \cosh(Z+\omega) + (Z_z + 1/z)\sinh(Z+\omega)}{2e^{\Omega}C_1^{(1)}}, \tag{31d}$$

$$w_2^*[2] = d_2(y)\frac{\cosh(Z+\Omega)}{2e^{\Omega}C_1^{(1)}}, \tag{31e}$$

where $c_j(y)d_j(y)=(d/dy)\log a(y)$. Equations (31) represent two types of solution for 2DTLSCS depending on the choice of ω and $\alpha(y)$: when $\omega,\alpha(y)\in\mathbb{R}$, it denotes a negaton solution, while when $\omega,\alpha(y)\in i\mathbb{R}$, it denotes a positon solution.

In general, ω can be any complex number, but the outcome may not be a real valued solution. Therefore, in the general case, we adopt the following method to get the real solution: Suppose $\omega=\xi+i\eta$. Then the real (or imaginary) part of f, g and derivatives of the real (or imaginary) part of f and g are all eigenfunctions of (30). So letting $a(y)\in\mathbb{R}$, we can get the real solution of 2DTLSCS by using (for instance) $\mathrm{Re}(f)$, $\mathrm{Re}(f)_\xi$ and

$$\mathrm{Re}(h) = \mathrm{Re}(f) + a(y)\mathrm{Re}(g), \quad \mathrm{Re}(h)_\xi = \mathrm{Re}(f)_\xi + a(y)\mathrm{Re}(g)_\xi$$

in (28). For example, the real part of h can be written as

$$\mathrm{Re}(h) = 2e^{\Omega_1}[\cosh Z_1 \cos Z_2 \cos \Omega_2 - \sinh Z_1 \sin Z_2 \sin \Omega_2],$$

where

$$\Omega_1 = (x+y)\cosh \xi \cos \eta + \alpha/2,$$

$$\Omega_2 = (x+y)\sinh \xi \sin \eta,$$

$$Z_1 = n\xi + (x-y)\sinh \xi \cos \eta - \alpha/2,$$

$$Z_2 = n\eta + (x-y)\cosh \xi \sin \eta.$$

According to this expression, the real solution obtained will involve exponential and trigonometric functions and correspond to the complex eigenvalue of the associated Lax presentation. Thus it is a so-called one-complexiton solution.[30]

Furthermore, by using the series of eigenfunctions $h_1,\ldots,h_k$ and their derivatives $\partial_{z_i}^j h_i$ or $\partial_{\xi_i}^j \mathrm{Re}(h_i)$, etc., we can get higher order positons, negatons, or complexitons or interaction solutions.

V. CONCLUSION

We present an ex2DTLH by using squared eigenfunction symmetries. It helps us construct the two-dimensional Toda lattice equation with self-consistent sources in a different way from Refs. 2 and 25 as well as their Lax representations. Since the two-dimensional Toda lattice equation with self-consistent sources can be considered as a two-dimensional Toda lattice equation with nonhomogeneous terms, the method of VC can be applied to the ordinary DTs for 2DTLSCS to construct a nonauto-Bäcklund DT. Then it offers a different way to solve 2DTLSCS in contrast with Refs. 2 and 25.

The 2DTLH offers various types of reductions, for example, the periodic reductions in the sinh-Gordan equation and dimension reductions in the Toda lattice equation. Then the following question arises: does ex2DTLH offers similar reductions in the above equation with self-consistent sources? We may discuss such problems elsewhere.

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