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THE GENERALIZED KUPERSHMIDT DEFORMATION FOR CONSTRUCTING NEW DISCRETE INTEGRABLE SYSTEMS

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It is known that the KdV6 equation can be represented as the Kupershmidt deformation of the KdV equation. We propose a generalized Kupershmidt deformation for constructing new discrete integrable systems starting from the bi-Hamiltonian structure of a discrete integrable system. We consider the Toda, Kac–van Moerbeke, and Ablowitz–Ladik hierarchies and obtain Lax representations for these new deformed systems. The generalized Kupershmidt deformation provides a new way to construct discrete integrable systems.

Keywords: Kupershmidt deformation, bi-Hamiltonian system, discrete integrable system

1. Introduction

Deformations of integrable systems have recently attracted much attention [1]–[17]. As is known, a new integrable system can be constructed from a bi-Hamiltonian system [2], [3]. The KdV6 equation was derived in [10] by Painlevé analysis and has the form

$$u_t = u_{xxx} + 6uu_x - \omega_x, \qquad \omega_{xxx} + 4u\omega_x + 2u_x\omega = 0, \tag{1.1}$$

which could also be seen as a nonholonomic deformation of the KdV equation [10]. Many authors have studied various integrability properties of the KdV6 equation, the zero-curvature representation, the bi-Hamiltonian structure, conserved quantities, multisolitons, etc. [10]–[15]. Kupershimidt found that system (1.1) can be converted into

$$u_t = J\left(\frac{\delta H_3}{\delta u}\right) - J(\omega), \qquad K(\omega) = 0,$$

where $J = \partial = \partial_x$ and $K = \partial^3 + 2(u\partial + \partial u)$ are two standard Hamiltonian operators of the KdV hierarchy and $H_3 = u^3 - u_x^2/2$ is the Hamiltonian function. In general, for a bi-Hamiltonian system

$$u_{t_m} = J\left(\frac{\delta H_{m+1}}{\delta u}\right) = K\left(\frac{\delta H_m}{\delta u}\right)$$

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there is its nonholonomic deformation [1]

$$u_{t_m} = J\biggl(\frac{\delta H_{m+1}}{\delta u}\biggr) - J(\omega), \qquad K(\omega) = 0,$$

which is called the Kupershmidt deformation of bi-Hamiltonian systems.

In [15], we proved that the Kupershmidt deformation of the KdV equation could be seen as the Rosochatius deformation of the KdV equation with self-consistent sources and also obtained its bi-Hamiltonian structure. Further, in [16], [17], we constructed a generalized Kupershmidt deformation (GKD) for a certain bi-Hamiltonian system with purely differential Hamiltonian operators.

For an integrable system, we know that its two Hamiltonian operators are related by

$$K\biggl(\frac{\delta\lambda_j}{\delta u}\biggr) = \gamma_j J\biggl(\frac{\delta\lambda_j}{\delta u}\biggr),$$

where $\delta\lambda_j/\delta u$ can be taken from the spectral problem of the system and $\gamma_j = \lambda_j$ or $\gamma_j = \lambda_j^2$ depending on which integrable system we are studying. In an integrable system with self-consistent sources, eigenfunctions and adjoint eigenfunctions are components of the sources, and they satisfy the spectral problem and the adjoint spectral problem corresponding to λ_j. In the GKD case, we assume that nonholonomic terms satisfy $(K - \gamma_j J)(\delta\lambda_j/\delta u) = 0$ instead of the spectral problem, and they hence have more freedom because they satisfy higher-order equations. The GKD of the integrable system in the continuous case is therefore written as [16], [17]

$$u_{t_m} = J\biggl(\frac{\delta H_{m+1}}{\delta u} - \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\biggr), \qquad (K - \gamma_j J)\biggl(\frac{\delta\lambda_j}{\delta u}\biggr) = 0, \quad j = 1, 2, \ldots, N, \tag{1.2}$$

where γ_j equals λ_j or λ_j^2 depending on the integrable system.

The integrability of the generalized system can be proved by constructing its Lax representation. This paper is devoted to the GKD for the discrete integrable systems. As examples, we take the Toda lattice, Kac–van Moerbeke, and Ablowitz–Ladik hierarchies, for which we construct the GKD and also their Lax representation; we thus show that the new systems are integrable generalizations of the original discrete systems.

In [14], the concept of a mixed hierarchy of soliton equations was introduced by combining negative- and positive-order equations, and it was shown that Kupershmidt-deformed systems are particular representations of mixed soliton hierarchies. The mixed hierarchy of soliton equations in [14], which have one nonholonomic term, are certain special cases of nonholonomic deformed equations. Our GKD admits N nonholonomic terms.

This paper is organized as follows. In Sec. 2, we construct the GKD for the Toda hierarchy. In Sec. 3, we consider the GKD for the Kac–van Moerbeke hierarchy. In Sec. 4, we obtain the GKD for the Ablowitz–Ladik hierarchy. We present some conclusions in Sec. 5.

2. The generalized Kupershmidt deformation of the Toda hierarchy

We recall some fundamental concepts in the theory of discrete integrable systems. We take a function $f = f(n,t)$, where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. We define the shift operator E and the difference operator D as

$$(Ef)(n) = f(n+1), \quad (E^{-1}f)(n) = f(n-1), \quad (Df)(n) = (E-1)f(n), \quad n \in \mathbb{Z},$$

and set $f^{(j)} = E^j f$, $j \in \mathbb{Z}$.

We consider the discrete eigenvalue problem for the Toda hierarchy [18]

$$\Psi^{(-1)} = U(u, v, \lambda)\Psi, \qquad U(u, v, \lambda) = \begin{pmatrix} 0 & 1 \\ -v^{(1)} & \lambda - u \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi^{(1)} \\ \psi \end{pmatrix}. \tag{2.1}$$

We first consider the stationary zero-curvature equation for the generating function Γ,

$$\Gamma^{(-1)}U - U\Gamma = 0, \qquad \Gamma = \sum_{i=0}^{+\infty} \Gamma_i \lambda^{-i} = \sum_{i=0}^{+\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i},$$

where a_i, b_i, and c_i are functions of n and t. In the general case, we have the recurrence relation

$$Da_{i+1} = u^{(1)}Da_i - c_i - b_i^{(1)} v^{(2)}, \qquad b_{i+1} = u^{(1)} b_i - a_i^{(1)} - a_i, \qquad c_i = -v^{(1)} b_i^{(-1)},$$

where we use the initial data $a_0 = 1/2$ and $b_0 = c_0 = 0$.

We define the modification matrix $\Delta_m = \mathrm{diag}(b_{m+1} + \delta, \delta)$, where δ is an arbitrary constant. Let

$$\Psi_{t_m} = V_m \Psi, \qquad V_m = (\lambda^m \Gamma)_+ + \Delta_m = \sum_{i=0}^{m} \Gamma_i \lambda^{m-i} + \Delta_m. \tag{2.2}$$

The compatibility condition for (2.1) and (2.2) then yields the zero-curvature representation of the Toda lattice hierarchy

$$U_{t_m} = UV_m - V_m^{(-1)} U.$$

For $m = 1$, we have the famous Toda equation

$$v_t = v(u^{(-1)} - u), \qquad u_t = v - v^{(1)}.$$

The bi-Hamiltonian structure of the Toda hierarchy is (see, e.g., [19])

$$\begin{pmatrix} v \\ u \end{pmatrix}_{t_m} = J^{(\mathrm{T})} \begin{pmatrix} a_{m+1}^{(-1)}/v \\ -b_{m+1}^{(-1)} \end{pmatrix} = K^{(\mathrm{T})} \begin{pmatrix} a_m^{(-1)}/v \\ -b_m^{(-1)} \end{pmatrix},$$

where

$$J^{(\mathrm{T})} = \begin{pmatrix} 0 & v(E^{-1} - 1) \\ (1 - E)v & 0 \end{pmatrix}, \qquad K^{(\mathrm{T})} = \begin{pmatrix} -vEv + vE^{-1}v & -vu + vE^{-1}u \\ vu - uEv & vE^{-1} - Ev \end{pmatrix}$$

are two standard Hamiltonian operators.

We now derive $\delta\lambda_j/\delta v$ and $\delta\lambda_j/\delta u$ from (2.1) and its adjoint problem, setting $\lambda = \lambda_j$. We introduce the operator $L = v^{(1)}E + u + E^{-1}$ and its adjoint $L^* = vE^{-1} + u + E$. We then obtain $L\psi_j = \lambda_j \psi_j$ and $L^*\phi_j = \lambda_j \phi_j$. We assume that L, λ_j, ψ_j, and ϕ_j have a perturbation ε. We can then assume that they are functions of ε. Furthermore, we can choose ϕ_j such that $\langle \psi_j, \phi_j \rangle = 1$, where the inner product is defined as $\langle f, g \rangle = \sum_n f(n)g(n)$. Taking these assumptions into account, we obtain

$$\begin{aligned} \frac{d\lambda_j(\varepsilon)}{d\varepsilon} &= \frac{d\langle \lambda_j \psi_j, \phi_j \rangle}{d\varepsilon} = \frac{d\langle L\psi_j, \phi_j \rangle}{d\varepsilon} = \\ &= \left\langle \frac{dL}{d\varepsilon}\psi_j, \phi_j \right\rangle + \left\langle L\frac{d\psi_j}{d\varepsilon}, \phi_j \right\rangle + \left\langle L\psi_j, \frac{d\phi_j}{d\varepsilon} \right\rangle = \\ &= \left\langle \frac{dv^{(1)}}{d\varepsilon}\, \psi_j^{(1)} + \frac{du}{d\varepsilon}\, \psi_j, \phi_j \right\rangle + \left\langle \frac{d\psi_j}{d\varepsilon}, L^*\phi_j \right\rangle + \left\langle L\psi_j, \frac{d\phi_j}{d\varepsilon} \right\rangle = \\ &= \left\langle \frac{dv^{(1)}}{d\varepsilon}\, \psi_j^{(1)} + \frac{du}{d\varepsilon}\, \psi_j, \phi_j \right\rangle + \lambda_j \frac{d\langle \psi_j, \phi_j \rangle}{d\varepsilon} = \left\langle \frac{dv^{(1)}}{d\varepsilon}\, \psi_j^{(1)} + \frac{du}{d\varepsilon}\, \psi_j, \phi_j \right\rangle. \end{aligned}$$

We hence have

$$\begin{pmatrix} \delta\lambda_j/\delta v \\ \delta\lambda_j/\delta u \end{pmatrix} = \begin{pmatrix} \psi_j \phi_j^{(-1)} \\ \psi_j \phi_j \end{pmatrix}.$$

We set $\gamma_j = \lambda_j$, and (1.2) then yields a new generalized Toda lattice hierarchy

$$\begin{pmatrix} v \\ u \end{pmatrix}_{t_m} = J^{(\mathrm{T})} \left(\begin{pmatrix} a_{m+1}^{(-1)}/v \\ -b_{m+1}^{(-1)} \end{pmatrix} - \sum_{j=1}^{N} \begin{pmatrix} \bar\psi_j \bar\phi_j^{(-1)} \\ \bar\psi_j \bar\phi_j \end{pmatrix} \right), \tag{2.3a}$$

$$(K^{(\mathrm{T})} - \lambda_j J^{(\mathrm{T})}) \begin{pmatrix} \bar\psi_j \bar\phi_j^{(-1)} \\ \bar\psi_j \bar\phi_j \end{pmatrix} = 0, \quad j = 1, 2, \ldots, N. \tag{2.3b}$$

We note that in our new deformed system, $\bar\psi_j$ and $\bar\phi_j$ are no longer eigenfunctions and adjoint eigenfunctions of the spectral problem but are solutions of (2.3b). Simplifying that equation, we obtain

$$-v^{(1)}\bar\psi_j^{(1)}\bar\phi_j + v^{(-1)}\bar\psi_j^{(-1)}\bar\phi_j^{(-2)} - u\bar\psi_j\bar\phi_j + u^{(-1)}\bar\psi_j^{(-1)}\bar\phi_j^{(-1)} - \lambda_j\bar\psi_j^{(-1)}\bar\phi_j^{(-1)} + \lambda_j\bar\psi_j\bar\phi_j = 0,$$
$$vu\bar\psi_j\bar\phi_j^{(-1)} - uv^{(1)}\bar\psi_j^{(1)}\bar\phi_j + v\bar\psi_j^{(-1)}\bar\phi_j^{(-1)} - v^{(1)}\bar\psi_j^{(1)}\bar\phi_j^{(1)} + \lambda_j v^{(1)}\bar\psi_j^{(1)}\bar\phi_j - \lambda_j v\bar\psi_j\bar\phi_j^{(-1)} = 0.$$

Introducing the notation

$$f_j = u\bar\psi_j + v^{(1)}\bar\psi_j^{(1)} + \bar\psi_j^{(-1)} - \lambda_j\bar\psi_j, \qquad g_j = u\bar\phi_j + v\bar\phi_j^{(-1)} + \bar\phi_j^{(1)} - \lambda_j\bar\phi_j,$$

we bring these equations to the form

$$\bar\phi_j f_j - \bar\psi_j^{(-1)} g_j^{(-1)} = 0, \qquad v\bar\phi_j^{(-1)} f_j - v^{(1)}\bar\psi_j^{(1)} g_j = 0.$$

It hence follows that

$$f_j = \frac{\mu_j}{v\bar\psi_j\bar\phi_j\bar\phi_j^{(-1)}}, \qquad g_j = \frac{\mu_j}{v^{(1)}\bar\psi_j\bar\psi_j^{(1)}\bar\phi_j},$$

where μ_j are some arbitrary constants. Thus, having simplified (2.3b), we see that our new constraints are the original spectral problem with some additional terms involving some arbitrary constants μ_j. The GKD of the Toda lattice hierarchy is

$$\begin{pmatrix} v \\ u \end{pmatrix}_{t_m} = J^{(\mathrm{T})} \left(\begin{pmatrix} a_{m+1}^{(-1)}/v \\ -b_{m+1}^{(-1)} \end{pmatrix} - \sum_{j=1}^{N} \begin{pmatrix} \bar\psi_j \bar\phi_j^{(-1)} \\ \bar\psi_j \bar\phi_j \end{pmatrix} \right),$$
$$u\bar\psi_j + v^{(1)}\bar\psi_j^{(1)} + \bar\psi_j^{(-1)} = \lambda_j\bar\psi_j + \frac{\mu_j}{v\bar\psi_j\bar\phi_j\bar\phi_j^{(-1)}},$$
$$u\bar\phi_j + v\bar\phi_j^{(-1)} + \bar\phi_j^{(1)} = \lambda_j\bar\phi_j + \frac{\mu_j}{v^{(1)}\bar\psi_j\bar\psi_j^{(1)}\bar\phi_j}.$$

We can prove that in the GKD of the Toda hierarchy, the constraints are the spectral problem and its adjoint problem with some nonholonomic terms with arbitrary constants μ_j.

With $m=1$, the GKD of the Toda equation is

$$v_t = v\Big(u^{(-1)} + \sum_{j=1}^{N} \bar\psi_j^{(-1)}\bar\phi_j^{(-1)}\Big) - v\Big(u + \sum_{j=1}^{N}\bar\psi_j\bar\phi_j\Big), \tag{2.4a}$$

$$u_t = v\Big(1 + \sum_{j=1}^{N}\bar\psi_j\bar\phi_j^{(-1)}\Big) - v^{(1)}\Big(1 + \sum_{j=1}^{N}\bar\psi_j^{(1)}\bar\phi_j\Big), \tag{2.4b}$$

$$u\bar\psi_j + v^{(1)}\bar\psi_j^{(1)} + \bar\psi_j^{(-1)} = \lambda_j\bar\psi_j + \frac{\mu_j}{v\bar\psi_j\bar\phi_j\bar\phi_j^{(-1)}}, \tag{2.4c}$$

$$u\bar\phi_j + v\bar\phi_j^{(-1)} + \bar\phi_j^{(1)} = \lambda_j\bar\phi_j + \frac{\mu_j}{v^{(1)}\bar\psi_j\bar\psi_j^{(1)}\bar\phi_j}, \tag{2.4d}$$

and its Lax representation is

$$(v^{(1)}E + u + E^{-1})\psi = \lambda\psi,$$

$$-\psi_t = v^{(1)}\psi^{(1)} + \sum_{j=1}^{N}\frac{v^{(1)}\bar\phi_j}{\lambda-\lambda_j}(\psi\bar\psi_j^{(1)} - \psi^{(1)}\bar\psi_j).$$

The compatibility of this system under conditions (2.4c) and (2.4d) gives (2.4a) and (2.4b). It is an interesting question to find a Lax pair with μ_j inside to give system (2.4). This will be studied in the future.

If we take $\lambda_j = 0$, then (2.4) reduces to the Kupershmidt deformation of the Toda equation. If we take $\mu_j = 0$, then (2.4) reduces to the Toda equation with self-consistent sources [20].

3. The generalized Kupershmidt deformation of the Kac–van Moerbeke hierarchy

We consider the eigenvalue problem for the Kac–van Moerbeke hierarchy [21]

$$(E + vE^{-1})\psi = \lambda\psi.$$

Its adjoint equation is

$$(E^{-1} + Ev)\phi = \lambda\phi.$$

We can also write the eigenvalue problem in matrix form

$$E\begin{pmatrix}\psi^{(-1)}\\ \psi\end{pmatrix} = U\begin{pmatrix}\psi^{(-1)}\\ \psi\end{pmatrix}, \qquad U = \begin{pmatrix}0 & 1\\ -v & \lambda\end{pmatrix}. \tag{3.1}$$

To derive the hierarchy of evolution equations associated with the eigenvalue problem, we first solve the stationary discrete zero-curvature equation $\Gamma^{(1)}U - U\Gamma = 0$. Let

$$\Gamma = \begin{pmatrix}a & b\\ c & -a\end{pmatrix} = \sum_{i=0}^{+\infty}\Gamma_i\lambda^{-i} = \sum_{i=0}^{+\infty}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{-i}.$$

We choose initial data in the form $a_0 = 1/2$, $b_0 = 0$, and $b_1 = -1$. We can obtain the recurrence relation

$$b_{2i} = c_{2i} = a_{2i+1} = 0, \qquad c_{2i+1} = -vb_{2i+1}^{(1)},$$

$$b_{2i+1} = -(a_{2i}^{(-1)} + a_{2i}), \qquad c_{2i+1}^{(1)} + vb_{2i+1} = a_{2i+2}^{(1)} - a_{2i+2}, \quad i = 0, 1, \ldots.$$

As the auxiliary linear problem, we take the equation

$$\psi_{t_m} = V_{2m}\psi, \qquad V_{2m} = (\Gamma\lambda^{2m})_+ + \Delta_{2m}, \tag{3.2}$$

where

$$(\Gamma\lambda^{2m})_+ + \Delta_{2m} = \begin{pmatrix} \sum_{i=0}^{m} a_{2i}\lambda^{2m-2i} & \sum_{i=0}^{m-1} b_{2i+1}\lambda^{2m-2i-1} \\ \sum_{i=0}^{m-1} c_{2i+1}\lambda^{2m-2i-1} & -\sum_{i=0}^{m} a_{2i}\lambda^{2m-2i} \end{pmatrix} + \begin{pmatrix} b_{2m+1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The compatibility condition for (3.1) and (3.2) then yields the zero-curvature representation of the Kac–van Moerbeke hierarchy

$$U_{t_m} = V_{2m}^{(1)}U - UV_{2m}.$$

If $m = 1$, then we have the Kac–van Moerbeke equation $v_t = v(v^{(-1)} - v^{(1)})$.

The Hamiltonian operators of the Kac–van Moerbeke hierarchy are defined as

$$J^{(\mathrm{KvM})} = v(E^{-1} - E)v,$$
$$K^{(\mathrm{KvM})} = v(vE^{-1} + v^{(-1)}E^{-1} + v^{(-1)}E^{-2} - vE - v^{(1)}E^2 - v^{(1)}E)v.$$

The Kac–van Moerbeke hierarchy has the bi-Hamiltonian form

$$v_{t_m} = J^{(\mathrm{KvM})}\frac{\delta H_{2m}}{\delta v} = K^{(\mathrm{KvM})}\frac{H_{2m-2}}{\delta v},$$

where $H_{2m} = -b_{2m+1}/2m$ is the Hamiltonian function. It is easy to find that

$$\frac{\delta\lambda_j}{\delta v} = \psi_j^{(-1)}\phi_j.$$

The new generalized Kac–van Moerbeke hierarchy is constructed as

$$v_{t_m} = J^{(\mathrm{KvM})}\frac{\delta H_{2m}}{\delta v} + J^{(\mathrm{KvM})}\sum_{j=1}^{N}\bar{\psi}_j^{(-1)}\bar{\phi}_j, \tag{3.3a}$$

$$(K^{(\mathrm{KvM})} - \lambda_j^2 J^{(\mathrm{KvM})})(\bar{\psi}_j^{(-1)}\bar{\phi}_j) = 0. \tag{3.3b}$$

We assume that

$$(E + vE^{-1})\bar{\psi}_j = \lambda_j\bar{\psi}_j + f_j, \qquad (E^{-1} + Ev)\bar{\phi}_j = \lambda_j\bar{\phi}_j + g_j. \tag{3.4}$$

Simplifying (3.3b), we obtain

$$\begin{aligned}(vv^{(-1)}\bar{\psi}_j^{(-2)} - v^{(1)}\bar{\psi}_j)g_j + (v^{(-1)}\bar{\phi}_j^{(-1)} - vv^{(1)}\bar{\phi}_j^{(1)})f_j^{(-1)} + \\ + v^{(-1)}(\lambda_j\bar{\psi}_j^{(-2)} - \bar{\psi}_j^{(-1)})g_j^{(-1)} + v^{(-1)}(\lambda_j\bar{\phi}_j^{(-1)} - v\bar{\phi}_j)f_j^{(-2)} - \\ - v^{(1)}(\lambda_j\bar{\psi}_j - v\bar{\psi}_j^{(-1)})g_j^{(1)} - v^{(1)}(\lambda_j\bar{\phi}_j^{(1)} - \bar{\phi}_j)f_j + \\ + v^{(-1)}g_j^{(-1)}f_j^{(-2)} - v^{(1)}g_j^{(1)}f_j + (v^{(1)} - v^{(-1)})(\bar{\psi}_j\bar{\phi}_j^{(-1)} - v\bar{\psi}_j^{(-1)}\bar{\phi}_j) = 0.\end{aligned} \tag{3.5}$$

We multiply the first equation in (3.4) by $\bar{\phi}_j$ and the second by $\bar{\psi}_j$ and take the difference of the obtained expressions. But we first note that

$$\bar{\phi}_j f_j - \bar{\psi}_j g_j = (E-1)(\bar{\psi}_j \bar{\phi}_j^{(-1)} - v\bar{\psi}_j^{(-1)}\bar{\phi}_j).$$

Now assuming that

$$v\bar{\phi}_j^{(-1)} f_j^{(-1)} = v\bar{\psi}_j g_j, \tag{3.6}$$

we obtain

$$\bar{\psi}_j g_j = \bar{\psi}_j \bar{\phi}_j^{(-1)} - v\bar{\psi}_j^{(-1)}\bar{\phi}_j. \tag{3.7}$$

Substituting (3.7) in (3.5) and setting

$$v^{(1)}\bar{\phi}_j^{(1)} f_j = v\bar{\psi}_j^{(-1)} g_j, \tag{3.8}$$

we see that (3.5) is satisfied. Taking assumptions (3.6) and (3.8) into account, we write the solutions f_j and g_j as

$$f_j = \frac{\mu_j}{v^{(1)}\bar{\psi}_j\bar{\phi}_j\bar{\phi}_j^{(1)}}, \qquad g_j = \frac{\mu_j}{v\bar{\psi}_j^{(-1)}\bar{\psi}_j\bar{\phi}_j},$$

where μ_j are some arbitrary constants. Hence, the GKD of the Kac–van Moerbeke hierarchy is

$$v_{t_m} = J^{(\mathrm{KvM})}\frac{\delta H_{2m}}{\delta v} + J^{(\mathrm{KvM})}\sum_{j=1}^{N}\bar{\psi}_j^{(-1)}\bar{\phi}_j,$$

$$\bar{\psi}_j^{(1)} + v\bar{\psi}_j^{(-1)} = \lambda_j\bar{\psi}_j + \frac{\mu_j}{v^{(1)}\bar{\psi}_j\bar{\phi}_j\bar{\phi}_j^{(1)}},$$

$$\bar{\phi}_j^{(-1)} + v^{(1)}\bar{\phi}_j^{(1)} = \lambda_j\bar{\phi}_j + \frac{\mu_j}{v\bar{\psi}_j^{(-1)}\bar{\psi}_j\bar{\phi}_j}.$$

If $m=1$, then the GKD of the Kac–van Moerbeke equation is

$$v_t = v(v^{(-1)} - v^{(1)}) + \sum_{j=1}^{N}(vv^{(-1)}\bar{\psi}_j^{(-2)}\bar{\phi}_j^{(-1)} - vv^{(1)}\bar{\psi}_j\bar{\phi}_j^{(1)}), \tag{3.9a}$$

$$\bar{\psi}_j^{(1)} + v\bar{\psi}_j^{(-1)} = \lambda_j\bar{\psi}_j + \frac{\mu_j}{v^{(1)}\bar{\psi}_j\bar{\phi}_j\bar{\phi}_j^{(1)}}, \tag{3.9b}$$

$$\bar{\phi}_j^{(-1)} + v^{(1)}\bar{\phi}_j^{(1)} = \lambda_j\bar{\phi}_j + \frac{\mu_j}{v\bar{\psi}_j^{(-1)}\bar{\psi}_j\bar{\phi}_j} \tag{3.9c}$$

and has the Lax pair

$$\psi^{(1)} + v\psi^{(-1)} = \lambda\psi,$$

$$\psi_t = \lambda v\psi^{(-1)} - \left(\frac{1}{2}\lambda^2 + v\right)\psi + \sum_{j=1}^{N}\frac{\lambda_j^2}{\lambda^2-\lambda_j^2}v\bar{\psi}_j^{(-1)}\bar{\phi}_j\psi - \frac{\lambda\lambda_j}{\lambda^2-\lambda_j^2}v\bar{\psi}_j\bar{\phi}_j\psi^{(-1)}.$$

The compatibility of this system under conditions (3.9b) and (3.9c) gives (3.9a). It is also an interesting question to find a Lax pair with μ_j inside to give system (3.9). This will be studied in the future.

If we set $\lambda_j = 0$, then the GKD of Kac–van Moerbeke equation (3.9) reduces to the Kupershmidt deformation version. If we set $\mu_j = 0$, then (3.9) leads to the Kac–van Moerbeke equation with self-consistent sources.

4. The generalized Kupershmidt deformation of the Ablowitz–Ladik hierarchy

We consider the Ablowitz–Ladik discrete isospectral problem [22]

$$E\psi = U\psi, \qquad U = \begin{pmatrix} z & Q \\ R & 1/z \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.1}$$

where $Q = Q(n,t)$, $R = R(n,t)$, and z is the spectral parameter. The adjoint problem is

$$E^{(-1)}\phi = \phi U, \qquad \phi = (\phi_1, \phi_2).$$

To derive the Ablowitz–Ladik hierarchy, we first solve the discrete zero-curvature equation $(E\Gamma)U - U\Gamma = 0$. Setting

$$\Gamma = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{4.2}$$

we obtain

$$\begin{aligned} &A^{(1)}z + B^{(1)}R - Az - CQ = 0, \qquad A^{(1)}Q + B^{(1)}\frac{1}{z} - Bz + AQ = 0, \\ &C^{(1)}z - A^{(1)}R - AR - C\frac{1}{z} = 0, \qquad C^{(1)}Q - A^{(1)}\frac{1}{z} - BR + A\frac{1}{z} = 0. \end{aligned} \tag{4.3}$$

We can expand these relations in corresponding power series in z and $1/z$ [23].

4.1. Expansion of Γ in a power series in $1/z$. We assume that

$$\Gamma = \sum_{i=0}^{\infty} \begin{pmatrix} A_{2i}z^{-2i} & B_{2i+1}z^{-2i-1} \\ C_{2i+1}z^{-2i-1} & -A_{2i}z^{-2i} \end{pmatrix}. \tag{4.4}$$

Recurrence relation (4.3) then leads to

$$\begin{aligned} &A_0^{(1)} - A_0 = 0, \qquad B_1 = Q(A_0^{(1)} + A_0), \qquad C_1^{(1)} = R(A_0^{(1)} + A_0), \\ &A_{2i}^{(1)} - A_{2i} = QC_{2i-1} - RB_{2i-1}^{(1)} = QC_{2i+1}^{(1)} - RB_{2i+1}, \\ &B_{2i+1} = Q(A_{2i}^{(1)} + A_{2i}) + B_{2i-1}^{(1)}, \qquad C_{2i+1}^{(1)} = R(A_{2i}^{(1)} + A_{2i}) + C_{2i-1}, \quad i = 1, 2, \ldots, \end{aligned}$$

where we can take $A_0 = 1/2$ as the initial value.

We can write the Ablowitz–Ladik hierarchy in the Hamiltonian form [23]

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J^{(\mathrm{AL})}\frac{\delta H_m}{\delta u} = K^{(\mathrm{AL})}\frac{\delta H_{m-1}}{\delta u}, \tag{4.5}$$

where $H_m = -A_{2m}/m$ and the Hamiltonian operators are

$$J^{(\mathrm{AL})} = \begin{pmatrix} 0 & 1 - RQ \\ RQ - 1 & 0 \end{pmatrix}, \qquad K^{(\mathrm{AL})} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \tag{4.6}$$

where

$$K_{11} = QD^{-1}Q^{(1)}(1-RQ) + (1-RQ)Q^{(1)}D^{-1}EQ,$$

$$K_{12} = E(1-RQ) - QD^{-1}R^{(1)}E^{(2)}(1-RQ) - (1-RQ)Q^{(1)}D^{-1}ER,$$

$$K_{21} = -E^{(-1)}(1-RQ) - RD^{(-1)}QE^{(-1)}(1-RQ) + (RQ-1)R^{(-1)}D^{-1}Q,$$

$$K_{22} = RD^{-1}RE(1-RQ) + (1-RQ)R^{(-1)}D^{-1}R.$$

It is easy to show that

$$\frac{\delta z}{\delta Q} = \psi_2\phi_1, \qquad \frac{\delta z}{\delta R} = \psi_1\phi_2.$$

The GKD of the Ablowitz–Ladik hierarchy is

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J^{(\mathrm{AL})} \begin{pmatrix} \dfrac{\delta H_m}{\delta Q} - \displaystyle\sum_{j=1}^{N} \tilde\psi_{2,j}\tilde\phi_{1,j} \\ \dfrac{\delta H_m}{\delta R} - \displaystyle\sum_{i=1}^{N} \tilde\psi_{1,j}\tilde\phi_{2,j} \end{pmatrix}, \tag{4.7a}$$

$$(K^{(\mathrm{AL})} - z_j^2 J^{(\mathrm{AL})}) \begin{pmatrix} \tilde\psi_{2,j}\tilde\phi_{1,j} \\ \tilde\psi_{1,j}\tilde\phi_{2,j} \end{pmatrix} = 0. \tag{4.7b}$$

We assume that

$$\begin{aligned} E\tilde\psi_{1,j} &= z_j\tilde\psi_{1,j} + Q\tilde\psi_{2,j} + f_{1,j}, & E\tilde\psi_{2,j} &= R\tilde\psi_{1,j} + \frac{1}{z_j}\tilde\psi_{2,j} + f_{2,j}, \\ E^{(-1)}\tilde\phi_{1,j} &= z_j\tilde\phi_{1,j} + R\tilde\phi_{2,j} + g_{1,j}, & E^{(-1)}\tilde\phi_{2,j} &= Q\tilde\phi_{1,j} + \frac{1}{z_j}\tilde\phi_{2,j} + g_{2,j}. \end{aligned} \tag{4.8}$$

Simplifying (4.7b), we obtain

$$\begin{aligned} &QD^{-1}Q^{(1)}(1-RQ)\tilde\psi_{2,j}\tilde\phi_{1,j} + (1-RQ)Q^{(1)}D^{-1}EQ\tilde\psi_{2,j}\tilde\phi_{1,j} + E(1-RQ)\tilde\psi_{1,j}\tilde\phi_{2,j} - \\ &\quad - QD^{-1}R^{(1)}E^{(2)}(1-RQ)\tilde\psi_{1,j}\tilde\phi_{2,j} - (1-RQ)Q^{(1)}D^{-1}ER\tilde\psi_{1,j}\tilde\phi_{2,j} - z_j^2(1-RQ)\tilde\psi_{1,j}\tilde\phi_{2,j} = 0, \\ &E^{(-1)}(1-RQ)\tilde\psi_{2,j}\tilde\phi_{1,j} + RD^{-1}QE^{(-1)}(1-RQ)\tilde\psi_{2,j}\tilde\phi_{1,j} + (1-RQ)R^{(-1)}D^{-1}Q\tilde\psi_{2,j}\tilde\phi_{1,j} - \\ &\quad - RD^{-1}RE(1-RQ)\tilde\psi_{1,j}\tilde\phi_{2,j} - (1-RQ)R^{(-1)}D^{-1}R\tilde\psi_{1,j}\tilde\phi_{2,j} - z_j^2(1-RQ)\tilde\psi_{2,j}\tilde\phi_{1,j} = 0. \end{aligned}$$

Substituting (4.8) in these equations, we obtain

$$\begin{aligned} &z_j(1-RQ)(\tilde\phi_{2,j}f_{1,j} - \tilde\psi_{1,j}^{(1)}g_{2,j}^{(1)}) + \\ &\quad + (1-RQ)Q^{(1)}[\tilde\psi_{1,j}^{(1)}g_{1,j}^{(1)} + D^{-1}E(\tilde\psi_{1,j}g_{1,j} - \tilde\phi_{1,j}f_{1,j})] + \\ &\quad + QD^{-1}[(1-RQ)\tilde\psi_{2,j}(Q^{(1)}g_{1,j}^{(1)} - zg_{2,j}^{(1)}) - (1-RQ)^{(1)}\tilde\phi_{2,j}^{(1)}(Rf_{1,j} - z_jf_{2,j})] = 0, \\ &z_j(1-RQ)(\tilde\psi_{2,j}g_{1,j} - \tilde\phi_{1,j}^{(-1)}f_{2,j}^{(-1)}) + \\ &\quad + (1-RQ)R^{(-1)}[\tilde\phi_{1,j}^{(-1)}f_{1,j}^{(-1)} + D^{-1}(\tilde\psi_{1,j}g_{1,j} - \tilde\phi_{1,j}f_{1,j})] + \\ &\quad + RD^{-1}[(1-RQ)\tilde\psi_{2,j}(Q^{(1)}g_{1,j}^{(1)} - zg_{2,j}^{(1)}) - (1-RQ)^{(1)}\tilde\phi_{2,j}^{(1)}(Rf_{1,j} - z_jf_{2,j})] = 0. \end{aligned}$$

We first assume that

$$f_{2,j} = \frac{R}{z_j} f_{1,j}, \qquad g_{2,j} = \frac{Q}{z_j} g_{1,j}.$$

The above expressions are then simplified, and we find that

$$f_{1,j} = \frac{\mu_j}{\tilde{\phi}_{1,j} - z_j \tilde{\phi}_{2,j}/Q^{(-1)}}, \qquad g_{1,j} = \frac{\mu_j}{\tilde{\psi}_{1,j} - z_j \tilde{\psi}_{2,j}/R^{(-1)}},$$

where μ_j are some arbitrary constants. Hence, the GKD of the Ablowitz–Ladik hierarchy is

$$\begin{aligned}
\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} &= J^{(\mathrm{AL})} \begin{pmatrix} \dfrac{\delta H_m}{\delta Q} - \displaystyle\sum_{j=1}^{N} \tilde{\psi}_{2,j} \tilde{\phi}_{1,j} \\ \dfrac{\delta H_m}{\delta R} - \displaystyle\sum_{i=1}^{N} \tilde{\psi}_{1,j} \tilde{\phi}_{2,j} \end{pmatrix}, \\
E\tilde{\psi}_{1,j} &= z_j \tilde{\psi}_{1,j} + Q\tilde{\psi}_{2,j} + \frac{\mu_j}{\tilde{\phi}_{1,j} - z_j \tilde{\phi}_{2,j}/Q^{(-1)}}, \\
E\tilde{\psi}_{2,j} &= R\tilde{\psi}_{1,j} + \frac{1}{z_j}\tilde{\psi}_{2,j} + \frac{\mu_j R}{z_j(\tilde{\phi}_{1,j} - z_j \tilde{\phi}_{2,j}/Q^{(-1)})}, \\
E^{(-1)}\tilde{\phi}_{1,j} &= z_j \tilde{\phi}_{1,j} + R\tilde{\phi}_{2,j} + \frac{\mu_j}{\tilde{\psi}_{1,j} - z_j \tilde{\psi}_{2,j}/R^{(-1)}}, \\
E^{(-1)}\tilde{\phi}_{2,j} &= Q\tilde{\phi}_{1,j} + \frac{1}{z_j}\tilde{\phi}_{2,j} + \frac{\mu_j Q}{z_j(\tilde{\psi}_{1,j} - z_j \tilde{\psi}_{2,j}/R^{(-1)})}.
\end{aligned} \tag{4.9}$$

Its Lax representation is

$$E\psi = U\psi, \qquad \psi_t = \left(V_m + \sum_{j=1}^{N} X_j \right)\psi,$$

where

$$\begin{aligned}
V_m &= (\Gamma z^{2m})_+ + \begin{pmatrix} 0 & 0 \\ 0 & A_{2m} \end{pmatrix}, \\
X_j &= \frac{1}{z^2 - z_j^2} \left[\begin{pmatrix} z_j^2 \tilde{\psi}_{1,j} \tilde{\phi}_{1,j}^{(-1)} & z z_j \tilde{\psi}_{1,j} \tilde{\phi}_{2,j}^{(-1)} \\ z z_j \tilde{\psi}_{2,j} \tilde{\phi}_{1,j}^{(-1)} & z^2 \tilde{\psi}_{2,j} \tilde{\phi}_{2,j}^{(-1)} \end{pmatrix} + \right. \\
&\quad \left. + (\tilde{\psi}_{1,j} \tilde{\phi}_{1,j}^{(-1)} + \tilde{\psi}_{2,j} \tilde{\phi}_{2,j}^{(-1)}) \begin{pmatrix} (z^2 - 3z_j^2)/4 & 0 \\ 0 & (z_j^2 - 3z^2)/4 \end{pmatrix} \right].
\end{aligned}$$

If we take $\mu_j = 0$, then generalized system (4.9) reduces to the Ablowitz–Ladik hierarchy with self-consistent sources (corresponding to the generating matrix Γ in the power series in $1/z$).

4.2. Γ expanded in a power series in z. Similarly, we assume that Γ in (4.2) is

$$\bar{\Gamma} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & -\bar{A} \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} \bar{A}_{2i} z^{2i} & \bar{B}_{2i+1} z^{2i+1} \\ \bar{C}_{2i+1} z^{2i+1} & -\bar{A}_{2i} z^{2i} \end{pmatrix}.$$

We then have the recurrence relations

$$\bar{A}_0^{(1)} - \bar{A}_0 = 0, \qquad \bar{B}_1^{(1)} = -Q(\bar{A}_0^{(1)} + \bar{A}_0), \qquad \bar{C}_1 = -R(\bar{A}_0^{(1)} + \bar{A}_0),$$
$$\bar{A}_{2i}^{(1)} - \bar{A}_{2i} = Q\bar{C}_{2i-1}^{(1)} - R\bar{B}_{2i-1} = Q\bar{C}_{2i+1} - R\bar{B}_{2i+1}^{(1)},$$
$$\bar{B}_{2i+1}^{(1)} = -Q(\bar{A}_{2i}^{(1)} + \bar{A}_{2i}) + \bar{B}_{2i-1}, \qquad \bar{C}_{2i+1} = -R(\bar{A}_{2i}^{(1)} + \bar{A}_{2i}) + \bar{C}_{2i-1}^{(1)}, \quad i = 1, 2, \ldots,$$

where we choose $\bar{A}_0 = 1/2$ as the initial value.

We can write the Ablowitz–Ladik hierarchy in the Hamiltonian form [21]

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J^{(\mathrm{AL})} \frac{\delta \bar{H}_m}{\delta u} = \bar{K}^{(\mathrm{AL})} \frac{\delta \bar{H}_{m-1}}{\delta u},$$

where $\bar{H}_m = -\bar{A}_{2m}/m$, $J^{(\mathrm{AL})}$ is given by (4.6), and

$$\bar{K}^{(\mathrm{AL})} = \begin{pmatrix} \bar{K}_{11} & \bar{K}_{12} \\ \bar{K}_{21} & \bar{K}_{22} \end{pmatrix}, \tag{4.10}$$

where

$$\bar{K}_{11} = -QD^{-1}QE(1 - RQ) - (1 - RQ)Q^{(1)}D^{-1}Q,$$
$$\bar{K}_{12} = E^{(-1)}(1 - RQ) + QD^{-1}RE^{(-1)}(1 - RQ) + (1 - RQ)Q^{(-1)}D^{-1}R,$$
$$\bar{K}_{21} = -E(1 - RQ) + RD^{-1}Q^{(1)}E^{(2)}(1 - RQ) + (1 - RQ)R^{(1)}D^{-1}EQ,$$
$$\bar{K}_{22} = -RD^{-1}R^{(1)}(1 - RQ) - (1 - RQ)R^{(1)}D^{-1}ER.$$

We have thus obtained another type of GKD of the Ablowitz–Ladik hierarchy

$$\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J^{(\mathrm{AL})} \begin{pmatrix} \dfrac{\delta H_m}{\delta Q} - \displaystyle\sum_{j=1}^{N} \bar{\psi}_{2,j}\bar{\phi}_{1,j} \\ \dfrac{\delta H_m}{\delta R} - \displaystyle\sum_{i=1}^{N} \bar{\psi}_{1,j}\bar{\phi}_{2,j} \end{pmatrix}, \qquad \left(\bar{K}^{(\mathrm{AL})} - \frac{1}{z_j^2} J^{(\mathrm{AL})}\right) \begin{pmatrix} \bar{\psi}_{2,j}\bar{\phi}_{1,j} \\ \bar{\psi}_{1,j}\bar{\phi}_{2,j} \end{pmatrix} = 0.$$

We assume that

$$E\bar{\psi}_{1,j} = z_j\bar{\psi}_{1,j} + Q\bar{\psi}_{2,j} + \bar{f}_{1,j}, \qquad E\bar{\psi}_{2,j} = R\bar{\psi}_{1,j} + \frac{1}{z_j}\bar{\psi}_{2,j} + \bar{f}_{2,j},$$
$$E^{(-1)}\bar{\phi}_{1,j} = z_j\bar{\phi}_{1,j} + R\bar{\phi}_{2,j} + \bar{g}_{1,j}, \qquad E^{(-1)}\bar{\phi}_{2,j} = Q\bar{\phi}_{1,j} + \frac{1}{z_j}\bar{\phi}_{2,j} + \bar{g}_{2,j}.$$

As in the case with Γ expanded in a power series in $1/z$, with some simplification, we assume that $\bar{f}_{1,j} = zQ\bar{f}_j^2$ and $\bar{g}_{1,j} = zR\bar{g}_j^2$ and then have

$$\bar{f}_{2,j} = \frac{\mu_j}{\bar{\phi}_{2,j} + \bar{\phi}_{1,j}/z_jR^{(1)}}, \qquad \bar{g}_{2,j} = \frac{\mu_j}{\bar{\psi}_{2,j} + \bar{\psi}_{1,j}/z_jQ^{(1)}},$$

where μ_j are some arbitrary constants. Hence, another type of GKD of the Ablowitz–Ladik hierarchy is

$$\begin{aligned}
&\begin{pmatrix} Q \\ R \end{pmatrix}_{t_m} = J^{(\mathrm{AL})} \begin{pmatrix} \dfrac{\delta H_m}{\delta Q} - \displaystyle\sum_{j=1}^{N} \bar\psi_{2,j}\bar\phi_{1,j} \\ \dfrac{\delta H_m}{\delta R} - \displaystyle\sum_{i=1}^{N} \bar\psi_{1,j}\bar\phi_{2,j} \end{pmatrix}, \\
&E\bar\psi_{1,j} = z_j\bar\psi_{1,j} + Q\bar\psi_{2,j} + \frac{\mu_j z_j Q}{\bar\phi_{2,j} + \bar\phi_{1,j}/z_j R^{(1)}}, \\
&E\bar\psi_{2,j} = R\bar\psi_{1,j} + \frac{1}{z_j}\bar\psi_{2,j} + \frac{\mu_j}{\bar\phi_{2,j} + \bar\phi_{1,j}/z_j R^{(1)}}, \\
&E^{(-1)}\bar\phi_{1,j} = z_j\bar\phi_{1,j} + R\bar\phi_{2,j} + \frac{\mu_j z_j R}{\bar\psi_{2,j} + \bar\psi_{1,j}/z_j Q^{(1)}}, \\
&E^{(-1)}\bar\phi_{2,j} = Q\bar\phi_{1,j} + \frac{1}{z_j}\bar\phi_{2,j} + \frac{\mu_j}{\bar\psi_{2,j} + \bar\psi_{1,j}/z_j Q^{(1)}}.
\end{aligned} \tag{4.11}$$

Its Lax representation is

$$E\psi = U\psi, \qquad \psi_t = \left(\bar V_m + \sum_{j=1}^{N} X_j\right)\psi,$$

where

$$\begin{aligned}
\bar V_m &= (\bar\Gamma z^{-2m})_- + \begin{pmatrix} -\bar A_{2m} & 0 \\ 0 & 0 \end{pmatrix}, \\
X_j &= \frac{1}{z^2 - z_j^2}\left[\begin{pmatrix} z_j^2\bar\psi_{1,j}\bar\phi_{1,j}^{(-1)} & zz_j\bar\psi_{1,j}\bar\phi_{2,j}^{(-1)} \\ zz_j\bar\psi_{2,j}\bar\phi_{1,j}^{(-1)} & z^2\bar\psi_{2,j}\bar\phi_{2,j}^{(-1)} \end{pmatrix} + \right. \\
&\quad \left. + (\bar\psi_{1,j}\bar\phi_{1,j}^{(-1)} + \bar\psi_{2,j}\bar\phi_{2,j}^{(-1)}) \begin{pmatrix} (z^2 - 3z_j^2)/4 & 0 \\ 0 & (z_j^2 - 3z^2)/4 \end{pmatrix}\right].
\end{aligned}$$

If we choose $\mu_j = 0$, then generalized system (4.11) reduces to the Ablowitz–Ladik hierarchy with self-consistent sources (corresponding to the generating matrix Γ in the power series in z). We have thus obtained two types of GKD of the Ablowitz–Ladik hierarchy according to the two types of Hamiltonian operators (4.6) and (4.10).

5. Conclusion

We have constructed some new integrable discrete systems and their Lax representations using the GKD of bi-Hamiltonian systems. We studied the GKDs of the Toda, Kac–van Moerbeke, and Ablowitz–Ladik hierarchies and found that these new discrete systems have some relations to an equation with self-consistent sources [4], [5], [7], [20]. We showed that the method can be used to construct new discrete integrable systems in the (1+1)-dimensional case. The Kupershmidt deformation for the higher-dimensional case will be studied in the future.

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