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RESONANCE POCKETS OF HILL'S EQUATIONS WITH TWO-STEP POTENTIALS*

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Abstract. In this paper, we use the rotation number approach to study in detail the characteristic values of Hill's equations with two-step periodic potentials. As a result, the global structure of resonance pockets is described completely. The results in this paper show that resonance pockets behave in a sensible and fairly rich way even in this simplest case.

Key words. resonance pocket, Hill's equation, characteristic value, rotation number

AMS subject classifications. 34L40, 34D20, 47A10

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1. Introduction.

In this paper we are concerned with the global structure of resonance pockets of parameterized Hill's equations

$$\ddot{x} + (\lambda + \varepsilon p(t))x = 0, \tag{1.1}$$

where $p(t)$ is a 2π-periodic step potential of two steps. For a general 2π-periodic potential, the resonance region R of (1.1) means the set of those parameters (λ, ε) in the (λ, ε)-plane such that (1.1) admits solutions $x(t)$ which are unbounded. The resonance pockets of (1.1), which will be explained more clearly later, are "compact" or "closed" parts of R.

The resonance region R of (1.1) can be described completely in theory. For any fixed parameter ε, R consists of the complement of all spectrum intervals of (1.1). More precisely, let $q(t)$ be a 2π-periodic potential such that $q \in L^1(0, 2\pi)$. Consider the eigenvalue problem

$$\ddot{x} + (\lambda + q(t))x = 0. \tag{1.2}$$

By Theorem 2.1 of Magnus and Winkler [10] or Theorem 8.1, Chapter III of Hale [6], it is well known that problem (1.2) has a sequence of the periodic eigenvalues

$$\lambda_0^P(q) < \lambda_1^P(q) \le \lambda_2^P(q) < \cdots < \lambda_{2n-1}^P(q) \le \lambda_{2n}^P(q) < \cdots$$

with respect to the periodic boundary conditions (P): $x(0)-x(2\pi) = \dot{x}(0)-\dot{x}(2\pi) = 0$. Meanwhile, problem (1.2) also has a sequence of the antiperiodic eigenvalues

$$\lambda_1^A(q) \le \lambda_2^A(q) < \cdots < \lambda_{2n-1}^A(q) \le \lambda_{2n}^A(q) < \cdots$$

with respect to the antiperiodic boundary conditions (A): $x(0) + x(2\pi) = \dot{x}(0) + \dot{x}(2\pi) = 0$. Let us rewrite them as

$$\underline{\lambda}_n(q) = \lambda_n^A(q) \quad \text{and} \quad \overline{\lambda}_n(q) = \lambda_{n+1}^A(q) \qquad \text{when } n \text{ is odd},$$

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$$\underline{\lambda}_n(q) = \lambda^P_{n-1}(q) \quad \text{and} \quad \overline{\lambda}_n(q) = \lambda^P_n(q) \qquad \text{when } n \text{ is even.}$$

These eigenvalues, called characteristic values of (1.2) as a whole [10, p. 12], have the following order (see Theorem 2.1 of [10]):

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \le \overline{\lambda}_1(q) < \cdots < \underline{\lambda}_n(q) \le \overline{\lambda}_n(q) < \cdots .$$

Now the resonance region R of (1.1) is given by

$$R = \bigcup\nolimits_{n=0}^{\infty} R_n,$$

where

$$R_0 = \{(\lambda, \varepsilon) : \lambda < \overline{\lambda}_0(\varepsilon p)\}, \qquad R_n = \{(\lambda, \varepsilon) : \underline{\lambda}_n(\varepsilon p) < \lambda < \overline{\lambda}_n(\varepsilon p)\},\ n = 1, 2, \ldots .$$

A typical example is the Mathieu case: $p(t) = \cos t$. In this case, $\underline{\lambda}_n(p_\varepsilon) < \overline{\lambda}_n(p_\varepsilon)$ holds for all $\varepsilon \ne 0$, $n \in \mathbb{N}$. Thus each resonance region R_n is like a "tongue" which approaches to the point $((n/2)^2, 0)$ on the λ-axis. These are the so-called Arnold tongues (resonance tongues, instability tongues); see section 25, Chapter 5 of [1] and section III.8 of Hale [6]. However, for the near Mathieu case $p(t) = \cos t + \beta \cos 2t$ or the square wave case $p(t) = \text{sign} \cos t$, besides the resonance tongues, it is also observed that some resonance regions R_n would have some closed subregion, namely, $\underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p)$ for some nonzero parameter ε. These interesting phenomena are called resonance pockets; see [1, 4, 6]. One may find in [3] the historical development of the study for resonance regions of Hill's equations. For resonance tongues of certain nonlinear systems, one can refer to [2, 5, 7, 9, 13]. A geometric explanation using singularity theory to the appearance of resonance pockets is given in [3] and has been developed in [2, 4]. Such an idea is very fruitful in explaining the pockets near the λ-axis. However, so far as we know, the global structure for all resonance pockets are not available even for the simplest case—the square wave case.

Note that the problem of resonance pockets of the Hill's equations is just to study the coexistence problem [10, p. 90] of characteristic values:

$$\overline{\lambda}_n(\varepsilon p) = \underline{\lambda}_n(\varepsilon p). \tag{1.3}$$

Such a coexistence problem for general potentials $p(t)$ is extraordinarily difficult. A preliminary idea is to approximate general potentials by step ones. In doing so, we can give a complete analysis of the simplest case, i.e., the 2π-periodic two-step potentials:

$$p(t) = p_{c_1,c_2,t_1}(t) := \begin{cases} c_1 & \text{if } 0 \le t < t_1, \\ c_2 & \text{if } t_1 \le t < 2\pi, \end{cases} \tag{1.4}$$

where $c_1 \ne c_2$, $0 < t_1 < 2\pi$. Denote $t_2 = 2\pi - t_1$. Our result is the following theorem.

THEOREM 1.1. *Let $p(t)$ be given by (1.4). Then the number of resonance pockets in the nth resonance region R_n of (1.1) is exactly*

$$N_n = \begin{cases} n-2 & \text{if } \frac{nt_1}{2\pi} \text{ is an integer,} \\ n-1 & \text{if } \frac{nt_1}{2\pi} \text{ is not an integer.} \end{cases}$$

This result shows that the coexistence problem (1.3) and the global structure of the corresponding Hill's equations (1.1) depend on the ratio of $t_1/2\pi$ in a very sensible way, while the global structure of (1.1) behaves in an elegant way for "generic" two-step potentials.

Corollary 1.2. *When t_1 in (1.4) is incommensurable with π, i.e., t_1/π is irrational, the nth resonance region R_n of (1.1) contains exactly $n-1$ resonance pockets for each $n \in \mathbb{N}$. Moreover, all of resonance pockets are transversal.*

When the square wave potential $p(t)$ (i.e., $c_1 = -1$, $c_2 = +1$, and $t_1 = t_2 = \pi$) is considered, the structure of resonance pockets behaves as follows.

Corollary 1.3. *The number of resonance pockets in the nth resonance region R_n of (1.1) with the square wave potential $p(t)$ is exactly*

$$N_n = \begin{cases} n-2 & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Note that the problem for two-step potentials is not too difficult because (1.1) can be solved using trigonometric functions. In particular, the discriminate of (1.1) can be computed explicitly; cf. (1.5). Now characteristic values can be determined by

$$\begin{aligned} &\operatorname{tr} P_\lambda = 2\cos(t_1\sqrt{\lambda+\varepsilon c_1})\cos(t_2\sqrt{\lambda+\varepsilon c_2}) \\ &\quad -\left(\sqrt{\frac{\lambda+\varepsilon c_1}{\lambda+\varepsilon c_2}}+\sqrt{\frac{\lambda+\varepsilon c_2}{\lambda+\varepsilon c_1}}\right)\sin(t_1\sqrt{\lambda+\varepsilon c_1})\sin(t_2\sqrt{\lambda+\varepsilon c_2}) = \pm 2; \end{aligned} \tag{1.5}$$

cf. Lemma 2.3 and p. 116 of [10]. However, (1.5) is not easily analyzed. Due to the coexistence of characteristic values, there is some difficulty in solving (1.5) even numerically. Because of this reason, we adopt in this paper the rotation number approach to characteristic values [8, 11, 12].

The paper is organized as follows. In section 2, the rotation number approach to characteristic values with general periodic potentials is reviewed. Some results concerning the coexistence and the characterization of characteristic values using the solutions of (2.3) (see next section) are given. These results may be of some independent interest. In section 3, we obtain the coexistence conditions and the equations for characteristic values. The results on resonance pockets are proved in section 4.

2. Rotation number approach to characteristic values. Let $\mathcal{P}$ denote the collection of all 2π-periodic functions $q(t)$ such that $q \in L^1(0, 2\pi)$.

Assume that $q \in \mathcal{P}$ and consider eigenvalue problem (1.2). We intend to use the rotation number function to characterize all characteristic values $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$. Let $y = -\dot{x}$ in (1.2). Then (1.2) is equivalent to the following linear planar system:

$$\dot{x} = -y, \qquad \dot{y} = (\lambda + q(t))x. \tag{2.1}$$

In the polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$,

$$\dot{r} = (\lambda + q(t) - 1)r\cos\theta\sin\theta, \tag{2.2}$$

$$\dot{\theta} = (\lambda + q(t))\cos^2\theta + \sin^2\theta =: \Xi(t, \theta; \lambda). \tag{2.3}$$

Let $\Theta(t; \theta_0, \lambda)$ be the unique solution of (2.3) satisfying the initial condition: $\Theta(0; \theta_0, \lambda) = \theta_0$. As the vector field $\Xi(t, \theta; \lambda)$ is 2π-periodic in t and is π-periodic in θ, one has

$$\Theta(t + 2m\pi; \theta_0, \lambda) = \Theta(t; \Theta(2m\pi; \theta_0, \lambda), \lambda) \tag{2.4}$$

$$\Theta(t; \theta_0 + n\pi, \lambda) = \Theta(t; \theta_0, \lambda) + n\pi \tag{2.5}$$

for all $t,\ \theta_0,\ \lambda \in \mathbb{R}$ and $m,\ n \in \mathbb{Z}$. Thus the rotation number of (2.3)

$$\rho(\lambda) = \rho(\lambda; q) = \lim_{t\to\infty} \frac{\Theta(t; \theta_0, \lambda) - \theta_0}{t}$$

exists and is independent of θ_0; see Theorem 2.1, Chapter 2 of Hale [6].

The solutions $\Theta(t;\theta_0,\lambda)$ depend continuously on the parameter λ. As $\Xi(t,\theta;\lambda)$ is nondecreasing with respect to λ, then so does $\Theta(t;\theta_0,\lambda)$ according to the comparison theorem. From Corollary 2.1, Chapter 2 of Hale [6], one knows that the rotation number function $\rho(\lambda)$ is continuous and nondecreasing. Furthermore, it can be proved that $\rho(\lambda)=0$ for $\lambda \ll -1$, and $\lim_{\lambda\to+\infty}\rho(\lambda)=+\infty$. Now all characteristic values can be determined using $\rho(\lambda)$.

PROPOSITION 2.1. *$\underline{\lambda}_n(q)=\min\{\lambda\in\mathbb{R}:\rho(\lambda)=n/2\}$ for all $n\in\mathbb{N}$, and $\overline{\lambda}_n(q)=\max\{\lambda\in\mathbb{R}:\rho(\lambda)=n/2\}$ for all $n\in\mathbb{Z}^+$.*

Proof. The relationship between spectrum and rotation number has been well developed in [8, 11, 12]. This characterization of characteristic values using rotation number function is a classical result; cf. Theorems 4.3 and 4.4 of [11]. As a proof is not given in [11], we sketch here, for completeness, the proof based on Theorem 2.1 of [10].

Let P_λ be the Poincaré matrix associated with the system (2.1), i.e.,

$$P_\lambda(x_0,y_0)=(x(2\pi;x_0,y_0,\lambda),y(2\pi;x_0,y_0,\lambda)),$$

where $(x(t;x_0,y_0,\lambda),y(t;x_0,y_0,\lambda))$ is the solution of (2.1) satisfying

$$(x(0;x_0,y_0,\lambda),y(0;x_0,y_0,\lambda))=(x_0,y_0).$$

If $\underline{\lambda}_n(q)\le\lambda\le\overline{\lambda}_n(q)$ for some $n\in\mathbb{N}$, it follows from Theorem 2.1 of [10] that $|\operatorname{tr}P_\lambda|\ge 2$ and P_λ has real eigenvalues $\mu_{1,2}$: $P_\lambda v_i=\mu_i v_i$, $v_i\in\mathbb{R}^2\backslash\{0\}$, $i=1,\ 2$. Let $\theta_i\in\mathbb{R}$ be such that $v_i=r_i(\cos\theta_i,\sin\theta_i)$, $i=1,\ 2$. Then $\Theta(2\pi;\theta_i,\lambda)=\theta_i+k_i\pi$ and $\rho(\lambda)=k_1/2=k_2/2=k/2$, where $k=k_\lambda\in\mathbb{Z}$ for each $\lambda\in[\underline{\lambda}_n(q),\overline{\lambda}_n(q)]$. As $\rho(\lambda)$ is continuous, k_λ is independent of $\lambda\in[\underline{\lambda}_n(q),\overline{\lambda}_n(q)]$. In fact, it can be proved that

$$\rho(\lambda)=n/2 \qquad \text{for all}\ \ \lambda\in[\underline{\lambda}_n(q),\overline{\lambda}_n(q)]. \tag{2.6}$$

On the other hand, if $\lambda\in(\overline{\lambda}_n(q),\underline{\lambda}_{n+1}(q))$ for some $n\in\mathbb{Z}^+$, then $|\operatorname{tr}P_\lambda|<2$. Therefore eigenvalues $\mu_{1,2}$ of P_λ are on the unit circle: $\mu_1=\bar{\mu}_2=e^{\alpha\sqrt{-1}}$ for some $\alpha=\alpha_\lambda\in\mathbb{R}\backslash\pi\mathbb{Z}$. In this case, one has

$$\rho(\lambda)=\alpha/2\pi\ (\mathrm{mod}\ \mathbb{Z})\notin\frac{1}{2}\mathbb{Z}. \tag{2.7}$$

Now (2.6) and (2.7) show that $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are the endpoints of the interval $\rho^{-1}(n/2)\subset\mathbb{R}$. □

Let $h:\mathbb{R}\to\mathbb{R}$ be a homeomorphism such that

$$h(\vartheta+n\pi)\equiv h(\vartheta)+n\pi \tag{2.8}$$

for all $\vartheta\in\mathbb{R}$ and all $n\in\mathbb{Z}$. One can define the rotation number of h as

$$\rho(h)=\lim_{m\to\infty}\frac{h^m(\vartheta_0)-\vartheta_0}{2m\pi}$$

(independent of the choice of ϑ_0).

Let $h_\lambda:\mathbb{R}\to\mathbb{R}$ be the Poincaré map of (2.3), i.e., $h_\lambda(\vartheta)=\Theta(2\pi;\vartheta,\lambda)$ for $\vartheta\in\mathbb{R}$. By (2.5), h_λ satisfies (2.8). Now the rotation number $\rho(\lambda)$ is same as $\rho(h_\lambda)$.

PROPOSITION 2.2. *Let h be a homeomorphism of $\mathbb{R}$ satisfying* (2.8) *and n be an integer. Then*

(i) $\rho(h) \geq n/2$ *iff* $\max_{\vartheta\in\mathbb{R}}(h(\vartheta) - (\vartheta + n\pi)) \geq 0$.
(ii) $\rho(h) \leq n/2$ *iff* $\min_{\vartheta\in\mathbb{R}}(h(\vartheta) - (\vartheta + n\pi)) \leq 0$.

Proof. Let us prove (i). Assume that $h(\vartheta_0) \geq \vartheta_0 + n\pi$ for some $\vartheta_0 \in \mathbb{R}$. Using (2.8), it is easy to see that $h^m(\vartheta_0) \geq \vartheta_0 + mn\pi$ for all $m \in \mathbb{N}$. Thus

$$\rho(h) = \lim_{m\to+\infty} \frac{h^m(\vartheta_0) - \vartheta_0}{2m\pi} \geq \frac{n}{2}.$$

Conversely, let $M_0 = \max_{\vartheta\in\mathbb{R}}(h(\vartheta) - (\vartheta + n\pi))$. If $M_0 < 0$, we need to prove that $\rho(h) < n/2$. Notice that

$$h(\vartheta) \leq \vartheta + (n\pi + M_0) \qquad \text{for all } \vartheta \in \mathbb{R}$$

implies that

$$h^m(\vartheta) \leq \vartheta + m(n\pi + M_0)$$

for all $m \in \mathbb{N}$ and all $\vartheta \in \mathbb{R}$. Thus

$$\rho(h) = \lim_{m\to+\infty} \frac{h^m(\vartheta) - \vartheta}{2m\pi} \leq \frac{n}{2} + \frac{M_0}{2\pi} < \frac{n}{2}.$$

Conclusion (ii) can be proved similarly. □

PROPOSITION 2.3. *Let n be an integer. Then the following hold.*
(i) $\lambda = \underline{\lambda}_n(q)$ *iff* $\max_{\theta_0}(\Theta(2\pi;\theta_0,\lambda) - (\theta_0 + n\pi)) = 0$.
(ii) $\lambda = \overline{\lambda}_n(q)$ *iff* $\min_{\theta_0}(\Theta(2\pi;\theta_0,\lambda) - (\theta_0 + n\pi)) = 0$.

Proof. By the comparison theorem for solutions, it can be proved that $\Theta(2\pi;\theta_0,\lambda)$ is strictly increasing with respect to λ. Now the results follow from Propositions 2.1 and 2.2. □

It follows from Proposition 2.3 that the coexistence $\overline{\lambda}_n(q) = \underline{\lambda}_n(q)$ can be described using the solutions $\Theta(2\pi;\theta_0,\lambda)$ in the following way.

PROPOSITION 2.4. $\overline{\lambda}_n(q) = \underline{\lambda}_n(q)$ $(= \lambda)$ *iff* $\Theta(2\pi;\theta_0,\lambda) \equiv \theta_0 + n\pi$ *for all* θ_0.

It follows also from Proposition 2.3 that if $\lambda = \overline{\lambda}_n(q)$ or $\lambda = \underline{\lambda}_n(q)$, then it is necessary that there exists some $\vartheta_0 \in \mathbb{R}$ such that

$$\Theta(2\pi;\vartheta_0,\lambda) = \vartheta_0 + n\pi \qquad \text{and} \qquad \left.\frac{d\Theta(2\pi;\vartheta,\lambda)}{d\vartheta}\right|_{\vartheta=\vartheta_0} = 1. \tag{2.9}$$

We show using the Hamiltonian structure of (2.1) that condition (2.9) is also sufficient for λ to be a characteristic value.

PROPOSITION 2.5. $\lambda = \overline{\lambda}_n(q)$ *or* $\underline{\lambda}_n(q)$ *iff* λ *satisfies* (2.9) *for some* $\vartheta_0 \in \mathbb{R}$. $\theta_0 \in \mathbb{R}$.

Proof. For any fixed $\vartheta \in \mathbb{R}$, let $r = R(t;\vartheta,\lambda)$ and $\theta = \Theta(t;\vartheta,\lambda)$ be the solutions of (2.2) and (2.3) satisfying $R(0;\vartheta,\lambda) = 1$ and $\Theta(0;\vartheta,\lambda) = \vartheta$.

Let $P_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré map of (2.1). Then P_λ is area-preserving because (2.1) is a Hamiltonian system. Using the solutions $R(t;\vartheta,\lambda)$ and $\Theta(t;\vartheta,\lambda)$, P_λ is given by

$$P_\lambda(r\cos\vartheta,\, r\sin\vartheta) = rR(2\pi;\vartheta,\lambda)(\cos\Theta(2\pi;\vartheta,\lambda),\, \sin\Theta(2\pi;\vartheta,\lambda)) \tag{2.10}$$

for all $r \in \mathbb{R}$ and all ϑ.

Let ϑ_0 be any fixed real number. For any ϑ_1 near ϑ_0, consider the following sector:

$$S = \{(r\cos\vartheta,\, r\sin\vartheta) \in \mathbb{R}^2 : 0 \le r \le 1,\ \vartheta_0 \le \vartheta \le \vartheta_1\}.$$

Then S has area $\frac{1}{2}(\vartheta_1 - \vartheta_0)$. The image $S' = P_\lambda(S)$ is

$$\begin{aligned} S' = \{(r'\cos\vartheta',\, r'\sin\vartheta') \in \mathbb{R}^2 :\ & 0 \le r' \le R(2\pi; \Theta^{-1}(\vartheta'; \lambda), \lambda), \\ & \Theta(2\pi; \vartheta_0, \lambda) \le \vartheta' \le \Theta(2\pi; \vartheta_1, \lambda)\}, \end{aligned}$$

where $\Theta^{-1}(\cdot;\, \lambda)$ is the inverse of $\Theta(2\pi;\, \cdot,\, \lambda)$. Thus S' has area

$$\frac{1}{2}\int_{\Theta(2\pi;\vartheta_0,\lambda)}^{\Theta(2\pi;\vartheta_1,\lambda)} R^2(2\pi; \Theta^{-1}(\vartheta'; \lambda), \lambda)d\vartheta' = \frac{1}{2}\int_{\vartheta_0}^{\vartheta_1} R^2(2\pi; \vartheta, \lambda)\frac{d\Theta(2\pi; \vartheta, \lambda)}{d\vartheta}d\vartheta.$$

As P_λ is area-preserving,

$$\frac{1}{2}(\vartheta_1 - \vartheta_0) \equiv \frac{1}{2}\int_{\vartheta_0}^{\vartheta_1} R^2(2\pi; \vartheta, \lambda)\frac{d\Theta(2\pi; \vartheta, \lambda)}{d\vartheta}d\vartheta.$$

Thus

$$\frac{d\Theta}{d\vartheta}(2\pi; \vartheta, \lambda) \equiv \frac{1}{R^2(2\pi; \vartheta, \lambda)}. \tag{2.11}$$

Assume now that $\vartheta_0 \in \mathbb{R}$ satisfies (2.9). Then $\Theta(2\pi; \vartheta_0, \lambda) = \vartheta_0 + n\pi$. Moreover, by the second equality in (2.9) and by (2.11), $R(2\pi; \vartheta_0, \lambda) = 1$. Now we get from (2.10) that

$$\begin{aligned} P_\lambda(\cos\vartheta_0,\, \sin\vartheta_0) &= R(2\pi; \vartheta_0, \lambda)(\cos\Theta(2\pi; \vartheta_0, \lambda),\, \sin\Theta(2\pi; \vartheta_0, \lambda)) \\ &= (\cos(\vartheta_0 + n\pi),\, \sin(\vartheta_0 + n\pi)) \\ &= (-1)^n(\cos\vartheta_0,\, \sin\vartheta_0). \end{aligned}$$

This shows that P_λ has a nonzero fixed point $(\cos\vartheta_0,\, \sin\vartheta_0)$ if n is even, which yields a nonzero 2π-periodic solution of (2.1). Thus λ is a periodic eigenvalue of (1.2). The case that n is odd implies that λ is an antiperiodic eigenvalue of (1.2). ◻

3. Two classes of conditions. Let $q(t) \in \mathcal{P}$ be the 2π-periodic potential given by

$$q(t) = q_{b_1,b_2,t_1}(t) := \begin{cases} b_1 & \text{for } 0 \le t < t_1\ (< 2\pi), \\ b_2 & \text{for } t_1 \le t < 2\pi. \end{cases} \tag{3.1}$$

Denote $t_2 = 2\pi - t_1$. We consider the following linear equation:

$$\ddot{x} + q(t)x = 0,$$

or, its equivalent system

$$\dot{x} = -y, \qquad \dot{y} = q(t)x.$$

As in section 2, let $x = r\cos\theta,\ y = r\sin\theta$. Then θ satisfies

$$\dot{\theta} = q(t)\cos^2\theta + \sin^2\theta =: \Xi(t,\theta). \tag{3.2}$$

Let $\Theta(t;\theta_0)$ be the solution of (3.2) satisfying the initial condition $\Theta(0;\theta_0) = \theta_0$. Denote $\Theta(\theta_0) := \Theta(2\pi;\theta_0)$. For any fixed $n \in \mathbb{N}$, we want to find the explicit conditions on $b_1,\ b_2,\ t_1,\ t_2$ so that

$$\Theta(\theta_0) \equiv \theta_0 + n\pi \qquad \text{for all } \theta_0 \in \mathbb{R}. \tag{3.3}$$

By Proposition 2.4, condition (3.3) is related with the coexistence of characteristic values.

In order to study (3.3), we need not consider the trivial case $b_1 = b_2$. Hence we assume that $b_1 \neq b_2$ in (3.1).

PROPOSITION 3.1. *Condition* (3.3) *holds iff* $b_1,\ b_2,\ t_1,\ t_2$ *satisfy* $b_1 > 0,\ b_2 > 0,$ *and*

$$t_1\sqrt{b_1} = k\pi \qquad \textit{and} \qquad t_2\sqrt{b_2} = (n-k)\pi \tag{3.4}$$

for some integer k *with* $0 < k < n$.

Proof. Let $\Theta_1(\theta_0) := \Theta(t_1;\theta_0)$. We have four cases to be discussed.

Case 1. $b_1 = a_1^2 > 0$ and $b_2 = a_2^2 > 0$. Assume that (3.3) holds. In this case, by integrating (3.2) on $[0,t_1]$ and $[t_1, 2\pi]$, respectively, we have the following two equalities:

$$\int_{\theta_0}^{\Theta_1(\theta_0)} \frac{d\theta}{a_1^2\cos^2\theta + \sin^2\theta} = t_1, \tag{3.5}$$

$$\int_{\Theta_1(\theta_0)}^{n\pi+\theta_0} \frac{d\theta}{a_2^2\cos^2\theta + \sin^2\theta} = t_2 \tag{3.6}$$

for all θ_0. Differentiating (3.5) and (3.6) with respect to θ_0, one has

$$\frac{1}{a_1^2\cos^2\theta_0 + \sin^2\theta_0} = \frac{\Theta_1'(\theta_0)}{a_1^2\cos^2\Theta_1(\theta_0) + \sin^2\Theta_1(\theta_0)}, \tag{3.7}$$

$$\frac{1}{a_2^2\cos^2\theta_0 + \sin^2\theta_0} = \frac{\Theta_1'(\theta_0)}{a_2^2\cos^2\Theta_1(\theta_0) + \sin^2\Theta_1(\theta_0)} \tag{3.8}$$

for all $\theta_0 \in \mathbb{R}$. From these we obtain

$$\sin(\Theta_1(\theta_0) - \theta_0)\sin(\Theta_1(\theta_0) + \theta_0) \equiv 0.$$

As $\Theta_1(\theta_0)$ is continuous in θ_0, we have either

$$\Theta_1(\theta_0) - \theta_0 \equiv k\pi \qquad \text{for some } k \in \mathbb{Z} \tag{3.9}$$

or

$$\Theta_1(\theta_0) + \theta_0 \equiv k\pi \qquad \text{for some } k \in \mathbb{Z}. \tag{3.10}$$

If (3.9) holds, then k satisfies $0 < k < n$ because $\theta_0 < \Theta_1(\theta_0) < \theta_0 + n\pi$ in this case. Note that

$$\int_0^{\pi} \frac{d\theta}{a^2\cos^2\theta + \sin^2\theta} = \frac{\pi}{a} \qquad (a > 0).$$

It now follows from (3.5) and (3.6) that

$$a_1 t_1 = k\pi \qquad \text{and} \qquad a_2 t_2 = (n-k)\pi \qquad \text{for some } 0 < k < n. \tag{3.11}$$

Conversely, if (3.11) is satisfied for some $0 < k < n$, it is easy to see that $\Theta_1(\theta_0) \equiv \theta_0 + k\pi$ and $\Theta(\theta_0) \equiv \Theta_1(\theta_0) + (n-k)\pi \equiv \theta_0 + n\pi$, i.e., equality (3.3) holds for all θ_0.

Assume now that (3.10) is satisfied. Let $\theta_0 = \ell\pi + \alpha$, where $\ell \in \mathbb{Z}$ and $\alpha \in [-\pi/2, \pi/2)$. Thus, by (3.10), $\Theta_1(\theta_0) = (k-\ell)\pi - \alpha$. It follows from (3.5) that

$$\begin{aligned} t_1 &= \int_{\ell\pi+\alpha}^{(k-\ell)\pi-\alpha} \frac{d\theta}{a_1^2\cos^2\theta + \sin^2\theta} \\ &= \left\{ \int_{\ell\pi+\alpha}^{\ell\pi} + \int_{\ell\pi}^{(k-\ell)\pi} + \int_{(k-\ell)\pi}^{(k-\ell)\pi-\alpha} \right\} \frac{d\theta}{a_1^2\cos^2\theta + \sin^2\theta} \\ &= \frac{(k-2\ell)\pi}{a_1} - 2\int_0^\alpha \frac{d\theta}{a_1^2\cos^2\theta + \sin^2\theta} \\ &= \frac{(k-2\ell)\pi}{a_1} - \frac{2}{a_1}\arctan\left(\frac{1}{a_1}\tan\alpha\right). \end{aligned}$$

Namely,

$$k\pi - a_1 t_1 = 2\ell\pi + 2\arctan\left(\frac{1}{a_1}\tan\theta_0\right). \tag{3.12}$$

Note that equality (3.12) cannot hold for all $\theta_0 \in \mathbb{R}$. Thus (3.10) cannot happen in this case.

We remark here that if (3.6) is used, one can obtain

$$a_2 t_2 - (n-k)\pi = 2\ell\pi + 2\arctan\left(\frac{1}{a_2}\tan\theta_0\right). \tag{3.13}$$

This also implies that (3.10) cannot happen in this case.

Case 2. $b_1 \le 0$ and $b_2 = a_2^2 > 0$. As $\Psi(\theta) = b_1\cos^2\theta + \sin^2\theta$ has zeros $\theta = \theta_\pm = \pm\arctan\sqrt{-b_1} + j\pi$, $j \in \mathbb{Z}$, we have $\Theta_1(\theta_\pm) = \theta_\pm$. Let now $\theta_0 = \theta_\pm$ in (3.6). Then

$$t_2 = \int_{\theta_\pm}^{\theta_\pm + n\pi} \frac{d\theta}{a_2^2\cos^2\theta + \sin^2\theta} = \frac{n\pi}{a_2}.$$

Thus $a_2 t_2 = n\pi$. This condition, together with (3.6), implies that $\Theta_1(\theta_0) \equiv \theta_0$ for all θ_0, which is impossible because $\Theta_1(\theta_0) = \Theta(t_1;\theta_0)$ is determined by differential equation

$$\dot\theta = b_1\cos^2\theta + \sin^2\theta, \qquad t \in [0, t_1].$$

Case 3. $b_1 > 0$ and $b_2 \le 0$. As characteristic values are invariant under translations of potentials $q_s(t)$ $(= q(t+s))$, one can transfer this case to Case 2.

Case 4. $b_1 \le 0$ and $b_2 \le 0$. In this case the vector field $\Xi(t,\theta) = q(t)\cos^2\theta + \sin^2\theta \le \Psi(\theta) := -\beta^2\cos^2\theta + \sin^2\theta$, where $\beta = \min\{\sqrt{-b_1}, \sqrt{-b_2}\}$. Thus

$$\dot\theta = \Xi(t,\theta) \le -\beta^2\cos^2\theta + \sin^2\theta = \Psi(\theta).$$

As $\Psi(\theta)$ has zeros $\theta_\pm = \pm\arctan\beta + j\pi$, $j \in \mathbb{Z}$, the comparison theorem shows that $\Theta(2\pi;\theta_\pm) \le \theta_\pm$. As a result, (3.3) does not hold for all θ_0. □

Another class of conditions on $b_1,\ b_2,\ t_1,\ t_2$ is when the following holds:

$$\exists\ \theta_0 \text{ such that } \Theta(\theta_0) = \theta_0 + n\pi \text{ and } \left.\frac{d\Theta(\vartheta)}{d\vartheta}\right|_{\vartheta=\theta_0} = 1. \tag{3.14}$$

By Proposition 2.5, condition (3.14) is related with the determination of characteristic values.

PROPOSITION 3.2. *Condition* (3.14) *is equivalent to either*

$$a_1 \sin\frac{a_1 t_1}{2}\cos\frac{a_2 t_2 - n\pi}{2} + a_2 \cos\frac{a_1 t_1}{2}\sin\frac{a_2 t_2 - n\pi}{2} = 0 \tag{3.15}$$

or

$$a_1 \cos\frac{a_1 t_1}{2}\sin\frac{a_2 t_2 - n\pi}{2} + a_2 \sin\frac{a_1 t_1}{2}\cos\frac{a_2 t_2 - n\pi}{2} = 0, \tag{3.16}$$

where $a_1 = \sqrt{b_1}$ *and* $a_2 = \sqrt{b_2}$.

Proof. We consider the first case that $b_1 = a_1^2 > 0$ and $b_2 = a_2^2 > 0$ in the proof of Proposition 3.1. Note that the equalities (3.5) and (3.6) now read as

$$\int_{\vartheta}^{\Theta_1(\vartheta)} \frac{d\theta}{a_1^2\cos^2\theta + \sin^2\theta} = t_1$$

and

$$\int_{\Theta_1(\vartheta)}^{\Theta(\vartheta)} \frac{d\theta}{a_2^2\cos^2\theta + \sin^2\theta} = t_2$$

for all ϑ. Differentiating these equations with respect to ϑ at $\vartheta = \theta_0$, we can once again obtain equalities (3.7) and (3.8) for this specific θ_0 by simply noticing the conditions in (3.14). Now we can proceed as in the proof of Proposition 3.1 and conclude that either (3.11) holds or both of (3.12) and (3.13) hold for this specific θ_0.

Note that (3.11) is a special case of (3.12) and (3.13) with $\ell = 0$ and $\theta_0 = 0$. Eliminating θ_0 from (3.12) and (3.13), we arrive at

$$a_1 \tan\frac{k\pi - a_1 t_1}{2} = a_2 \tan\frac{a_2 t_2 - (n-k)\pi}{2}. \tag{3.17$_k$}$$

Observe that if $k' = k+2$ then $(3.17)_{k'}$ is the same as $(3.17)_k$. Thus $(3.17)_k$ yield actually only two equations:

$$a_1 \tan\frac{a_1 t_1}{2} + a_2 \tan\frac{a_2 t_2 - n\pi}{2} = 0,$$

and

$$a_1 \cot\frac{s a_1 t_1}{2} + a_2 \cot\frac{a_2 t_2 - n\pi}{2} = 0.$$

These are just the conditions (3.15) and (3.16), respectively, which are described in the proposition. The converse can also be proved. These prove the proposition for Case 1.

One can prove in the other cases similarly if the complex cosine and sine functions are used in (3.15) and (3.16). □

Let $q(t) = q_{b_1,b_2,t_1}(t)$ be given by (3.1). It follows from Proposition 3.1 that the coexistence $\overline{\lambda}_n(q_{b_1,b_2,t_1}) = \underline{\lambda}_n(q_{b_1,b_2,t_1})$ $(= \lambda)$ is determined by

$$t_1\sqrt{\lambda + b_1} = k\pi \qquad \text{and} \qquad t_2\sqrt{\lambda + b_2} = (n-k)\pi$$

for some $0 < k < n$. Namely, $b_1,\ b_2,\ t_1$ satisfy

$$H_{n,k}:\quad b_2 - b_1 = \left((n-k)\pi/t_2\right)^2 - \left(k\pi/t_1\right)^2, \qquad 0 < k < n. \tag{3.18}$$

We will see from the next section that these surfaces $H_{n,k}$ in the (b_1, b_2, t_1)-space play a fundamental role in analyzing resonance pockets.

4. Application to resonance pockets. Now we apply the results in section 3 to the resonance pockets of Hill's equations (1.1) with two-step potentials, where $p(t) = p_{c_1,c_2,t_1}(t)$ is given by (1.4). Correspondingly, the parameters (b_1, b_2, t_1) in (3.1) are $(c_1\varepsilon, c_2\varepsilon, t_1)$ in this case.

Fix an integer $n \geq 2$. Starting from $\varepsilon = 0$ where $\underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p) = (n/2)^2$, if $\varepsilon \neq 0$ is such that $(c_1\varepsilon, c_2\varepsilon, t_1)$ hits $H_{n,k}$ for some $0 < k < n$, then one gets a resonance pocket inside R_n of (1.1). Explicitly, $(c_1\varepsilon, c_2\varepsilon, t_1) \in H_{n,k}$ is given by

$$\varepsilon = \varepsilon_{n,k} := \frac{1}{c_2 - c_1}\left(((n-k)\pi/t_2)^2 - (k\pi/t_1)^2\right), \tag{4.1}$$

where $\lambda = \underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p)$ is

$$\lambda = \lambda_{n,k} := (k\pi/t_1)^2 - c_1\varepsilon_{n,k} = \frac{1}{c_2 - c_1}\left(c_2\left((n-k)\pi/t_2\right)^2 - c_1\left(k\pi/t_1\right)^2\right). \tag{4.2}$$

Now we can complete the proof of Theorem 1.1. We need only to analyze (4.1). Note that $\varepsilon_{n,k}$ is decreasing when k runs from 1 to $n-1$. If t_1 is such that $nt_1/2\pi$ is not an integer, then all $\varepsilon_{n,k} \neq 0$ for $k = 1, \dots, n-1$. Note that $\underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p) = (n/2)^2$ when $\varepsilon = 0$. Thus $\underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p)$ iff $\varepsilon = \varepsilon_{n,k}$, $k = 1, \dots, n-1$, or $\varepsilon = 0$. As a result, R_n contains exactly $n-1$ pockets. When $nt_1/2\pi = k_0$ is an integer, then $0 < k_0 < n$ and $\varepsilon_{n,k_0} = 0$. As a result, $\underline{\lambda}_n(\varepsilon p) = \overline{\lambda}_n(\varepsilon p)$ iff $\varepsilon = \varepsilon_{n,k}$, $k = 1, \dots, n-1$. Thus R_n contains exactly $n-2$ pockets. This completes the proof of Theorem 1.1. $\square$

We remark that by Proposition 3.2, characteristic values $\lambda = \underline{\lambda}_n(\varepsilon p)$ and $\lambda = \overline{\lambda}_n(\varepsilon p)$ of (1.1) are determined by

$$\begin{aligned}&\sqrt{\lambda + c_1\varepsilon}\sin\frac{t_1\sqrt{\lambda+c_1\varepsilon}}{2}\cos\frac{t_2\sqrt{\lambda+c_2\varepsilon}-n\pi}{2}\\ &+\sqrt{\lambda + c_2\varepsilon}\cos\frac{t_1\sqrt{\lambda+c_1\varepsilon}}{2}\sin\frac{t_2\sqrt{\lambda+c_2\varepsilon}-n\pi}{2} = 0,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\sqrt{\lambda + c_1\varepsilon}\cos\frac{t_1\sqrt{\lambda+c_1\varepsilon}}{2}\sin\frac{t_2\sqrt{\lambda+c_2\varepsilon}-n\pi}{2}\\ &+\sqrt{\lambda + c_2\varepsilon}\sin\frac{t_1\sqrt{\lambda+c_1\varepsilon}}{2}\cos\frac{t_2\sqrt{\lambda+c_2\varepsilon}-n\pi}{2} = 0;\end{aligned} \tag{4.4}$$

see (3.15) and (3.16).

Let $\lambda = \Lambda_1(\varepsilon)$ and $\lambda = \Lambda_2(\varepsilon)$ be the solutions of (4.3) and (4.4) starting at $\Lambda_1(0) = \Lambda_2(0) = (n/2)^2$, respectively. At $(\lambda, \varepsilon) = (\lambda_{n,k}, \varepsilon_{n,k})$, we have

$$\frac{d\Lambda_1}{d\varepsilon} = -\frac{c_1t_1^3(n-k)^2 + c_2t_2^3k^2}{t_1^3(n-k)^2 + t_2^3k^2}, \tag{4.5}$$

$$\frac{d\Lambda_2}{d\varepsilon} = -\frac{c_1t_1 + c_2t_2}{2\pi}, \tag{4.6}$$

when k is odd, and

$$\frac{d\Lambda_1}{d\varepsilon} = -\frac{c_1t_1 + c_2t_2}{2\pi}, \tag{4.7}$$

$$\frac{d\Lambda_2}{d\varepsilon} = -\frac{c_1t_1^3(n-k)^2 + c_2t_2^3k^2}{t_1^3(n-k)^2 + t_2^3k^2}, \tag{4.8}$$

when k is even. Similarly, at the point $(\lambda, \varepsilon) = ((n/2)^2, 0)$, we get from (4.3) and (4.4) that

$$\frac{d\Lambda_1}{d\varepsilon} = -\frac{c_1(\frac{nt_1}{2} + \sin\frac{nt_1}{2}) + c_2(\frac{nt_2}{2} - \sin\frac{nt_1}{2})}{n\pi}, \tag{4.9}$$

$$\frac{d\Lambda_2}{d\varepsilon} = -\frac{c_1(\frac{nt_1}{2} - \sin\frac{nt_1}{2}) + c_2(\frac{nt_2}{2} + \sin\frac{nt_1}{2})}{n\pi}. \tag{4.10}$$

From (4.5)–(4.10), it is easy to check that

$$\left.\frac{d\Lambda_1}{d\varepsilon}\right|_{\varepsilon=\varepsilon_{n,k}} = \left.\frac{d\Lambda_2}{d\varepsilon}\right|_{\varepsilon=\varepsilon_{n,k}} \iff \varepsilon_{n,k} = 0 \tag{4.11}$$

and

$$\left.\frac{d\Lambda_1}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{d\Lambda_2}{d\varepsilon}\right|_{\varepsilon=0} \iff \sin\frac{nt_1}{2} = 0. \tag{4.12}$$

Proof of Corollary 1.2. Assume that t_1 is such that t_1/π is irrational. Then $\varepsilon_{n,k} \neq 0$ and $\sin\frac{nt_1}{2} \neq 0$. By (4.11) and (4.12), we have

$$\frac{d\Lambda_1}{d\varepsilon} \neq \frac{d\Lambda_2}{d\varepsilon}$$

at all $(\lambda, \varepsilon) = (\lambda_{n,k}, \varepsilon_{n,k})$ and at $((n/2)^2, 0)$. This means that all resonance pockets in this case are transversal in the (λ, ε)-plane. □

A typical combinatorics structure in this case is plotted in Figure 4.1. The characteristic values are found by solving (4.3) and (4.4) numerically. We remark that suitable ratios for sizes of resonance pockets need a careful choice of the irrational number $t_1/2\pi$ such that it is badly approximated by rational numbers. In Figure 4.1, one of the pockets in R_5 near the λ-axis is very small and is almost invisible.

Assume now that $t_1/2\pi$ is rational. There are two cases to be discussed. The first one is when $n \in \mathbb{N}$ is such that $nt_1/2\pi$ is not an integer. By Theorem 1.1, the nth resonance region R_n of (1.1) has $n-1$ resonance pockets, which are all transversal by (4.11) and (4.12). The second case is when $nt_1/2\pi = k_0$ is an integer. Then all resonance pockets inside R_n, except the two pockets

$$\left\{(\lambda, \varepsilon) : \underline{\lambda}_n(\varepsilon p) < \lambda < \overline{\lambda}_n(\varepsilon p),\ \varepsilon \in (0, \varepsilon_{n,k_0-1})\right\}$$

and

$$\left\{(\lambda, \varepsilon) : \underline{\lambda}_n(\varepsilon p) < \lambda < \overline{\lambda}_n(\varepsilon p),\ \varepsilon \in (\varepsilon_{n,k_0+1}, 0)\right\},$$

are also transversal.

In particular, for the square wave case, i.e., $c_1 = -1$, $c_2 = +1$ and $t_1 = t_2 = \pi$, we know that all pockets inside R_n are transversal if n is odd, and all pockets, except the two pockets

$$\left\{(\lambda, \varepsilon) : \underline{\lambda}_n(\varepsilon p) < \lambda < \overline{\lambda}_n(\varepsilon p),\ 0 < \varepsilon < n\right\}$$

and

$$\left\{(\lambda, \varepsilon) : \underline{\lambda}_n(\varepsilon p) < \lambda < \overline{\lambda}_n(\varepsilon p),\ -n < \varepsilon < 0\right\},$$

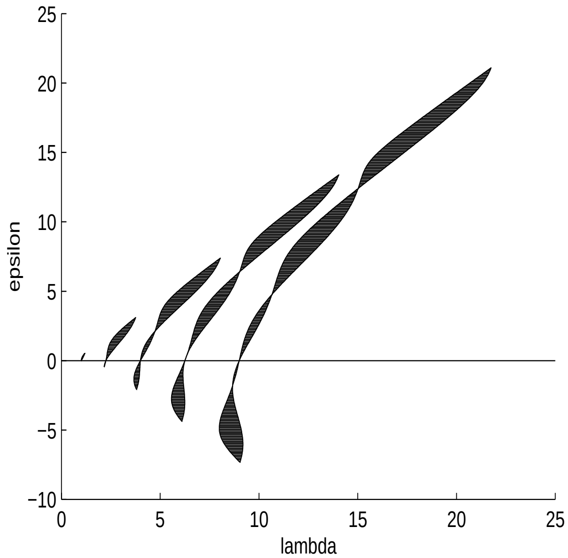

FIG. 4.1. *Resonance pockets for "generic" two-step potentials. Here* $c_1 = -1$, $c_2 = +1$, *and* $t_1 = (\sqrt{5} - 1)\pi$.

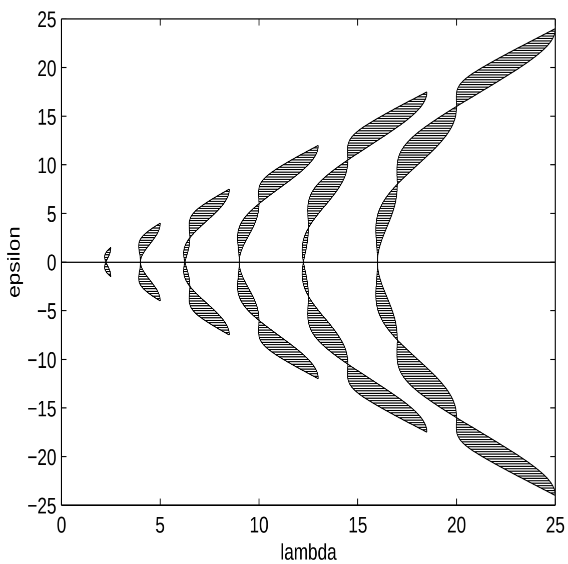

FIG. 4.2. *Resonance pockets for the square wave potential.*

are transversal when n is even. As $-\varepsilon \operatorname{sign} \cos t \equiv \varepsilon \operatorname{sign} \cos(t + \pi)$, the resonance pockets in the square wave case are symmetric with respect to the λ-axis, because characteristic values are invariant when the potentials are translated. This proves Corollary 1.3.

In Figure 4.2, the resonance pockets inside R_n, $n = 3, \ldots, 8$, of the square wave Hill's equations are plotted.

Theorem 1.1 shows that, when $p_\varepsilon(t) = \varepsilon p_{c_1,c_2,t_1}(t)$ is dependent on ε in a linear way, each resonance region of

$$\ddot{x} + (\lambda + p_\varepsilon(t))x = 0 \tag{4.13}$$

contains at most finitely many resonance pockets. However, when general families of two-step potentials

$$p_\varepsilon(t) = p_{b_1(\varepsilon),b_2(\varepsilon),t_1(\varepsilon)}(t)$$

are considered (which depend on ε in a nonlinear way), some resonance regions R_n of (4.13) may contain infinitely many resonance pockets. One example presenting infinitely many resonance pockets inside R_2 is given in [14]. In fact, one can use (3.15), (3.16), and (3.18) to give a global description to all resonance pockets inside all resonance regions R_n of (4.13).

When t_1 is such that t_1/π is irrational, it follows from Corollary 1.2 that all resonance pockets are transversal. This implies that the global structure of resonance pockets of (1.1) is preserved when $p(t)$ has certain kind of smooth perturbations.

Finally, we remark that even for step potentials, the structure of resonance pockets is not easily analyzed. It seems that our approach here is not applicable even to the case $p(t)$ is a three-step potential.

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