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Optimal bounds for bifurcation values of a superlinear periodic problem

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In this paper we will show that the optimal bounds for certain static and dynamic bifurcation values of periodic solutions of some superlinear differential equations can be expressed explicitly using Sobolev constants.

1. Introduction

Given an interval I of length $|I| = 1$ and an exponent $\sigma \in [1, \infty]$, the inequality

$$C\|\varphi\|^2_{L^\sigma(I)} \leqslant \|\varphi'\|^2_{L^2(I)}$$

holds for some $C > 0$ and each function φ in the Sobolev space $H^1_0(I)$. We refer to [2] for more information on this type of inequality. The largest admissible C was computed by Talenti [11, p. 357]. It will be referred to as the Sobolev constant and denoted by

$$S(\sigma) := \min\{\|\varphi'\|^2_{L^2(I)} | \varphi \in H^1_0(I), \|\varphi\|_{L^\sigma(I)} = 1\}.$$

The purpose of this paper is to show that this number plays a role in the study of certain nonlinear boundary-value problems. To illustrate the results of the paper, we consider a concrete example, namely the periodic problem for

$$x'' + [x]^p_+ = \tilde{f}(t) + s, \tag{1.1}$$

where $p \in (1, \infty)$, $[x]_+ = \max\{x, 0\}$, $s \in \mathbb{R}$ is a parameter and the function $\tilde{f}$ is locally integrable, T-periodic and has zero average; in short, $\tilde{f} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$. Also, for $p = \infty$, we consider

$$x'' + \mathrm{e}^x = \tilde{f}(t) + s. \tag{1.2}$$

Equations (1.1) and (1.2) are of Landesman–Lazer type. The results by Ward in [12] imply that (1.1) has a T-periodic solution if $s \geqslant 0$ for $p \in (1, \infty)$ and $s > 0$ for

$p = \infty$. Under the more restrictive condition

$$0 < s < \Sigma(p),$$

we can prove that (1.1), or (1.2), has a unique T-periodic solution. Here,

$$\Sigma(p) := \begin{cases} \left[\dfrac{4}{pT^2} S(2p)\right]^{p^*} & \text{if } p \in (1, \infty), \\ \dfrac{4}{T^2} S(\infty) & \text{if } p = \infty. \end{cases} \tag{1.3}$$

From now on, p^* is the conjugate exponent of p:

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

We also prove that there exist sequences $\tilde{f}_n \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$ and $s_n \in \mathbb{R}$, with $s_n \to \Sigma(p)$, such that equation (1.1) has at least two T-periodic solutions if $\tilde{f} = \tilde{f}_n$ and $s = s_n$. This shows that the number $\Sigma(p)$ is sharp.

The dynamical properties of periodic solutions are also linked to Sobolev constants. In fact, if $s \in (0, \Sigma(p)/4^{p^*})$, the unique T-periodic solution of (1.1) is *linearly stable*. Again, $\Sigma(p)/4^{p^*}$ is optimal in the same sense as before.

We can also obtain similar results for equations without uniqueness of periodic solutions, like the superlinear Ambrosetti–Prodi problem,

$$x'' + |x|^p = \tilde{f}(t) + s. \tag{1.4}$$

In any case, the class of nonlinearities considered in this paper is dictated by the method of proof. Fortunately, this method has some flexibility and we have adapted it to a problem of more applied nature: the system of prey and predator under seasonal effects. In this case, we need to employ a weighted Sobolev inequality.

The rest of the paper is organized in four sections. First we discuss in § 2 some properties of linear equations that will be useful later. In the next section we consider a general class of nonlinearities and obtain results on the exact number of T-periodic solutions. Section 4 is devoted to the study of (1.4); in the process we show the sharpness of $\Sigma(p)$. Finally, § 5 deals with the prey–predator system.

2. Hill's equation and Sobolev constants

To each function q in $L^1(\mathbb{R}/T\mathbb{Z})$, we associate a linear equation, or a Hill equation [6],

$$x'' + q(t)x = 0. \tag{2.1}$$

The next result is crucial for the paper.

PROPOSITION 2.1. *Assume that* $q \in L^\sigma(\mathbb{R}/T\mathbb{Z})$ *for some* $\sigma \in [1, \infty]$ *and*

$$\|[q]_+\|_{L^\sigma(0,T)} < 4S(2\sigma^*)/T^{1+1/\sigma^*}. \tag{2.2}$$

Then every non-trivial T-periodic solution $\varphi(t)$ of (2.1) satisfies $\varphi(t) \neq 0$ for each $t \in \mathbb{R}$.

This result is a consequence of theorem 3.7 in [14], but we include a proof for completeness.

Proof. After a rescaling in Sobolev inequality, we obtain

$$S(\sigma)|I|^{-1-2/\sigma}\|\varphi\|^2_{L^\sigma(I)} \leqslant \|\varphi'\|^2_{L^2(I)} \quad \text{for all } \varphi \in H^1_0(I),$$

where I is an interval of arbitrary length $|I|$.

Assume by contradiction that $\varphi(t)$ is a non-trivial T-periodic solution of (2.1) vanishing at some τ. Then $\varphi(\tau) = \varphi(\tau + T) = 0$ and $\varphi'(\tau) = \varphi'(\tau + T) \neq 0$. This implies the existence of $\hat{\tau} \in (\tau, \tau + T)$ with $\varphi(\hat{\tau}) = 0$. Define $I_1 = (\tau, \hat{\tau})$ and $I_2 = (\hat{\tau}, \tau + T)$. We multiply (2.1) by φ and integrate over I_i to obtain

$$\int_{I_i} (\varphi')^2 = \int_{I_i} q\varphi^2 \leqslant \int_{I_i} [q]_+\varphi^2 \leqslant \|[q]_+\|_{L^\sigma(I_i)}\|\varphi\|^2_{L^{2\sigma^*}(I_i)}.$$

From the inequality above,

$$\|[q]_+\|_{L^\sigma(I_i)} \geqslant S(2\sigma^*)|I_i|^{-1-1/\sigma^*}.$$

Thus

$$\begin{aligned}\|[q]_+\|_{L^\sigma(0,T)} &= [\|[q]_+\|^\sigma_{L^\sigma(I_1)} + \|[q]_+\|^\sigma_{L^\sigma(I_2)}]^{1/\sigma} \\ &\geqslant S(2\sigma^*)[|I_1|^{-(1+1/\sigma^*)\sigma} + |I_2|^{-(1+1/\sigma^*)\sigma}]^{1/\sigma}.\end{aligned}$$

The equality in brackets reaches its minimum at $|I_1| = |I_2| = \frac{1}{2}T$ and we are led to an inequality incompatible with (2.2). □

It can be proved that the constant in (2.2) is optimal, but we will not use this fact directly.

Equation (2.1) is said to be *degenerate* if it has non-trivial T-periodic solutions. The next result about degeneracy is an easy consequence of the previous proposition (see [7, lemma 2.2] and [9, lemma 5.4] for details).

Corollary 2.2. *Assume that q_1 and q_2 are two different functions in $L^\sigma(\mathbb{R}/T\mathbb{Z})$ and both of them satisfy (2.2). In addition, $q_1 \leqslant q_2$, with strict inequality on a set of positive measure. Then at least one of the equations $x'' + q_i(t)x = 0$, $i = 1, 2$, is non-degenerate.*

Next we recall some standard notions in the theory of Hill's equations. The eigenvalues of any monodromy matrix of (2.1) are called the Floquet multipliers and denoted by μ_1 and μ_2. They satisfy $\mu_1 \cdot \mu_2 = 1$. The equation (2.1) is called *elliptic* if $|\mu_1| = |\mu_2| = 1$, $\mu_i \neq \pm 1$, *hyperbolic* if $|\mu_1| < 1 < |\mu_2|$ and *parabolic* if $\mu_1 = \mu_2 = \pm 1$.

The periodic and anti-periodic eigenvalues associated to $q(t)$ are a sequence of real numbers

$$\bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leqslant \bar{\lambda}_1(q) < \underline{\lambda}_2(q) \leqslant \bar{\lambda}_2(q) < \cdots,$$

with $\underline{\lambda}_n(q) \to +\infty$ as $n \to \infty$. For $\lambda = \underline{\lambda}_n(q)$ or $\bar{\lambda}_n(q)$ and n even (respectively, n odd), the equation

$$x'' + (\lambda + q(t))x = 0 \tag{2.3}$$

has non-trivial periodic (respectively, anti-periodic) solutions of period T. Moreover, these solutions have exactly n zeros in $[0,T)$. In the intervals between two consecutive eigenvalues, equation (2.3) is alternatively hyperbolic or elliptic. For instance, it satisfies $0<\mu_1<1<\mu_2$ if $\lambda\in(\underline{\lambda}_2(q),\bar{\lambda}_2(q))$. The eigenvalues depend continuously on q with respect to reasonable topologies, say, $L^\sigma(\mathbb{R}/T\mathbb{Z})$, and there is a comparison result: if $q_1\leqslant q_2$, with a strict inequality on a set of positive measure, then $\underline{\lambda}_n(q_1)>\underline{\lambda}_n(q_2)$ and $\bar{\lambda}_n(q_1)>\bar{\lambda}_n(q_2)$. We refer to [6] for more details.

Given $p\in(1,\infty)$, the unique solution of

$$x''+x^{2p-1}=0,\quad x(t)>0\quad\text{in }(0,\tfrac{1}{2}T),\quad x(0)=x(\tfrac{1}{2}T)=0$$

will be denoted by φ_*. For this function, the Sobolev inequality for $\sigma=2p$ and the interval $I=(0,\frac{1}{2}T)$ becomes an identity and, from the equation defining φ_*, one can deduce that

$$\int_0^{T/2}\varphi_*^{2p}=\tfrac{1}{2}T\left[\frac{4S(2p)}{T^2}\right]^{p^*}.$$

We also introduce the function

$$\begin{gathered}q_*(t)=|\varphi_*(t)|^{2p-2}\quad\text{if }t\in[0,\tfrac{1}{2}T),\\ q_*(t+\tfrac{1}{2}T)=q_*(t).\end{gathered}$$

Its minimal period is $\frac{1}{2}T$, but we will look at it as a function in $C(\mathbb{R}/T\mathbb{Z})$.

Lemma 2.3. *There exists $\delta>0$ such that, if $q\in C(\mathbb{R}/T\mathbb{Z})$ satisfies*

$q\geqslant q_*$, with strict inequality somewhere, and $q(t)\leqslant q_*(t)+\delta$ for each $t\in\mathbb{R}$,

then $\underline{\lambda}_2(q)<0<\bar{\lambda}_2(q)$. In particular, $0<\mu_1<1<\mu_2$.

Proof. Consider the function ϕ given by

$$\begin{gathered}\phi(t)=\begin{cases}\varphi_*(t) & \text{if }t\in[0,\tfrac{1}{2}T],\\ -\varphi_*(-t) & \text{if }t\in[-\tfrac{1}{2}T,0),\end{cases}\\ \phi(t+T)=\phi(t)\quad\text{everywhere.}\end{gathered}$$

It is a periodic solution of the autonomous equation

$$x''+|x|^{2p-2}x=0$$

and also satisfies

$$x''+q_*(t)x=0.$$

Since ϕ has two zeros in $[0,T)$, we deduce that either $\underline{\lambda}_2(q_*)=0$ or $\bar{\lambda}_2(q_*)=0$.

The derivative $\phi'(t)$ is a solution of

$$x''+(2p-1)q_*(t)x=0.$$

This function also has two zeros in $[0,T)$, and so

$$\underline{\lambda}_2((2p-1)q_*)=0\quad\text{or}\quad\bar{\lambda}_2((2p-1)q_*)=0.$$

Since $2p-1>1$, by the comparison of eigenvalues, we conclude that

$$\lambda_2(q_*) = 0 < \bar{\lambda}_2(q_*).$$

The lemma then follows from comparison and continuous dependence of eigenvalues. □

We conclude this section with another result that will be useful in treating the nonlinear problem with $p=\infty$.

Lemma 2.4. *There exists a sequence of functions* $q_n \in C^\infty(\mathbb{R}/T\mathbb{Z})$, *with* $q_n > 0$ *everywhere,* $T\int_0^T q_n(t)\,\mathrm{d}t \to 16$ *and such that the Floquet multipliers of (2.1) with* $q = q_n$ *satisfy* $0 < \mu_1 < 1 < \mu_2$.

Proof. The optimality of Lyapunov criterion for stability (see [6]), applied to the period $\frac{1}{2}T$, allows us to find $q_n \in C^\infty(\mathbb{R}/T\mathbb{Z})$, $q_n > 0$, such that q_n also has the period $\frac{1}{2}T$ and the Floquet multipliers (of period $\frac{1}{2}T$) satisfy $\hat{\mu}_1 < -1 < \hat{\mu}_2 < 0$. The required example has been constructed since the multipliers (of period T) are $\mu_i = \hat{\mu}_i^2$. □

The Sobolev constant $S(\infty)$ is reached at the function $\varphi(t) = \min\{t, 1-t\}$. From here, it follows that $S(\infty) = 4$ and $(1/T)\int_0^T q_n \to \Sigma(\infty)$.

3. The class of nonlinearities

In this section we consider a general nonlinear equation of the type

$$x'' + g(x) = f(t), \tag{3.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and $f \in L^1(\mathbb{R}/T\mathbb{Z})$.

Given $\sigma \in [1,\infty)$ and $A, B \in [0,\infty)$, we say that g satisfies the condition $\mathcal{C}(\sigma; A, B)$ if

$$\left[\frac{g(x_1) - g(x_2)}{x_1 - x_2}\right]_+^\sigma \leqslant A(\tfrac{1}{2}(g(x_1) + g(x_2))) + B$$

for every $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$.

The functions $g(x) = |x|^p$ and $g(x) = [x]_+^p$, $p \in (1,\infty)$, satisfy $\mathcal{C}(\sigma; A, B)$ with $\sigma = p^*$, $A = p^{p^*}$ and $B = 0$. This is a consequence of the following inequality.

Lemma 3.1. *Given* $p \in (1,\infty)$, $1/p + 1/p^* = 1$,

$$\left|\frac{|x_1|^p - |x_2|^p}{x_1 - x_2}\right|^{p^*} \leqslant p^{p^*}(\tfrac{1}{2}(|x_1|^p + |x_2|^p)), \quad x_1 \neq x_2.$$

Moreover, the constant p^{p^*} *is optimal.*

Proof. The identity

$$\begin{aligned}|x_1|^p - |x_2|^p &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}(|\lambda x_1 + (1-\lambda)x_2|^p)\,\mathrm{d}\lambda \\ &= p(x_1 - x_2)\int_0^1 |\lambda x_1 + (1-\lambda)x_2|^{p-1}\operatorname{sgn}(\lambda x_1 + (1-\lambda)x_2)\,\mathrm{d}\lambda\end{aligned}$$

leads to

$$\left|\frac{|x_1|^p - |x_2|^p}{x_1 - x_2}\right|^{p^*} \leqslant p^{p^*}\left[\int_0^1 |\lambda x_1 + (1-\lambda)x_2|^{p-1}\,\mathrm{d}\lambda\right]^{p^*}$$

$$\leqslant p^{p^*}\int_0^1 |\lambda x_1 + (1-\lambda)x_2|^{(p-1)p^*}\,\mathrm{d}\lambda,$$

where the Hölder inequality has been used. Note that $(p-1)p^* = p$ and the function $\xi \mapsto |\xi|^p$ is convex. So

$$\int_0^1 |\lambda x_1 + (1-\lambda)x_2|^p\,\mathrm{d}\lambda \leqslant \tfrac{1}{2}(|x_1|^p + |x_2|^p),$$

and the inequality is proved. The optimality of p^{p^*} is obtained by letting x_2 converge to x_1. □

As an example for $\sigma = 1$, $g(x) = \mathrm{e}^x$ satisfies $\mathcal{C}(1;1,0)$. A perturbation like $g(x) = \mathrm{e}^x + \sin x$ satisfies $\mathcal{C}(1;1,2)$.

THEOREM 3.2. *Assume that g satisfies $\mathcal{C}(\sigma;A,B)$ and $f(t)$ satisfies*

$$A\bar{f} + B < (4S(2\sigma^*)/T^2)^\sigma, \tag{3.2}$$

with $\bar{f} = (1/T)\int_0^T f(t)\,\mathrm{d}t$, the average of $f(t)$. Then T-periodic solutions of (3.1) do not cross. In other words, if $\varphi_1(t)$ and $\varphi_2(t)$ are different T-periodic solutions, $\varphi_1(t) \neq \varphi_2(t)$ everywhere.

Proof. First of all, we integrate the equation over a period and observe that

$$\int_0^T g(\varphi_i(t))\,\mathrm{d}t = \int_0^T f(t)\,\mathrm{d}t, \quad i = 1,2. \tag{3.3}$$

The difference $\varphi = \varphi_1 - \varphi_2$ is a non-trivial T-periodic solution of the Hill's equation $x'' + q(t)x = 0$ with

$$q(t) = \frac{g(\varphi_1(t)) - g(\varphi_2(t))}{\varphi_1(t) - \varphi_2(t)}.$$

This function is well defined almost everywhere and belongs to $L^\infty(\mathbb{R}/T\mathbb{Z})$. The assumption $\mathcal{C}(\sigma;A,B)$ and (3.3) imply that

$$\|[q]_+\|^\sigma_{L^\sigma(0,T)} \leqslant \tfrac{1}{2}A\left(\int_0^T g(\varphi_1) + \int_0^T g(\varphi_2)\right) + BT = (A\bar{f} + B)T.$$

The condition (3.2) implies that (2.2) holds and so, by proposition 2.1, φ does not vanish anywhere. □

Once we know that the set of T-periodic solutions is ordered, it is possible to obtain precise information on the number of periodic solutions if g is monotone or convex.

COROLLARY 3.3. *Assume that g satisfies $\mathcal{C}(\sigma;A,B)$ and (3.2) holds. Then we have the following.*

(i) *If g is strictly increasing, then equation (3.1) has at most one T-periodic solution.*

(ii) *If $g \in C^1(\mathbb{R})$ and g' is strictly increasing, then equation (3.1) has at most two T-periodic solutions.*

Proof. (i) Assume that (3.1) has two T-periodic solutions φ_1 and φ_2, with, say, $\varphi_1 > \varphi_2$. Then

$$\int_0^T g(\varphi_1) > \int_0^T g(\varphi_2),$$

and this is not compatible with the identity (3.3), which must hold for all T-periodic solutions.

(ii) The proof uses theorem 3.2 and corollary 2.2 and is essentially the same as the proof of [7, theorem 2.1] or [9, proposition 5.2]. □

Statement (i) in the previous Corollary applies to e^x, and statement (ii) to $|x|^p$, $p \in (1, \infty)$. For $[x]_+^p$ and $\bar{f} > 0$, one can modify the proof of (i), because periodic solutions must be positive somewhere.

We finish this section with a result about the linear stability of T-periodic solutions. Assume that g is C^1. A T-periodic solution $\varphi(t)$ of (3.1) is said to be elliptic if the linearized equation

$$x'' + g'(\varphi(t))x = 0 \tag{3.4}$$

is elliptic. The ellipticity of φ implies that (3.4) is stable, but it is not sufficient to guarantee that $\varphi(t)$ is Lyapunov stable as a solution of the original nonlinear equation (3.1).

THEOREM 3.4. *Assume $g \in C^1(\mathbb{R})$ is strictly increasing and satisfies $\mathcal{C}(\sigma; A, B)$. In addition, $f(t)$ satisfies*

$$A\bar{f} + B < (S(2\sigma^*)/T^2)^\sigma. \tag{3.5}$$

Then, if (3.1) has a non-constant T-periodic solution, it will be elliptic.

Proof. We proceed as in the proof of theorem 3.2, with $q(t) = g'(\varphi(t))$, and arrive at the inequalities

$$q \geqslant 0, \qquad T^{1+1/\sigma^*}\|q\|_{L^\sigma(0,T)} < S(2\sigma^*).$$

Moreover, q is strictly positive on a set of positive measure. We can now apply theorem 1 in [15]. The statement of this theorem says that φ is linearly stable, but the proof shows that it is indeed elliptic. □

The previous result applies to e^x, but not to $[x]_+^p$. Again, the proof can be adapted for this nonlinearity if $\bar{f} > 0$.

4. The optimality of Σ

This section is devoted to the study of the equation

$$x'' + |x|^p = \tilde{f}(t) + s, \tag{4.1}$$

with $p \in (1, \infty)$. The results in [4] imply that, for each $\tilde{f} \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$, there exists $\hat{s} = \hat{s}(\tilde{f}) \geqslant 0$ such that (4.1) has at least two T-periodic solutions if $s > \hat{s}$, at least one if $s = \hat{s}$ and none if $s < \hat{s}$. We will apply corollary 3.3 to this equation to obtain the following result. Note that $|x|^p$ satisfies $\mathcal{C}(p^*; p^{p^*}, 0)$ and the parameter s is the average of $\tilde{f}(t) + s$. So the condition (3.2) reads as

$$p^{p^*} s < (4S(2p)/T^2)^{p^*}$$

or

$$s < \Sigma(p),$$

with the constants $\Sigma(p)$ defined by (1.3).

Proposition 4.1. *Assume that $s < \Sigma(p)$. Then (4.1) has exactly two T-periodic solutions if $s > \hat{s}$ and exactly one if $s = \hat{s}$.*

Proof. The only point that is not a direct consequence of previous results is the uniqueness for $s = \hat{s}$. The key observation is that, in this case, any T-periodic solution should be degenerate, since otherwise it could be continued to s close to $\hat{s}$ with $s < \hat{s}$. If φ_1 and φ_2 were two different T-periodic solutions, they should be ordered according to theorem 3.2, say, $\varphi_1 > \varphi_2$. Hence $g'(\varphi_1) > g'(\varphi_2)$, with $g(x) = |x|^p$. Letting x_2 tend to x_1 in condition $\mathcal{C}(p^*; p^{p^*}, 0)$, or just computing, we deduce that

$$[g'(\varphi_i)]_+^{p^*} \leqslant p^{p^*} g(\varphi_i), \quad i = 1, 2.$$

From $s < \Sigma(p)$, we deduce that $q = g'(\varphi_i)$ is in the condition of corollary 2.2 with $\sigma = p^*$. Then either φ_1 or φ_2 should be non-degenerate, but this is inconsistent with $s = \hat{s}$. □

The previous proposition was already proved in [9, proposition 5.2] for the case $p = 2$ and $s < 64/T^4$, a constant smaller than $\Sigma(2)$. In the next proposition we show that $\Sigma(p)$ is optimal. By the formula in [11, p. 357], the optimal constant $\Sigma(p)$ for $p = 2$ is

$$\Sigma(2) = \frac{4S^2(4)}{T^4} = \frac{\Gamma^8(\frac{1}{4})}{12\pi^2 T^4} \approx \frac{252}{T^4},$$

where $\Gamma(\cdot)$ is the Euler function.

Proposition 4.2. *There exist sequences of functions $\tilde{f}_n \in \tilde{L}^1(\mathbb{R}/T\mathbb{Z})$ and of numbers $s_n \to \Sigma(p)$ such that equation (4.1) has at least four T-periodic solutions for $f = \tilde{f}_n$ and $s = s_n$.*

In the proof we employ some facts that can be deduced from the proofs in [4]. Given an isolated T-periodic solution φ, its index of period T will be denoted by $\gamma(\varphi)$. We refer to [5], and also to [8], for more information on this notion. Assuming that (4.1) has a finite number of T-periodic solutions, $\varphi_1, \dots, \varphi_n$, it is known that

$$\sum_{i=1}^{n} \gamma(\varphi_i) = 0. \tag{4.2}$$

Let $F(t)$ be a T-periodic function with $F''(t)=\tilde{f}(t)$. Then $-c+F(t)$ is a strict lower solution of (4.1) for c large enough. On the other hand, T-periodic solutions for $s=\hat{s}$ are strict upper solutions for $s>\hat{s}$. From these facts, we deduce that if (4.1) has a finite number of T-periodic solutions, then at least one of them, $\hat{\varphi}$, satisfies

$$\bar{\lambda}_0(\hat{q})\geqslant 0 \quad \text{with } \hat{q}(t)=p|\hat{\varphi}(t)|^{p-1}\operatorname{sgn}\hat{\varphi}(t), \qquad \gamma(\hat{\varphi})=-1.$$

This is a consequence of the following result.

Lemma 4.3. *Assume that* $g\in C^1(\mathbb{R})$, $f\in L^1(\mathbb{R}/T\mathbb{Z})$ *and that there exist functions* $\phi_1,\phi_2\in W^{2,1}(\mathbb{R}/T\mathbb{Z})$, *with* $\phi_1>\phi_2$ *everywhere, and a number* $\varepsilon>0$ *such that*

$$\phi_1''(t)+g(\phi_1(t))\leqslant f(t)-\varepsilon, \qquad \phi_2''(t)+g(\phi_2(t))\geqslant f(t)+\varepsilon$$

almost everywhere. In addition, it is assumed that the equation

$$x''+g(x)=f(t)$$

has a finite number of T*-periodic solutions between* ϕ_2 *and* ϕ_1. *Then there exists* $\varphi(t)$, *a* T*-periodic solution lying between* ϕ_2 *and* ϕ_1, *such that*

$$\bar{\lambda}_0(g'(\varphi))\geqslant 0 \quad \textit{and} \quad \gamma(\varphi)=-1.$$

This result is well known, but, for completeness, we present a proof at the end of the section.

Now we are ready to prove the multiplicity of periodic solutions.

Proof of proposition 4.2. Throughout the proof it is assumed that the number of T-periodic solutions is finite. We start with the function φ_*, which was defined in § 2. The domain of φ_* is $[0,\frac{1}{2}T]$, but we extend it to $\mathbb{R}$ by periodicity, $\varphi_*(t+\frac{1}{2}T)=\varphi_*(t)$, $t\in\mathbb{R}$. We notice that φ_* is in $H^1(\mathbb{R}/T\mathbb{Z})$ but not in $C^1(\mathbb{R}/T\mathbb{Z})$. Also,

$$\int_0^T \varphi_*^{2p}=2\int_0^{T/2}\varphi_*^{2p}=T\left[\frac{4S(2p)}{T^2}\right]^{p^*}.$$

Next we define

$$\psi_*(t)=\frac{1}{p^{1/(p-1)}}(\varphi_*(t))^2.$$

This is a non-negative function that only vanishes at $\frac{1}{2}T+\mathbb{Z}$. Also,

$$\int_0^T \psi_*^p=T\Sigma(p).$$

For each $\varepsilon>0$, we construct a function $\psi_\varepsilon\in C^\infty(\mathbb{R}/T\mathbb{Z})$ such that

$$\psi_\varepsilon>0 \quad \text{everywhere}, \qquad \psi_\varepsilon\geqslant\psi_*, \qquad \psi_\varepsilon\to\psi_* \quad \text{uniformly}.$$

We define $\tilde{f}_\varepsilon\in\tilde{L}^1(\mathbb{R}/T\mathbb{Z})$ and s_ε by the equation

$$\psi_\varepsilon''(t)+(\psi_\varepsilon(t))^p=\tilde{f}_\varepsilon(t)+s_\varepsilon.$$

Our task is to prove that (4.1) has at least four T-periodic solutions for $\tilde{f} = \tilde{f}_\varepsilon$, $s = s_\varepsilon$ and ε small. Notice that

$$s_\varepsilon = \frac{1}{T}\int_0^T \psi_\varepsilon^p \to \frac{1}{T}\int_0^T \psi_*^p = \Sigma(p).$$

By construction, we know that ψ_ε is a T-periodic solution of (4.1). The linearization

$$x'' + q_\varepsilon(t)x = 0, \qquad q_\varepsilon(t) = p(\psi_\varepsilon(t))^{p-1}$$

is in the conditions of lemma 2.3 for small ε. Hence $\underline{\lambda}_2(q_\varepsilon) < 0 < \bar{\lambda}_2(q_\varepsilon)$ and $\gamma(\psi_\varepsilon) = -1$. The value of the index is computed by the formula

$$\gamma(\psi_\varepsilon) = \operatorname{sgn}\{(1-\mu_1)(1-\mu_2)\}, \quad \text{valid if } \mu_i \neq 1.$$

Since ψ_ε is non-degenerate, we deduce that $s_\varepsilon > \hat{s}(\tilde{f}_\varepsilon)$ and, from the discussion before the proof, we obtain another T-periodic solution $\hat{\varphi}$ with

$$\bar{\lambda}_0(\hat{q}) \leqslant 0 \quad \text{and} \quad \gamma(\hat{\varphi}) = -1.$$

Both solutions have the same index, but the position of the eigenvalues is different. It is known that the index of any T-periodic solution satisfies $\gamma(\varphi) \leqslant 1$ (see [8, 10]). Formula (4.2) implies the existence of at least two T-periodic solutions with index 1. □

The results stated in § 1 about the optimality of Σ for (1.1) are proven in a similar way. Now the sum of the indexes of all T-periodic solutions must be 1. For $p = \infty$, one uses lemma 2.4.

This proves the optimality of the constants $\Sigma(p)$ for the existence of T-periodic solutions of equations (1.1), (1.2) or (1.4) in § 1. The optimality of the constants $\Sigma(p)/4^{p^*}$ for the ellipticity of T-periodic solutions of equations (1.1) and (1.2) can be proved similarly. In the previous construction, the interval $(0, \frac{1}{2}T)$ is replaced by $(0, T)$.

Remark 4.4. For equation (1.4), if $\hat{s} < s < \Sigma(p)/4^{p^*}$, it has exactly two T-periodic solutions. It is possible to prove that one of these must be hyperbolic, and another is elliptic and so linearly stable. The proofs are similar to those in [9, proposition 5.5]. It is interesting to notice that, for the quadratic case ($p = 2$), when $\hat{s} < s < \Sigma^*$ for some constant $\Sigma^* < \frac{1}{8}\Sigma(2)$, the elliptic T-periodic solution of (1.4) is actually of twist type, and so *Lyapunov stable* (see [9, theorem 5.1]). However, the optimal bound for Σ^* is unknown to the authors even for the case $p = 2$. Some preliminary results can be found in [9, § 5] (also see [13] for some improvement).

Proof of lemma 4.3. The functional

$$\mathcal{A} : H^1(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}, \qquad \mathcal{A}[x] = \int_0^T \{\tfrac{1}{2}x'(t)^2 - G(x(t)) + f(t)x(t)\}\,\mathrm{d}t$$

is lower semi-continuous with respect to the weak topology (as usual, G is a primitive of g). Let φ be a minimizer of $\mathcal{A}$ on the closed and convex set

$$\Omega = \{x \in H^1(\mathbb{R}/T\mathbb{Z}) \mid \phi_2 \leqslant x \leqslant \phi_1\}.$$

Working with test functions, one can prove that φ is in the interior of Ω and so φ is a local minimizer of $\mathcal{A}$ and a solution of the equation. The properties of minimizers imply that, if φ is isolated as a T-periodic solution, then $\gamma(\varphi) = -1$ (see [1, 8]). Also, the second derivative $D^2\mathcal{A}[\varphi]$ must be a non-negative quadratic form. This implies that $\bar{\lambda}_0(g'(\varphi)) \geqslant 0$. □

5. The prey–predator system

Consider

$$\left.\begin{aligned} u' &= (a(t) - b(t)v)u, \qquad u > 0,\\ v' &= (-c(t) + d(t)u)v, \quad v > 0,\end{aligned}\right\} \tag{5.1}$$

where $a, c \in L^1(\mathbb{R}/T\mathbb{Z})$ and $b, d \in L^\infty_+(\mathbb{R}/T\mathbb{Z})$. We employ the notation

$$L^\infty_+(\mathbb{R}/T\mathbb{Z}) = \{f \in L^\infty(\mathbb{R}/T\mathbb{Z}) \mid \text{ess inf}\, f > 0\}.$$

The existence of T-periodic solutions, sometimes called *coexistence states*, has been discussed by several authors (see, for instance, [3]). The necessary and sufficient condition for the existence of such solutions is

$$\bar{a} > 0 \quad \text{and} \quad \bar{c} > 0. \tag{5.2}$$

THEOREM 5.1. *Assume that (5.2) holds and*

$$\bar{a} \cdot \bar{c} \leqslant (4/T)^2. \tag{5.3}$$

Then (5.1) has a unique T-periodic solution. Moreover, this solution is elliptic if

$$\bar{a} \cdot \bar{c} \leqslant (2/T)^2. \tag{5.4}$$

Conditions (5.3) and (5.4) are optimal.

The proof relies on a result for the linear periodic system

$$\xi' = -\beta(t)\eta, \qquad \eta' = \delta(t)\xi, \tag{5.5}$$

with $\beta, \delta \in L^\infty_+(\mathbb{R}/T\mathbb{Z})$.

LEMMA 5.2. *Assume that $\bar{\beta} \cdot \bar{\delta} \leqslant (4/T)^2$. Then (5.5) has no T-periodic solutions (different from $\xi = \eta = 0$). Moreover, if $\bar{\beta} \cdot \bar{\delta} \leqslant (2/T)^2$, then system (5.5) is elliptic.*

On the other hand, there exist sequences of functions $\beta_n, \delta_n \in C^\infty(\mathbb{R}/T\mathbb{Z})$, $\beta_n > 0$, $\delta_n > 0$, with $\bar{\beta}_n \cdot \bar{\delta}_n \to (4/T)^2$ (respectively, $(2/T)^2$) such that the Floquet multipliers of (5.5) for $\beta = \beta_n$, $\delta = \delta_n$ satisfy $0 < \mu_1 < 1 < \mu_2$ (respectively, $\mu_2 < -1 < \mu_1 < 0$).

This result is a consequence of the very classical ideas due to Lyapunov. In fact, for $\beta = 1$, it is just a classical result for Hill's equations. At the end of the section we sketch a proof emphasizing the connection with Sobolev inequalities.

Proof of theorem 5.1. The change of variables $U = \ln u$, $V = \ln v$ transforms (5.1) into

$$\left.\begin{aligned} U' &= a(t) - b(t)\mathrm{e}^V,\\ V' &= -c(t) + d(t)\mathrm{e}^U.\end{aligned}\right\} \tag{5.6}$$

If $(U(t), V(t))$ is a T-periodic solution, we integrate over a period and obtain

$$\int_0^T b\mathrm{e}^V = \int_0^T a, \qquad \int_0^T d\mathrm{e}^U = \int_0^T c. \tag{5.7}$$

We prove the uniqueness by a contradiction argument. Assume that $(U_i(t), V_i(t))$, $i = 1, 2$, are two different T-periodic solutions of (5.6). The difference, $\xi = U_1 - U_2$, $\eta = V_1 - V_2$, is a T-periodic solution of (5.5) with

$$\beta = b\frac{\mathrm{e}^{V_1} - \mathrm{e}^{V_2}}{V_1 - V_2}, \qquad \delta = d\frac{\mathrm{e}^{U_1} - \mathrm{e}^{U_2}}{U_1 - U_2}$$

(the quotient $(\mathrm{e}^x - \mathrm{e}^y)/(x - y)$ is replaced by e^x if $x = y$). The exponential function satisfies $\mathcal{C}(1; 1, 0)$ and so

$$0 < \beta \leqslant b(\tfrac{1}{2}(\mathrm{e}^{V_1} + \mathrm{e}^{V_2})), \qquad 0 < \delta \leqslant d(\tfrac{1}{2}(\mathrm{e}^{U_1} + \mathrm{e}^{U_2})).$$

From (5.7), one obtains

$$0 < \bar{\beta} \leqslant \bar{a}, \qquad 0 < \bar{\delta} \leqslant \bar{c},$$

and we are led to a contradiction with lemma 5.2. The proof of the ellipticity is similar, because one can use now the linearized equation as system (5.5).

Finally, we prove the optimality of the constants. The proof of the existence of periodic solution under (5.2) implies that if (5.6) has a finite number of T-periodic solutions, $(U_1, V_1), \dots, (U_n, V_n)$, then

$$\sum_{i=1}^n \gamma(U_i, V_i) = 1$$

(see [3]). In particular, if (5.6) has a unique T-periodic solution, its index is one.

Let β_n and δ_n be the functions appearing in lemma 5.2. We define

$$V_n = \ln \beta_n, \qquad U_n = \ln \delta_n.$$

By construction, (U_n, V_n) is a solution of (5.6) with

$$b = d = 1, \qquad a = U_n' + \mathrm{e}^{V_n}, \qquad c = -V_n' + \mathrm{e}^{U_n}.$$

The index $\gamma(U_n, V_n) = \operatorname{sgn}\{(1 - \mu_1)(1 - \mu_2)\} = -1$, and so the uniqueness is lost.

The proof of the optimality of $(2/T)^2$ is similar. The same construction now leads to a system having a unique T-periodic solution which is hyperbolic. □

It remains to prove the lemma. To this end, we recall the inequality

$$\frac{4}{\int_I 1/p} \|\varphi\|^2_{L^\infty(I)} \leqslant \int_I p(t)(\varphi'(t))^2 \,\mathrm{d}t,$$

which is valid for $p \in L^\infty_+(I)$ and $\varphi \in H^1_0(I)$. This inequality is strict unless $\varphi = c\varphi_\tau$, with $c \in \mathbb{R}$, $\tau \in I = (a, b)$ and

$$\varphi_\tau(t) = \begin{cases} \left(\displaystyle\int_\tau^b \frac{1}{p}\right) \displaystyle\int_a^t \frac{1}{p} & \text{if } a \leqslant t \leqslant \tau, \\ \left(\displaystyle\int_a^\tau \frac{1}{p}\right) \displaystyle\int_t^b \frac{1}{p} & \text{if } \tau \leqslant t \leqslant b. \end{cases}$$

One way to arrive at this inequality is to maximize the linear functional $\varphi \in H_0^1(I) \mapsto \varphi(\tau)$ constrained by the $\int_I p\varphi'^2 = 1$. The method of Lagrange multipliers leads to the equation (understood in the sense of distributions)

$$(p(t)\varphi')' = \lambda\delta_\tau, \quad \lambda \in \mathbb{R},$$

where δ_τ is the Dirac measure concentrated at τ. The solutions in $H_0^1(I)$ are of the type $c\varphi_\tau$.

Proof of lemma 5.2. We prove the result stated for $(4/T)^2$. The other case is similar.

Given $\xi(t)$, $\eta(t)$, non-trivial T-periodic solution of (5.5), we notice that $\int_0^T \delta\xi = 0$, and so ξ must vanish somewhere. If $\xi(\tau) = 0$, by uniqueness, $\eta(\tau) \neq 0$. From the first equation, we have that

$$\xi(t) = -\int_\tau^t \beta(s)\eta(s)\,\mathrm{d}s,$$

and so ξ changes sign at τ. Thus we find $\tau < \hat{\tau} < \tau + T$ such that

$$\xi(\tau) = \xi(\hat{\tau}) = \xi(\tau + T) = 0.$$

Define $I_1 = (\tau, \hat{\tau})$ and $I_2 = (\hat{\tau}, \tau + T)$. We multiply the first equation by η and the second by ξ. After integrating over I_i,

$$\int_{I_i} \frac{1}{\beta}\xi'^2 = \int_{I_i} \beta\eta^2 = \int_{I_i} \delta\xi^2.$$

These imply

$$\left(\int_{I_i} \delta\right) \|\xi\|^2_{L^\infty(I_i)} \geqslant \int_{I_i} \frac{1}{\beta}\xi'^2.$$

We apply the weighted Sobolev inequality with $p = 1/\beta$ to deduce that

$$\left(\int_{I_i} \beta\right)\left(\int_{I_i} \delta\right) > 4.$$

Here there is a subtle point. In principle, we should have

$$\left(\int_{I_i} \beta\right)\left(\int_{I_i} \delta\right) \geqslant 4,$$

but the non-strict case would imply that $\xi = c\varphi_{\tau_*}$ for some $\tau_* \in I_i$. From the first equation of (5.5), one deduces that η should be a piecewise constant function with a jump at τ. Of course, this is not possible since η is Lipschitz continuous.

The inequalities $A_iB_i > 4$, with $A_i = \int_{I_i} \beta$, $B_i = \int_{I_i} \delta$, imply that

$$(A_1 + A_2)(B_1 + B_2) > 16.$$

Thus the assumption $\bar{\beta} \cdot \bar{\delta} \leqslant (4/T)^2$ does not hold.

Finally, we construct β_n and δ_n. We take $\beta_n = 1$ so that the system is equivalent to Hill's equation,

$$\xi'' + \delta_n(t)\xi = 0.$$

The condition $\bar{\beta}_n \cdot \bar{\delta}_n \to (4/T)^2$ becomes $T\int_0^T \delta_n \to 16$ and we can invoke lemma 2.4. □

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