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Best Estimates of Weighted Eigenvalues of One-dimensional p-Laplacian*

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Abstract: In this paper, we determine the infimum and the supremum of the Dirichlet eigenvalues $\lambda_n(\rho)$ $(n = 1, 2, \cdots)$ of the problem $(|x'|^{p-2}x')' + \lambda\rho(t)|x|^{p-2}x = 0$, $t \in [0, T]$, where $1 < p < \infty$, and the weights ρ are nonnegative and are subject to conditions $\int_0^T \rho(t)dt = M$ and $\max_{t\in[0,T]} \rho(t) = H$. It is also explained for what weights ρ the infimum and the supremum will be attained.

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1 Introduction

Eigenvalue theory, including the existence, the distribution and estimates of eigenvalues, always remains as a central topic in many literature. Some well studied classes of problems include the Sturm-Liouville operators

$$(q(t)x')' + p(t)x + \lambda\rho(t)x = 0 \tag{1.1}$$

and the higher dimensional Laplacian operator, with respect to various boundary conditions.

The purpose of this paper is to find the best estimates of the weighted eigenvalues of the one-dimensional p-Laplacian $(1 < p < \infty)$

$$(|x'|^{p-2}x')' + \lambda\rho(t)|x|^{p-2}x = 0 \qquad (t \in [0, T]) \tag{1.2}$$

with the Dirichlet boundary condition

$$x(0) = x(T) = 0, \tag{D}$$

when the weights ρ are in $L^1(0, T)$ with $\rho \succ 0$, and

$$\int_0^T \rho(t)dt \qquad \text{and} \qquad \max_{t\in[0,T]} \rho(t)$$

are given. Here the notation $\rho \succ 0$ means that $\rho(t) \geq 0$ for a.e. t and $\int_0^T \rho(t)dt > 0$.

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Such an estimate problem is motivated by the following works. When the coefficients in (1.1) are periodic, the estimates of the periodic and anti-periodic eigenvalues go back to the Lyapunov time, which yield the famous Lyapunov stability criterion. It was Krein[1] who first gave the best estimates to the weighted eigenvalues of the linear problem of (1.2):

$$x'' + \lambda\rho(t)x = 0 \tag{1.3}$$

with (D), where the weight $\rho(t)$, which can be explained as the density of a string, is nonnegative and the integral $\int_0^T \rho(t)dt$, which can be explained as the mass of the string, is given. He then applied the results to the stability problem and obtained various stability criteria of (1.3). Other best estimates can be found in [2, 3]. 40 years after Krein's work, a very nice paper by Karaa[4] has obtained the best estimates both from below and from upper to all eigenvalues of (1.1)+(D) for various classes of $q(t)$, $p(t)$ and $\rho(t)$. A very recent work by Li and the second author[5] has obtained the best lower bound for the first anti-periodic eigenvalue of (1.3) using the L^α $(1 \le \alpha \le \infty)$ norms of $\rho(t)$. Such an estimate can yield many interesting applications, including the stability criteria which 'interpolated' the famous Lyapunov criterion (the L^1 case) and the Zukovskii criterion (the L^∞ case) and the monotone iterative method (see [6]).

In the recent two decades, a nonlinear eigenvalue problem, which is called the p-Laplacian and is a generalization of the Laplacian, has received more and more attention in literature because of its applications in non-Newtonian fluids, reaction-diffusion problems, nonlinear elasticity, and glaciology etc. (see [7, 8]). The eigenvalues of the p-Laplacian with the Dirichlet boundary condition are those λ such that the problem

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has nontrivial solutions, where $1 < p < \infty$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and $\Omega \subset \mathbb{R}^N$ is a domain. As usual, a nontrivial solution of (1.4) will be referred to as an eigenfunction corresponding to λ. One can use various variational techniques to find certain eigenvalues of (1.4) which are called *variational eigenvalues* for general domains Ω. However, due to the nonlinearity of the operator Δ_p $(p \ne 2)$, the complete structure of eigenvalues of (1.4) has not been understood completely even when Ω is a ball or a cube, where $N \ge 2$. See [8] for the survey of this nonlinear operator and also [9, 10, 11, 12, 13] for the related results and problems about this topic.

If the dimension $N = 1$ and $\Omega = (0, T)$, the complete set of eigenvalues of (1.4) is characterized by a sequence $\{\lambda_n\}$ given by $\lambda_n = (n\pi_p/T)^p$, $n = 1, 2, \cdots$, where

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}.$$

In this case, even a weight is considered as in (1.1), the set of eigenvalues of (1.2)+(D) is discrete and is given by a sequence

$$0 < \lambda_1(\rho) < \lambda_2(\rho) < \cdots < \lambda_n(\rho) < \cdots \tag{1.5}$$

going to $+\infty$ when $n \to \infty$ (see [14, 15]).

In this paper, we will follow from some ideas in Krein[1] and Karaa[4] for the linear problem to give some best estimates for those Dirichlet eigenvalues. To be specific, let

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$M > 0$, $H > 0$, $T > 0$ be numbers such that $M \le HT$. Define

$$E = E(M,H,T) := \left\{ \rho \in L^1[0,T]; \int_0^T \rho(t)dt = M, 0 \le \rho(t) \le H, \text{ a.e. } t \in [0,T] \right\}. \quad (1.6)$$

The main results of this paper are: for any $n \in \mathbb{N}$,

$$\mu_n := \inf_{\rho \in E} \lambda_n(\rho) = \min_{\rho \in E} \lambda_n(\rho) = \frac{n^p 2^p H^{p-1}}{M^p} \chi_p \left(\frac{M}{HT} \right), \quad (1.7)$$

$$\Lambda_n := \sup_{\rho \in E} \lambda_n(\rho) = \max_{\rho \in E} \lambda_n(\rho) = \frac{n^p \pi_p^p H^{p-1}}{M^p}, \quad (1.8)$$

where the function χ_p will be explained in Section 3. See Theorems 3.2 and 4.2. It will also be explained when the minimum μ_n and when the maximum Λ_n in (1.7) and (1.8) will be attained. These estimates can yield some best estimates to the periodic and anti-periodic eigenvalues of (1.2) which are recently defined using rotation number by the second author[16]. See Remark 4.1.

The rest of this paper is organized as follows. In Section 2, a comparison result for eigenvalues of one-dimensional p-Laplacian will be proved. We determine μ_n in Section 3. The analysis on μ_1 is fundamental and the determination of μ_n is based on the results of μ_1 and our comparison theorem. Section 4 is devoted to the determination of Λ_n. The results on μ_n and Λ_n are generalized in Section 5 where ρ runs through another class of weights $E^*(M,H,L,T)$.

2 Preliminaries

Let $1 < p < \infty$ be fixed. Denote $\phi_p(x) = |x|^{p-2}x$ for all $x \neq 0$ and $\phi_p(0) = 0$. We introduce some functions which are called the p-cosine and the p-sine as in [17]. Consider the equation

$$(\phi_p(x'))' + \phi_p(x) = 0. \quad (2.1)$$

Let $y = -\phi_p(x')$. Equation (2.1) is then equivalent to the following system:

$$x' = -\phi_q(y), \quad y' = \phi_p(x), \quad (2.2)$$

where q is the conjugate exponent of p: $p^{-1} + q^{-1} = 1$. Note that (2.2) is a Hamiltonian system with the Hamiltonian $H(x,y) = p^{-1}|x|^p + q^{-1}|y|^q$. For any (x_0, y_0), the initial value problem of (2.2) with $(x(0), y(0)) = (x_0, y_0)$ has a unique solution $(x(t), y(t))$ which is well defined on the whole line $\mathbb{R}$. All the solutions of (2.2) are periodic of the same period $2\pi_p$. Let $(x,y) = (C_p(t), S_p(t))$ be the unique solution of (2.2) verifying $(C_p(0), S_p(0)) = (1,0)$. The functions $C_p(t)$ and $S_p(t)$, which are called *p-cosine* and *p-sine* respectively, behave much like the cosine and sine functions.

Lemma 2.1[17] (1) *Both $C_p(t)$ and $S_p(t)$ are $2\pi_p$-periodic;*

(2) *$C_p(t)$ is even in t and $S_p(t)$ is odd in t;*

(3) *$C_p(t+\pi_p) = -C_p(t)$, $S_p(t+\pi_p) = -S_p(t)$;*

(4) *$C_p(t) = 0$ if and only if $t = \pi_p/2 + n\pi_p$, $n \in \mathbb{Z}$; and $S_p(t) = 0$ if and only if $t = n\pi_p$, $n \in \mathbb{Z}$;*

(5) *$C_p'(t) = -\phi_q(S_p(t))$ and $S_p'(t) = \phi_p(C_p(t))$; and*

(6) *$p^{-1}|C_p(t)|^p + q^{-1}|S_p(t)|^q \equiv p^{-1}$.*

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The *p-tangent* and *p-cotangent* functions are also introduced in [17]:

$$T_p(t) = \pm\frac{|S_p(t)|}{|C_p(t)|^{p-1}}, \quad G_p(t) = \pm\frac{|C_p(t)|}{|S_p(t)|^{q-1}},$$

where the signs $\pm$ are determined by what quadrant on which $(C_p(t), S_p(t))$ lies.

The following result is a generalization of the comparison theorem for one-dimensional Schödinger operator.

Lemma 2.2 *Given two equations*

$$(\phi_p(x'))' + f(t)\phi_p(x) = 0, \tag{2.3}$$

$$(\phi_p(z'))' + g(t)\phi_p(z) = 0, \tag{2.4}$$

where $f,\ g \in L^1(0,T)$. If $f(t) \le g(t)$ on $[0,T]$, then each solution of (2.4) has at least one zero between any two zeros of any nontrivial solution of (2.3).

Lemma 2.3[15] *Assume that $\rho(t) \in L^1(0,T)$, $\rho(t) \succ 0$. For any $n \in \mathbb{N}$, the nth eigenfunction of (1.2)+(D) has and has only $n+1$ zeros on the interval $[0,T]$.*

Now we give a comparison between the Dirichlet eigenvalues, which is important in the next section.

Lemma 2.4 *Suppose that the interval $I = [0,T]$ is divided into $I = \bigcup_{i=1}^n I_i$, where $I_i = [a_{i-1}, a_i]$ and $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = T$. Then*

$$\max_{1\le i\le n} \lambda_1(\rho|I_i) \ge \lambda_n(\rho|I). \tag{2.5}$$

Moreover, the equality holds if and only if a_i, $i = 0, 1, \cdots, n$, are the $n+1$ zeros of the nth eigenfunction of (1.2)+(D) corresponding to $\lambda_n(\rho|I)$, and under such a condition, $\lambda_1(\rho|I_i) = \lambda_n(\rho|I)$ for any $1 \le i \le n$.

Proof. Let x, corresponding to $\lambda_n(\rho|I)$, be the nth eigenfunction of (1.2)+(D) with the zeros $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$. Then there must be some $1 \le i_0,\ j_0 \le n$ such that $I_{i_0} \subset [t_{j_0-1}, t_{j_0}] =: T_{j_0}$. It follows from Lemma 2.2 that

$$\lambda_1(\rho|I_{i_0}) \ge \lambda_1(\rho|T_{j_0}) = \lambda_n(\rho|I).$$

So (2.5) is proved. Furthermore, the equality in (2.5) holds only if $a_{i_0-1} = t_{j_0-1}$ and $a_{i_0} = t_{j_0}$ in the above inequality. Repeating the reasoning as above, one has actually that the other intervals I_i must also coincide with the nodal intervals, i.e., $[a_{i-1}, a_i] = [t_{i-1}, t_i]$ for all $1 \le i \le n$. Thus $\lambda_1(\rho|I_i) = \lambda_n(\rho|I)$ for $1 \le i \le n$ and the lemma is proved.

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3 The Infimum

3.1 μ_1 can be attained for some $\rho \in E(M, H, T)$

This can be proved by using the compactness technique. Denote by $W_0^{1,p}(0,T)$ the usual Sobolev space. It is well-known that the smallest eigenvalue of (1.2)+(D) is

$$\lambda_1(\rho) = \min_{y \in W_0^{1,p}\setminus\{0\}} \frac{\int_0^T |y'(t)|^p dt}{\int_0^T |y(t)|^p \rho(t) dt}, \tag{3.1}$$

where the minimum is attained only for eigenfunctions $y(t)$ corresponding to $\lambda_1(\rho)$.

Let $\mu_1 = \inf_{\rho \in E} \lambda_1(\rho)$ so that there exists a sequence $\{\rho_n\}$ in E such that $\lambda_1(\rho_n) \to \mu_1$ as $n \to \infty$. Suppose that y_n is the eigenfunction corresponding to $\lambda_1(\rho_n)$, i.e.,

$$\begin{cases} (\phi_p(y_n'))' + \lambda_1(\rho_n)\rho_n(t)\phi_p(y_n) = 0, \\ y_n(0) = y_n(T) = 0. \end{cases} \tag{3.2}$$

Without loss of generality, assume that

$$\int_0^T |y_n(t)|^p \rho_n(t) dt = 1 \tag{3.3}$$

for all n. Integrating (3.2) by part, we get from (3.3) that

$$\int_0^T |y_n'|^p dt = \lambda_1(\rho_n) \tag{3.4}$$

for all n. By the Hölder inequality, we know from (3.4) that

$$\begin{aligned} |y_n(t_2) - y_n(t_1)| &\le \left| \int_{t_1}^{t_2} |y_n'(s)| ds \right| \le |t_2 - t_1|^{1/q} \left| \int_{t_1}^{t_2} |y_n'(s)|^p ds \right|^{1/p} \\ &\le (\lambda_1(\rho_n))^{1/p} |t_2 - t_1|^{1/q}, \end{aligned}$$

for any $t_1, t_2 \in [0,T]$. In particular, $|y_n(t)| \le (\lambda_1(\rho_n))^{1/p} T^{1/q}$ for all $t \in [0,T]$. Recall that $\lambda_1(\rho_n)$ converges to μ_1. Thus the sequence $\{y_n\}$ is uniformly bounded and equi-continuous. Without loss of generality, we assume that the sequence $\{y_n\}$ uniformly converges to a certain function y_0 with $y_0(0) = y_0(T) = 0$.

Let $\sigma_n(t) = \int_0^t \rho_n(s) ds$, $n \in \mathbb{N}$. Since $\rho_n \in E(M, H, T)$, it is obvious that the sequence $\{\sigma_n\}$ is uniformly bounded and equi-continuous. Without loss of generality, we assume that $\{\sigma_n\}$ converges uniformly to some non-decreasing function σ_0 such that

$$0 \le \sigma_0(t+h) - \sigma_0(t) \le Hh \quad (0 \le t \le t+h \le T),$$

and $\sigma_0(0) = 0$, $\sigma_0(T) = M$. It then follows that a measurable function $\rho_0(t) \in E(M, H, T)$ can be found such that

$$\sigma_0(t) = \int_0^t \rho_0(s) ds.$$

Let v be any C^1 function with compact support in $(0,T)$. It follows from (3.2) that

$$\int_0^T \phi_p(y_n'(t)) v'(t) dt = \lambda_1(\rho_n) \int_0^T \rho_n(t) \phi_p(y_n(t)) v(t) dt.$$

The convergence results of $y_n \to y_0$ uniformly, $\sigma_n \to \sigma_0$ uniformly and $\lambda_1(\rho_n) \to \mu_1$ show that

$$\int_0^T \rho_0(t) |y_0(t)|^p dt = 1,$$

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(thus $y_0(t)$ is not identically zero), and

$$\int_0^T \phi_p(y_0'(t))v'(t)dt = \mu_1 \int_0^T \rho_0(t)\phi_p(y_0(t))v(t)dt$$

for any C^1 function v with compact support in $(0,T)$, which implies by the regularity theory that

$$\begin{cases} (\phi_p(y_0'))' + \mu_1\rho_0(t)\phi_p(y_0) = 0, \\ y_0(0) = y_0(T) = 0, \end{cases} \tag{3.5}$$

and $\mu_1 = \lambda_1(\rho_0)$. Thus $\mu_1 = \inf_{\rho\in E}\lambda_1(\rho)$ is attained for this $\rho_0 \in E(M,H,T)$.

3.2 The value of μ_1

Let $\rho_0(t)$ and $y_0(t)$ be as above. Let ρ be any function from $E(M,H,T)$. Then

$$\frac{\int_0^T |y_0'(t)|^p dt}{\int_0^T \rho_0(t)|y_0(t)|^p dt} = \lambda_1(\rho_0) = \mu_1 \le \lambda_1(\rho) \le \frac{\int_0^T |y_0'(t)|^p dt}{\int_0^T \rho(t)|y_0(t)|^p dt},$$

where the last inequality follows from the characterization (3.1). Hence

$$\int_0^T \rho_0(t)|y_0(t)|^p dt \ge \int_0^T \rho(t)|y_0(t)|^p dt, \quad \forall\ \rho \in E(M,H,T). \tag{3.6}$$

The function $y_0(t)$, being the first eigenfunction of (1.2)+(D) corresponding to $\mu_1(\rho_0)$ and ρ_0, has no zeros in $(0,T)$ (see Lemma 2.3). We may assume that it is positive. Consider the function

$$f(t) = y_0(t+d) - y_0(t),$$

where $d = M/H$. Since $f(0) = y_0(d) > 0$ and $f(T-d) = -y_0(T-d) < 0$, we can find a number $\xi \in (0, T-d)$ such that $f(\xi) = 0$. Let $y_0(\xi+d) = y_0(\xi) = h > 0$. In view of (3.5), $\phi_p(y_0')$ and consequently y_0' are nonincreasing in $[0,T]$. Thus y_0 is convex, and consequently

$$0 \le y_0(t) \le h \quad \text{for } t \in (0,\xi)\cup(\xi+d,T),$$
$$y_0(t) \ge h \quad \text{for } t \in (\xi,\xi+d).$$

Define

$$\rho_\xi(t) = \begin{cases} H & \text{for } t \in (\xi,\xi+d), \\ 0 & \text{for } t \in (0,\xi)\cup(\xi+d,T). \end{cases} \tag{3.7}$$

Then

$$\begin{aligned} & \int_0^T |y_0(t)|^p \rho_\xi(t)dt - \int_0^T |y_0(t)|^p\rho_0(t)dt \\ = & \int_\xi^{\xi+d} |y_0(t)|^p(H-\rho_0(t))dt - \left(\int_0^\xi + \int_{\xi+d}^T\right)|y_0(t)|^p\rho_0(t)dt \\ \ge & \ h^p\int_\xi^{\xi+d}(H-\rho_0(t))dt - h^p\left(\int_0^\xi + \int_{\xi+d}^T\right)\rho_0(t)dt \\ = & \ h^p\left(Hd - \int_0^T \rho_0(t)dt\right) = 0. \end{aligned} \tag{3.8}$$

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Comparing (3.6) with (3.8) we know that the equality holds in (3.8), and $y_0(t)$ is the first eigenfunction of (1.2)+(D) corresponding to $\lambda_1 = \lambda_1(\rho_\xi)$ and ρ_ξ. Since (3.8) is an equality, we know from the proof that $\rho_0(t) = \rho_\xi(t)$ for a.e. t. Now (3.5) becomes

$$\begin{cases} (\phi_p(y_0'(t)))' + \mu_1 H\phi_p(y_0(t)) = 0 & \text{for} \quad t \in (\xi, \xi+d), \\ (\phi_p(y_0'(t)))' = 0 & \text{for} \quad t \in (0,\xi) \cup (\xi+d, T), \\ y_0(0) = y_0(T) = 0, \\ y_0(\xi) = y_0(\xi+d) = h. \end{cases} \tag{3.9}$$

Since equation (3.9) is piecewise integrable, one can find that the unique solution of (3.9) is given by

$$y_0(t) = \begin{cases} h\left(1 + \alpha\frac{\phi_q(S_p(\alpha d/2))}{C_p(\alpha d/2)}(t-\xi)\right), & t \in [0,\xi], \\ h\frac{C_p(\alpha(t-\xi-d/2))}{C_p(\alpha d/2)}, & t \in [\xi, \xi+d], \\ h\left(1 - \alpha\frac{\phi_q(S_p(\alpha d/2))}{C_p(\alpha d/2)}(t-\xi-d)\right), & t \in [\xi+d, T], \end{cases} \tag{3.10}$$

where $\alpha = (\mu_1 H)^{1/p}$ and the functions S_p and C_p are as in Section 2. Set $\chi^{1/p} = \alpha d/2$. Then $\mu_1 = \frac{2^p}{Md^{p-1}}\chi$. Since $y_0(0) = y_0(T) = 0$, we get from (3.10) that $\xi = (T-d)/2$ and

$$\chi^{1/p}\frac{\phi_q(S_p(\chi^{1/p}))}{C_p(\chi^{1/p})} = \frac{d}{T-d} \ (>0).$$

In terms of the p-tangent function T_p in Section 2, χ satisfies

$$\chi^{1/p}\left(T_p(\chi^{1/p})\right)^{q-1} = \frac{d/T}{1-d/T}. \tag{3.11}$$

Suggested by (3.11), we introduce a function $\chi_p(t)$. For any $t \in (0,1)$, $\chi = \chi_p(t)$ is the least positive root of the following equation

$$\chi^{1/p}\left(T_p(\chi^{1/p})\right)^{q-1} = \frac{t}{1-t}. \tag{3.12}$$

In fact, $\chi_p(t)$ is the unique solution χ of (3.12) in the interval $(0, (\pi_p/2)^p)$. The function $\chi_p(t)$ is strictly increasing in $(0,1)$. Moreover, $\chi(0) = 0$ and $\lim_{t\to 1^-}\chi(t) = (\pi_p/2)^p$. So $\chi_p(t)$ can be extended to $t \in [0,1]$. Now we can summarize our conclusions into the following theorem.

Theorem 3.1 *Let* $\mu_1 = \inf_{\rho\in E}\lambda_1(\rho)$, *where* $E = E(M,H,T)$ *is as in (1.6). Then*

$$\mu_1 = \frac{2^p}{Hd^p}\chi_p(d/T) = \frac{2^pH^{p-1}}{M^p}\chi_p(M/HT),$$

where $d = M/H$. *Moreover,* μ_1 *is attained only when* $\rho(t) = \rho_\xi(t)$ *given by (3.7), where* $\xi = (T-d)/2$.

3.3 The value of μ_n

Denote by $c_j^{(n)}$ the midpoints of the intervals $\left(\frac{(j-1)T}{n}, \frac{jT}{n}\right)$, $j = 1, \cdots, n$. The function $\rho_n^\mu \in E(M,H,T)$ is defined by

$$\rho_n^\mu(t) = \begin{cases} H & t \in (c_j - \frac{d}{2n}, c_j + \frac{d}{2n}),\ j = 1,2,\cdots,n, \\ 0 & \text{otherwise}. \end{cases} \tag{3.13}$$

Theorem 3.2 *Let* $\mu_n = \inf_{\rho\in E}\lambda_n(\rho)$, *where* $E = E(M,H,T)$ *is as in (1.6). Then*

$$\mu_n = n^p\mu_1 = \frac{n^p2^p}{Hd^p}\chi_p(d/T) = \frac{n^p2^pH^{p-1}}{M^p}\chi_p(M/HT),$$

and μ_n *is attained uniquely for the function* $\rho = \rho_n^\mu$ *given by (3.13).*

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Proof. The case $n = 1$ is Theorem 3.1. Suppose that $n \geq 2$. Similar argument as in Section 3.1 shows that μ_n can be attained by some $\rho \in E$. Let y be the corresponding nth eigenfunction, i.e.,

$$(\phi_p(y'))' + \mu_n \rho(t)\phi_p(y) = 0, \tag{3.14}$$

and $y(0) = y(T) = 0$. Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be the zeros of $y(t)$. Set

$$T_i = t_i - t_{i-1}, \quad M_i = \int_{t_{i-1}}^{t_i} \rho(t)dt, \quad d_i = M_i/H, \quad i = 1, 2, \cdots, n.$$

So $\mu_n = \lambda_1(\rho|I_i)$, $I_i = (t_{i-1}, t_i)$, for any $1 \leq i \leq n$. It follows from Theorem 3.1 that

$$\mu_n \geq \frac{2^p}{Hd_i^p}\chi_p(d_i/T_i), \quad i = 1, 2, \cdots, n, \tag{3.15}$$

where for any given i the equality holds if and only if the function $\rho|I_i$ coincides with

$$\hat{\rho}_i(t) = \begin{cases} H & t \in (\gamma_i - d_i/2, \gamma_i + d_i/2), \\ 0 & \text{otherwise}, \end{cases}$$

where γ_i is the midpoint of the interval (t_{i-1}, t_i). We introduce the function $\hat{\rho}(t) \in E$, which coincides, in each of the intervals (t_{i-1}, t_i), with the function $\hat{\rho}_i(t)$, $i = 1, 2, \cdots, n$. Combining the definition of μ_n, Lemma 2.4, Theorem 3.1 and (3.15), we obtain

$$\mu_n \leq \lambda_n(\hat{\rho}) \leq \max_{1\leq i\leq n} \lambda_1(\hat{\rho}|(t_{i-1}, t_i)) = \max_{1\leq i\leq n} \frac{2^p}{Hd_i^p}\chi_p(d_i/T_i) \leq \mu_n.$$

Thus

$$\frac{2^p}{Hd_i^p}\chi_p(d_i/T_i) = \mu_n, \quad i = 1, 2, \cdots, n, \tag{3.16}$$

and $\rho(t) = \hat{\rho}(t)$ for a.e. $t \in [0, T]$. The proof of the theorem will be completed in the following Lemma 3.1 because (3.16) imply that

$$\frac{n^p 2^p H^{p-1}}{M^p}\chi_p(M/HT) = \frac{n^p 2^p}{H}F(d,\, T) = \frac{2^p}{H}F\left(\sum_{i=1}^n \frac{1}{n}d_i,\, \sum_{i=1}^n \frac{1}{n}T_i\right) \leq \mu_n,$$

and the equality holds if and only if

$$d_1 = d_2 = \cdots = d_n = d/n, \quad T_1 = T_2 = \cdots = T_n = T/n.$$

Here the function $F(u, v)$ is defined by

$$F(u, v) = \frac{1}{u^p}\chi_p(u/v), \qquad 0 < u < v.$$

Lemma 3.1 *Assume that (u_i, v_i), $i = 1, 2, \cdots, n$, are such that $F(u_i, v_i) = A^p$ for some $A > 0$ and for all $1 \leq i \leq n$. Then, for any $\tau_1,\ \tau_2,\ \cdots,\ \tau_n > 0$ with $\sum_{i=1}^n \tau_i = 1$,*

$$F\left(\sum_{i=1}^n \tau_i u_i, \sum_{i=1}^n \tau_i v_i\right) \leq A^p,$$

and the equality holds if and only if all (u_i, v_i) are equal.

Proof. The domain $\{(u, v);\, 0 < u \leq v\}$ of F is fully filled by the level curves L_α:

$$F(u, v) = \frac{1}{u^p}\chi_p(u/v) = \frac{1}{\alpha^p}, \qquad \alpha > 0.$$

If $(u, v) \in L_\alpha$, then

$$u/\alpha = (\chi_p(u/v))^{1/p} \quad (\in (0, \pi_p/2)).$$

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By the definition of the function $\chi_p(\cdot)$,

$$\frac{u}{\alpha}T_p^{q-1}(u/\alpha) = \frac{u/v}{1-u/v} = \frac{u}{v-u},$$

and consequently, the curves L_α can be rewritten as

$$v = u + \alpha\frac{C_p(u/\alpha)}{S_p^{q-1}(u/\alpha)}.$$

Applying Lemma 2.1, one has

$$\frac{dv}{du} = 1 - \frac{q-1}{S_p^q(u/\alpha)} < 0,$$

and

$$\frac{d^2v}{du^2} = \frac{q(q-1)}{\alpha}\frac{\phi_p(C_p(u/\alpha))}{S_p^{q+1}(u/\alpha)} > 0.$$

Thus the curves L_α are convex. It is obvious that L_β lies strictly above L_α when $\beta > \alpha$.

Let now (u_i, v_i) and τ_i be as in the lemma. Then (u_i, v_i) are all on $L_{1/A}$. As $L_{1/A}$ is convex, the point $P = (\sum_{i=1}^n \tau_i u_i, \sum_{i=1}^n \tau_i v_i)$ must lie strictly above $L_{1/A}$ until all (u_i, v_i) are equal. If not all (u_i, v_i) are the same, the point P lies on some curve L_α where $\alpha > 1/A$, i.e.,

$$F\left(\sum_{i=1}^n \tau_i u_i, \sum_{i=1}^n \tau_i v_i\right) = \frac{1}{\alpha^p} < A^p.$$

This proves the lemma.

4 The Supremum

For any $\rho \in E(M, H, T)$, denote by $\tilde{\rho}(t)$ the function symmetric to $\rho(t)$:

$$\tilde{\rho}(t) = \rho(T-t), \quad \forall\, t \in [0, T].$$

Then

$$\lambda_n(\tilde{\rho}) = \lambda_n(\rho), \quad n = 1, 2, \cdots$$

The following elementary result can be proved by using the characterization (3.1).

Lemma 4.1 *For any $\rho(t) \in E(M, H, T)$, it is necessary that*

$$\lambda_1((\rho+\tilde{\rho})/2) \geq \lambda_1(\rho),$$

where the equality holds if and only if $\rho(t) = \tilde{\rho}(t)$ is symmetric.

In view of Lemma 4.1, we pay attention only to the symmetric functions in $E(M, H, T)$ when we consider the maximum of $\lambda_1(\rho)$, $\rho \in E$.

Theorem 4.1 *Let $\Lambda_1 = \sup_{\rho\in E}\lambda_1(\rho)$, where $E = E(M, H, T)$ is defined in (1.6). Then*

$$\Lambda_1 = \frac{\pi_p^p}{Hd^p} = \frac{\pi_p^p H^{p-1}}{M^p},$$

and Λ_1 is attained uniquely for the function $\rho = \rho^\Lambda$:

$$\rho^\Lambda(t) = \begin{cases} H & t \in (0, d/2) \cup (T-d/2, T), \\ 0 & t \in (d/2, T-d/2). \end{cases} \tag{4.1}$$

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Proof. It is easy to check that $\Lambda_1 = \pi_p^p H^{p-1}/M^p$ is the first eigenvalue of

$$\begin{cases} (\phi_p(x'))' + \lambda\rho^{\Lambda}(t)\phi_p(x) = 0, \\ x(0) = x(T) = 0 \end{cases} \tag{4.2}$$

with the first eigenfunction

$$x(t) = x(T-t) = \begin{cases} S_p(\alpha t) & t \in [0, d/2], \\ S_p(\alpha d/2) & t \in [d/2, T/2], \end{cases}$$

where $\alpha = (\Lambda_1 H)^{1/p}$ and $d = M/H$ (Note $\alpha d/2 = \pi_p/2$ and thus $S_p(\alpha d/2) = \sqrt[q]{q/p}$).

For any symmetric function $\rho = \tilde{\rho} \in E(M, H, T)$, since

$$\begin{aligned} \int_0^T \rho(t)|x(t)|^p dt - \int_0^T \rho^{\Lambda}(t)|x(t)|^p dt \;&=\; 2\int_0^{T/2} (\rho(t) - \rho^{\Lambda}(t))|x(t)|^p dt \\ =\; 2\int_0^{d/2} (\rho(t) - H)|x(t)|^p dt + 2\int_{d/2}^{T/2} \rho(t)|x(t)|^p dt & \\ \geq\; 2S_p(\alpha d/2)\int_0^{d/2} (\rho(t) - H) dt + 2S_p(\alpha d/2)\int_{d/2}^{T/2} \rho(t) dt \;&=\; 0, \end{aligned}$$

we get

$$\lambda_1(\rho) \leq \frac{\int_0^T |x'(t)|^p dt}{\int_0^T \rho(t)|x(t)|^p dt} \leq \frac{\int_0^T |x'(t)|^p dt}{\int_0^T \rho^{\Lambda}(t)|x(t)|^p dt} = \lambda_1(\rho^{\Lambda}) = \Lambda_1,$$

and $\lambda_1(\rho) = \Lambda_1$ if and only if $x(t)$ is the first eigenfunction of (1.2)+(D) corresponding to Λ_1 and ρ, and consequently, for almost all $t \in [0, T]$, $\rho(t) = \rho^{\Lambda}(t)$.

Theorem 4.2 *Let $\Lambda_n = \sup_{\rho \in E} \lambda_n(\rho)$, where $E = E(M, H, T)$ is as in (1.6). Then*

$$\Lambda_n = n^p \Lambda_1 = \frac{n^p \pi_p^p H^{p-1}}{M^p}.$$

When $n > 1$, Λ_n is attained for an infinite set of the functions ρ.

Proof. The case $n = 1$ is Theorem 4.1. Suppose $n \geq 2$. For any $\rho \in E(M, H, T)$, let $x(t)$ be the corresponding nth eigenfunction of (1.2)+(D) with zeros $0 = t_0 < t_1 < \cdots < t_n = T$. There must be a j_0, $1 \leq j_0 \leq n$, such that

$$M_{j_0} = \int_{t_{j_0-1}}^{t_{j_0}} \rho(t) dt \geq M/n.$$

Combining Lemma 2.3 and Theorem 4.1, we get

$$\lambda_n(\rho) = \lambda_1(\rho|(t_{j_0-1}, t_{j_0})) \leq \frac{\pi_p^p H^{p-1}}{M_{j_0}^p} \leq \frac{n^p \pi_p^p H^{p-1}}{M^p} = \Lambda_n,$$

and $\lambda_n(\rho) = \Lambda_n$ if and only if, for any $1 \leq j \leq n$, $M_j = M/n$ and the function $\rho(t)$ in the interval (t_{j-1}, t_j) is given by

$$\rho(t) = \begin{cases} H & t \in (t_{j-1}, t_{j-1} + \frac{d}{2n}) \cup (t_j - \frac{d}{2n}, t_j), \\ 0 & t \in (t_{j-1} + \frac{d}{2n}, t_j - \frac{d}{2n}). \end{cases}$$

Let $0 = \beta_0 < \beta_1 \cdots < \beta_{n-1} < \beta_n = T$ be any sequence satisfying $\beta_j - \beta_{j-1} \geq d/n$, $j = 1, 2, \cdots n$. Define the function $\rho_\beta(t)$ in each of the intervals (β_{j-1}, β_j), $j = 1, 2, \cdots, n$, by

$$\rho_\beta(t) = \begin{cases} H & t \in (\beta_{j-1}, \beta_{j-1} + \frac{d}{2n}) \cup (\beta_j - \frac{d}{2n}, \beta_j), \\ 0 & t \in (\beta_{j-1} + \frac{d}{2n}, \beta_j - \frac{d}{2n}). \end{cases}$$

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Then $\lambda_n(\rho) = \Lambda_n$ if and only if $\rho(t)$ coincides with some $\rho_\beta(t)$.

Remark 4.1 A very recent work of the second author[16] shows that one can use the Dirichlet eigenvalues to construct *certain* periodic eigenvalues and anti-periodic eigenvalues of (1.2) in the following way. For any $n \in \mathbb{N}$, let

$$\underline{\lambda}_n(\rho) = \min\{\lambda_n(\rho_s);\ s \in \mathbb{R}\} \quad \text{and} \quad \overline{\lambda}_n(\rho) = \max\{\lambda_n(\rho_s);\ s \in \mathbb{R}\}, \tag{4.3}$$

where $\rho_s(t) = \rho(t+s)$. (We always understand a function from $L^1(0,T)$ as a T-periodic one by its T-periodic extension.) Then $\underline{\lambda}_n(\rho)$ and $\overline{\lambda}_n(\rho)$ are eigenvalues of (1.2) with the periodic boundary condition (P): $x(0) - x(T) = x'(0) - x'(T) = 0$ when n is even, and $\underline{\lambda}_n(\rho)$ and $\overline{\lambda}_n(\rho)$ are eigenvalues of (1.2) with the anti-periodic boundary condition (A): $x(0) + x(T) = x'(0)+x'(T) = 0$ when n is odd. Note that the complete structure of eigenvalues of (1.2)+(P) and (1.2)+(A) is not known even for such a $1\frac{1}{2}$-dimensional eigenvalue problem (see [16]). As a corollary of Theorems 3.2 and 4.2, we can obtain the following best estimates for the periodic eigenvalues: For any $n \in \mathbb{N}$,

$$\min_{\rho \in E(M,H,T)} \underline{\lambda}_n(\rho) = \mu_n, \qquad \max_{\rho \in E(M,H,T)} \overline{\lambda}_n(\rho) = \Lambda_n. \tag{4.4}$$

For some possible applications of those estimates (4.4), we refer to [15].

5 Generalizations of the Preceding Results

Let $M,\ H,\ L,\ T$ be positive numbers satisfying $LT \le M \le HT$. Define

$$E^*(M,H,L,T) =: \left\{\rho \in L^1(0,T);\ \int_0^T \rho(t)dt = M,\ L \le \rho(t) \le H,\ \text{a.e. } t \in [0,T]\right\}. \tag{5.1}$$

Denote by $w_p(s,t)$ $(0 < s,\ t < \infty)$ the least positive root w of the equation

$$T_p(w^{1/p}) = s^{1/p} G_p(t(sw)^{1/p}), \tag{5.2}$$

where the functions T_p and G_p are as in Section 2. Then $0 < w_p(s,t) < (\pi_p/2)^p$ and $0 < t^p s w_p(s,t) < (\pi_p/2)^p$.

Let d and D be such that $d + D = T$ and $Hd + LD = M$. Then

$$d = \frac{M - LT}{H - L}, \quad D = \frac{HT - M}{H - L}.$$

Define the function $\rho_1^L \in E^*(M,H,L,T)$ by

$$\rho_1^L(t) = \begin{cases} L & t \in (0, D/2) \cup (T - D/2, T), \\ H & t \in (D/2, d + D/2). \end{cases} \tag{5.3}$$

For any $n \in \mathbb{N}$, the function ρ_n^L is defined as follows: Divide the interval $(0,T)$ into n equal parts and in each of them specify ρ_n^L analogously to (5.3), using d/n, D/n instead of d, D. We define the function ρ_n^U by switching d with D and L with H in the definition of ρ_n^L.

Similar arguments as in the previous sections bring out the following results.

Theorem 5.1 *For any $\rho \in E^*(M,H,L,T)$ and any $n \in \mathbb{N}$, there hold the following inequalities:*

$$\frac{n^p 2^p}{H d^p} w_p\left(\frac{L}{H}, \frac{D}{d}\right) \le \lambda_n(\rho) \le \frac{n^p 2^p}{L D^p} w_p\left(\frac{H}{L}, \frac{d}{D}\right)$$

where the lower bound is attained uniquely by the function ρ_n^L, while the upper bound is attained uniquely by the function ρ_n^U.

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