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A p-Laplacian problem with a multi-point boundary condition

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Abstract

In this paper we will use the degree theory to give an existence result to the boundary value problem of nonlinear perturbations of the scalar p-Laplacian with the multi-point boundary condition involving the Lebesgue-Stieltjes integral. Among the boundary conditions considered in this paper, we will find also the optimal bounds on the growth of the nonlinear perturbations in our existence result.

Keywords: p-Laplacian; Multi-point boundary condition; Lebesgue-Stieltjes integral; Degree theory; Eigenvalue; L^p norms

1. Introduction

In recent years, the boundary value problems (BVPs) of nonlinear differential equations with the following multi-point boundary condition (over the interval

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$J=[a,b]$, $-\infty<a<b<\infty$) have received many studies in literature:

$$u(a)=0, \quad u(b)=\sum_{i=1}^{m}\eta_i u(\tau_i), \tag{1.1}$$

where $m\in\mathbb{N}$, $a<\tau_1<\cdots<\tau_m<b$, and all η_i are positive constants. See, for example, [2,6,8,9,11,12]. However, as the differential operator $u\mapsto -u''$ is not symmetric with respect to (1.1), many problems related with (1.1) such as the structure of eigenvalues, remain open. As for the solvability of BVPs, we say that (1.1) is *nonresonant* if

$$\sum_{i=1}^{m}\eta_i(\tau_i-a)\neq b-a. \tag{1.2}$$

(This will be explained later.) A particular case of interest is the case

$$\sum_{i=1}^{m}\eta_i\leqslant 1. \tag{1.3}$$

This condition has connection with the maximum principle of $-u''$ with respect to (1.1).

In a recent work [7], García-Huidobro, Manásevich and Zhang used the observation that if $u(t)$ satisfies (1.1) then

$$u(\tau_i)=\int_a^{\tau_i}u'(s)\,\mathrm{d}s, \quad i=1,2,\ldots,m,m+1,$$

where $\tau_{m+1}=b$. Thus, (1.1) can be rewritten as

$$u(a)=0, \quad \int_J\chi(s)u'(s)\,\mathrm{d}s=0, \tag{1.4}$$

where $\chi(s)$ is determined from the boundary data η_i and τ_i. It is easy to see that when (and only when) condition (1.3) is satisfied, the kernel function $\chi(s)$ is nonnegative (and is a step function in this case). Motivated by (1.4), they then studied more general multi-point boundary condition like

$$u(a)=0, \quad \int_J u'(s)\,\mathrm{d}\xi(s)=0, \tag{1.5}$$

where $\xi=\xi(s):J\to\mathbb{R}$ is a nondecreasing and not identically a constant function, which may be normalized as $\xi(a)=0$ and $\xi(b)=1$. The integral in (1.5) is in the Lebesgue–Stieltjes sense [1]. In order to avoid the meaningless case, it is also assumed that $\xi\neq\xi_a$, where

$$\xi_a(s)=\begin{cases}0, & s=a,\\ 1, & s\in]a,b].\end{cases}$$

Such a boundary condition (BC) (1.5) may involve not only finitely many, but infinitely many points in J. The boundary condition (1.5) may be referred as *Stieltjes BC*. With respect to Stieltjes BC, they studied BVP of the vector p-Laplacian-type nonlinear differential equation

$$(\phi(u'))' + f(t, u, u') = 0. \tag{1.6}$$

Here the continuous mapping $\phi : \mathbb{R}^N \mapsto \mathbb{R}^N$, which generates the differential operator, satisfies $\phi(0) = 0$ and

(H_1) For any $x, y \in \mathbb{R}^N$, $x \neq y$, $\langle \phi(x) - \phi(y), x - y\rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, and

(H_2) there exists a function $\alpha : [0, \infty[\to [0, \infty[$, $\alpha(s) \to \infty$ as $s \to \infty$, such that

$$\langle \phi(x), x\rangle \geqslant \alpha(|x|)|x| \quad \text{for all } x \in \mathbb{R}^N.$$

It is known that ϕ is an homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ and $|\phi^{-1}(y)| \to \infty$ as $|y| \to \infty$. The function $f = f(t, x, y) : J \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ in (1.6) is assumed to satisfy the Carathéodory conditions.

By a solution to problem (1.6), (1.5) we understand a continuously differentiable function $u(t)$ defined in J, with $\phi(u')$ absolutely continuous which satisfies Eq. (1.6) for a.e. $t \in J$ and the BC (1.5). Note that the nonresonant condition (1.2) means that the equation

$$(\phi(u'))' = 0$$

has only the trivial solution verifying (1.1).

The main result in [7] is as follows. Suppose that $f(t, x, y)$ satisfies, for some $q(t)$, $\tilde{q}(t)$, $\rho(t) \in L^1(J, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty[$,

$$|f(t, x, y)| \leqslant q(t)|\phi(x)| + \tilde{q}(t)|\phi(y)| + \rho(t)$$

for a.e. t and all x, y. Then there exist some constants C_1, C_2 (depending upon ϕ, N, and the boundary data only) such that if the L^1 norms satisfy

$$\|q\|_{L^1(J)} < C_1, \quad \|\tilde{q}\|_{L^1(J)} < C_2,$$

problem (1.6), (1.5) has at least one solution.

In this paper, we will give some sharp results to the *scalar*, *p-Laplacian*, *conservative* case of (1.6), (1.5), i.e., $N = 1$, $\phi(x) = \phi_p(x) := |x|^{p-2}x$, $f = f(t, x)$, where $1 < p < \infty$ is a fixed exponent. That is to say, we are considering the following nonlinear scalar equation

$$(\phi_p(u'))' + f(t, u) = 0, \quad t \in J, \quad u \in \mathbb{R}, \tag{1.7}$$

with the BC (1.5).

Two main results for (1.7), (1.5) are proved in this paper. The first one is an existence result, see Theorem 3.1, in which $f(t, x)$ satisfies, for some $q_i \in L^1(J)$,

$$q_1(t) \leqslant \liminf_{|x|\to\infty} \frac{f(t, x)}{\phi_p(x)} \leqslant \limsup_{|x|\to\infty} \frac{f(t, x)}{\phi_p(x)} \leqslant q_2(t) \tag{1.8}$$

uniformly in a.e. $t \in J$. The Theorem asserts the existence of solutions when the pair (q_1, q_2) has the so-called property P [3,10,17], which means that for any $q \in L^1(J)$ with

$$q_1(t) \leqslant q(t) \leqslant q_2(t) \quad \text{a.e. } t \in J, \tag{1.9}$$

the equation

$$(\phi_p(u'))' + q(t)\phi_p(u) = 0 \tag{1.10}$$

has only the trivial solution satisfying (1.5). The result will be proved using the degree method. However, in finding the a priori estimates, our approach is quite different from the usual ones in the literature. It is based on a principle of Zhang [17] on perturbations of families of positively homogeneous operators and all estimates will be reduced to elementary inequalities. This principle was used successfully by Yan [16] for the periodic problem. Note also that such a nonresonant condition expressed using the property P is optimal in some sense.

The second result of this paper is some quantitative results, see Theorems 4.1 and 4.2, in which $f(t, x)$ satisfies, for some nonnegative functions $q(t)$, $\rho(t)$,

$$|f(t, x)| \leqslant q(t)|x|^{p-1} + \rho(t) \tag{1.11}$$

for a.e. $t \in J$ and all x. Then, for any exponent $1 \leqslant \alpha \leqslant \infty$, we will find the *optimal* bound $R(\alpha, p, J)$, expressed explicitly, such that if

$$\|q\|_{\alpha,J} < R(\alpha, p, J),$$

then problem (1.7), (1.5) has at least one solution. The idea is from [20] for the linear case and [19] for general p. Note that such a condition on $q(t)$ guarantees that (1.7) has at least one solution with each of the Stieltjes BCs (1.5) (depending on ξ). In this sense, we are considering in this paper a family of boundary value problems as a whole.

We end the introduction with some notation. Let $J = [a, b]$ be a closed interval. Then $C(J)$ denotes the space of all continuous functions from J to $\mathbb{R}$, endowed with the sup-norm denoted by $\|\cdot\|_\infty$. For $1 \leqslant \alpha \leqslant \infty$, $L^\alpha(J)$ denotes the Lebesgue space with the corresponding norm denoted by $\|\cdot\|_{\alpha,J}$. In some cases, we need also the Sobolev spaces such as $W_0^{1,p}(J)$.

2. Reduction to fixed point problem and homogeneity

We begin this section with the auxiliary problem

$$(\phi_p(u'))' + h(t) = 0, \tag{2.1}$$

with the BC (1.5), where $h \in L^1(J)$. It is well known that there are several methods in reducing problem (2.1), (1.5) to a fixed point problem in some Banach space. However, we need a reduction which will fit with the perturbation principle in [17]. To this end, let us write down the details so that the homogeneity can be analyzed explicitly.

Given $h \in L^1(J)$. Define

$$\mathscr{H}(h)(t) = \int_a^t h(s)\,\mathrm{d}s, \qquad t \in J.$$

Suppose that $u(t)$ is a solution of (2.1), (1.5). Then we get from (2.1),

$$\phi_p(u'(t)) = c - \mathscr{H}(h)(t),$$

where $c=\phi_p(u'(a))$. Write $p^*=p/(p-1)$ the conjugate exponent of p. Note that $\phi_p^{-1}=\phi_{p^*}$. So we have

$$u'(t) = \phi_{p^*}(c - \mathscr{H}(h)(t)).$$

Then the boundary condition (1.5) means that

$$\int_J \phi_{p^*}(c - \mathscr{H}(h)(s))\,\mathrm{d}\xi(s) = 0. \tag{2.2}$$

It is easy to see that the left-hand side of (2.2), viewed as a function of $c \in \mathbb{R}$, is a self-homeomorphism of $\mathbb{R}$. Thus there exists a unique $c = c(\mathscr{H}(h)) =: \mathscr{L}(h) \in \mathbb{R}$ satisfying (2.2). Using the condition $u(a) = 0$ in (1.5), we obtain

$$u(t) = \int_a^t \phi_{p^*}(\mathscr{L}(h) - \mathscr{H}(h)(s))\,\mathrm{d}s =: (\mathscr{K}u)(t).$$

The operator $\mathscr{K}$ maps $L^1(J)$ into $C(J)$. We summarize these in the following:

Proposition 2.1. *For any* $h \in L^1(J)$, (2.1), (1.5) *has a unique solution which is given by* $u = \mathscr{K}(h)$.

Following the arguments of [13] or [14], one can prove that the operator $\mathscr{K}$ is continuous and sends equi-integrable sets in $L^1(J)$ into relatively compact sets in $C(J)$. See also [7].

We have introduced a linear operator $\mathscr{H} : (L^1(J), \|\cdot\|_1) \mapsto (C(J), \|\cdot\|_\infty)$, a nonlinear functional $\mathscr{L} : (L^1(J), \|\cdot\|_1) \mapsto \mathbb{R}$, and a nonlinear operator $\mathscr{K} : (L^1(J), \|\cdot\|_1) \mapsto (C(J), \|\cdot\|_\infty)$. Some properties of these are collected in the following

Proposition 2.2. (i) *The function* $\phi_{p^*} : \mathbb{R} \to \mathbb{R}$ *is* (p^*-1)-*homogenous*:

$$\phi_{p^*}(kx) = |k|^{p^*-2}k\,\phi_{p^*}(x), \quad k, x \in \mathbb{R}.$$

(ii) *The linear operator* $\mathscr{H} : (L^1(J), \|\cdot\|_1) \mapsto (C(J), \|\cdot\|_\infty)$ *has norm* $\|\mathscr{H}\| = 1$.

(iii) *The functional* $\mathscr{L} : L^1(J) \mapsto \mathbb{R}$ *is homogenous*:

$$\mathscr{L}(kh) = k\mathscr{L}(h), \quad k \in \mathbb{R},\ h \in L^1(J).$$

(iv) *The operator* $\mathscr{K} : (L^1(J), \|\cdot\|_1) \mapsto (C(J), \|\cdot\|_\infty)$ *is* (p^*-1)-*homogenous*:

$$\mathscr{K}(kh) = |k|^{p^*-2}k\,\mathscr{K}(h), \quad k \in \mathbb{R},\ h \in L^1(J).$$

(v) $\mathscr{L}$ *satisfies*

$$\min_{s\in J}(\mathscr{H}(h_1) - \mathscr{H}(h_2))(s) \leqslant \mathscr{L}(h_1) - \mathscr{L}(h_2) \leqslant \max_{s\in J}(\mathscr{H}(h_1) - \mathscr{H}(h_2))(s). \tag{2.3}$$

In particular,

$$|\mathscr{L}(h_1)-\mathscr{L}(h_2)|\leqslant \|\mathscr{H}(h_1)-\mathscr{H}(h_2)\|_\infty.$$

Proof. We need only to prove (2.3). Let $H_i(s)=\mathscr{H}(h_i)(s)\in C(J)$, $i=1,2$. Then the constants $c_i=\mathscr{L}(h_i)$ are determined by

$$\int_J \phi_{p^*}(c_i-H_i(s))\,\mathrm{d}\xi(s)=0,\quad i=1,2.$$

See (2.2). Thus

$$\int_J (\phi_{p^*}(c_1-H_1(s))-\phi_{p^*}(c_2-H_2(s)))\,\mathrm{d}\xi(s)=0.$$

Since ξ is nondecreasing and $\xi\neq\xi_a$, we know that there exists $s_0\in]a,b]$ such that

$$\phi_{p^*}(c_1-H_1(s_0))-\phi_{p^*}(c_2-H_2(s_0))=0,$$

i.e.,

$$c_1-H_1(s_0)=c_2-H_2(s_0),$$

or

$$c_1-c_2=H_1(s_0)-H_2(s_0),$$

which proves (2.3). □

Next, let us consider the pair (q_1,q_2) in $L^1(J)$ such that $q_1\leqslant q_2$. Denote

$$\mathscr{I}(q_1,q_2):=\{q\in L^1(J): q \text{ satisfies (1.9)}\}\subset L^1(J).$$

We endow $\mathscr{I}(q_1,q_2)$ with the topology of weak convergence. Then $\mathscr{I}(q_1,q_2)$ is sequentially compact. Consider now the family of equations (1.10) with a ‘parameter’ $q\in\mathscr{I}(q_1,q_2)$. Using the reduction above, $u(t)$ is a solution of (1.10), (1.5) if and only if $u\in C(J)$ satisfies

$$u=\mathscr{T}(q,u), \tag{2.4}$$

where the operator $\mathscr{T}:\mathscr{I}(q_1,q_2)\times C(J)\to C(J)$ is defined by

$$\mathscr{T}(q,u):=\mathscr{K}(q(\cdot)\phi_p(u(\cdot))).$$

Using the reduction above, it can be proved that $\mathscr{T}:\mathscr{I}(q_1,q_2)\times C(J)\to C(J)$ is uniformly completely continuous in the sense defined in [17, p. 342]. This can done as in [17, Proposition 5.1] and [16], where the periodic problem is studied.

By Proposition 2.2, we know that for any given $q\in\mathscr{I}(q_1,q_2)$, the operator $u\mapsto\mathscr{T}(q,u)$ is homogenous, i.e.,

$$\mathscr{T}(q,ku)=k\mathscr{T}(q,u)$$

for all $k\in\mathbb{R}$ and $u\in C(J)$.

When the pair (q_1, q_2) has property P (see Section 1), each Eq. (2.4) has a unique fixed point $u = 0$ in $C(J)$. Thus, by [17, Theorem 3.1], we have

Proposition 2.3. *Suppose the pair* (q_1, q_2) *has property* P. *Then there exists a positive constant* $c_0 > 0$ *such that*

$$\|u - \mathscr{T}(q, u)\|_\infty \geqslant c_0 \|u\|_\infty \tag{2.5}$$

for all $(q, u) \in \mathscr{I}(q_1, q_2) \times C(J)$. *Moreover, for any* $q_0 \in \mathscr{I}(q_1, q_2)$, *the Leray-Schauder degree*

$$\deg_{LS}(I - \mathscr{T}(q_0, \cdot), B_r, 0) \neq 0, \tag{2.6}$$

where $r > 0$ *and* B_r *is the ball of* $C(J)$ *centered at* 0 *with radius* r.

Proof. Estimate (2.5) follows directly from [17, Theorem 3.1]. Since the operator $\mathscr{T}(q, \cdot)$ is odd, (2.6) follows from the Borsuk theorem. Actually, $\deg_{LS}(I - \mathscr{T}(q_0, \cdot), B_r, 0)$ is an odd number. □

3. An existence result

In this section, we consider the BVP (1.7), (1.5). Let $\mathscr{F} : C(J) \mapsto L^1(J)$ be the operator defined by

$$\mathscr{F}(u)(t) := f(t, u(t)), \quad t \in J.$$

Following the construction in Section 2, we define an operator $\mathscr{T}_0 : C(J) \mapsto C(J)$ by

$$\mathscr{T}_0(u)(t) := \mathscr{K}(\mathscr{F}(u))(t).$$

Note that $\mathscr{T}_0$ is completely continuous. It is clear from Proposition 2.1 that $u \in C(J)$ will be a solution of problem (1.7), (1.5) if and only if $u \in C(J)$ is a fixed point of $\mathscr{T}_0$:

$$u = \mathscr{T}_0(u). \tag{3.1}$$

The main result in this section is

Theorem 3.1. *Assume that* $f(t, x)$ *satisfies* (1.8) *for some* $q_1, q_2 \in L^1(J)$. *If* (q_1, q_2) *has property* P, *then* (1.7), (1.5) *has at least one solution*.

To prove the theorem, we need some elementary inequalities in [16] concerning with the functions ϕ_p.

Lemma 3.1. (i) *If* $1 < \beta \leqslant 2$, *then*

$$|\phi_\beta(u + v) - \phi_\beta(u)| \leqslant 2^{2-\beta} |v|^{\beta - 1} \tag{3.2}$$

for all $u, v \in \mathbb{R}$.

(ii) *If* $\beta > 2$, *then*

$$|\phi_\beta(u+v) - \phi_\beta(u)| \leqslant (\beta-1)(|u|+|v|)^{\beta-2}|v| \tag{3.3}$$

for all $u, v \in \mathbb{R}$.

Proof of Theorem 3.1. Fix a $q_0 \in \mathscr{I}(q_1, q_2)$. So (1.10) with $q = q_0$ has only the trivial solution satisfying (1.5). Consider then the following homotopy equation:

$$(\phi_p(u'))' + f_\lambda(t,u) = 0 \quad (\lambda \in [0,1]),$$
$$f_\lambda(t,x) = \lambda q_0(t)\phi_p(x) + (1-\lambda) f(t,x). \tag{3.4}$$

By the construction in Section 2, $u \in C(J)$ is a solution of (3.4), (1.5) if and only if u is a fixed point of the following problem in the space $C(J)$:

$$u = \mathscr{T}_\lambda(u), \tag{3.5}$$

where $\mathscr{T}_\lambda : C(J) \to C(J)$ is defined by

$$\mathscr{T}_\lambda(u) := \mathscr{K}(f_\lambda(\cdot, u(\cdot))).$$

Step 1. *Decomposition of the operator* $\mathscr{T}_\lambda$. From the assumption (1.8), we know that for any given $0 < \varepsilon < 1$, it is always possible to decompose the nonlinear function $f(t,x)$ as

$$f(t,x) = m_\varepsilon(t,x)\phi_p(x) + \rho_\varepsilon(t,x),$$

where both $m_\varepsilon(t,x)$ and $\rho_\varepsilon(t,x)$ are Carathéodory functions such that

$$q_1(t) \leqslant m_\varepsilon(t,x) \leqslant q_2(t), \quad \forall t \in J, \ \forall x \in \mathbb{R}$$

and there exists $l_\varepsilon(t) \in L^1(0,T;\mathbb{R}_+)$ with the property

$$|\rho_\varepsilon(t,x)| \leqslant \varepsilon|\phi_p(x)| + l_\varepsilon(t), \quad \forall t \in J, \ \forall x \in \mathbb{R}. \tag{3.6}$$

Given $\lambda \in [0,1]$ and $u(\cdot) \in C(J)$. Then

$$\begin{aligned} f_\lambda(t,u(t)) &= \lambda q_0(t)\phi_p(u(t)) + (1-\lambda) f(t,u(t)) \\ &= m(t)\phi_p(u(t)) + (1-\lambda)\rho_\varepsilon(t,u(t)) \\ &=: h_1(t) + h_2(t), \end{aligned}$$

where

$$m(t) = \lambda q_0(t) + (1-\lambda) m_\varepsilon(t,u(t)). \tag{3.7}$$

It is obvious that $m \in \mathscr{I}(q_1,q_2)$. Thus

$$|h_1(t)| = |m(t)\phi_p(u(t))| \leqslant \max\{|q_1(t)|, |q_2(t)|\}|u(t)|^{p-1}.$$

By (3.6),

$$|h_2(t)| \leqslant \varepsilon|u(t)|^{p-1} + l_\varepsilon(t).$$

From these, we have

$$\|\mathscr{H}(h_1)\|_\infty \leqslant C_0\|u\|_\infty^{p-1}, \tag{3.8}$$

where $C_0 = \int_J \max\{|q_1(t)|, |q_2(t)|\}\,\mathrm{d}t$, and

$$\|\mathscr{H}(h_2)\|_\infty \leqslant |J|\,\varepsilon\|u\|_\infty^{p-1} + C_\varepsilon \quad (C_\varepsilon = \|l_\varepsilon\|_1). \tag{3.9}$$

We can rewrite the operator $\mathscr{T}_\lambda(u)$ as

$$\begin{aligned}(\mathscr{T}_\lambda u)(t) &= \int_a^t \phi_{p^*}(\mathscr{L}(h_1+h_2) - \mathscr{H}(h_1)(s) - \mathscr{H}(h_2)(s))\,\mathrm{d}s\\ &= \mathscr{T}(m,u)(t) + G(t),\end{aligned} \tag{3.10}$$

where $m(\cdot)$ is given by (3.7) and $G(t)$ is

$$\begin{aligned}G(t) = \int_a^t &(\phi_{p^*}(\mathscr{L}(h_1+h_2) - \mathscr{H}(h_1)(s) - \mathscr{H}(h_2)(s))\\ &- \phi_{p^*}(\mathscr{L}(h_1) - \mathscr{H}(h_1)(s)))\,\mathrm{d}s.\end{aligned} \tag{3.11}$$

Step 2: *Estimates of the term* G. For any $s \in J$, we obtain from Proposition 2.2(v) that

$$\begin{aligned}&|(\mathscr{L}(h_1+h_2) - \mathscr{H}(h_1)(s) - \mathscr{H}(h_2)(s)) - (\mathscr{L}(h_1) - \mathscr{H}(h_1)(s))|\\ &\quad = |(\mathscr{L}(h_1+h_2) - \mathscr{L}(h_1)) - \mathscr{H}(h_2)(s)|\\ &\quad \leqslant |\mathscr{L}(h_1+h_2) - \mathscr{L}(h_1)| + |\mathscr{H}(h_2)(s)|\\ &\quad \leqslant \|\mathscr{H}(h_2)\|_\infty + |\mathscr{H}(h_2)(s)| \leqslant 2\|\mathscr{H}(h_2)\|_\infty\end{aligned} \tag{3.12}$$

and

$$|\mathscr{L}(h_1) - \mathscr{H}(h_1)(s)| \leqslant 2\|\mathscr{H}(h_1)\|_\infty. \tag{3.13}$$

In case $p \geqslant 2$, i.e., $1 < p^* \leqslant 2$, applying (3.2) to $\beta = p^*$, $u = \mathscr{L}(h_1) - \mathscr{H}(h_1)(s)$, and $v = (\mathscr{L}(h_1+h_2) - \mathscr{L}(h_1)) - \mathscr{H}(h_2)(s)$, we obtain from (3.9) and (3.12),

$$\begin{aligned}&|\phi_{p^*}(\mathscr{L}(h_1+h_2) - \mathscr{H}(h_1)(s) - \mathscr{H}(h_2)(s)) - \phi_{p^*}(\mathscr{L}(h_1) - \mathscr{H}(h_1)(s))|\\ &\quad \leqslant 2^{2-p^*}(2\|\mathscr{H}(h_2)\|_\infty)^{p^*-1} \leqslant 2(|J|\varepsilon\|u\|_\infty^{p-1} + C_\varepsilon)^{p^*-1}.\end{aligned}$$

Thus, by (3.11),

$$\|G\|_\infty \leqslant 2|J|(|J|\varepsilon\|u\|_\infty^{p-1} + C_\varepsilon)^{p^*-1}. \tag{3.14}$$

In case $p < 2$, i.e., $p^* > 2$, we will apply (3.3) to $\beta = p^*$, $u = \mathscr{L}(h_1) - \mathscr{H}(h_1)(s)$, and $v = (\mathscr{L}(h_1+h_2) - \mathscr{L}(h_1)) - \mathscr{H}(h_2)(s)$. Note that, by (3.13) and (3.8),

$$|u| \leqslant 2\|\mathscr{H}(h_1)\|_\infty \leqslant 2C_0\|u\|_\infty^{p-1} \tag{3.15}$$

and, by (3.12) and (3.9),

$$|v| \leqslant 2\|\mathscr{H}(h_2)\|_\infty \leqslant 2(|J|\,\varepsilon\|u\|_\infty^{p-1} + C_\varepsilon). \tag{3.16}$$

Thus, it follows from (3.3), (3.15), and (3.16) that

$$\begin{aligned}|\phi_{p^*}(\mathscr{L}(h_1+h_2)-\mathscr{H}(h_1)(s)-\mathscr{H}(h_2)(s))-\phi_{p^*}(\mathscr{L}(h_1)-\mathscr{H}(h_1)(s))|\\ \leqslant (p^*-1)[2((C_0+|J|\,\varepsilon)\|u\|_\infty^{p-1}+C_\varepsilon)]^{p^*-2}[2(|J|\,\varepsilon\|u\|_\infty^{p-1}+C_\varepsilon)]\\ =(p^*-1)2^{p^*-1}[(C_0+|J|\,\varepsilon)\|u\|_\infty^{p-1}+C_\varepsilon]^{p^*-2}[|J|\,\varepsilon\|u\|_\infty^{p-1}+C_\varepsilon]\\ \leqslant (p^*-1)2^{p^*-1}[(C_0+|J|)\|u\|_\infty^{p-1}+C_\varepsilon]^{p^*-2}[|J|\,\varepsilon\|u\|_\infty^{p-1}+C_\varepsilon],\end{aligned}$$

because $0<\varepsilon<1$. Thus, by (3.11),

$$\begin{aligned}\|G\|_\infty \leqslant |J|(p^*-1)2^{p^*-1}[(C_0+|J|)\|u\|_\infty^{p-1}\\ +C_\varepsilon]^{p^*-2}[|J|\,\varepsilon\|u\|_\infty^{p-1}+C_\varepsilon].\end{aligned} \tag{3.17}$$

Note that $(p-1)(p^*-1)=1$. Considering the right-hand sides of (3.14) and (3.17) as functions of $\|u\|_\infty$, we know that the leading terms are, respectively,

$$2|J|^{p^*}\varepsilon^{p^*-1}\|u\|_\infty$$

and

$$|J|^2\varepsilon(p^*-1)2^{p^*-1}(C_0+|J|)^{p^*-2}\|u\|_\infty.$$

Thus we conclude that there are some constants $A>0$ (independent of $\varepsilon\in(0,1)$, $\lambda\in[0,1]$, and $u\in C(J)$) and $B_\varepsilon>0$ (independent of λ and u) such that

$$\|G\|_\infty \leqslant A\max\{\varepsilon^{p^*-1},\varepsilon\}\|u\|_\infty+B_\varepsilon. \tag{3.18}$$

In fact, we can take

$$A=2\max\{2|J|^{p^*},|J|^2(p^*-1)2^{p^*-1}(C_0+|J|)^{p^*-2}\}$$

by (3.14) and (3.17).

Step 3: *Boundedness of the fixed points of* $\mathscr{T}_\lambda$. Suppose that $u\in C(J)$ is a solution of (3.4), (1.5) for some $\lambda\in[0,1]$. That is, u satisfies (3.5). Using the decomposition (3.10) for $\mathscr{T}_\lambda$, we can find some $m\in\mathscr{I}(q_1,q_2)$, which depends upon u and λ, such that

$$u-\mathscr{T}(m,u)=G \tag{3.19}$$

for some $G\in C(J)$ which satisfies the estimate (3.18) for any $0<\varepsilon<1$.

Since the pair (q_1,q_2) has property P, we know from Proposition 2.3 that there exists a positive constant c_0 such that (2.5) holds on $C(J)$. In particular, we have

$$\|u-\mathscr{T}(m,u)\|_\infty \geqslant c_0\|u\|_\infty. \tag{3.20}$$

By estimates (3.18) and (3.20), we obtain from (3.19) that

$$A\max\{\varepsilon,\varepsilon^{p^*-1}\}\|u\|_\infty+B_\varepsilon \geqslant \|G\|=\|u-\mathscr{T}(m,u)\|_\infty \geqslant c_0\|u\|_\infty.$$

Let us take, in Step 1, that $\varepsilon\in(0,1)$ satisfying

$$A\max\{\varepsilon,\varepsilon^{p^*-1}\}<c_0.$$

Then u must satisfy

$$\|u\|_\infty \leqslant \frac{B_\varepsilon}{c_0 - A\max\{\varepsilon, \varepsilon^{p^*-1}\}} =: R_0.$$

This proves that all possible solutions u of (3.5) in $C(J)$ are a priori bounded.

Step 4: *Existence of solutions.* By the homotopy invariance of Leray–Schauder degree,

$$\deg(I - \mathscr{T}_0, B(0, 2R_0), 0) = \deg(I - \mathscr{T}_1, B(0, 2R_0), 0),$$

where $B(0, 2R_0) = \{u \in C(J) : \|u\|_\infty < 2R_0\}$. Note that the operator $\mathscr{T}_1 : C(J) \mapsto C(J)$ is the same as $\mathscr{T}(q_0, \cdot)$ in the notation of Proposition 2.3. Thus the Borsuk theorem implies (2.6). So Eq. (3.1), which is equivalent to problem (1.7), (1.5), has necessarily at least one solution u in $B(0, 2R_0)$. Theorem 3.1 is thus proved. □

Remark 3.1. The method in Steps 1–3 for estimating solutions of the homotopy problem (3.4), (1.5) is different from the usual ones in the literature. It is used in [16] for the periodic problem. Besides the perturbation principle of [17, Theorem 3.1], one sees that all estimates in Steps 1–3 are elementary. In some sense, the principle of [17, Theorem 3.1] is more convenient than that given by Fučík [5].

Remark 3.2. Assume that q_1 and q_2 in (1.8) are equal, $q_1 = q_2 = q$. That is to say, $f(t, x)$ is asymptotically $(p-1)$-homogeneous near $x = \infty$. Suppose that (1.10) has only the trivial solution verifying (1.5) for this$q$. Then (1.7), (1.5) has at least one solution. Such a solvability condition corresponds to the well-known Fredholm Principle for the linear operators. Thus our solvability condition, property P, is a generalization of this principle to nonlinear problems. Such a solvability condition was first found by Fonda and Mawhin [4] for the periodic problem (with $p = 2$) and then has been extended to many different kinds of BVPs [16–19].

4. An optimal bound on growth

For $1 < p < \infty$ and $1 \leqslant \gamma \leqslant \infty$, let $K(\gamma, p, J)$ be the best Sobolev constant in the following inequality:

$$C\|u\|_{\gamma,J}^p \leqslant \|u'\|_{p,J}^p, \quad \text{for all } u \in W_0^{1,p}(J).$$

That is,

$$K(\gamma, p, J) = \inf_{u \in W_0^{1,p}(J)\setminus\{0\}} \frac{\|u'\|_{p,J}^p}{\|u\|_{\gamma,J}^p}. \tag{4.1}$$

The explicit formula for $K(\gamma, p, J)$ is given by Talenti [15, p. 357]. That is,

$$K(\gamma, p, J) = K(\gamma, p)/|J|^{p-1+p/\gamma}, \tag{4.2}$$

where $|J|$ is the length of the interval J and

$$K(\gamma, p) = \begin{cases} \left(\dfrac{2(1+\gamma/p^*)^{1/\gamma} B(1/\gamma, 1/p^*)}{\gamma(1+p^*/\gamma)^{1/p}} \right)^p & \text{if } 1 \leqslant \gamma < \infty, \\ 2^p & \text{if } \gamma = \infty, \end{cases} \tag{4.3}$$

where $B(\cdot,\cdot)$ is the Beta function of Euler. Moreover, if $1 \leqslant \gamma < \infty$, the infimum (4.1) can be attained by some function $u_\gamma(t) \in W_0^{1,p}(J)\backslash\{0\}$ which satisfies $u_\gamma(t) > 0$ for all $t \in \mathrm{int}(J)$, the interior of J. In fact, let $J = [a, b]$ and define the function $u_\gamma(t)$ by

$$u_\gamma(t) = \begin{cases} E_\gamma^{-1}(2(t-a)E_\gamma(1)/|J|) & \text{if } t \in [a, (a+b)/2], \\ E_\gamma^{-1}(2(b-t)E_\gamma(1)/|J|) & \text{if } t \in [(a+b)/2, b], \end{cases}$$

where $E_\gamma : [0, 1] \to \mathbb{R}$ is given by

$$E_\gamma(u) = \int_0^u \frac{\mathrm{d}u}{(1-u^\gamma)^{1/p}}.$$

Then we have

$$K(\gamma, p, J) = \|u'\|_{p,J}^p / \|u\|_{\gamma,J}^p$$

for any $u(t) = cu_\gamma(t)$, $c \neq 0$. However, if $\gamma = \infty$, infimum (4.1) is not attainable. For these results, see [15] and also [19].

In the following we give some explicit conditions on q_1, q_2 so that they have property P. To this end, we need to prove a preliminary result for the gap of zeros of solutions of the following differential equation

$$(\phi_p(u'))' + w(t)\phi_p(u) = 0, \quad t \in I, \tag{4.4}$$

where $I \subset \mathbb{R}$ is an interval, and the coefficient $w(t)$ is locally integrable in I. It is well-known that the solutions of initial value problems for (4.4) are well defined on I.

Proposition 4.1. *Suppose that the potential $w(t)$ from* (4.4) *is locally L^α integrable for some $1 \leqslant \alpha \leqslant \infty$. Let $t_1 < t_2$ be two consecutive zeros of any* (*non-trivial*) *solution $u(t)$ of* (4.4). *Then*

$$\|w_+\|_{\alpha,[t_1,t_2]} \geqslant K(p\alpha^*, p)/(t_2 - t_1)^{(p\alpha-1)/\alpha}, \tag{4.5}$$

where $K(p\alpha^*, p,)$ is given by (4.3) *and* $w_+(t) = \max(w(t), 0)$.

Proof. Let $u(t)$ be a solution of (4.4) such that $u(t_1) = u(t_2) = 0$. Multiplying the equation with $u(t)$ and then integrating over $I = [t_1, t_2]$, we obtain

$$\begin{aligned} \int_I |u'|^p \,\mathrm{d}t &= -\int_I (\phi_p(u'))' u \,\mathrm{d}t = \int_I w(t)|u|^p \,\mathrm{d}t \\ &\leqslant \int_I w_+(t)|u|^p \,\mathrm{d}t \leqslant \|w_+\|_{\alpha,I} \| |u|^p \|_{\alpha^*,I} \\ &= \|w_+\|_{\alpha,I} \|u\|_{p\alpha^*,I}^p \leqslant \frac{\|w_+\|_{\alpha,I}}{K(p\alpha^*, p, I)} \|u'\|_{p,I}^p, \end{aligned} \tag{4.6}$$

where the Hölder inequality is used. This yields the desired inequality (4.5) by noticing that

$$K(p\alpha^*, p, I) = K(p\alpha^*, p)/|I|^{p-1+p/(p\alpha^*)} = K(p\alpha^*, p)/|I|^{p-1/\alpha}. \qquad \square$$

As a consequence, we have the following nonresonant result.

Proposition 4.2. *Suppose that the potential* $q(t)$ *from* (1.10) *is in* $L^\alpha(J)$ *for some* $1 \leqslant \alpha \leqslant \infty$. *If*

$$\|q_+\|_{\alpha,J} < 2^{-p} K(p\alpha^*, p, J), \tag{4.7}$$

then (1.10) *has only the trivial solution verifying the Stieltjes BC* (1.5).

Proof. Suppose that (1.10) has a nonzero solution $u(t)$ satisfying (1.5). It follows from (1.5) that $u(t)$ satisfies

$$u(a) = 0 \text{ and } u'(\tau) = 0 \quad \text{for some } \tau \in]a, b].$$

Define then a potential $w : I \to \mathbb{R}$ and a function $\tilde{u} : I \to \mathbb{R}$, where $I = [a, 2\tau - a]$, by

$$w(t) = \begin{cases} q(t) & \text{if } t \in [a, \tau], \\ q(2\tau - t) & \text{if } t \in [\tau, 2\tau - a] \end{cases}$$

and

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [a, \tau], \\ u(2\tau - t) & \text{if } t \in [\tau, 2\tau - a]. \end{cases}$$

($w(t)$ and $\tilde{u}(t)$ are symmetric with respect to $t=\tau$.) Then $w \in L^\alpha(I)$ and $u(a)=u(2\tau-a)=0$. Since $u'(\tau) = 0$, we know that $\tilde{u}(t)$ is C^1 on I. Moreover, $\tilde{u}(t)$ and $w(t)$ satisfy (4.4) on the interval I. By Proposition 4.1, we know that

$$\begin{aligned}\|w_+\|_{\alpha,I} &= 2^{1/\alpha}\|w_+\|_{\alpha,[a,\tau]} \geqslant K(p\alpha^*, p)/|I|^{(p\alpha-1)/\alpha} \\ &= K(p\alpha^*, p)/(2(\tau - a))^{(p\alpha-1)/\alpha}.\end{aligned}$$

This implies that

$$\begin{aligned}\|w_+\|_{\alpha,J} \geqslant \|w_+\|_{\alpha,[a,\tau]} &\geqslant 2^{-p} K(p\alpha^*, p)/(\tau - a)^{(p\alpha-1)/\alpha} \\ &\geqslant 2^{-p} K(p\alpha^*, p)/|J|^{(p\alpha-1)/\alpha}.\end{aligned}$$

This is a contradiction to assumption (4.7). $\square$

Remark 4.1. Inequality (4.5) is strict when $\alpha = 1$, because the Sobolev inequality (4.1) cannot be attained in this case. Thus the strict inequality $<$ in (4.7) can be improved as $\leqslant$ when $\alpha = 1$.

Note that the bounds in (4.7) are independent of the boundary conditions (1.5), i.e., of the functions ξ in defining the boundary conditions.

Remark 4.2. The bounds in (4.7) are optimal among all boundary conditions of type (1.5), in both cases $\alpha = 1$ and $1 < \alpha \leqslant \infty$.

To see this, let us consider, for instance, the case $1 < \alpha \leqslant \infty$. The extremal boundary condition is the case $\xi = \xi_b$, where

$$\xi_b(s) = \begin{cases} 0, & s \in [a, b[, \\ 1, & s = b. \end{cases}$$

Now (1.5) is

$$u(a) = 0, \quad u'(b) = 0. \tag{4.8}$$

In the proof of Proposition 4.1, let $t_1 = a$ and $t_2 = b + |J| = 2b - a$, where $[a, b] = J$. Set $I = [t_1, t_2]$. One can find functions $q \in L^\alpha(I)$ and $u \in C(I)$, $u \neq 0$, such that $\|q_+\|_{\alpha,I} = K(p\alpha^*, p, I)$ and all inequalities in (4.6) become equalities. For the details of the construction of these q and u, we refer to [20] for $p = 2$ and to [19] for general p. Note that $u(t)$ is actually a minimizer of (4.1) with J replaced by I and $q(t)$ is related with some power of $u(t)$. Thus $q(t)$ $(\geqslant 0)$ and $u(t)$ are symmetric with respect to $t = b$. So $u(t)$ satisfies also the boundary condition (4.8). The equalities in (4.6) mean that $u(t)$ satisfies (1.10) on I, while the condition $\|q_+\|_{\alpha,I} = K(p\alpha^*, p, I)$ is equivalent to the equality in (4.7).

Let us consider again the problem (1.7), (1.5). As a corollary of Theorem 3.1 and Proposition 4.2, we have

Theorem 4.1. *Assume that* $f(t, x)$ *satisfies* (1.8) *for some* $q_1, q_2 \in L^\alpha(J)$, $1 \leqslant \alpha \leqslant \infty$. *Suppose that* q_2 *satisfies* (4.7), *i.e.*,

$$\|(q_2)_+\|_{\alpha,J} < 2^{-p} K(p\alpha^*, p, J). \tag{4.9}$$

Then, *for any boundary condition* (1.5), *Eq*. (1.7) *has at least one solution satisfying* (1.5).

A special case of Theorem 4.1 is

Theorem 4.2. *Assume that* $f(t, x)$ *satisfies* (1.11) *for some* $q \in L^\alpha(J, \mathbb{R}_+)$ *and* $\rho \in L^\alpha(J, \mathbb{R}_+)$, *where* $1 \leqslant \alpha \leqslant \infty$. *Suppose that* q *satisfies*

$$\|q\|_{\alpha,J} < 2^{-p} K(p\alpha^*, p, J). \tag{4.10}$$

Then, *for any boundary condition* (1.5), *Eq*. (1.7) *has at least one solution satisfying* (1.5).

Remark 4.3. Conditions (4.9) and (4.10) are improvements of the results in [7] in this case. Moreover, as mentioned in Remarks 4.1 and 4.2, the bounds in (4.9) and (4.10) are optimal among all boundary conditions (1.5). However, for a specific boundary condition (1.5), finding the corresponding optimal bounds like in (4.9) and (4.10) appeals the study of the eigenvalue problem

$$(\phi_p(u'))' + (\lambda + q(t))\phi_p(u) = 0$$

with the boundary condition (1.5), which is not clear even for the linear case, i.e., $p = 2$.

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