

## Certain classes of potentials for $p$ -Laplacian to be non-degenerate

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Given a positive integer  $n$  and an exponent  $1 \leq \alpha \leq \infty$ . We will find explicitly the optimal bound  $r_n$  such that if the  $L^\alpha$  norm of a potential  $q(t)$  satisfies  $\|q\|_{L^\alpha(I)} < r_n$  then the  $n^{\text{th}}$  Dirichlet eigenvalue of the one-dimensional  $p$ -Laplacian with the potential  $q(t)$ :  $(|u'|^{p-2}u')' + (\lambda + q(t))|u|^{p-2}u = 0$  ( $1 < p < \infty$ ) will be positive. Using these bounds, we will construct, for the Dirichlet, the Neumann, the periodic or the anti-periodic boundary conditions, certain classes of potentials  $q(t)$  so that the  $p$ -Laplacian with the potential  $q(t)$  is non-degenerate, which means that the above equation with  $\lambda = 0$  has only the trivial solution verifying the corresponding boundary condition.

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### 1 Introduction

Let  $1 < p < \infty$  be a fixed exponent,  $I$  be a finite closed interval. Given  $q(t) \in L^1(I)$ . We say that the one-dimensional  $p$ -Laplacian with the potential  $q(t)$ :

$$(|u'|^{p-2}u')' + q(t)|u|^{p-2}u = 0, \quad \text{a.e. } t \in I, \quad (1.1)$$

is *non-degenerate* with respect to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial I, \quad (1.2)$$

if Eq. (1.1) has only the trivial solution verifying Eq. (1.2). In some literature, this is also referred as *non-resonant* [15].

Such a notion of non-degeneracy is very important in many problems. For example, if Eq. (1.1) is non-degenerate, then for any  $f(t) \in L^1(I)$ , the following non-homogeneous equation

$$(|u'|^{p-2}u')' + q(t)|u|^{p-2}u = f(t), \quad \text{a.e. } t \in I,$$

has at least one solution verifying the boundary condition (1.2), see, e.g., [16] for the Dirichlet problems and [12] for the periodic problem. When  $p = 2$ , the linear equation has only exactly one solution verifying Eq. (1.2) by the Fredholm Alternative Principle.

In this paper, the problem we are studying is to find certain classes of potentials  $q(t)$  which make Eq. (1.1) be non-degenerate with respect to various boundary conditions, including the Dirichlet, the Neumann, and the periodic boundary conditions. These classes of potentials will be expressed explicitly using the  $L^\alpha$ ,  $1 \leq \alpha \leq \infty$ , norms, and the case for  $\alpha = \infty$  is well-known.

Theoretically, the non-degeneracy problem (1.1)–(1.2) can be solved using the eigenvalue theory for  $p$ -Laplacian. For simplicity, define the mapping  $\phi_p : \mathbb{R} \rightarrow \mathbb{R}$  by  $\phi_p(s) = |s|^{p-2}s$ . Consider then the eigenvalue problem of

$$(\phi_p(u'))' + (\lambda + q(t))\phi_p(u) = 0, \quad \text{a.e. } t \in I, \quad (1.3)$$

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with Eq. (1.2). It is well-known that the problem (1.3)–(1.2) have a sequence of (simple) eigenvalues

$$-\infty < \lambda_1^D(q) < \lambda_2^D(q) < \dots < \lambda_n^D(q) < \dots, \quad \lambda_n^D(q) \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

See, e.g., [17]. Then the problem(1.1)–(1.2) is non-degenerate if and only if  $\lambda_k^D(q) \neq 0$  for all  $k \geq 1$ , or, equivalently, there exists some positive integer  $n$  such that

$$\lambda_{n-1}^D(q) < 0 \tag{1.4}$$

and

$$\lambda_n^D(q) > 0. \tag{1.5}$$

Here  $\lambda_0^D(q)$  is understood as  $-\infty$ . In this sense, to study non-degeneracy is the same as the estimates of eigenvalues both from above and from below.

From this point of view, some classes of potentials for the problem (1.1)–(1.2) to be non-degenerate can be obtained easily. We use  $\mathbb{R}$ ,  $\mathbb{N}$  and  $\mathbb{Z}^+$  to denote the sets of all real numbers, all positive integers and all nonnegative integers, respectively. Let

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}.$$

When the potential  $q(t) = c$  is constant, the eigenvalues  $\lambda_n^D(q)$  are known explicitly:

$$\lambda_n^D(c) = (n\pi_p/|I|)^p - c, \quad n \in \mathbb{N}.$$

By the comparison results (see, e.g., [17, Remark 4.3]) for eigenvalues of the problem (1.3)–(1.2), we know

$$(k\pi_p/|I|)^p < q(t) \text{ a.e. } t \implies \lambda_k^D(q) < 0, \tag{1.6}$$

and

$$q(t) < (k\pi_p/|I|)^p \text{ a.e. } t \implies \lambda_k^D(q) > 0. \tag{1.7}$$

Moreover, the constants  $(k\pi_p/|I|)^p$  in Eqs. (1.6) and (1.7) are optimal for any given  $k \in \mathbb{N}$ . Combining Eqs. (1.6) and (1.7) with Eqs. (1.4) and (1.5), we know that if  $q(t)$  satisfies for some  $n \in \mathbb{N}$ ,

$$((n-1)\pi_p/|I|)^p < q(t) < (n\pi_p/|I|)^p \quad (n \geq 2), \tag{1.8}$$

$$q(t) < (\pi_p/|I|)^p \quad (n = 1), \tag{1.9}$$

then the problem (1.1)–(1.2) is non-degenerate. The criteria (1.8) and (1.9) for non-degeneracy are very simple, but yet used in many applications of eigenvalues.

The main work of this paper is to extend Eqs. (1.8) and (1.9) to the  $L^\alpha$ ,  $1 \leq \alpha < \infty$ , cases. However, it is known that  $\lambda_k^D(q)$  cannot be estimated from above only the  $L^\alpha$ ,  $1 \leq \alpha < \infty$ , of  $q(t)$  is given. See [5, 6, 13]. That is to say, Eq. (1.6) cannot be generalized to the  $L^\alpha$ ,  $1 \leq \alpha < \infty$ , cases. In order to describe our generalization of Eq. (1.7) to the  $L^\alpha$  cases, we will use the following notations throughout this paper.

For an interval  $I$  and an exponent  $1 \leq \alpha \leq \infty$ , we use  $L^\alpha(I)$  to denote the usual  $L^\alpha$  Lebesgue space with the usual norm denoted by  $\|\cdot\|_{\alpha, I}$ . For  $q \in L^1(I)$ ,  $q_+(t) := \max\{q(t), 0\}$  and  $q_-(t) := \max\{-q(t), 0\}$  are the positive part and the negative part of  $q(t)$ , respectively.

In Section 2, for any  $1 \leq \alpha \leq \infty$  and any  $1 < p < \infty$ , we will find the optimal bound  $r_n = r_n(\alpha, p, I)$  such that

$$\|q_+\|_{\alpha, I} < r_n \implies \lambda_n^D(q) > 0. \tag{1.10}$$

See Theorem 2.9. The explicit formulas for  $r_n(\alpha, p, I)$  can be obtained using the best Sobolev constants in [10]. When  $\alpha = \infty$ ,  $r_n(\alpha, p, I)$  returns to the constant  $(n\pi_p/|I|)^p$  in Eq. (1.7).

Combining Eq. (1.10) with Eq. (1.6), in Section 3 we will assign to each pair  $(\alpha, p) \in [1, \infty] \times (1, \infty)$  an integer  $N(\alpha, p)$ ,  $3 \leq N(\alpha, p) \leq \infty$ , and then introduce for each  $1 \leq n < N(\alpha, p)$  a subset  $\mathcal{Q}_n^D(\alpha, p, I)$  of

$L^\alpha(I)$ , from which each potential  $q(t)$  ensures that the problem (1.1)–(1.2) is non-degenerate. When  $\alpha = \infty$ , the classes  $\mathcal{Q}_n^D(\alpha, p, I)$  coincide with those defined by Eqs. (1.8) and (1.9). See Theorem 3.2. The corresponding classes of non-degenerate potentials for Eq. (1.1) with respect to the Neumann and to the periodic, the anti-periodic boundary conditions will also be constructed.

As an application of these classes, in Section 4 we will obtain some optimal estimates of the higher order stability zones of the Hill equations.

We end this introduction with the following remarks. The non-degenerate potentials (1.8) and (1.9) for the  $L^\infty$  case are obtained based on the comparison results of eigenvalues. However, the classes of potentials constructed in this paper using the  $L^\alpha$ ,  $1 \leq \alpha < \infty$ , norms are based on certain Sobolev inequalities, which yield some optimal bounds on the gap of zeros of solutions of Eq. (1.1). These ideas are used first in [18] and are very useful in many problems [2, 9, 11].

## 2 Some optimal results for the Dirichlet eigenvalues

In this section, we will give some lower bounds of eigenvalues of Eq. (1.3) with the Dirichlet boundary condition (1.2), and then find the optimal bounds  $r_n(\alpha, p, I)$  in Eq. (1.10).

The following Sobolev constants are fundamental. For an exponent  $1 \leq \gamma \leq \infty$ , the conjugate exponent of  $\gamma$  is  $\gamma^* := \gamma/(\gamma - 1) \in [1, \infty]$ . Let  $K(\gamma, p, I)$  be the best Sobolev constant in the following inequality

$$C \|u\|_{\gamma, I}^p \leq \|u'\|_{p, I}^p \quad \text{for all } u \in \mathcal{W} := W_0^{1,p}(I),$$

where  $I \subset \mathbb{R}$  is a finite interval. Namely,

$$K(\gamma, p, I) = \inf_{u \in \mathcal{W} \setminus \{0\}} \|u'\|_{p, I}^p / \|u\|_{\gamma, I}^p. \tag{2.1}$$

**Lemma 2.1** *The constants  $K(\gamma, p, I)$  are given by*

$$K(\gamma, p, I) = K(\gamma, p) / |I|^{p-1+p/\gamma}, \tag{2.2}$$

where

$$K(\gamma, p) = \begin{cases} \left( \frac{2(1 + \gamma/p^*)^{1/\gamma} B(1/\gamma, 1/p^*)}{\gamma(1 + p^*/\gamma)^{1/p}} \right)^p, & \text{if } 1 \leq \gamma < \infty, \\ 2^p, & \text{if } \gamma = \infty, \end{cases}$$

and  $B(\cdot, \cdot)$  is the Beta function of Euler.

*Proof.* For the case  $1 \leq \gamma < \infty$ , see [10, p. 357]. The case  $\gamma = \infty$  is obtained by a limiting procedure.  $\square$

**Remark 2.2** (i) Let  $1 \leq \gamma < \infty$ . Then the infimum in Eq. (2.1) can be attained. In fact, let  $I = [a, b]$  and define the function  $u_\gamma(t)$  by

$$u_\gamma(t) = \begin{cases} E_\gamma^{-1}(2(t - a)E_\gamma(1)/|I|), & \text{if } t \in [a, (a + b)/2], \\ E_\gamma^{-1}(2(b - t)E_\gamma(1)/|I|), & \text{if } t \in [(a + b)/2, b], \end{cases}$$

where  $E_\gamma : [0, 1] \rightarrow \mathbb{R}$  is given by

$$E_\gamma(u) = \int_0^u \frac{du}{(1 - u^\gamma)^{1/p}}.$$

Then we have

$$K(\gamma, p, I) = \|u'\|_{p, I}^p / \|u\|_{\gamma, I}^p$$

for any  $u(t) = c u_\gamma(t)$ ,  $c \neq 0$ . However, if  $\gamma = \infty$ , the infimum in Eq. (2.1) is not attainable by any function  $u(t)$  from  $\mathcal{W}$ .

(ii) When the exponent  $\gamma$  is  $p$ , the Sobolev constant  $K(p, p, I) = (\pi_p/|I|)^p$  is the first Dirichlet eigenvalue of the  $p$ -Laplacian (with the potential  $q(t) = 0$ ).

**Lemma 2.3**  $K(\gamma, p)$  is decreasing as a function of  $\gamma \in [1, \infty]$ .

*Proof.* By Eqs. (2.1) and (2.2), if  $I$  is an interval of length 1, e.g.,  $I = [0, 1]$ , then

$$K(\gamma, p) = \inf_{u \in \mathcal{W} \setminus \{0\}} \|u'\|_{p,I}^p / \|u\|_{\gamma,I}^p.$$

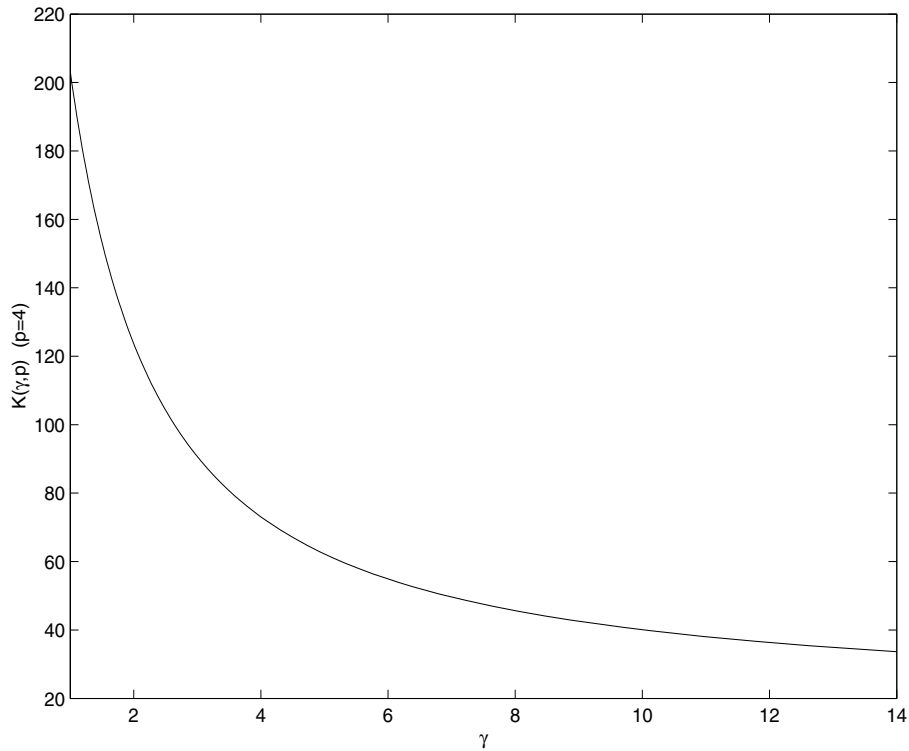
Consider exponents  $1 \leq \gamma < \tilde{\gamma} \leq \infty$ . Let  $u_\gamma(t)$  be the minimizer of Eq. (2.1) with the exponent  $\gamma$ . See Remark 2.2 (i). Then  $u_\gamma(t)$  is not a constant. By the Hölder inequality,

$$\|u_\gamma\|_{\gamma,I} < |I|^{\tilde{\gamma}/(\tilde{\gamma}-\gamma)} \|u_\gamma\|_{\tilde{\gamma},I} = \|u_\gamma\|_{\tilde{\gamma},I}.$$

Thus

$$K(\gamma, p) = \|u'_\gamma\|_{p,I}^p / \|u_\gamma\|_{\gamma,I}^p > \|u'_\gamma\|_{p,I}^p / \|u_\gamma\|_{\tilde{\gamma},I}^p \geq K(\tilde{\gamma}, p). \quad \square$$

A typical graph of  $K(\cdot, p)$  is shown in Figure 1, where  $p = 4$ .



**Fig. 1** Graph of  $K(\cdot, p)$  with  $p = 4$

Now we establish a lower bound for the first Dirichlet eigenvalue  $\lambda_1^D(q)$  of Eq. (1.3).

**Theorem 2.4** Suppose that the potential  $q(t)$  is in  $L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ . If

$$\|q_+\|_{\alpha,I} \leq K(p\alpha^*, p, I), \tag{2.3}$$

then

$$\lambda_1^D(q) \geq \left(\frac{\pi_p}{|I|}\right)^p \left(1 - \frac{\|q_+\|_{\alpha,I}}{K(p\alpha^*, p, I)}\right) (\geq 0). \tag{2.4}$$

*Proof.* Let  $u \in \mathcal{W}$ . By the Hölder inequality and definition of  $K(\gamma, p, I)$ , we have

$$\begin{aligned} \int_I q(t) |u|^p dt &\leq \int_I q_+(t) |u|^p dt \\ &\leq \|q_+\|_{\alpha,I} \| |u|^p \|_{\alpha^*,I} = \|q_+\|_{\alpha,I} \|u\|_{p\alpha^*,I}^p \leq \frac{\|q_+\|_{\alpha,I}}{K(p\alpha^*, p, I)} \|u'\|_{p,I}^p. \end{aligned} \tag{2.5}$$

Let now  $u(t)$  be an eigenfunction associated with  $\lambda_1 := \lambda_1^D(q)$ , i.e.,  $u(t)$  satisfies

$$(\phi_p(u'))' + (\lambda_1 + q(t))\phi_p(u) = 0, \quad \text{a.e. } t \in I,$$

and Eq. (1.2). Multiplying this equation by  $u(t)$  and then integrating over  $I$ , we obtain

$$\begin{aligned} \lambda_1 \|u\|_{p,I}^p &= - \int_I (\phi_p(u'))' u \, dt - \int_I q(t) |u|^p \, dt \\ &= \int_I |u'|^p \, dt - \int_I q(t) |u|^p \, dt \\ &\geq \left(1 - \frac{\|q_+\|_{\alpha,I}}{K(p\alpha^*, p, I)}\right) \|u'\|_{p,I}^p, \end{aligned} \tag{2.6}$$

where Eq. (2.5) is used. Thus

$$\lambda_1 \geq \left(1 - \frac{\|q_+\|_{\alpha,I}}{K(p\alpha^*, p, I)}\right) \frac{\|u'\|_{p,I}^p}{\|u\|_{p,I}^p} \geq \left(1 - \frac{\|q_+\|_{\alpha,I}}{K(p\alpha^*, p, I)}\right) \left(\frac{\pi_p}{|I|}\right)^p. \quad \square$$

Theorem 2.4 gives some lower bounds for  $\lambda_1^D(q)$  using only the  $L^\alpha$  norms of the potentials  $q(t)$ . Generally speaking, these estimates are not optimal. To find the optimal lower estimates using only the  $L^\alpha$  norms of the potentials  $q(t)$ , one needs some sophisticated techniques such as those in [4, 5, 6, 13]. However, in some applications later, Theorem 2.4 can yield some optimal results.

Note that if Eq. (2.3) is strict, then  $\lambda_1^D(q) > 0$ . This can be improved a little bit more when  $\alpha = 1$  or  $\alpha = \infty$ . To this end, we need some notations. Let  $q \in L^1(I)$ . The mean value of  $q$  over the interval  $I$  is denoted by  $[q]_I := (1/|I|) \int_I q(t) \, dt$ . For  $q_i \in L^1(I)$ ,  $i = 1, 2$ , we write  $q_1 \succ (<) q_2$  if  $q_1(t) \geq (\leq) q_2(t)$  for a.e.  $t \in I$ , and  $[q_1]_I > (<) [q_2]_I$ . Now the comparison results (see, e.g., [17, Remark 4.3]) for the Dirichlet eigenvalues can be written as

$$q_1 \prec q_2 \implies \lambda_n^D(q_1) > \lambda_n^D(q_2) \text{ for all } n \in \mathbb{N}. \tag{2.7}$$

**Corollary 2.5** *Suppose that the potential  $q(t)$  is in  $L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ . If*

$$\begin{cases} q \prec K(p, p, I) = (\pi_p/|I|)^p, & \alpha = \infty, \\ \|q_+\|_{\alpha,I} < K(p\alpha^*, p, I), & 1 < \alpha < \infty, \\ \|q_+\|_{\alpha,I} \leq K(\infty, p, I), & \alpha = 1, \end{cases} \tag{2.8}$$

then  $\lambda_1^D(q) > 0$ . Moreover, those bounds in Eq. (2.8) are optimal in the following sense: There exists some  $q(t)$  such that  $\lambda_1^D(q) < 0$ , while  $\|q_+\|_{\alpha,I}$  is greater than but near  $K(p\alpha^*, p, I)$ .

**Proof.** Note that if  $q(t)$  satisfies Eq. (2.8), then  $\lambda_1^D(q) \geq 0$  by Eq. (2.4). In case  $1 < \alpha < \infty$ ,  $\lambda_1^D(q) > 0$  by Theorem 2.4.

If  $\alpha = 1$ , it suffices to prove that  $\lambda_1^D(q) > 0$  even when  $\|q_+\|_{1,I} = K(\infty, p, I)$ . Otherwise, we would have  $\lambda_1^D(q) = 0$ . Then Eq. (2.6) would be an equality for the first eigenfunction  $u(t)$ . Thus all inequalities in Eq. (2.5) must be equalities. In particular,  $u(t)$  must satisfy  $\|u\|_{\infty,I}^p = K(\infty, p, I) \|u'\|_{p,I}^p$ , which is impossible by Remark 2.2 (i).

When  $\alpha = \infty$  and Eq. (2.8) is satisfied, the fact  $\lambda_1^D(q) > 0$  follows from Eq. (2.7) directly.

Let us now prove that the bounds in Eq. (2.8) are optimal. Assume that  $1 < \alpha \leq \infty$ . Take a minimizer  $u_{p\alpha^*}(t)$  of Eq. (2.1) with  $\gamma = p\alpha^* \in [p, \infty)$ . Consider the potential

$$q(t) = q_\eta(t) := \eta (u_{p\alpha^*}(t))^{p/(\alpha-1)},$$

where  $\eta > 0$  is determined by

$$\|(q_\eta)_+\|_{\alpha,I} = \|q_\eta\|_{\alpha,I} = K(p\alpha^*, p, I).$$

(Those functions ensure that all inequalities in Eq. (2.5) are actually equalities.) It can be checked that such  $u(t)$  and  $q_\eta(t)$  satisfy Eq. (1.1) on  $I$  (with  $q(t) = q_\eta(t)$ ) and  $u = 0$  on  $\partial I$ . This means that  $\lambda_1^D(q_\eta) = 0$ . If  $\eta' > \eta$ , the potential  $q_{\eta'}(t) = \eta' (u_{p\alpha^*}(t))^{p/(\alpha-1)}$  no longer satisfies Eq. (2.8). Since  $q_{\eta'} \succ q_\eta$ ,  $\lambda_1^D(q_{\eta'}) < \lambda_1^D(q_\eta) = 0$  by Eq. (2.7). This proves the optimality.

The case  $\alpha = 1$  can be proved by a slight modification.  $\square$

Theorem 2.4 can be rephrased in several ways. For example, it gives also some optimal estimates for the gap of zeros of non-trivial solutions of the following differential equation

$$(\phi_p(u'))' + q(t)\phi_p(u) = 0, \quad t \in J, \quad (2.9)$$

where  $J \subset \mathbb{R}$  is an interval, and the coefficient  $q(t)$  is locally integrable in  $J$ . It is well-known that the solutions of initial value problems for Eq. (2.9) are well defined on  $J$ .

**Corollary 2.6** *Suppose that the potential  $q(t)$  in Eq. (2.9) is locally  $L^\alpha$  integrable for some  $1 \leq \alpha \leq \infty$ . Let  $t_1 < t_2$  be two consecutive zeros of any (non-trivial) solution  $u(t)$  of Eq. (2.9). Then*

$$t_2 - t_1 \geq \left( \frac{K(p\alpha^*, p)}{\|q_+\|_{\alpha, [t_1, t_2]}} \right)^{\alpha/(p\alpha-1)}. \quad (2.10)$$

Furthermore, if  $\alpha = 1$ , the inequality is strict.

**Proof.** Suppose that  $t_1 < t_2$  are two consecutive zeros of a (non-trivial) solution  $u(t)$  of Eq. (2.9). This means that 0 is the first Dirichlet eigenvalue of Eq. (1.3), where the interval  $I$  is  $[t_1, t_2]$  and the potential is  $q|_I$ , the restriction of  $q(t)$  to  $I$ . By Corollary 2.5,

$$\|q_+\|_{\alpha, I} \geq K(p\alpha^*, p, I) = K(p\alpha^*, p)/|I|^{(p\alpha-1)/\alpha}, \quad (2.11)$$

which is just Eq. (2.10). If  $\alpha = 1$ , inequality (2.10) must be strict as in Corollary 2.5.  $\square$

**Remark 2.7** By the Sturm Comparison Theorem for linear equations ( $p = 2$ ), it is known that if  $q(t) \leq M$ ,  $M > 0$ , then two consecutive zeros  $t_1 < t_2$  of any (non-trivial) solution  $u(t)$  satisfy

$$t_2 - t_1 \geq \pi/\sqrt{M}. \quad (2.12)$$

This corresponds to Eq. (2.10) with  $p = 2$  and  $\alpha = \infty$ . Thus Corollary 2.6 is a generalization of Eq. (2.12) to the  $p$ -Laplacian using  $L^\alpha$  norms of potentials,  $1 \leq \alpha \leq \infty$ .

Next we consider the case of  $(n + 1)$  consecutive zeros.

**Theorem 2.8** *Assume that  $q(t)$  is as in Corollary 2.6. Let  $t_0 < t_1 < \dots < t_n$  be  $(n + 1)$  consecutive zeros of any (non-trivial) solution  $u(t)$  of Eq. (2.9). Denote  $I = [t_0, t_n]$ . Then*

$$\|q_+\|_{\alpha, I} \geq n^p K(p\alpha^*, p, I).$$

Furthermore, if  $\alpha = 1$ , the inequality is strict.

**Proof.** On each sub-interval  $I_i := [t_{i-1}, t_i]$ ,  $1 \leq i \leq n$ , we have the estimate like Eq. (2.11), that is,

$$\|q_+\|_{\alpha, I_i} \geq K(p\alpha^*, p)/|I_i|^{(p\alpha-1)/\alpha}.$$

Thus

$$\|q_+\|_{\alpha, I} = \left( \sum_{i=1}^n \|q_+\|_{\alpha, I_i}^\alpha \right)^{1/\alpha} \geq K(p\alpha^*, p) \left( \sum_{i=1}^n \frac{1}{|I_i|^{p\alpha-1}} \right)^{1/\alpha}. \quad (2.13)$$

Note that

$$\sum_{i=1}^n |I_i| = |I|.$$

The right-hand side of (2.13) attains its minimum at  $(|I_1|, \dots, |I_n|) = (|I|/n, \dots, |I|/n)$ . Thus

$$\|q_+\|_{\alpha, I} \geq K(p\alpha^*, p) \left( n / (|I|/n)^{p\alpha-1} \right)^{1/\alpha} = n^p K(p\alpha^*, p) / |I|^{(p\alpha-1)/\alpha}. \quad \square$$

For the higher order Dirichlet eigenvalues  $\lambda_n^D(q)$  of Eq. (1.3), we have

**Theorem 2.9** *Suppose that the potential  $q(t)$  is in  $L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ . If*

$$\|q_+\|_{\alpha, I} \leq n^p K(p\alpha^*, p, I), \tag{2.14}$$

then the  $n^{\text{th}}$  Dirichlet eigenvalue  $\lambda_n^D(q)$  of Eq. (1.3) satisfies  $\lambda_n^D(q) \geq 0$ . Moreover, if

$$\begin{cases} q \prec n^p K(p, p, I) = (n\pi_p/|I|)^p, & \alpha = \infty, \\ \|q_+\|_{\alpha, I} < n^p K(p\alpha^*, p, I), & 1 < \alpha < \infty, \\ \|q_+\|_{\alpha, I} \leq n^p K(\infty, p, I), & \alpha = 1, \end{cases} \tag{2.15}$$

then  $\lambda_n^D(q) > 0$ .

Furthermore, the bounds in Eqs. (2.14) and (2.15) are also optimal.

**Proof.** Let  $\lambda_n := \lambda_n^D(q)$  be the  $n^{\text{th}}$  Dirichlet eigenvalue of Eq. (1.3). We need to prove that  $\lambda_n \geq 0$  under assumption (2.14). Otherwise, suppose that  $\lambda_n < 0$ . Let  $u(t)$  be an eigenfunction associated with  $\lambda_n$ , i.e.,

$$(\phi_p(u'))' + (\lambda_n + q(t))\phi_p(u) = 0$$

a.e. on  $I$ , and  $u = 0$  on  $\partial I$ . It is well-known ([17]) that  $u(t)$  has exactly  $(n + 1)$  zeros on  $I$ , including the two end-points of  $I$ . By Theorem 2.8, we have

$$\|(\lambda_n + q)_+\|_{\alpha, I} \geq n^p K(p\alpha^*, p, I).$$

Since  $\lambda_n < 0$ , we have

$$\|q_+\|_{\alpha, I} > \|(\lambda_n + q)_+\|_{\alpha, I} \geq n^p K(p\alpha^*, p, I),$$

which contradicts with assumption (2.14).

As in the previous results, one can obtain the strict inequality  $\lambda_n^D(q) > 0$  under further assumption (2.15), and the bounds in Eqs. (2.14) and (2.15) are optimal. □

### 3 Classes of non-degenerate potentials

In this section, we use the results in Section 2 to construct certain classes of potentials  $q(t)$  for which Eq. (1.1) is non-degenerate with respect to various boundary conditions.

#### 3.1 The Dirichlet problem

Let us consider the Dirichlet problem (1.1)–(1.2). Based on the optimal estimates (1.6) and (2.15), in order to realize the non-degeneracy conditions (1.4) and (1.5), we can introduce the following classes of potentials.

Let  $1 < p < \infty$ ,  $1 \leq \alpha \leq \infty$ , and  $n \in \mathbb{N}$ . If  $n \geq 2$ ,

$$\begin{aligned} \mathcal{Q}_n^D(\infty, p, I) &= \{q \in L^\infty(I) : q \succ ((n - 1)\pi_p/|I|)^p \text{ and } q \prec (n\pi_p/|I|)^p\}, \\ \mathcal{Q}_n^D(\alpha, p, I) &= \{q \in L^\alpha(I) : q \succ ((n - 1)\pi_p/|I|)^p \text{ and } \|q\|_{\alpha, I} < n^p K(p\alpha^*, p, I)\} \\ &\hspace{15em} (1 < \alpha < \infty), \\ \mathcal{Q}_n^D(1, p, I) &= \{q \in L^1(I) : q \succ ((n - 1)\pi_p/|I|)^p \text{ and } \|q\|_{1, I} \leq n^p K(\infty, p, I)\}. \end{aligned}$$

If  $n = 1$ ,

$$\begin{aligned} \mathcal{Q}_1^D(\infty, p, I) &= \{q \in L^\infty(I) : q \prec (\pi_p/|I|)^p\}, \\ \mathcal{Q}_1^D(\alpha, p, I) &= \{q \in L^\alpha(I) : \|q_+\|_{\alpha, I} < K(p\alpha^*, p, I)\} \quad (1 < \alpha < \infty), \\ \mathcal{Q}_1^D(1, p, I) &= \{q \in L^1(I) : \|q_+\|_{1, I} \leq K(\infty, p, I)\}. \end{aligned}$$

It is easy to check that  $\mathcal{Q}_1^D(\alpha, p, I)$  is always non-empty for all  $1 \leq \alpha \leq \infty$ .

If  $n \geq 2$ , then  $\mathcal{Q}_n^D(\alpha, p, I)$  is non-empty if and only if the potential  $q_0(t) \equiv ((n - 1)\pi_p/|I|)^p$  satisfies

$$\|q_0\|_{\alpha, I} = ((n - 1)\pi_p/|I|)^p |I|^{1/\alpha} < n^p K(p\alpha^*, p, I) = n^p K(p\alpha^*, p)/|I|^{p-1/\alpha}.$$

That is,

$$1 - n^{-1} < (K(p\alpha^*, p))^{1/p} / \pi_p = (K(p\alpha^*, p)/K(p, p))^{1/p}. \tag{3.1}$$

Note that for  $\alpha = \infty$ , the right-hand side of Eq. (3.1) is 1 and it is satisfied for all  $n \in \mathbb{N}$ . If  $1 \leq \alpha < \infty$ , the right-hand side of Eq. (3.1) is strictly less than 1 by Lemma 2.3. Thus Eq. (3.1) is

$$n < \frac{1}{1 - (K(p\alpha^*, p)/K(p, p))^{1/p}} =: K_0(\alpha, p) (< \infty). \tag{3.2}$$

Using again Lemma 2.3, we have a very rough estimate to  $K_0(\alpha, p)$ :

$$K_0(\alpha, p) \geq K_0(1, p) = \frac{1}{1 - (K(\infty, p)/K(p, p))^{1/p}} = \frac{\pi_p}{\pi_p - 2} \geq \frac{\pi}{\pi - 2} > 2 \tag{3.3}$$

for all  $1 \leq \alpha < \infty$  and all  $1 < p < \infty$ , because  $2 < \pi_p \leq \pi$  for all  $1 < p < \infty$ .

Let us assign to each pair  $(\alpha, p) \in [1, \infty] \times (1, \infty)$  a number  $N(\alpha, p)$  as follows:

$$N(\alpha, p) = \begin{cases} \infty & \text{if } \alpha = \infty, \\ [K_0(\alpha, p)] & \text{if } 1 \leq \alpha < \infty \text{ and } K_0(\alpha, p) \text{ is an integer,} \\ [K_0(\alpha, p)] + 1 & \text{if } 1 \leq \alpha < \infty \text{ and } K_0(\alpha, p) \text{ is not an integer.} \end{cases} \tag{3.4}$$

Here  $[x]$  stands for the integer part of  $x$ . Then Eq. (3.2) is satisfied if and only if  $n < N(\alpha, p)$  in all cases. Some properties of  $N(\alpha, p)$  are collected in the following lemma.

**Lemma 3.1** *The numbers defined by Eq. (3.4) have the following properties.*

- (i) *As a function of  $\alpha \in [1, \infty]$ ,  $N(\alpha, p)$  is non-decreasing.*
- (ii)  *$N(\alpha, p) \geq 3$  for all  $(\alpha, p) \in [1, \infty] \times (1, \infty)$ , and  $N(\infty, p) = \infty$ .*
- (iii)  *$N(\alpha, p) < \infty$  when  $1 \leq \alpha < \infty$ , and  $N(\alpha, p) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ .*
- (iv)  *$N(\alpha, p) \rightarrow \infty$  as  $p \rightarrow 1$  or  $p \rightarrow \infty$ .*

Note that the inequality  $N(\alpha, p) \geq 3$  follows simply from Eq. (3.3).

The graph of  $N(\cdot, p)$  is shown in Figure 2, where  $p = 4$ . In this case,

$$N(\alpha, 4) \geq N(1, 4) = [\pi_4/(\pi_4 - 2)] + 1 = 4 \quad \text{for all } 1 \leq \alpha \leq \infty.$$

Combining Theorem 2.9 with Eq. (1.6), we have

**Theorem 3.2** *Let  $\alpha, p$  and  $I$  be as above.*

- (i) *When  $1 \leq n < N(\alpha, p)$ , the set  $\mathcal{Q}_n^D(\alpha, p, I)$  is a non-empty convex set in  $L^\alpha(I)$ .*
- (ii) *If  $1 \leq m, n < N(\alpha, p)$  and  $m \neq n$ , then  $\mathcal{Q}_m^D(\alpha, p, I) \cap \mathcal{Q}_n^D(\alpha, p, I) = \emptyset$ .*
- (iii) *Let  $n$  be as in (i). If  $q \in \mathcal{Q}_n^D(\alpha, p, I)$ , then  $\lambda_{n-1}^D(q) < 0$  and  $\lambda_n^D(q) > 0$ . Consequently, the problem (1.1)–(1.2) is non-degenerate.*

### 3.2 The Neumann problem

We will construct in this subsection the corresponding classes of potentials for which Eq. (1.1) is non-degenerate with respect to the Neumann boundary condition

$$u' = 0 \quad \text{on } \partial I. \tag{3.5}$$

Let the eigenvalues of Eq. (1.3) with respect to Eq. (3.5) be denoted by

$$-\infty < \lambda_0^N(q) < \lambda_1^N(q) < \dots < \lambda_n^N(q) < \dots, \quad \lambda_n^N(q) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Comparing with the Dirichlet problem, the Neumann problem has a zeroth eigenvalue  $\lambda_0^N(q)$  with a nowhere vanishing eigenfunction. In the next lemma, we give an upper bound for  $\lambda_0^N(q)$ , which is a generalization of [8, Theorem 4.4].

**Lemma 3.3** *Let  $q \in L^1(I)$ . Then  $\lambda_0^N(q) \leq -[q]_I$ . Moreover, the equality holds when and only when the potential  $q(t)$  is a constant.*

**Proof.** Let  $u(t)$  be an eigenfunction associated with  $\lambda_0 := \lambda_0^N(q)$ . That is,

$$(\phi_p(u'))' + (\lambda_0 + q(t))\phi_p(u) = 0.$$

It is known that  $u(t) \neq 0$  for all  $t$ . Let us assume that  $u(t) > 0$  for all  $t$ . Define a function

$$r(t) = \frac{|u'(t)|^{p-2}u'(t)}{(u(t))^{p-1}}.$$

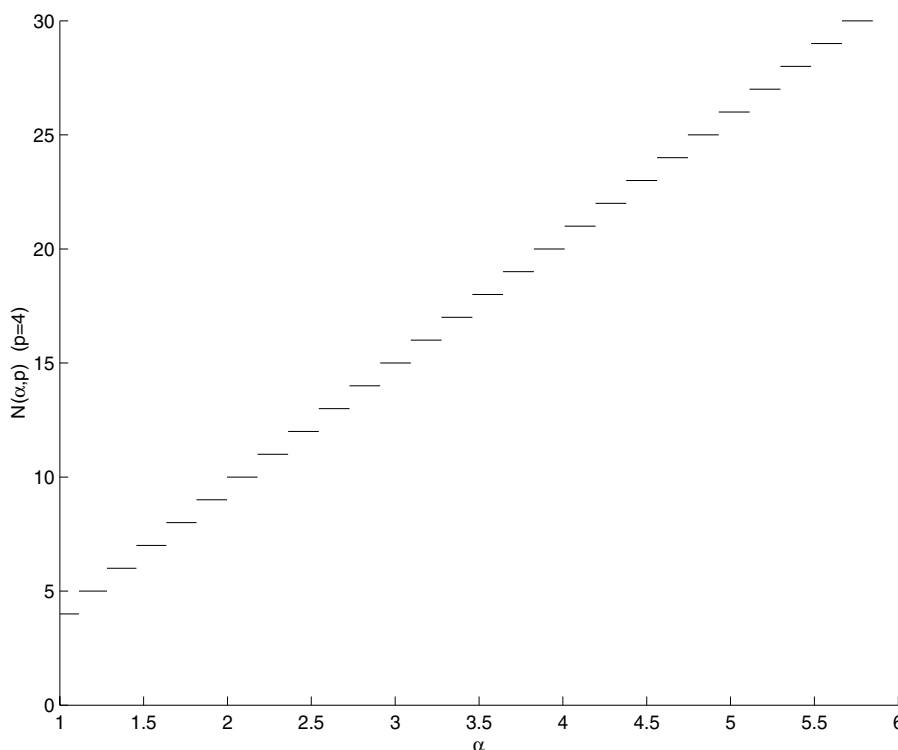
Then

$$r' = \frac{(|u'|^{p-2}u')'u^{p-1} - (p-1)u^{p-2}u'|u'|^{p-2}u'}{u^{2(p-1)}} = -(\lambda_0 + q(t)) - (p-1)|r|^{p^*}.$$

Note that  $r(a) = r(b) = 0$  by Eq. (3.5). Integrating the above equation over  $I$ , we have

$$\lambda_0 + [q]_I = -\frac{p-1}{|I|} \int_I |r(t)|^{p^*} dt \leq 0.$$

Moreover, if  $\lambda_0 + [q]_I = 0$  holds, we must have  $r(t) \equiv 0$ , i.e.,  $u'(t) \equiv 0$ . Thus  $u(t) \equiv c$  and  $q(t)$  is also a constant. □



**Fig. 2** Graph of  $N(\cdot, p)$  with  $p = 4$

In the next theorem, we consider the  $n^{\text{th}}$ ,  $n \geq 1$ , Neumann eigenvalue  $\lambda_n^N(q)$  of the problem (1.3)–(3.5).

**Theorem 3.4** *Suppose that the potential  $q(t)$  is in  $L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ .*

(i) *Let  $n \in \mathbb{N}$ . If  $q(t)$  satisfies Eq. (2.14), then  $\lambda_n^N(q) \geq 0$ . If  $q(t)$  satisfies Eq. (2.15), then  $\lambda_n^N(q) > 0$ . Furthermore, those bounds in Eqs. (2.14) and (2.15) are also optimal for the Neumann problem.*

(ii) *Let  $n \in \mathbb{Z}^+$ . If  $q \succ (n\pi_p/|I|)^p$ , then  $\lambda_n^N(q) < 0$ .*

**Proof.** Let  $u(t)$ ,  $t \in I$ , be an eigenfunction associated with  $\lambda_n := \lambda_n^N(q)$ . Extend the potential  $q(t)$  to the whole  $\mathbb{R}$  by the  $|I|$ -periodicity. We use  $\tilde{q}(t)$  to denote such an extension.

Extend the eigenfunction  $u(t)$  to the whole  $\mathbb{R}$  in the following way. Let  $\tilde{u}(t) = u(t)$  for  $t \in I = [a, b]$ . For  $t \in (a + |I|, b + |I|]$ , where  $a + |I| = b$ , define

$$\tilde{u}(t) = \frac{u(b)}{u(a)} u(t - |I|).$$

Then  $\tilde{u}(t)$  is continuous on  $[a, b + |I|]$ . Using the boundary condition  $u'(a) = u'(b) = 0$ ,  $\tilde{u}(t)$  is actually  $C^1$  on  $[a, b + |I|]$ . By the  $|I|$ -periodicity of  $\tilde{q}(t)$  and the homogeneity of Eq. (1.3), we know that, for a.e.  $t \in [a, b + |I|]$ ,

$$(\phi_p(\tilde{u}'(t)))' + (\lambda_n + \tilde{q}(t))\phi_p(\tilde{u}(t)) = 0. \tag{3.6}$$

In a similar way, one can use  $\tilde{u}|_{[a+|I|, b+|I|]}$  to obtain the extension  $\tilde{u}|_{[a, b+2|I|]}$  so that Eq. (3.6) is satisfied for a.e.  $t \in [a, b + 2|I|]$ , etc. So we can obtain the extension  $\tilde{u}|_{[a, \infty)}$  which satisfies Eq. (3.6) for a.e.  $t \in [a, \infty)$ . The extension  $\tilde{u}(t)$  of  $u(t)$  to the left side of the interval  $I$  is similar. In this way, we have obtained an extension  $\tilde{u}$  of  $u$  to  $\mathbb{R}$  which satisfies Eq. (3.6) for a.e.  $t \in \mathbb{R}$ . Note from the above extension that if  $\tilde{u}(t_0) = 0$  then  $\tilde{u}(t_0 + k|I|) = 0$  for all  $k \in \mathbb{Z}$ .

By the results in [17],  $\tilde{u}$  has exactly  $(n + 1)$  zeros on any interval like  $[t_0, t_0 + |I|]$ ,  $t_0 \in \mathbb{R}$ . Let us take a zero  $t_0$  of  $\tilde{u}$ . Then  $\tilde{u}$  has exactly  $(n + 1)$  zeros

$$t_0 < t_1 < \dots < t_n = t_0 + |I|$$

on the interval  $[t_0, t_0 + |I|]$ . Define then a function  $\hat{u} : I \rightarrow \mathbb{R}$  and a potential  $\hat{q} : I \rightarrow \mathbb{R}$  by

$$\hat{u}(t) = \tilde{u}(t - t_0 + a) \quad \text{and} \quad \hat{q}(t) = \tilde{q}(t - t_0 + a).$$

Then  $\hat{u}(a) = \hat{u}(b) = 0$  and  $\hat{u}(t) (\neq 0)$  satisfies, for a.e.  $t \in I$ ,

$$(\phi_p(\hat{u}'(t)))' + (\lambda_n + \hat{q}(t))\phi_p(\hat{u}(t)) = 0.$$

Since  $\hat{u}$  has exactly  $(n + 1)$  zeros on  $I$ , we know from the above equation that  $\lambda_n^N(q) = \lambda_n = \lambda_n^D(\hat{q})$ . Note that  $\|q_+\|_{\alpha, I} = \|\hat{q}_+\|_{\alpha, I}$ . Now the conclusions in this theorem follow from Theorem 2.9 and Eq. (1.6).  $\square$

Like the Dirichlet problem, we can use Lemma 3.3 and Theorem 3.4 to introduce the following classes of potentials for the Neumann problem. Let  $1 < p < \infty$ ,  $1 \leq \alpha \leq \infty$ , and  $n \in \mathbb{Z}^+$ . If  $n \geq 2$ ,

$$\mathcal{Q}_n^N(\alpha, p, I) = \mathcal{Q}_n^D(\alpha, p, I).$$

If  $n = 1$ ,

$$\begin{aligned} \mathcal{Q}_1^N(\infty, p, I) &= \{q \in L^\infty(I) : [q]_I > 0 \text{ and } q \prec (\pi_p/|I|)^p\}, \\ \mathcal{Q}_1^N(\alpha, p, I) &= \{q \in L^\alpha(I) : [q]_I > 0 \text{ and } \|q_+\|_{\alpha, I} < K(p\alpha^*, p, I)\} \quad (1 < \alpha < \infty), \\ \mathcal{Q}_1^N(1, p, I) &= \{q \in L^1(I) : [q]_I > 0 \text{ and } \|q_+\|_{1, I} \leq K(\infty, p, I)\}. \end{aligned}$$

Moreover, if  $n = 0$ ,

$$\mathcal{Q}_0^N(\alpha, p, I) = \{q \in L^\alpha(I) : q \prec 0\} \quad \text{for all } 1 \leq \alpha \leq \infty.$$

**Theorem 3.5** *Suppose that  $0 \leq n < N(\alpha, p)$ . Then  $\mathcal{Q}_n^N(\alpha, p, I)$  is a non-empty convex set in  $L^\alpha(I)$ . Moreover,  $\mathcal{Q}_m^N(\alpha, p, I) \cap \mathcal{Q}_n^N(\alpha, p, I) = \emptyset$  if  $m \neq n$ . If  $q \in \mathcal{Q}_n^N(\alpha, p, I)$ , then  $\lambda_{n-1}^N(q) < 0$  and  $\lambda_n^N(q) > 0$ . (Here  $\lambda_{-1}^N(q) = -\infty$ .) Consequently, the problem (1.1)–(3.5) is non-degenerate.*

### 3.3 The periodic and anti-periodic problems

In this subsection, we consider the periodic and anti-periodic problems and construct the corresponding classes of potentials.

Recall that so far the complete set of eigenvalues of Eq. (1.3) with the periodic boundary condition

$$u(b) - u(a) = 0 = u'(b) - u'(a) \quad (I = [a, b]) \tag{3.7}$$

or, with the anti-periodic boundary condition

$$u(b) + u(a) = 0 = u'(b) + u'(a), \tag{3.8}$$

is not known yet for the case  $p \neq 2$ . A partial result to the structure of those eigenvalues is recently given by the present author in [17] using the rotation number approach. The main result there is that there exist sequences  $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$  and  $\{\bar{\lambda}_n(q) : n \in \mathbb{Z}^+\}$  with

$$-\infty < \bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \bar{\lambda}_1(q) < \dots < \underline{\lambda}_n(q) \leq \bar{\lambda}_n(q) < \dots$$

such that  $\underline{\lambda}_n(q)$  and  $\bar{\lambda}_n(q)$  are eigenvalues of Eqs. (1.3)–(3.7) when  $n$  is even, while  $\underline{\lambda}_n(q)$  and  $\bar{\lambda}_n(q)$  are eigenvalues of the problem (1.3)–(3.8) when  $n$  is odd.

These periodic and anti-periodic eigenvalues can be characterized using the Dirichlet and the Neumann eigenvalues of Eq. (1.3) with suitable choice of potentials [17]. They are as follows. Let us still use  $q(t)$  to denote the periodic extension of  $q|_I$  with the period  $|I|$ . For  $s \in \mathbb{R}$ ,  $q_s : \mathbb{R} \rightarrow \mathbb{R}$  is the translation of  $q(t)$ :  $q_s(t) \equiv q(t + s)$ . Then we have, for any  $n \in \mathbb{N}$ ,

$$\underline{\lambda}_n(q) = \min \{ \lambda_n^\sigma(q_s) : s \in \mathbb{R} \}, \tag{3.9}$$

$$\bar{\lambda}_n(q) = \max \{ \lambda_n^\sigma(q_s) : s \in \mathbb{R} \}, \tag{3.10}$$

where  $\sigma$  stands for  $D$  or  $N$ , and

$$\bar{\lambda}_0(q) = \max \{ \lambda_0^N(q_s) : s \in \mathbb{R} \}. \tag{3.11}$$

As mentioned above, it is an open problem if these eigenvalues  $\underline{\lambda}_n(q)$  and  $\bar{\lambda}_n(q)$  give all eigenvalues of the problems (1.3)–(3.7) and (1.3)–(3.8). This is like the higher dimensional eigenvalue problem of the  $p$ -Laplacian with the Dirichlet boundary condition [7]. However, we can find some interesting applications even using those eigenvalues  $\underline{\lambda}_n(q)$  and  $\bar{\lambda}_n(q)$ .

Note that  $\|(q_s)_+\|_{\alpha, I} = \|q_+\|_{\alpha, I}$  and  $[q_s]_I = [q]_I$  for all  $s \in \mathbb{R}$ . We can use the characterizations (3.9)–(3.11) to obtain from Lemma 3.3 the following results.

**Lemma 3.6** *Let  $q \in L^1(I)$ . Then the zeroth periodic eigenvalue satisfies  $\bar{\lambda}_0(q) \leq -[q]_I$ . Moreover, the equality holds when and only when the potential  $q(t)$  is a constant.*

By Theorems 2.9 and 3.4, we have

**Theorem 3.7** *Suppose that  $q$  is in  $L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ .*

(i) *Let  $n \in \mathbb{N}$ . If  $q(t)$  satisfies Eq. (2.14), then  $\bar{\lambda}_n(q) \geq \underline{\lambda}_n(q) \geq 0$ . If  $q(t)$  satisfies Eq. (2.15), then  $\bar{\lambda}_n(q) \geq \underline{\lambda}_n(q) > 0$ .*

*Furthermore, those bounds in Eqs. (2.14) and (2.15) are also optimal for the periodic and the anti-periodic problems.*

(ii) *Let  $n \in \mathbb{Z}^+$ . If  $q \succ (n\pi_p/|I|)^p$ , then  $\underline{\lambda}_n(q) \leq \bar{\lambda}_n(q) < 0$ . (Here  $\underline{\lambda}_0(q) = -\infty$ .)*

By Lemma 3.6 and Theorem 3.7, we can now introduce the following classes of potentials for the periodic problem. Let  $1 < p < \infty$ ,  $1 \leq \alpha \leq \infty$ , and  $n \in \mathbb{Z}^+$ . If  $n \geq 2$ ,

$$\begin{aligned} \mathcal{Q}_n^P(\infty, p, I) &= \{q \in L^\infty(I) : q \succ (2(n-1)\pi_p/|I|)^p \text{ and } q \prec (2n\pi_p/|I|)^p\}, \\ \mathcal{Q}_n^P(\alpha, p, I) &= \{q \in L^\alpha(I) : q \succ (2(n-1)\pi_p/|I|)^p \text{ and } \|q\|_{\alpha, I} < (2n)^p K(p\alpha^*, p, I)\} \\ &\quad (1 < \alpha < \infty), \\ \mathcal{Q}_n^P(1, p, I) &= \{q \in L^1(I) : q \succ (2(n-1)\pi_p/|I|)^p \text{ and } \|q\|_{1, I} \leq (2n)^p K(\infty, p, I)\}. \end{aligned}$$

If  $n = 1$ ,

$$\begin{aligned}\mathcal{Q}_1^P(\infty, p, I) &= \{q \in L^\infty(I) : [q]_I > 0 \text{ and } q \prec (2\pi_p/|I|)^p\}, \\ \mathcal{Q}_1^P(\alpha, p, I) &= \{q \in L^\alpha(I) : [q]_I > 0 \text{ and } \|q_+\|_{\alpha, I} < 2^p K(p\alpha^*, p, I)\} \quad (1 < \alpha < \infty), \\ \mathcal{Q}_1^P(1, p, I) &= \{q \in L^1(I) : [q]_I > 0 \text{ and } \|q_+\|_{1, I} \leq 2^p K(\infty, p, I)\}.\end{aligned}$$

Moreover, if  $n = 0$ ,

$$\mathcal{Q}_0^P(\alpha, p, I) = \{q \in L^\alpha(I) : q \prec 0\} \quad \text{for all } 1 \leq \alpha \leq \infty.$$

**Theorem 3.8** *Let  $0 \leq n < N(\alpha, p)$ . Then  $\mathcal{Q}_n^P(\alpha, p, I)$  is a non-empty convex set of  $L^\alpha(I)$ . Moreover,  $\mathcal{Q}_m^P(\alpha, p, I) \cap \mathcal{Q}_n^P(\alpha, p, I) = \emptyset$  when  $m \neq n$ . If  $q \in \mathcal{Q}_n^P(\alpha, p, I)$ , then, with the void explanation to the notations  $\underline{\lambda}_{-2}(q)$ ,  $\bar{\lambda}_{-2}(q)$  and  $\underline{\lambda}_0(q)$ , we have  $\underline{\lambda}_{2(n-1)}(q) \leq \bar{\lambda}_{2(n-1)}(q) < 0$  and  $\bar{\lambda}_{2n}(q) \geq \underline{\lambda}_{2n}(q) > 0$ . Consequently, the problem (1.3)–(3.7) is non-degenerate.*

For the anti-periodic problem, the classes of potentials are as follows. Let  $n \in \mathbb{N}$ . If  $n \geq 2$ ,

$$\begin{aligned}\mathcal{Q}_n^A(\infty, p, I) &= \{q \in L^\infty(I) : q \succ ((2n-3)\pi_p/|I|)^p \text{ and } q \prec ((2n-1)\pi_p/|I|)^p\}, \\ \mathcal{Q}_n^A(\alpha, p, I) &= \{q \in L^\alpha(I) : q \succ ((2n-3)\pi_p/|I|)^p \text{ and } \|q\|_{\alpha, I} < (2n-1)^p K(p\alpha^*, p, I)\} \\ &\quad (1 < \alpha < \infty), \\ \mathcal{Q}_n^A(1, p, I) &= \{q \in L^1(I) : q \succ ((2n-3)\pi_p/|I|)^p \text{ and } \|q\|_{1, I} \leq (2n-1)^p K(\infty, p, I)\}.\end{aligned}$$

If  $n = 1$ ,

$$\begin{aligned}\mathcal{Q}_1^A(\infty, p, I) &= \{q \in L^\infty(I) : q \prec (\pi_p/|I|)^p\}, \\ \mathcal{Q}_1^A(\alpha, p, I) &= \{q \in L^\alpha(I) : \|q_+\|_{\alpha, I} < K(p\alpha^*, p, I)\} \quad (1 < \alpha < \infty), \\ \mathcal{Q}_1^A(1, p, I) &= \{q \in L^1(I) : \|q_+\|_{1, I} \leq K(\infty, p, I)\}.\end{aligned}$$

The condition for  $\mathcal{Q}_n^A(\alpha, p, I)$  to be non-empty is

$$n < 1/2 + K_0(\alpha, p),$$

where  $K_0(\alpha, p)$  is as in Eq. (3.2). Define the numbers  $\tilde{N}(\alpha, p)$ ,  $(\alpha, p) \in [1, \infty] \times (1, \infty)$ , as follows:

$$\tilde{N}(\alpha, p) = \begin{cases} \infty & \text{if } \alpha = \infty, \\ [K_0(\alpha, p) + 1/2] & \text{if } 1 \leq \alpha < \infty \text{ and } K_0(\alpha, p) + 1/2 \text{ is an integer,} \\ [K_0(\alpha, p) + 1/2] + 1 & \text{if } 1 \leq \alpha < \infty \text{ and } K_0(\alpha, p) + 1/2 \text{ is not an integer.} \end{cases} \quad (3.12)$$

As in Lemma 3.1, we have

**Lemma 3.9** *The numbers defined by Eq. (3.3) have the following properties.*

- (i) *As a function of  $\alpha \in [1, \infty]$ ,  $\tilde{N}(\alpha, p)$  is non-decreasing.*
- (ii)  *$\tilde{N}(\alpha, p) \geq 4$  for all  $(\alpha, p) \in [1, \infty] \times (1, \infty)$ , and  $\tilde{N}(\infty, p) = \infty$ .*
- (iii)  *$\tilde{N}(\alpha, p) < \infty$  when  $\alpha < \infty$ , and  $\tilde{N}(\alpha, p) \rightarrow \infty$  as  $\alpha \rightarrow \infty$ .*
- (iv)  *$\tilde{N}(\alpha, p) \rightarrow \infty$  as  $p \rightarrow 1$  or  $p \rightarrow \infty$ .*

Now we have

**Theorem 3.10** *Let  $1 \leq n < \tilde{N}(\alpha, p)$ . Then  $\mathcal{Q}_n^A(\alpha, p, I)$  is a non-empty convex set of  $L^\alpha(I)$ . Moreover,  $\mathcal{Q}_m^A(\alpha, p, I) \cap \mathcal{Q}_n^A(\alpha, p, I) = \emptyset$  when  $m \neq n$ . If  $q \in \mathcal{Q}_n^A(\alpha, p, I)$ , then, with the void explanation to the notations  $\underline{\lambda}_{-1}(q)$  and  $\bar{\lambda}_{-1}(q)$ , we have  $\underline{\lambda}_{2n-3}(q) \leq \bar{\lambda}_{2n-3}(q) < 0$  and  $\bar{\lambda}_{2n-1}(q) \geq \underline{\lambda}_{2n-1}(q) > 0$ . Consequently, the problem (1.3)–(3.8) is non-degenerate.*

#### 4 Stability zones of the Hill equations

The classes of potentials constructed in Sections 2 and 3 have applications in many problems, such as the solvability of boundary value problems of nonlinear equations [1, 3, 12, 13, 14, 15, 16]. We will not develop this here.

In this section we will give an application of the classes of potentials to the Lyapunov stability of the Hill equation

$$u'' + q(t)u = 0, \quad (4.1)$$

where  $q(t)$  is periodic of the period  $T > 0$  and  $q(t) \in L^1[0, T]$ . In the following we use  $I$  to denote the interval  $[0, T]$ .

Introduce a parameter  $\mu \in \mathbb{R}$  for Eq. (4.1),

$$u'' + (\mu + q(t))u = 0. \quad (4.2)$$

Recall from [6, 8] that Eq. (4.2) is stable if

$$\mu \in (\bar{\lambda}_{n-1}(q), \underline{\lambda}_n(q))$$

for some  $n \in \mathbb{N}$ , which is called the  $n^{\text{th}}$  stability zone of Eq. (4.2). Thus Eq. (4.1) is stable if there exists  $n \in \mathbb{N}$  such that

$$\bar{\lambda}_{n-1}(q) < 0 \quad \text{and} \quad \underline{\lambda}_n(q) > 0. \quad (4.3)$$

In case  $n = 1$ , there is a famous Lyapunov stability criterion (of the first stability zone):

$$[q]_{[0, T]} > 0 \quad \text{and} \quad \|q_+\|_{1, [0, T]} \leq 4/T. \quad (4.4)$$

By the results in previous sections, Eq. (4.4) implies that Eq. (4.3) holds for  $n = 1$ . Very recently, this has been generalized to the  $L^\alpha$ ,  $1 \leq \alpha \leq \infty$ , cases by Zhang and Li [18].

Using the classes of potentials constructed in this paper, the work of Zhang and Li [18] for  $n = 1$  can be generalized to higher order cases.

Note that the classes  $\mathcal{Q}_n^N(\alpha, 2, I)$  for the Neumann problems fulfill the requirement (4.3) for  $n \geq 1$ . Thus we have the following stability criteria for the stability of Eq. (4.1), which may be called the  $L^\alpha$ -Lyapunov stability criteria for higher order stability zones.

**Theorem 4.1** *Suppose that  $q(t)$  is  $T$ -periodic and  $q \in L^\alpha(I)$  for some  $1 \leq \alpha \leq \infty$ , where  $I = [0, T]$ . If there exists  $n \in \mathbb{N}$  with*

$$1 \leq n < N(\alpha, 2) \quad (4.5)$$

such that

$$q \in \mathcal{Q}_n^N(\alpha, 2, I), \quad (4.6)$$

then 0 is in the  $n^{\text{th}}$  stability zone and the Hill equation (4.1) is stable.

Note that  $N(\alpha, 2) \geq 3$  for all  $\alpha$ . Thus, given any  $\alpha \in [1, \infty]$ , Eqs. (4.5) and (4.6) give more than one stability criteria. In fact, when  $\alpha$  increases, Eq. (4.6) gives more stability criteria because  $N(\alpha, 2) \uparrow \infty$  as  $\alpha \uparrow \infty$ .

Finally, the constants in defining the classes  $\mathcal{Q}_n^N(\alpha, 2, I)$  are optimal.

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