Linearized Stability and Instability of Nonconstant Periodic Solutions of Lagrangian Equations

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Abstract
This paper is motivated by the stability problem of nonconstant periodic solutions of time-periodic Lagrangian equations, like the swing and the elliptic Sitnikov problem. As a beginning step, we will study the linearized stability and instability of nonconstant periodic solutions which are bifurcated from those of autonomous Lagrangian equations. Applying the theory for Hill’s equations, we will establish a criterion for linearized stability. The criterion shows that the linearized stability depends on the temporal frequencies of the perturbed systems in a delicate way.

Keywords: Nonconstant periodic solution; linearized stability; linearized instability; Hill’s equation; Lagrangian equation; Sitnikov problem.

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1 Introduction
This paper is motivated by the study for the stability and instability of nonconstant periodic solutions of the elliptic Sitnikov problem [24, 8, 10] and the swing, which contain parameters like the eccentricity in the elliptic Sitnikov problem. There are many analytic and topological methods to establish the existence and the continuation of periodic solutions. However, as Lagrangian equations are conservative, and, moreover, we are considering nonconstant periodic solutions, the classical Lyapunov’s methods cannot give too much for the Lyapunov stability/instability. In fact, for conservative systems, the Lyapunov stability/instability is involved of deep theories like the Birkhoff normal forms and Moser’s twist theorem, even for systems of lower degrees of freedom [14, 23]. Moreover, even for the equilibria of non-autonomous systems, it is also a difficult problem to study the Lyapunov stability/instability.
For example, in a series of papers [17, 18, 19], Ortega has successfully characterized the Lyapunov stability for the equilibrium $x(t) \equiv 0$ of the swing or the pendulum of variable length

$$\ddot{x} + l(t) \sin x = 0, \quad \text{(1.1)}$$

where the variable length $l(t) > 0$ is $2\pi$-periodic. Roughly speaking, if the linearized equation of (1.1) along $x(t) = 0$

$$\ddot{x} + l(t)x = 0$$

is stable, then, as a $2\pi$-periodic solution, the equilibrium $x(t) \equiv 0$ of (1.1) is also Lyapunov stable. In doing so, Ortega has systematically developed the third-order approximation method, which has further been profound later as in [7, 25, 27]. See also [1, 3, 4, 5, 15, 16].

As a beginning step towards nonconstant periodic solutions, in this paper we will study the linearized stability/instability. The choice for the problem to be studied is very initiative, i.e., we are considering periodic solutions which are bifurcated from nonconstant (parabolic) periodic solutions of autonomous Lagrangian equations.

To simplify the argument and notation, let us start with an autonomous Lagrange equation of degree-one-of-freedom

$$\ddot{x} + f(x) = 0. \quad \text{(1.2)}$$

Here $\dot{} = \frac{d}{dt}$ and $f(x)$ is a smooth, odd function such that

$$xf(x) > 0 \quad \forall x \neq 0.$$ 

Hence $x = 0$ is the unique equilibrium of Eq. (1.2). Then we consider the bifurcation problem

$$\ddot{x} + F(x, t, e) = 0. \quad \text{(1.3)}$$

Here the parameter $e \in [0, +\infty)$ is assumed to be one-sided. Motivated by the symmetries in the Sitnikov problem, we assume that $F(x, t, e)$ is a smooth function of $(x, t, e) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty)$ such that $F(x, t, 0) \equiv f(x)$ and

$$F(x, t + 2k\pi, e) \equiv F(x, t, e) \quad \forall k \in \mathbb{Z}, \quad \text{(1.4)}$$

$$F(-x, t, e) \equiv -F(x, t, e) \quad \text{and} \quad F(x, -t, e) \equiv F(x, t, e). \quad \text{(1.5)}$$

For autonomous equation (1.2), there are two conventional methods to obtain nonconstant periodic solutions. The first one is based on the period function. Define

$$E(x) := \int_0^x f(u) du, \quad x \in \mathbb{R},$$

an even function. Because of the oddness of $f(x)$, we choose initial values

$$x(0) = \xi \in (0, +\infty), \quad \dot{x}(0) = 0. \quad \text{(1.6)}$$

Let us use $x = X(t, \xi)$ to denote the unique solution of problem (1.2)-(1.6). Then $x = X(t, \xi)$ satisfies

$$\dot{x}^2/2 + E(x) \equiv E(\xi). \quad \text{(1.7)}$$

For each $\xi \in (0, +\infty)$, $X(t, \xi)$ is a nonconstant periodic solution of Eq. (1.2) with the minimal period

$$T(\xi) = 2 \int_0^\xi \frac{du}{\sqrt{2(E(\xi) - E(u))}} = 2\sqrt{2} \int_0^\xi \frac{du}{\sqrt{E(\xi) - E(u)}}. \quad \text{(1.8)}$$
To study Eq. (1.3), we start with a nonconstant $2\pi$-periodic solution of Eq. (1.2). However, the minimal period will not be restricted to be $2\pi$, but $2\pi/p$ for some $p \in \mathbb{N}$, i.e., by assuming that

$$T(\xi) = 2\pi/p$$

has a solution $\xi = \xi_p \in (0, +\infty)$, $\phi(t) = \phi_p(t) := X(t, \xi_p)$ is a periodic solution of Eq. (1.2) of the minimal period $2\pi/p$. Moreover, $\phi(t)$ possesses the following properties

$$\phi(-t) \equiv \phi(t) \quad \text{and} \quad \phi(t + k\pi/p) \equiv (-1)^k \phi(t) \quad \forall k \in \mathbb{Z},$$

and $\phi(t)$ has precisely $p$ zeros in each interval $[t_0, t_0 + \pi)$. As $\phi_p(t)$ ‘looks’ like $\cos(pt)$, these will be called even $2\pi$-periodic solutions of Eq. (1.2).

These periodic solutions can also be obtained using the shooting method, as did in [9]. Using the solutions $X(t, \xi)$ of initial value problems, the velocity at $t = \pi$ defines a function

$$\left. \frac{\partial X}{\partial t} \right|_{(\pi, \xi)} = 0, \quad \xi \in (0, +\infty).$$

Then solutions of (1.9) and (1.11) are the same. At any positive solution $\xi = \xi_p$ of (1.11), we also assume that system (1.2) satisfies the following non-degeneracy condition

$$\hat{c}_p := \left. \frac{\partial^2 X}{\partial \xi \partial t} \right|_{(\pi, \xi_p)} \neq 0.$$  (1.12)

For Eq. (1.3), we use $X(t, \xi, e)$ to denote the unique solution with the initial condition (1.6). Thus $X(t, \xi, 0) \equiv X(t, \xi)$. Let us consider the shooting equation for (1.3)

$$\left. \frac{\partial X}{\partial t} \right|_{(\pi, \xi, e)} = 0, \quad \xi \in (0, +\infty).$$

When $e = 0$, (1.13) returns to Eq. (1.11). Under the non-degeneracy condition (1.12), the Implicit Function Theorem implies that there exists some $e^*_p > 0$ and a smooth function $\Xi_p(e)$ of $e \in [0, e^*_p)$ such that $\Xi_p(0) = \xi_p$ and $\xi = \Xi_p(e)$ satisfies Eq. (1.13) for all $e \in [0, e^*_p)$. Using these $\Xi_p(e)$ as initial values, for each $e \in [0, e^*_p)$,

$$\phi(t, e) = \phi_p(t, e) := X(t, \Xi_p(e), e)$$

is an even $2\pi$-periodic solution of Eq. (1.3) such that it has precisely $p$ zeros in each interval $[t_0, t_0 + \pi)$. These periodic solutions $\phi(t, e)$ are bifurcated from $\phi(t) = \phi(t, 0)$. Usually, when $e > 0$, the minimal period $\phi(t, e)$ is $2\pi$, not $2\pi/p$.

The main concern of this paper is the linearized stability/instability of these nonconstant periodic solutions $\phi(t, e)$ for $0 < e \ll 1$, i.e., whether the corresponding linearization equation

$$\ddot{y} + Q(t, e)y = 0, \quad Q(t, e) := \left. \frac{\partial F}{\partial x} \right|_{(\phi(t, e), t, e)},$$

is (Lyapunov) stable. Since (1.14) is a Hill’s equation, the stability is determined by the trace $\tau(e) = \tau_p(e)$ of the corresponding $2\pi$-periodic Poincaré matrix. For more details, see the next section. For $e = 0$, as (1.14) is the linearization of autonomous equation (1.2), one
has $\tau(0) = 2$. In Theorem 3.1, we will derive the formula for $\tau'(0)$. In case $\tau'(0) \neq 0$, the sign of $\tau'(0)$ will yield the linearized stability/instability of these nonconstant periodic solutions $\phi(t, e)$ for $0 < e \ll 1$. See Theorem 3.2. These can be considered as an extension of the stability criterion of the Hopf bifurcation [2] to nonconstant periodic solutions. However, the content is much complicated. In Section 4, we will analyze the sign of $\tau'(0)$ for some typical examples. From these examples, it will be found that $\tau'(0)$ is dependent on the temporal frequencies of the perturbed systems in a delicate way. In particular, after analyzing the temporal frequencies of the elliptic Sitnikov problem, we find that $\tau'(0) = 0$ for those $2\pi$-periodic solutions constructed in [9]. In order to obtain linearized stability/instability for the elliptic Sitnikov problem, further analysis on higher-order derivatives of $\tau(e)$ is needed and will be undertaken in a future work.

Besides the contents above, in Section 2, we will give an extensive study on the linearization equation of (1.2). In particular, the relation between the non-degeneracy condition (1.12), the period function $T(\xi)$ in (1.8) and the unstable-parabolicity of solution $\phi(t)$ will be established. In Section 3, we will study Eq. (1.14) and the main result (Theorem 3.1) will be proved.

We end the introduction by the following remarks. In the monograph [11], Loud has systematically studied the linearized stability/instability of nonconstant periodic solutions of second-order ordinary differential equations by finding some formulas similar to $\tau'(0)$. However, his main concern is on equations with damping. On the other hand, our formula of $\tau'(0)$ for conservative Lagrangian equations is involved of relatively less information on the nonconstant periodic solutions of $\phi(t)$ of the autonomous equations. Hence the formula is more convenient in applications, as seen from the examples in Section 4.

### 2 Hill’s Equations and Linearization Equations

#### 2.1 Hill’s equations

Let us consider general Hill’s equation [12]

$$\ddot{y} + q(t)y = 0,$$  \hspace{1cm} (2.1)

where $q(t)$ is $T$-periodic and has some integrability, say that $q(t)$ is continuous. The fundamental solutions $y = \psi_i(t) = \psi_i(t, q)$, $i = 1, 2$, of Eq. (2.1) are those solutions of (2.1) satisfying initial conditions $(\psi_1(0), \dot{\psi}_1(0)) = (1, 0)$ and $(\psi_2(0), \dot{\psi}_2(0)) = (0, 1)$, respectively. The fundamental matrix solution is

$$M(t) := \begin{pmatrix}
\psi_1(t) & \psi_2(t) \\
\dot{\psi}_1(t) & \dot{\psi}_2(t)
\end{pmatrix}.$$

The Liouville law for Eq. (2.1) is

$$\det M(t) = \psi_1(t)\dot{\psi}_2(t) - \dot{\psi}_1(t)\psi_2(t) \equiv +1.$$  \hspace{1cm} (2.2)

The $T$-periodic Poincaré matrix of Eq. (2.1) is

$$P = P_T := \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
\psi_1(T) & \psi_2(T) \\
\dot{\psi}_1(T) & \dot{\psi}_2(T)
\end{pmatrix}.$$
The Floquet multipliers $\mu_{1,2}$ are eigenvalues of the Poincaré matrix $P$. As a consequence of the Liouville law (2.2), one has $\mu_1 \cdot \mu_2 = 1$. According to the distribution of $\mu_{1,2}$, Hill’s equation (2.1) is classified as

- hyperbolic, if $|\mu_{1,2}| \neq 1$,
- elliptic, if $|\mu_{1,2}| = 1$ and $\mu_{1,2} \neq \pm 1$, or
- parabolic, if $\mu_1 = \mu_2 = \pm 1$.

It is well known that (2.1) is Lyapunov stable (Lyapunov unstable respectively) if it is elliptic (hyperbolic respectively). In case (2.1) is parabolic, it is Lyapunov stable if $P = \pm I$ and Lyapunov unstable if $P \neq \pm I$, where $I$ is the identity matrix. For the former case, (2.1) is called stable-parabolic, and for the latter, it is called unstable-parabolic. The trace of the Poincaré matrix is

$$\tau := \text{tr}(P) = a + d = \psi_1(T) + \psi_2(T).$$

It is well known that (2.1) is hyperbolic, elliptic or parabolic if $|\tau| > 2$, $|\tau| < 2$ or $|\tau| = 2$ respectively. In order to emphasize their dependence on potential $q$, we will write these as $P(q), a(q), b(q), \mu_i(q), \tau(q)$ etc, which can be considered as nonlinear functionals of $q$.

We need the following formulas in [22].

**Lemma 2.1** ([22]) Given $t \in [0, T]$, by considering the fundamental solutions $\psi_i(t; q)$ and their derivatives $\dot{\psi}_i(t; q)$ as nonlinear functionals of potentials $q \in L^1(\mathbb{R}/T\mathbb{Z})$, the Lebesgue space endowed with the $L^1$ norm $\| \cdot \|_1 = \| \cdot \|_{L^1(\mathbb{R}/T\mathbb{Z})}$, they are continuously Fréchet differentiable in $q$. Moreover, the Fréchet derivatives are given by

$$\partial_q \dot{\psi}_i(t; q) \cdot h = \int_0^T (\psi_1(t; q)\dot{\psi}_2(s; q) - \dot{\psi}_2(t; q)\psi_1(s; q))\dot{\psi}_i(s; q)h(s) \, ds, \quad (2.3)$$

$$\partial_q \dot{\psi}_i(t; q) \cdot h = \int_0^T (\dot{\psi}_1(t; q)\psi_2(s; q) - \dot{\psi}_2(t; q)\dot{\psi}_1(s; q))\dot{\psi}_i(s; q)h(s) \, ds, \quad (2.4)$$

for $h \in L^1(\mathbb{R}/T\mathbb{Z})$ and $i = 1, 2$. Here the right-hand sides of (2.3) and (2.4) are understood as bounded linear functionals on the space $(L^1(\mathbb{R}/T\mathbb{Z}), \| \cdot \|_1)$.

Besides the Poincaré matrix $P$, let us introduce the following symmetric matrix

$$N = N(q) := \begin{pmatrix} a & b \\ b & -d \end{pmatrix} = \begin{pmatrix} a & d \\ d & -a \end{pmatrix},$$

and the following kernel function

$$K(s) = K(s; q) := (\psi_1(s), \psi_2(s))N(\psi_1(s), \psi_2(s))^{\top}$$

$$= -b\psi_1^2(s) + (a - d)\psi_1(s)\psi_2(s) + c\psi_2^2(s). \quad (2.5)$$

Such a matrix $N$ is related with the Green function for the Hill’s equations. Moreover, it has been proved that $K(s)$ is $T$-periodic in $s$. See [26, Lemma 3.4].

**Lemma 2.2** As a nonlinear functional of potential $q \in (L^1(\mathbb{R}/T\mathbb{Z}), \| \cdot \|_1)$, the trace $\tau(q)$ of the Hill’s equation (2.1) is continuously Fréchet differentiable in $q$. Moreover, the Fréchet derivative is

$$\partial_q \tau(q) \cdot h = \int_0^T K(s; q)h(s) \, ds, \quad h \in L^1(\mathbb{R}/T\mathbb{Z}). \quad (2.6)$$
Proof  As \( \tau(q) = \psi_1(T; q) + \dot{\psi}_2(T; q) \), the continuous differentiability of \( \tau(q) \) in \( q \) follows immediately from Lemma 2.1. By (2.3) and (2.4), the directional derivative along any \( h \in L^1(\mathbb{R}/T\mathbb{Z}) \) is

\[
\partial_q \tau(q) \cdot h = \partial_q \psi_1(T; q) \cdot h + \partial_q \dot{\psi}_2(T; q) \cdot h
= \int_0^T (\dot{\psi}_1(T; q)\psi_2(s; q) - \psi_2(T; q)\dot{\psi}_1(s; q))\psi_1(s; q)h(s) \, ds
+ \int_0^T (\dot{\psi}_1(T; q)\psi_2(s; q) - \dot{\psi}_2(T; q)\dot{\psi}_1(s; q))\psi_2(s; q)h(s) \, ds,
\]

which can be rewritten as in (2.5) and (2.6). \( \square \)

2.2 Linearization of autonomous equations

We consider autonomous equation (1.2) with the even 2\( \pi \)-periodic solution \( \phi(t) = \phi_p(t) = X(t, \xi_p) \). The linearization equation of (1.2) along \( \phi(t) \) is a 2\( \pi \)-periodic Hill’s equation

\[
\ddot{y} + Q(t)y = 0, \quad Q(t) := f'(\phi(t)). \tag{2.7}
\]

This corresponds to Eq. (1.14) with \( \epsilon = 0 \). As before, we use \( \psi_1(t) \) and \( M(t) \) to denote the fundamental solutions and the fundamental matrix solution of Eq. (2.7). Since Eq. (1.2) is autonomous, we have the following important observations.

Lemma 2.3  Using \( X(t, \xi) \), the fundamental solutions of Eq. (2.7) are

\[
\psi_1(t) = \frac{\partial X}{\partial \xi} \bigg|_{(t, \xi_p)} \quad \text{and} \quad \dot{\psi}_1(t) = \frac{\partial^2 X}{\partial t \partial \xi} \bigg|_{(t, \xi_p)}, \tag{2.8}
\]

\[
\psi_2(t) = \frac{\dot{\phi}(t)}{\phi(0)} = \frac{1}{-f(\xi_p)} \frac{\partial X}{\partial t} \bigg|_{(t, \xi_p)} \quad \text{and} \quad \dot{\psi}_2(t) = \frac{\dot{\phi}(t)}{\phi(0)} = \frac{f(X(t, \xi_p))}{f(\xi_p)}. \tag{2.9}
\]

Proof  Recall that \( X(t, \xi) \) is the solution of Eq. (1.2) with the initial condition (1.6). By considering \( \xi \) as a parameter, it is well known that the variational equation for \( y(t) := \frac{\partial X}{\partial \xi} \bigg|_{(t, \xi_p)} \) is just Eq. (2.7) and the initial value is just \((y(0), \dot{y}(0)) = (1, 0)\). Thus \( \psi_1(t) \equiv \left. \frac{\partial X}{\partial \xi} \bigg|_{(t, \xi_p)} \right. \). Hence we have (2.8).

On the other hand, \( \dot{\phi}(t) = X(t, \xi_p) \) satisfies Eq. (1.2), i.e.,

\[
\ddot{\phi}(t) + f(\phi(t)) = 0.
\]

Differentiating it with respect to \( t \), we know that \( y(t) := \dot{\phi}(t) \) is a solution of Eq. (2.7), with \( y(0) = \dot{\phi}(0) = 0 \) and \( \dot{y}(0) = \ddot{\phi}(0) = -f(\phi(0)) = -f(\xi_p) \). Hence \( \psi_2(t) \equiv -\dot{\phi}(t)/f(\xi_p) \). As \( \dot{\phi}(t) = X(t, \xi_p) \), we have the relations in (2.9). \( \square \)

Remark 2.4  Formulas (2.8) and (2.9) show that the fundamental solutions of (2.7) can be deduced from solutions \( X(t, \xi) \) of initial value problems of (1.2). As \( \dot{\phi}(t) \) is 2\( \pi \)-periodic, we conclude from (2.9) that Eq. (2.7) is necessarily parabolic with the Floquet multipliers +1.
As $f'(x)$ is even, we know from properties in (1.10) that $Q(t)$ of (2.7) is actually $\pi/p$-periodic. Hence (2.7) can be considered as Hill’s equations of different periods. In particular, we are interested in the periods $\pi/p$, $\pi$ and $2\pi$, and their Poincaré matrixes

$$\hat{P} := M(\pi/p), \quad \check{P} := M(\pi) \quad \text{and} \quad P := M(2\pi).$$

One has relations

$$\hat{P} = \hat{P}^p, \quad P = \check{P}^2 \quad \text{and} \quad P = \hat{P}^{2p}. \quad (2.10)$$

Let us define

$$\hat{c}_p := \psi_1(\pi/p), \quad \check{c}_p := \psi_1(\pi) \quad \text{and} \quad c_p := \psi_1(2\pi). \quad (2.11)$$

Here, by (2.8), $\check{c}_p$ is the same as in the non-degeneracy condition (1.12).

**Lemma 2.5** There hold

$$\hat{c}_p = p(-1)^{p+1}\check{c}_p \quad \text{and} \quad c_p = 2(-1)^p\check{c}_p = -2\check{c}_p. \quad (2.12)$$

Moreover, the $2\pi$-periodic Poincaré matrix of Eq. (2.7) is

$$P = \begin{pmatrix} 1 & 0 \\ c_p & 1 \end{pmatrix}. \quad (2.13)$$

In particular, under assumption (1.12), Eq. (2.7) is unstable-parabolic and admits only one linearly independent $2\pi$-periodic solution, say $y = \psi_2(t)$ or $y = \phi(t)$.

**Proof** Note that $\phi(t)$ satisfies $(\phi(0), \dot{\phi}(0)) = (\xi_p, 0)$ and by Eq. (1.2) itself, $\dot{\phi}(0) = -f(\xi_p)$. Due to (1.10), one has $\dot{\phi}(\pi/p) = -\xi_p$, $\ddot{\phi}(\pi/p) = 0$ and $\ddot{\phi}(\pi/p) = -f(-\xi_p) = f(\xi_p)$. By (2.9), we have $\dot{\psi}_2(\pi/p) = 0$ and $\dot{\psi}_2(\pi/p) = -1$. Moreover, it follows from (2.2) that $\psi_1(\pi/p) = -1$. Hence

$$\hat{P} = \begin{pmatrix} -1 & 0 \\ \hat{c}_p & -1 \end{pmatrix},$$

where $\hat{c}_p$ is as in (2.11). Now (2.10) shows that

$$\hat{P} = \begin{pmatrix} -1 & 0 \\ \hat{c}_p & -1 \end{pmatrix}^p = \begin{pmatrix} (-1)^p & 0 \\ p(-1)^{p+1}\hat{c}_p & (-1)^p \end{pmatrix}.$$  

This gives the first relation of (2.12). By using (2.10) again,

$$P = \begin{pmatrix} (-1)^p & 0 \\ \hat{c}_p & (-1)^p \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2(-1)^p\hat{c}_p & 1 \end{pmatrix}.$$  

Hence we have the second relation of (2.12) and result (2.13) is obtained. 

Next we give a relation between non-degeneracy condition (1.12) and the period function $T(\xi)$.

**Lemma 2.6** Suppose that the period function $T(\xi)$ is differentiable at $\xi = \xi_p$. Then

$$\hat{c}_p = -f(\xi_p)T'(\xi_p)/2 \quad \text{and} \quad \check{c}_p = p(-1)^p f(\xi_p)T'(\xi_p)/2. \quad (2.14)$$
Proof By using the solutions $X(t, \xi)$ and the periodic function $T(\xi)$, one has
\[
\frac{\partial X}{\partial t} \bigg|_{(T(\xi)/2, \xi)} \equiv 0.
\]

Differentiating with respect to $\xi$, we have
\[
0 = \frac{d}{d\xi} \left( \frac{\partial X}{\partial t} \bigg|_{(T(\xi)/2, \xi)} \right) = \frac{\partial^2 X}{\partial t^2} \bigg|_{(T(\xi)/2, \xi)} + \frac{\partial^2 X}{\partial \xi \partial t} \bigg|_{(T(\xi)/2, \xi)} = -f \left( X(T(\xi)/2, \xi) \right) \frac{T'(\xi)}{2} + \frac{\partial^2 X}{\partial \xi \partial t} \bigg|_{(T(\xi)/2, \xi)}.
\]

(2.15)

At $\xi = \xi_p$, one has $T(\xi_p)/2 = \pi/p$, $X(T(\xi_p)/2, \xi_p) = X(\pi/p, \xi_p) = -\xi_p$. Moreover, by (2.8),
\[
\frac{\partial^2 X}{\partial t^2} \bigg|_{(\pi/p, \xi_p)} = \frac{\partial^2 X}{\partial \xi \partial t} \bigg|_{(\pi/p, \xi_p)} = \psi_1(\pi/p) = \psi_p.
\]

Now the first equality of (2.14) follows from (2.15), while the second can be deduced from relation (2.12).

Remark 2.7 Due to relation (2.12), the non-degeneracy (1.12) for $\phi(t)$ is equivalent to the linearized instability of $\phi(t)$. On the other hand, by assuming that the period function $T(\xi)$ has a non-zero derivative at $\xi_p$, relation (2.14) implies that the non-degeneracy (1.12) for $\phi(t)$ is satisfied, and, moreover, $T(\xi)$ will be non-constant near $\xi = \xi_p$ and the solution $\phi(t)$ is also necessarily Lyapunov unstable.

3 A Criterion for Linearized Stability/Instability

Let us consider the family $\phi(t, e) = \phi_p(t, e)$ of nonconstant even $2\pi$-periodic solutions of Eq. (1.3) bifurcated from $\phi(t) = \phi_p(t)$ of Eq. (1.2). We will give a criterion for the linearized stability/instability of $\phi(t, e)$ for $0 < e \ll 1$.

For $e \in [0, e^*_p)$, the linearization equation of Eq. (1.3) along $x = \phi(t, e)$ is the Hill’s equation (1.14). The fundamental solutions of Eq. (1.14) are denoted by $\psi_i(t, e)$, which yield the corresponding trace
\[
\tau(e) = \tau_p(e) := \psi_1(2\pi, e) + \psi_2(2\pi, e).
\]

By (2.13), one has $\tau(0) = 2$. Due to the smoothness of $F(x, t, e)$, all objects above are smooth in $e$.

Theorem 3.1 Consider the $2\pi$-periodic solutions $\phi(t)$ and $\phi(t, e)$. Let $F_{23}(t)$ be
\[
F_{23}(t) := \frac{\partial^2 F}{\partial t^2}(\phi(t), t, 0).
\]

Then
\[
\tau'(0) = \tau'_p(0) := \frac{d\tau_p}{de} \bigg|_{e=0} = \frac{\dot{\psi}_1(2\pi)}{f(\xi_p)} \langle F_{23}, \psi_2 \rangle = -\frac{c_p}{f(\xi_p)} \langle F_{23}, \dot{\phi} \rangle.
\]

(3.1)

Here $\psi_i(t)$’s are fundamental solutions of Eq. (2.7) and
\[
\langle u, v \rangle := \int_0^{2\pi} u(t)v(t) \, dt.
\]

(3.2)
Proof With the potential $q = Q$, one has from (2.13) and (2.5)

$$N = \begin{pmatrix} 0 & 0 \\ 0 & c_p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \psi_1(2\pi) \end{pmatrix}, \quad K(t) = c_p\psi_2^2(t) = \psi_1(2\pi)\psi_2^2(t).$$

By considering $e$ as a parameter, potentials $Q(\cdot,e)$ of (1.14) can be considered as a smooth curve in the space $L^1(\mathbb{R}/2\pi\mathbb{Z})$. Hence we obtain from (2.6)

$$\tau'(0) = c_p\int_0^{2\pi} \psi_2^2(t)h(t)\,dt. \quad (3.3)$$

Here, following from (1.14),

$$h(t) = \left. \frac{\partial Q}{\partial e} \right|_{(t,0)} = \left. \frac{\partial^2 F}{\partial x^2} \right|_{(\phi(t),t,0)} \varphi(t) + F_{13}(t) = f''(\phi(t))\varphi(t) + F_{13}(t), \quad (3.4)$$

where

$$\varphi(t) := \left. \frac{\partial \phi}{\partial e} \right|_{(t,0)} \quad \text{and} \quad F_{13}(t) := \left. \frac{\partial^2 F}{\partial e \partial x} \right|_{(\phi(t),t,0)}. \quad (3.5)$$

The function $\varphi(t)$ can be determined as follows. On one hand, as $\phi(t,e)$ is $2\pi$-periodic in $t$, so is $\varphi(t)$. On the other hand, $\phi(t,e) = X(t,\Xi(e),e)$ is the solutions of initial value problems of Eq. (1.3). By considering $(\xi,e)$ as parameters for problem (1.3)-(1.6), $\varphi(t)$ satisfies the variational equation of Eq. (1.3) at $(\xi,e) = (\xi_p,0)$

$$\ddot{\varphi} + Q(t)\varphi + F_3(t) = 0, \quad (3.6)$$

where $Q(t)$ is as in (2.7) and

$$F_3(t) := \left. \frac{\partial F}{\partial e} \right|_{(\phi(t),t,0)}. \quad (3.7)$$

To obtain $\tau'(0)$, let us first compute

$$\tau := \int_0^{2\pi} \varphi(t)\psi_2^2(t)f''(\phi(t))\,dt \quad (3.8)$$

$$= \frac{1}{\phi(0)} \int_0^{2\pi} \varphi(t)\psi_2(t) \cdot f''(\phi(t))\dot{\phi}(t)\,dt \quad \text{(by (2.9))}$$

$$= \frac{1}{\phi(0)} \int_0^{2\pi} \varphi(t)\psi_2(t)\,dQ(t) \quad \text{(as } f''(\phi(t))\dot{\phi}(t)\,dt = dF'(\phi(t)) = dQ(t))$$

$$= \frac{1}{\phi(0)} \int_0^{2\pi} \left( -Q(t)\psi_2(t)\ddot{\varphi} + Q(t)\varphi(t)\dot{\psi}_2(t) \right)\,dt,$$

by integrating by parts and noticing the condition $\psi_2(0) = \psi_2(2\pi) = 0$. Using Eq. (2.7) for $\psi_1$ and Eq. (3.6) for $\varphi$, one has

$$-Q\psi_2\ddot{\varphi} - Q\dot{\varphi}\dot{\psi}_2 = \dot{\psi}_2\ddot{\varphi} + (\ddot{\varphi} + F_3)\dot{\psi}_2 = (\psi_2\dddot{\varphi}) + F_3\dot{\psi}_2.$$

As both $\psi_2$ and $\varphi$ are $2\pi$-periodic, we obtain

$$\tau = \frac{1}{\phi(0)} \int_0^{2\pi} F_3(t)\dot{\psi}_2(t)\,dt = \frac{1}{\phi(0)} \left( F_3, \psi_2 \right).$$
It can be simplified further. Note from (3.7) that $F_3(t)$ and $\psi_2(t)$ are $2\pi$-periodic. Moreover,

$$\dot{F}_3(t) = \frac{d}{dt} \left( \frac{\partial F}{\partial e}_{(\phi(t), t, 0)} \right) = \frac{\partial^2 F}{\partial x \partial e}_{(\phi(t), t, 0)} \frac{\partial F}{\partial e}_{(\phi(t), t, 0)} + \frac{\partial^2 F}{\partial t \partial e}_{(\phi(t), t, 0)}$$

which is $\phi(0) F_{13}(t) \psi_2(t) + F_{23}(t)$.

where $F_{23}(t)$ and $F_{13}(t)$ are as in (3.1) and (3.5). Hence, integrating by parts, we obtain

$$\dot{\tau} = \frac{1}{\phi(0)} \langle F_3, \psi_2 \rangle = -\frac{1}{\phi(0)} \langle \dot{F}_3, \psi_2 \rangle = -\frac{1}{\phi(0)} \langle \dot{\phi}(0) F_{13} \psi_2 + F_{23}, \psi_2 \rangle$$

$$= -\langle F_{13}, \psi_2 \rangle - \frac{1}{\phi(0)} \langle F_{23}, \psi_2 \rangle.$$

(3.9)

It follows from (3.3), (3.4) and (3.8), (3.9) that

$$\tau'(0) = c_p \left( \dot{\tau} + \langle F_{13}, \psi_2 \rangle \right) = -\frac{c_p}{\phi(0)} \langle F_{23}, \psi_2 \rangle = \frac{c_p}{f(\xi_p)} \langle F_{23}, \psi_2 \rangle.$$

This is the desired formula (3.2).

As a corollary of Theorem 3.1, we have the following criterion on linearized stability/instability of the bifurcated periodic solutions $\phi(t, e)$.

**Theorem 3.2** If $\tau'(0) < 0$, then $\phi(t, e)$ is elliptic and is linearized stable for $0 < e \ll 1$, and, if $\tau'(0) > 0$, then $\phi(t, e)$ is hyperbolic and is Lyapunov unstable for $0 < e \ll 1$.

We end this section with two remarks on formula (3.2).

Formula (3.2) is only involved of the $2\pi$-periodic solution $\phi(t)$ of the autonomous equation (1.2) and the fundamental solutions $\psi_i(t)$ of the linearization equation (2.7) for $\phi(t)$.

Formula (3.2) is reasonable. From (3.1), if $F(x, t, e)$ satisfies

$$\frac{\partial^2 F}{\partial t \partial e}_{(x, t, e)} = 0,$$

we have $F(x, t, e) = \dot{F}(x, e) + \hat{F}(x, t)$. As $F(x, t, 0) = f(x)$, we have $\dot{F}(x, 0) + \hat{F}(x, t) = f(x)$. Therefore

$$F(x, t, e) \equiv f(x) + \hat{F}(x, e) - \hat{F}(x, 0)$$

is independent of time $t$. As in Remark 2.4, one has $\tau(e) \equiv 2$ and $\tau'(0) = 0$. This is consistent with formula (3.2) because $F_{23}(t) \equiv 0$ in this case.

4 Examples and Conclusions

Consider the following Lagrangian equation

$$\ddot{x} + f(x) = e h(t) g(x),$$

(4.1)

where $f(x)$ is as in (1.2), $g(x)$ is a smooth odd function, and $h(t)$ is a smooth, even, $2\pi$-periodic function. System (4.1) fulfills all requirements (1.4)–(1.5). Here the parameter $e$ can be considered as in $\mathbb{R}$. 

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For later use, we introduce some notation for \( h(t) \). As \( h(t) \) is even and \( 2\pi \)-periodic, it can be expanded as the Fourier series

\[
h(t) = \sum_{n=0}^{\infty} a_n \cos(nt).
\]

Then \( \bar{h} := a_0 \) is called the constant part of \( h(t) \), while

\[
\tilde{h}(t) := h(t) - \bar{h} = \sum_{n=1}^{\infty} a_n \cos(nt)
\]

is called the oscillatory part of \( h(t) \). For any \( p \in \mathbb{N} \), let us define

\[
\tilde{h}_p(t) := \sum_{n=1}^{\infty} a_{2pn} \cos(2pnt), \tag{4.2}
\]

called the \( \pi/p \)-periodic oscillatory part of \( h(t) \). For the function \( \cos(nt) \), \( n \in \mathbb{N} \), one has

\[
\frac{1}{2p} \sum_{k=0}^{2p-1} \cos(n(t + k\pi/p)) = \frac{1}{2p} \Re \left( \sum_{k=0}^{2p-1} e^{in\pi/p} \right) = \begin{cases} 
\cos(nt) & \text{if } n/p \text{ is even,} \\
0 & \text{else.}
\end{cases}
\]

Thus the function \( \tilde{h}_p(t) \) of (4.2) can also be characterized as

\[
\tilde{h}_p(t) \equiv \frac{1}{2p} \sum_{k=0}^{2p-1} \tilde{h}(t + k\pi/p). \tag{4.3}
\]

Let \( p \in \mathbb{N} \) and \( \phi_p(t) \) be an even \( 2\pi \)-periodic solution of Eq. (1.2) with \( \phi_p(0) = \xi_p > 0 \). For Eq. (4.1), one has \( F_{23}(t) = -g(\phi_p(t))\bar{h}(t) \). Thus (3.2) is

\[
\tau'_p(0) = \frac{c_p}{(f(\xi_p))^2} \int_0^{2\pi} g(\phi_p) \dot{\phi}_p \dot{\bar{h}} \, dt \\
= \frac{c_p}{(f(\xi_p))^2} \int_0^{2\pi} g(\phi_p) \dot{\phi}_p \dot{\bar{h}} \, dt \\
= -\frac{c_p}{(f(\xi_p))^2} \int_0^{2\pi} \left( g'(\phi_p) \ddot{\phi}_p^2 + g(\phi_p) \dddot{\phi}_p \right) \dot{\bar{h}} \, dt \\
= -\frac{c_p}{(f(\xi_p))^2} \int_0^{2\pi} \Phi_p(t) \dot{\bar{h}}(t) \, dt, \tag{4.4}
\]

where, by using equations (1.2) and (1.7) for \( \phi_p \),

\[
\Phi_p = 2E(\xi_p)g'(\phi_p) - 2E(\phi_p)g'(\phi_p) - f(\phi_p)g(\phi_p). \tag{4.5}
\]

**Lemma 4.1** By using \( \Phi_p \) and \( \bar{h}_p \), we can rewrite \( \tau'_p(0) \) as

\[
\tau'_p(0) = \frac{4p^2 c_p}{(f(\xi_p))^2} \int_0^{\pi/p} \Phi_p(t) \bar{h}_p(t) \, dt, \tag{4.6}
\]

\[
\tau'_p(0) = \frac{8p^2 c_p}{(f(\xi_p))^2} \int_0^{\pi/(2p)} \Phi_p(t) \bar{h}_p(t) \, dt. \tag{4.7}
\]
Proof Let us notice from (1.5) that the function \( \Phi_p(t) \) of (4.5) is actually \( \pi/p \)-periodic. Thus

\[
\int_0^{2\pi} \Phi_p(t)q(t) \, dt = \sum_{k=0}^{2p-1} \int_{k\pi/p}^{(k+1)\pi/p} \Phi_p(t)q(t) \, dt
\]

\[
= \sum_{k=0}^{2p-1} \int_0^{\pi/p} \Phi_p(t)q(t) \, dt = \int_0^{\pi/p} \Phi_p(t)q(t) \, dt.
\]

See (4.3). Now (4.6) can be obtained by exploiting (2.12) and (4.4).

Note that \( \Phi_p(t) \) is even and \( \pi/p \)-periodic. Moreover, (4.2) shows that \( \tilde{h}p(t) \) is also even and \( \pi/p \)-periodic. Hence

\[
\int_0^{\pi/p} \Phi_p(t)\tilde{h}p(t) \, dt = \int_{-\pi/(2p)}^{\pi/(2p)} \Phi_p(t)\tilde{h}p(t) \, dt
\]

\[
= 2 \int_0^{\pi/(2p)} \Phi_p(t)\tilde{h}p(t) \, dt.
\]

Thus one has (4.7).

Formula (4.6) shows that the constant part and the non-\( \pi/p \)-periodic oscillatory parts of \( h(t) \) have no effect on \( \tau'_p(0) \), i.e., the temporal frequencies of the perturbation play an important role in the linearized stability problem.

4.1 A spatial-linear perturbation

In the following, we consider Eq. (4.1) by choosing

\[
f(x) = x^3, \quad g(x) \equiv x \quad \text{and} \quad h(t) = a_0 + \cos(2p_0t),
\]

where \( a_0 \in \mathbb{R} \) and \( p_0 \in \mathbb{N} \), i.e., the equation

\[
\ddot{x} + x^3 = e(a_0 + \cos(2p_0t))x.
\]

Here the temporal frequency of the perturbation is \( 2p_0 \).

For system (4.9), one has \( E(x) = x^4/4 \) and, by (1.8), (4.2) and (4.5),

\[
T(\xi) = \frac{\sqrt{2}B(1/4, 1/2)}{\xi} \quad \forall \xi > 0,
\]

\[
\Phi_p(t) = (\xi_p^4 - 3\phi_p^4(t))/2,
\]

\[
\tilde{h}(t) = \cos(2p_0t),
\]

\[
\tilde{p}(t) = \begin{cases} 
0 & \text{if } p \not| p_0, \\
\cos(2p_0t) & \text{if } p|p_0.
\end{cases}
\]

Hence we conclude from (4.6) that

\[
\tau'_p(0) = 0 \quad \text{if } p \not| p_0,
\]

(4.11)
and
\[ \tau'_p(0) = \frac{2p^2 \xi_p}{\xi_p^2} \int_0^{\pi/p} (\xi_p^4 - 3\phi_p^4(t)) \cos(2p_0 t) \, dt = \frac{6p^2 \xi_p}{\xi_p^2} C_{p,p_0} \quad \text{if } p | p_0. \] (4.12)

Here
\[ C_{p,p_0} := -\int_0^{\pi/p} (\phi_p(t)/\xi_p)^4 \cos(2p_0 t) \, dt. \]

Note that (2.15) implies that \( \hat{c}_p > 0 \) and (4.10) can yield
\[ \xi_p = \frac{pB(1/4,1/2)}{\sqrt{2\pi}}. \]

Again, (4.11) and (4.12) show that the temporal frequencies of the perturbation have important effect on \( \tau'_p(0) \).

Numerically, with the choice of \( p_0 = 1, 2, 3, 4 \), one has \( C_{p,p_0} = 0 \), except the following cases

\begin{align*}
C_{1,1} &= -0.7483, \\
C_{1,2} &= -0.2597, \quad C_{2,2} = -0.3742, \\
C_{1,3} &= -0.0405, \quad C_{2,3} = -0.2494, \\
C_{1,4} &= -0.0077, \quad C_{2,4} = -0.1300, \quad C_{4,4} = -0.1871
\end{align*}

In conclusion, for these choices of \((p,p_0)\), the bifurcated families of nonconstant periodic solutions of Eq. (4.9) are elliptic and linearized stable (hyperbolic and Lyapunov unstable, respectively) for \( 0 < e \ll 1 \) (for \( -1 \ll e < 0 \), respectively). For other cases, the analysis for linearized stability/instability appeals for further discussion on \( \tau''_p(0) \), \( \tau'''_p(0) \) etc.

### 4.2 The swing

Let us consider nonconstant periodic solutions of the swing (1.1). Suppose that the variable length is
\[ l(t) := a^2 + eh(t), \]
where \( a > 0 \) and \( h(t) \) is even \( 2\pi \)-periodic. Hence (1.1) is
\[ \ddot{x} + (a^2 + eh(t)) \sin x = 0, \] (4.13)
which is Eq. (4.1) with
\[ f(x) = a^2 \sin x, \quad g(x) = -\sin x. \]

When \( e = 0 \), (4.13) is the pendulum
\[ \ddot{x} + a^2 \sin x = 0. \] (4.14)

In order that (4.14) has nonconstant even \( 2\pi \)-periodic solution \( \phi_p(t) \), it is necessary and sufficient that \( a > p \). For (4.14), one has \( E(x) \equiv a^2(1 - \cos x) \). By (4.5),
\[ \Phi_p = a^2 \left(1 + 2 \cos \xi_p \cos \phi_p - 3 \cos^2 \phi_p \right). \]
Now (4.4) is
\[
\tau_p'(0) = - \frac{c_p}{a^2 \sin^2 \xi_p} \int_0^{2\pi} \left( 1 + 2 \cos \xi_p \cos \phi_p(t) - 3 \cos^2 \phi_p(t) \right) \tilde{h}(t) \, dt
\]
\[
= - \frac{c_p}{a^2 \sin^2 \xi_p} \int_0^{2\pi} \left( 2 \cos \xi_p \cos \phi_p(t) - 3 \cos^2 \phi_p(t) \right) \tilde{h}(t) \, dt
\]
\[
= \frac{4p^2 (-\tilde{c}_p)}{a^2 \sin^2 \xi_p} \int_0^{\pi/p} (3 \cos^2 \phi_p(t) - 2 \cos \xi_p \cos \phi_p(t)) \tilde{h}_p(t) \, dt,
\] (4.15)

as did in (4.6). By relation (2.14), we know that \(-\tilde{c}_p > 0\) in (4.15).

With the choice \(h(t) = \cos(2p_0 t)\), \(p_0 \in \mathbb{N}\), we have from (4.15) that \(\tau_p'(0) = 0\) if \(p \not| p_0\), and

\[
\begin{cases}
\tau_p'(0) = \frac{4p^2 (-\tilde{c}_p)}{a^2 \sin^2 \xi_p} C_{p,p_0} \\
C_{p,p_0} := \int_0^{\pi/p} (3 \cos^2 \phi_p(t) - 2 \cos \xi_p \cos \phi_p(t)) \cos(2p_0 t) \, dt
\end{cases}
\] (4.16)

if \(p|p_0\). Here one has \(-\tilde{c}_p > 0\). For example, let \(a = 5/2\). Hence \(p = 1, 2\). With the choice of \(p_0 = 1, 2\), the constants \(C_{p,p_0}\) in (4.16) are

\[
C_{1,1} = -0.7946, \quad C_{2,1} = 0, \quad C_{1,2} = +1.6712, \quad C_{2,2} = -1.2200
\]

The linearized stability/instability for nonconstant periodic solutions of the swing (4.13) depends on \((p,p_0)\).

### 4.3 The Sitnikov problems

The circular and the elliptic Sitnikov problems are respectively described by the second-order autonomous and non-autonomous Lagrangian equations

\[
\ddot{x} + f(x) = 0,
\] (4.17)
\[
\ddot{x} + F(x,t,e) = 0,
\] (4.18)

where \(e \in [0, 1)\) is the eccentricity and, with \(r_0 := 1/2\),

\[
f(x) = \frac{x}{(x^2 + r_0^2)^{3/2}},
\]
\[
F(x,t,e) = \frac{x}{(x^2 + r^2(t,e))^3},
\]
\[
r(t,e) = r_0(1 - e \cos u(t,e)),
\]

where, after some translation of time, \(u = u(t,e)\) is the solution of the Kepler’s equation

\[
u - e \sin u = t.
\]

Systems (4.17) and (4.18) fulfill the requirements of this paper. There are extensive studies on this simplest three-body problem. A rigorous mathematical study on the existence and continuation of nonconstant periodic solutions of (4.18) can be found in [9, 20, 21]. In recent years, the stability problem has been also studied as in [6, 13].
Since we are only interested in even periodic solutions, let us describe some existence results in [9]. For Eq. (4.17), the period function \( T(\xi) \) is strictly increasing in \( \xi \in (0, +\infty) \) and
\[
T_{\min} = \lim_{\xi \to 0^+} T(\xi) = 2\pi/\sqrt{8} \quad \text{and} \quad T_{\max} = \lim_{\xi \to +\infty} T(\xi) = +\infty. \tag{4.19}
\]
Given \( N \in \mathbb{N} \), the circular Sitnikov problem (4.17) has nonconstant \( 2N\pi \)-periodic solutions \( \phi_{p,N}(t) \) of the minimal period \( 2N\pi/p \) if and only if
\[
p \in \{1, 2, \cdots, \nu_N\}, \quad \text{where} \quad \nu_N := [\sqrt{8}N] \in \mathbb{N}. \tag{4.20}
\]
See (4.19). As before, we can take \( \phi_{p,N}(t) \) so that it is also even. Bifurcated from these solutions, one has the following results on the elliptic Sitnikov problem.

**Lemma 4.2** ([9, Theorem 1]) Let \( N \in \mathbb{N} \) and \( p \) be as in (4.20). Then there exists some \( e^{*}_{p,N} \in (0,1] \) such that for any \( e \in [0,e^{*}_{p,N}) \), Eq. (4.18) has an even \( 2N\pi/p \)-periodic solution \( \phi_{p,N}(t,e) = X(t,\xi_{p,N}(e),e) \).

Now we apply Theorem 3.1 to these \( 2\pi \)-periodic solutions \( \phi_{p,1}(t), p = 1, 2 \). From the defining equalities for \( f(x) \) and \( F(x,t,e) \), one has
\[
\tau_p'(0) = \frac{-r_0^2 c_p}{(f(\xi_p))^2} \int_0^{2\pi} \frac{-3\phi_p(t)\phi'_p(t)}{(\phi_p^2(t) + r_0^2)^{5/2}} \sin t \, dt. \tag{4.21}
\]
This shows that the temporal frequency of the elliptic Sitnikov problem is 1. For \( p = 1, 2 \), we have from (3.2) and (4.21)
\[
\tau_p'(0) = \frac{-r_0^2 c_p}{(f(\xi_p))^2} \int_0^{2\pi} \frac{-3\phi_p(t)\phi'_p(t)}{(\phi_p^2(t) + r_0^2)^{5/2}} \sin t \, dt. \tag{4.22}
\]
By introducing the \( \pi/p \)-periodic function
\[
\Phi_p(t) := 1/(\phi_p^2(t) + r_0^2)^{3/2},
\]
once has
\[
\Phi_p(t) = -3\phi_p(t)\phi'_p(t)/(\phi_p^2(t) + r_0^2)^{5/2}.
\]
By integrating (4.22) by parts, we obtain
\[
\tau_p'(0) = \frac{-r_0^2 c_p}{(f(\xi_p))^2} \int_0^{2\pi} \Phi_p(t) \cos t \, dt.
\]
By taking \( \tilde{h}(t) := \cos t \) and arguing as in (4.8), we have
\[
\tau_p'(0) = \frac{2pr_0^2 c_p}{(f(\xi_p))^2} \int_0^{\pi/p} \Phi_p(t) \tilde{h}_p(t) \, dt = 0, \quad p = 1, 2, \tag{4.23}
\]
because \( \tilde{h}_p(t) = (\cos t)_p \equiv 0 \), following simply from the defining equality (4.2).

Result (4.23) shows that, in order to detect the linearized stability/instability of \( 2\pi \)-periodic solutions \( \phi_{p,1}(t) \) of Eq. (4.18), one needs to find higher-order derivatives \( \tau_p''(0) \) etc.
4.4 Conclusions

Let us go back to system (1.3). Note that \( \frac{\partial^2 F}{\partial t \partial e} \) is odd in \( t \) and can be expanded as the Fourier expansion

\[
\frac{\partial^2 F}{\partial t \partial e}(x,t,0) = \sum_{n=1}^{\infty} f_n(x) \sin(nt),
\]
where \( f_n(x) \)'s are odd in \( x \). By defining

\[
E_n(x) := \int_0^x f_n(u) \, du,
\]
which is even in \( x \), one has from (3.2)

\[
\tau_p'(0) = -\frac{c_p}{(f(\xi_p))^2} \int_0^{2\pi} \phi_p(t) \left( \sum_{n=1}^{\infty} f_n(\phi_p(t)) \sin(nt) \right) \, dt
\]
\[
= \frac{c_p}{(f(\xi_p))^2} \sum_{n=1}^{\infty} n \int_0^{2\pi} E_n(\phi_p(t)) \cos(nt) \, dt.
\]

Note that \( E_n(\phi_p(t)) \) is \( \pi/p \)-periodic in \( t \). As did in (4.6), one has

\[
\tau_p'(0) = -\frac{8p^3c_p}{(f(\xi_p))^2} \sum_{n=1}^{\infty} n \int_0^{\pi/p} E_{2pn}(\phi_p(t)) \cos(2pnt) \, dt.
\]  \( (4.25) \)

Now (4.25) shows that only the terms of frequencies \( 2pn \), \( n \in \mathbb{N} \), in (4.24), have effect on \( \tau_p'(0) \). However, the effects of terms of other frequencies on higher-order derivatives \( \tau_p^{(k)}(0) \) remain open.

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