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Higher order non-resonance for differential equations with singularities

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SUMMARY

In this paper we prove an existence result of positive periodic solutions to second order differential equations with certain strong repulsive singularities near the origin and with some semilinear growth near infinity. Different from the nonsingular case, the result in this paper shows that both of the periodic and the antiperiodic eigenvalues play the same role in such an existence result. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: singular equation; repulsive singularity; positive periodic solution; eigenvalue

1. INTRODUCTION

In this paper we are concerned with the existence of (strictly) positive T-periodic solutions of the equation

$$x'' + f(t,x) = 0 \tag{1}$$

where $f:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$, $\mathbb{R}_+=(0,\infty)$, is continuous and T-periodic in the first variable, and $f(t,x)$ exhibits a repulsive singularity near the origin $x=0$, i.e., $\lim_{x\to 0+} f(t,x) = -\infty$. We also assume that $f(t,x)$ grows semilinearly at $x=\infty$ in the sense that there exist positive T-periodic continuous functions ϕ, Φ such that

$$\phi(t) \leqslant \liminf_{x\to+\infty} \frac{f(t,x)}{x} \leqslant \limsup_{x\to+\infty} \frac{f(t,x)}{x} \leqslant \Phi(t) \tag{S_∞}$$

uniformly in t.

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Equation (1) describes the motion of some interesting problems in applications such as the Brillouin focusing system [1–6], where the equation is a singular perturbation of the Mathieu equation

$$x'' + a(1 + \varepsilon \cos t)x = b/x, \quad a, b > 0, \ \varepsilon \in [-1, 1]$$

Another example is the parametric resonances of certain nonlinear Schrödinger systems. This is also related with the singular equation (1). See Reference [7].

A special case of (1) is the so-called Ermakov–Pinney equation [8]:

$$r'' + a(t)r = K/r^3 \tag{2}$$

where $a(t)$ is T-periodic and K is a positive constant. Note that (2) is related to the Hill equation

$$x'' + a(t)x = 0 \tag{3}$$

in the following way: If one tries to find the (complex) solutions $x(t) = r(t)\exp(i\psi(t))$ of (3), then $r(t)$ is a solution of the Ermakov–Pinney equation (2). In fact, there is a one-to-one correspondence between the solutions of (3) and that of (2). In this connection, let us recall a result of Ortega [9]. Suppose that the Hill equation (3) is elliptic, which means that the Floquet multipliers of (3) have absolute values 1 and are not reals. Then the Poincaré matrix P of the equivalent planar system of (3) is necessarily *conjugate to a rotation* (different from $\pm I$). By Reference [9, Proposition 7], if (3) is elliptic, there must be some transformation of the time t: $\tau = (t - t_0)/\alpha$ where $t_0 \in \mathbb{R}$ and $\alpha > 0$, such that the transformed equation of (3) is R-elliptic, which means that the corresponding Poincaré matrix is simply a *rotation*. From this fact one can easily derive that the Ermakov–Pinney equation (2) of (3) has necessarily a positive T-periodic solution provided that (3) is elliptic.

The main purpose of this paper is to prove that such an existence result of positive T-periodic solutions to (2) also holds for general equations (1) if the repulsive singularity $x = 0$ of (1) is strong enough, e.g. as in Reference [10], there exist positive constants c_1, c_2, δ, v such that $v \geqslant 1$ and

$$c_1/x^v < -f(t,x) < c_2/x^v \quad \text{for all } t \text{ and all } 0 < x \leqslant \delta \tag{S_0}$$

See Theorem 2. Of course, the ellipticity of (1) reads now as that (3) is elliptic for every measurable T-periodic function $a(t)$ satisfying

$$\phi(t) \leqslant a(t) \leqslant \Phi(t), \quad \forall t \tag{4}$$

Such a non-resonance condition at $x = \infty$ can be simplified using the comparison result for eigenvalues. See conditions (N_k) of Theorem 2. In particular, we find that both the periodic and the antiperiodic eigenvalues play the same role in these non-resonance conditions (N_k).

We remark that some preliminary results on singular equation (1) can be found in References [6,10]. Our Theorem 2 may be viewed as an extension of the result in Reference [10] from the point of view of non-autonomous equations.

2. STATEMENT OF MAIN RESULT

First we introduce some notation on eigenvalues. Let $q(t)$ be a T-periodic potential such that $q \in L^1_{\rm loc}(\mathbb{R})$. Consider the eigenvalue problems of

$$x'' + (\lambda + q(t))x = 0 \tag{5}$$

with the periodic boundary condition (P): $x(0) - x(T) = x'(0) - x'(T) = 0$, or, with the antiperiodic boundary condition (A): $x(0)+x(T) = x'(0)+x'(T) = 0$. We use $\lambda_1^{\rm D}(q) < \lambda_2^{\rm D}(q) < \cdots < \lambda_n^{\rm D}(q) < \cdots$ to denote all eigenvalues of (5) with the Dirichlet boundary condition (D): $x(0) = x(T) = 0$.

The following are standard results for eigenvalues. See, e.g. Reference [11]. A partial generalization of these results to the one-dimensional p-Laplacian with periodic potentials is given in Reference [12].

Theorem 1
There exist two sequences $\{\underline{\lambda}_n(q):\ n \in \mathbb{N}\}$ and $\{\bar{\lambda}_n(q):\ n \in \mathbb{Z}^+\}$ of the reals such that

(E_1) They have the ordering

$$\bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leqslant \bar{\lambda}_1(q) < \underline{\lambda}_2(q) \leqslant \bar{\lambda}_2(q) < \cdots < \underline{\lambda}_n(q) \leqslant \bar{\lambda}_n(q) < \cdots$$

(E_2) λ is an eigenvalue of $(5)+(P)$ if and only if $\lambda = \underline{\lambda}_n(q)$ or $\bar{\lambda}_n(q)$ for some even integer n; and λ is an eigenvalue of $(5)+(A)$ if and only if $\lambda = \underline{\lambda}_n(q)$ or $\bar{\lambda}_n(q)$ for some odd integer n.

(E_3) $\lambda_n^{\rm D}(q)$, $\underline{\lambda}_n(q)$, and $\bar{\lambda}_n(q)$ are continuous functions of q with respect to the L^1-metric on q's: $d(q_1, q_2) = \int_0^T |q_1(t) - q_2(t)|\,{\rm d}t$.

(E_4) The eigenvalues $\underline{\lambda}_n(q)$ and $\bar{\lambda}_n(q)$ can be recovered from the Dirichlet eigenvalues in the following way: For any $n \in \mathbb{N}$,

$$\underline{\lambda}_n(q) = \min\{\lambda_n^{\rm D}(q_{t_0}):\ t_0 \in \mathbb{R}\}, \quad \bar{\lambda}_n(q) = \max\{\lambda_n^{\rm D}(q_{t_0}):\ t_0 \in \mathbb{R}\}$$

Here $q_{t_0}(t)$ denotes the translation of $q(t)$: $q_{t_0}(t) \equiv q(t + t_0)$.

(E_5) The comparison results hold for all of these eigenvalues. If $q_1 \geqslant q_2$ then

$$\underline{\lambda}_n(q_1) \leqslant \underline{\lambda}_n(q_2), \quad \bar{\lambda}_n(q_1) \leqslant \bar{\lambda}_n(q_2), \quad \lambda_n^{\rm D}(q_1) \leqslant \lambda_n^{\rm D}(q_2) \tag{6}$$

for any $n \in \mathbb{N}$. If $q_1(t) \geqslant q_2(t)$ for all t, and $q_1(t) > q_2(t)$ for t in a subset of positive measure, then all of the inequalities in (6) are strict.

Our main result of this paper is:

Theorem 2
Suppose that f satisfies the strong force condition (S_0) near 0 and the semilinearity condition (S_∞) near infinity. Suppose that ϕ and Φ in (S_∞) satisfy for some positive constant γ that

$$\phi(t) \leqslant \gamma \leqslant \Phi(t) \quad \text{for all } t \tag{G}$$

If there exists some $k \in \mathbb{N}$ such that

$$\bar{\lambda}_k(\phi) < 0, \quad \underline{\lambda}_{k+1}(\Phi) > 0 \tag{N_k}$$

then Equation (1) has at least one positive T-periodic solution.

Before giving the proof, let us clarify the meaning of (N_k).

Lemma 1
Let $a(t)$ be an measurable T-periodic function satisfying (4). The following statements are equivalent:

(i) Condition (N_k) holds for some $k \in \mathbb{N}$.
(ii) $\lambda_1^{\mathrm{D}}(\Phi_{t_0}) < 0$ for all $t_0 \in \mathbb{R}$, and $\lambda_n^{\mathrm{D}}(a_{t_0}) \neq 0$ for all $n \in \mathbb{N}$ and all $t_0 \in \mathbb{R}$.
(iii) The Hill equation (3) is elliptic.

Proof
Let us prove that (i) is equivalent to (ii). Assume that (N_k) is satisfied for some k and $a(t)$ is as in the lemma. By the comparison results in Theorem 1, we have

$$\lambda_k^{\mathrm{D}}(a_{t_0}) \leqslant \lambda_k^{\mathrm{D}}(\phi_{t_0}) \leqslant \bar{\lambda}_k(\phi) < 0, \quad \lambda_{k+1}^{\mathrm{D}}(a_{t_0}) \geqslant \lambda_{k+1}^{\mathrm{D}}(\Phi_{t_0}) \geqslant \underline{\lambda}_{k+1}(\Phi) > 0$$

Thus (ii) holds. Conversely, assume that (ii) is satisfied for all $a(t)$ and t_0 as in the lemma. In particular, we have, for any n and any t_0,

$$\lambda_n^{\mathrm{D}}(\tau\phi_{t_0} + (1-\tau)\Phi_{t_0}) \neq 0$$

for all $\tau \in [0,1]$. As a function of τ, the left-hand side in the above inequality is continuous, cf. (E_3). Thus there must be some $k \in \mathbb{N}$ satisfying for all $\tau \in [0,1]$ that

$$\lambda_k^{\mathrm{D}}(\tau\phi_{t_0} + (1-\tau)\Phi_{t_0}) < 0 \tag{7}$$

$$\lambda_{k+1}^{\mathrm{D}}(\tau\phi_{t_0} + (1-\tau)\Phi_{t_0}) > 0 \tag{8}$$

because $\lambda_n^{\mathrm{D}}(q) \to +\infty$ as $n \to \infty$. Let $\tau = 1$ in (7). We have

$$\bar{\lambda}_k(\phi) = \max_{t_0} \lambda_k^{\mathrm{D}}(\phi_{t_0}) < 0$$

Let $\tau = 0$ in (8). We have

$$\underline{\lambda}_{k+1}(\Phi) = \min_{t_0} \lambda_{k+1}^{\mathrm{D}}(\Phi_{t_0}) > 0$$

Thus (N_k) is satisfied for this k. The equivalence between (i) and (iii) can be proved similarly. □

Remark
When $f = f(t,x): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in Equation (1) is non-singular and is semilinear at ∞:

$$\phi(t) \leqslant \liminf_{|x| \to \infty} \frac{f(t,x)}{x} \leqslant \limsup_{|x| \to \infty} \frac{f(t,x)}{x} \leqslant \Phi(t)$$

the non-resonance conditions in Reference [13] are: For any T-periodic measurable $a(t)$ satisfying (4), Equation (3) has only the trivial T-periodic solution. As in Lemma 1, this fact can be rewritten as that there exists some $k \in \mathbb{Z}^+$ such that

$$\text{either, } \underline{\lambda}_{2k}(\phi) < 0 \text{ and } \bar{\lambda}_{2k}(\Phi) > 0, \quad \text{or, } \bar{\lambda}_{2k}(\phi) < 0 \text{ and } \underline{\lambda}_{2k+2}(\Phi) > 0 \tag{9}$$

Note that both non-resonance conditions (9) involve only the periodic eigenvalues.

3. PROOF OF MAIN RESULT

Let

$$G(x) = \int^x -\frac{\mathrm{d}x}{x^\nu} = \begin{cases} \dfrac{1}{(\nu-1)x^{\nu-1}} & \text{if } \nu > 1 \\ -\log x & \text{if } \nu = 1 \end{cases}$$

Note that $G(x) \to +\infty$ when $x \to 0+$.

Our main result is proved using the degree method. To this end, we give the estimates for all possible positive T-periodic solutions of (1) and its homotopy equations. In order to simplify the notation, let us concentrate on Equation (1) itself.

The following result, called the elasticity property, can be proved by analysing the change of the energy of (1) within one period. We omit its proof because it is standard.

Lemma 2

Assume that (S_0) and (S_∞) are satisfied. Then there exist positive numbers $0 < \rho_0 < 1 < \rho_\infty$, c'_1, c'_2 with the following properties. Let $x(t)$ be a positive T-periodic solution of Equation (1) and $t_0, t_1, t_0 < t_1$, be two consecutive critical points of $x(t)$.

(i) If $x(t_0) \leqslant \rho_0$, then t_0 is an isolated minimum and t_1 is an isolated maximum and

$$c'_1 G(x(t_0)) \leqslant x(t_1)^2 \leqslant c'_2 G(x(t_0))$$

(ii) If $x(t_0) \geqslant \rho_\infty$, then t_0 is an isolated maximum and t_1 is an isolated minimum and

$$c'_1 G(x(t_1)) \leqslant x(t_0)^2 \leqslant c'_2 G(x(t_1))$$

Now we give the *a priori* estimates for all possible positive T-periodic solutions of (1).

Lemma 3

Assume that $(S_0), (S_\infty)$ and (N_k) are satisfied. Then all positive T-periodic solutions of (1) are *a priori* bounded in the following sense: There exist constants $0 < \varepsilon_0 < 1 < \varepsilon_\infty$ such that for any T-periodic solution $x(t)$ of (1) it holds that

$$\varepsilon_0 < x(t) < \varepsilon_\infty \quad \text{for all } t \tag{10}$$

Proof

Suppose the assertion is not true. Then there exists a sequence $x_n \in C_T$ (the space of all continuous T-periodic functions) such that $x_n(t) > 0$,

$$x''_n + f(t, x_n) = 0 \tag{11}$$

and either

$$\min_t x_n(t) \to 0 \quad \text{as } n \to \infty \tag{12}$$

or

$$\max_t x_n(t) \to \infty \quad \text{as } n \to \infty \tag{13}$$

From Lemma 2 we know that (12) and (13) are equivalent so that both (12) and (13) hold.

Let $z_n(t) = x_n(t)/\|x_n\|_\infty$, $n = 1, 2, \ldots$, where $\|\cdot\|_\infty$ denotes the supremum norm. Then $\|z_n\|_\infty = 1$ for all n. Multiplying (11) by $z_n(t)/\|x_n\|_\infty$ and integrating by parts, we get

$$\int_0^T |z_n'(t)|^2 \, dt = \int_0^T \frac{f(t, x_n(t))}{\|x_n\|_\infty} z_n(t) \, dt \tag{14}$$

By (S_0) and (S_∞), we can fix numbers $c, d > 0$ such that

$$f(t, x) \leqslant cx + d$$

for all $x > 0$ and all t. So

$$\frac{f(t, x_n(t))}{\|x_n\|_\infty} z_n(t) \leqslant \frac{cx_n(t) + d}{\|x_n\|_\infty} z_n(t) \leqslant c + \frac{d}{\|x_n\|_\infty}$$

is bounded from above. It thus follows from (14) that the L^2 norms $\|z_n'\|_2$ are bounded. Consequently, $\{z_n\}$ is a bounded sequence in H_T^1 (the Sobolev space of T-periodic functions). Passing to a subsequence if necessary, we can assume that $z_n \to z$ weakly in H_T^1 and $z_n \to z$ strongly in C_T. Obviously, $z(t) \geqslant 0$ for all t. As $\|z_n\|_\infty \equiv 1$, $\|z\|_\infty = 1$ and $z \neq 0$.

As in Lemma 2.2 of Reference [10], it can be proved that $z(t)$ has only isolated zeros. So let us assume this. Let $\Omega = \{t \in \mathbb{R}\colon\ z(t) > 0\} = \bigcup_I I$, where I's are disjoint open intervals.

Let now $I = (\alpha, \beta)$ be such an interval from Ω. Let ω be an arbitrary closed subinterval of I. Since $\min_{t \in \omega} z(t) > 0$, $x_n(t) \to +\infty$ uniformly in $t \in \omega$. Let

$$a_n(t) = \frac{f(t, x_n(t))}{x_n(t)}, \quad t \in \omega$$

By (S_∞), for any given $\varepsilon > 0$, we have

$$\phi(t) - \varepsilon < a_n(t) < \Phi(t) + \varepsilon$$

for all n sufficiently large and all $t \in \omega$. In particular the sequence $\{a_n(t)\}$ is bounded in $L^2(\omega)$. We can assume, passing to a subsequence if necessary, that $a_n(t) \to a_\omega(t)$ weakly in $L^2(\omega)$ to some function $a_\omega(t)$. Moreover, a_ω satisfies

$$\phi(t) \leqslant a_\omega(t) \leqslant \Phi(t) \tag{15}$$

for almost all $t \in \omega$. As ω is arbitrary, let us define the function $a^I : I \to \mathbb{R}$ by

$$a^I(t) = a_\omega(t) \quad \text{whenever } t \in \omega$$

Note that $a^I(t)$ is well defined on I by the uniqueness of weak limits.

For any C^1 function v whose support contained in ω, it follows from (11) that

$$\int_\omega z_n' v' \, dt = \int_\omega a_n(t) z_n(t) v(t) \, dt$$

because $v = 0$ on $\partial\omega$. Using the convergence results $z_n \rightharpoonup z$ in H_T^1, $a_n \rightharpoonup a^I$ in $L^2(\omega)$ and $z_n \to z$ in C_T, we have

$$\int_\omega z' v' \, dt = \int_\omega a^I(t) z(t) v(t) \, dt$$

for all C^1 functions v with supports in ω. As ω is an arbitrary closed interval in I, the regularity theory shows that $z(t)$ is a classical solution of the following linear equation:

$$z'' + a^I(t)z = 0, \quad t \in I \tag{16}$$

Note that $z(t) > 0$ for $t \in I = (\alpha, \beta)$. It is easy to verify from (16) that both the limits $\lim_{t\to\alpha+} z'(t)$ and $\lim_{t\to\beta-} z'(t)$ exist and are non-zero.

As I is an arbitrary interval contained in Ω and $\mathbb{R}\backslash\Omega$ contains only countable points, we can define a measurable T-periodic function $a(t)$ by

$$a(t) = a^I(t) \quad \text{whenever } t \in I$$

By (15), $a(t)$ satisfies (4) for almost all $t \in \mathbb{R}$. Moreover, $z(t)$ satisfies for a.e. $t \in \mathbb{R}$ the following equation:

$$z'' + a(t)z = 0 \tag{17}$$

Note that $z(t)$ is only a classical solution of (17) on each interval $I \subset \Omega$ because $z(t)$ is not even C^1 on $\mathbb{R}$.

Now we use the non-resonance condition (N_k) to complete the proof of the lemma. Note that $z(t+T) \equiv z(t)$. Let $0 \leqslant t_0 < t_1 < \cdots < t_{n-1} < t_n = t_0 + T$ be zeros of $z(t)$ within one period. As all the limits $\lim_{t\to t_i\pm} z'(t)$ exist and are non-zero, one can then choose non-zero constants ξ_i so that the following function:

$$\tilde{z}(t) = \xi_i z(t), \quad t \in [t_{i-1}, t_i], \quad i = 1, 2, \ldots, n$$

is C^1 on $[t_0, t_n]$. Moreover, $\tilde{z}(t)$ satisfies (17) for a.e. $t \in [t_0, t_n]$. Generally speaking, $\tilde{z}(t)$ has different derivatives at t_0 and $t_n = t_0 + T$. Let $y(t)$ be the function

$$y(t) \equiv \tilde{z}(t + t_0), \quad t \in [0, T]$$

Then $y(t)$ is a non-zero solution of the following equation:

$$y'' + a_{t_0}(t)y = 0, \quad t \in [0, T]$$

with the Dirichlet boundary condition (D) satisfied. This means that 0 is a Dirichlet eigenvalue with the potential a_{t_0}, which contradicts the non-resonance condition (N_k) by Lemma 1. The lemma is thus proved. □

Now we give the proof of Theorem 2. This will be completed using coincidence degree theory [14]. To this end, we simply deform Equation (1) to the following simple autonomous singular equation:

$$x'' + \gamma x - x^{-\nu} = 0 \tag{18}$$

where $\gamma > 0$ is as in the condition (G). The homotopy equation is

$$x'' + f_\lambda(t, x) = 0 \tag{19}$$

where $\lambda \in [0, 1]$ and $f_\lambda(t, x) = \lambda f(t, x) + (1 - \lambda)(\gamma x - x^{-\nu})$.

Note that f_λ satisfies all conditions in Theorem 2 uniformly with respect to $\lambda \in [0, 1]$. As in Lemma 3, there exist constants $0 < \varepsilon_0 < 1 < \varepsilon_\infty$ (independent of λ) such that all possible T-periodic solutions $x(t)$ of (19) satisfy (10).

Let

$$\Omega = \{x \in C_T:\ \varepsilon_0 < x(t) < \varepsilon_\infty\}$$

which is an open subset in C_T. Now the invariance of degree shows that the problem $(1)+(P)$ has the same degree as that of $(18)+(P)$ on Ω. As (18) is an autonomous equation, the result in Reference [15] shows that the absolute value of the degree of (18) is equal to the Brouwer degree

$$|\deg(\gamma x - x^{-\nu}, (\varepsilon_0, \varepsilon_\infty), 0)|$$

which is just 1. Thus $(1)+(P)$ has at least one solution in Ω. Theorem 2 is proved. □

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