Measure Differential Equations, II. Continuity of Eigenvalues in Measures with Weak* Topology

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Abstract

In this paper we continue to study second-order linear measure differential equations (MDE), following the line of the recent work [13], where the dependence of solutions of MDE on measures with the weak* topology is studied. In this part, we consider the Dirichlet and the Neumann eigenvalues of MDE. It will be proved that these eigenvalues are continuous in measures with the weak* topology. Such a result extends recent works on eigenvalues of Sturm-Liouville operators with potentials or weights in [27]. As an application, we will give a natural, simple explanation to extremal problems of eigenvalues of Sturm-Liouville operators studied in [23, 28].

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1 Introduction

This paper is a continuation of the recent work [13] on second-order linear measure differential equations (MDE) where the dependence of solutions of MDE on measures is studied. In this paper we will develop the basic theory for eigenvalues of MDE and then give a deep result on the dependence of eigenvalues of MDE on measures.

Let $I = [0, 1]$ be the unit interval and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We use $(\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$ to denote the Banach space of continuous $\mathbb{K}$-valued functions of I with the supremum norm $\|\cdot\|_\infty$. By the Riesz representation theorem, the dual space $\mathcal{M}_0(I, \mathbb{K}) := (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)^*$ is the space of $\mathbb{K}$-valued measures of I. Given $\mu \in \mathcal{M}_0(I, \mathbb{K})$. The second-order linear MDE with the measure μ is written as

$$\mathrm{d}\overset{\bullet}{y} + y\,\mathrm{d}\mu(t) = 0, \qquad t \in I. \tag{1.1}$$

With any initial condition

$$(y(0), \overset{\bullet}{y}(0)) = (y_0, z_0) =: u_0 \in \mathbb{K}^2, \tag{1.2}$$

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Eq. (1.1) has the unique solution $y(t, u_0, \mu)$ with the generalized right-derivative or the velocity $\overset{\bullet}{y}(t, u_0, \mu)$. Both are defined on $t \in I$. See [13]. Moreover, $y(\cdot, u_0, \mu)$ is absolutely continuous, while $\overset{\bullet}{y}(\cdot, u_0, \mu) \in \mathcal{M}(I, \mathbb{K})$, the space of non-normalized $\mathbb{K}$-valued measure on I. In general, $\overset{\bullet}{y}(\cdot, u_0, \mu)$ has discontinuity.

In the space $\mathcal{M}_0(I, \mathbb{K})$ of measures, one has the topology induced by the norm $\|\cdot\|_{\mathbf{V}}$ of total variations. On the other hand, as the dual space of $(\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$, one has also in $\mathcal{M}_0(I, \mathbb{K})$ the weak* topology w^* defined using the weak* convergence in $\mathcal{M}_0(I, \mathbb{K})$. The main results of [13] can be stated as follows.

Theorem 1.1 [13, Theorem 1.2 and Theorem 1.3] (i) *Given $u_0 \in \mathbb{K}^2$ and $t \in [0, 1]$. The following functionals*

$$(\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}}) \to \mathbb{K}, \qquad \mu \to y(t, u_0, \mu), \quad \mu \to \overset{\bullet}{y}(t, u_0, \mu),$$

are continuously Fréchet differentiable.

(ii) *Given $u_0 \in \mathbb{K}^2$. The solution mapping*

$$(\mathcal{M}_0(I, \mathbb{K}), w^*) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty), \qquad \mu \to y(\cdot, u_0, \mu)$$

is continuous. Moreover, the ending velocity functional

$$(\mathcal{M}_0(I, \mathbb{K}), w^*) \to \mathbb{K}, \qquad \mu \to \overset{\bullet}{y}(1, u_0, \mu) \tag{1.3}$$

is also continuous.

Differentials of $y(t, u_0, \mu)$ and $\overset{\bullet}{y}(t, u_0, \mu)$ in μ have been obtained in [13]. Examples show that continuity result (1.3) for velocities cannot be improved as the continuity of $\overset{\bullet}{y}(t, u_0, \mu)$ for other times $t \in (0, 1)$.

In this paper, our concern is on eigenvalues of MDE and their dependence on measures. Given a real measure $\mu \in \mathcal{M}_0(I, \mathbb{R})$. We consider eigenvalue problem

$$\mathrm{d}\overset{\bullet}{y} + \lambda y \,\mathrm{d}t + y \,\mathrm{d}\mu(t) = 0, \qquad t \in I. \tag{1.4}$$

with the Dirichlet or the Neumann boundary condition. By the shooting method as in [15], it can be proved that eigenvalues of (1.4) are well-defined. Note that when the measure $\mu(t)$ is absolutely continuous (with respect to) the Lebesgue measure with the density $\dot{\mu}(t) = q(t) \in \mathcal{L}^p(I, \mathbb{R})$, $1 \le p \le \infty$, Eq. (1.1) returns to ODE

$$\ddot{y} + q(t)y = 0, \qquad t \in I, \tag{1.5}$$

while eigenvalue problem (1.4) returns to Sturm-Liouville problem of

$$\ddot{y} + (\lambda + q(t))y = 0, \qquad t \in I. \tag{1.6}$$

In [15], by expanding solutions of (1.6) as power series of potentials, it is proved that the Dirichlet eigenvalues of (1.6) with potentials $q \in \mathcal{L}^2(I, \mathbb{R})$ are continuous in q endowed with the weak topology of $\mathcal{L}^2(I, \mathbb{R})$. Recently, using some type of families of Fredholm operators, such a continuity result has been extended in [27] to $\mathcal{L}^1(I, \mathbb{R})$ potentials with the corresponding weak topology. As observed in [13, Remark 5.5], different from the weak topologies of

$\mathcal{L}^p(I,\mathbb{R})$, the global weak* convergence of measures does not imply the local weak* convergence. It seems to the authors that the approach of power series in [15] cannot be adopted to the MDE in a direct way.

After introducing the arguments of MDE using a topological lifting, we will exploit the techniques in [27] for problem (1.6) to establish the following continuity result.

Theorem 1.2 *Let $\lambda_m^\sigma(\mu)$ denote the mth Dirichlet or Neumann eigenvalues of MDE* (1.4) *with the measure μ. Then eigenvalues $\lambda_m^\sigma(\mu)$ are continuous in measures $\mu \in (\mathcal{M}_0(I,\mathbb{R}), w^*)$.*

Note that continuity results of eigenvalues in weak topologies are important in the inverse spectral problems [15, 22]. Since we are using the weak* topology for measures, the continuity result of Theorem 1.2 is strongest in some sense. In fact, Theorem 1.2 has generalized the continuity results in [12, 15, 27] for ODE. For some related results on PDE, see [1, 3, 5].

In the usual topology $\|\cdot\|_{\mathbf{V}}$ for measures, one has the following result.

Theorem 1.3 *As nonlinear functionals of measures,*

$$(\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \to \mathbb{R}, \qquad \mu \to \lambda_m^\sigma(\mu),$$

are continuously Fréchet differentiable. Moreover, Fréchet derivatives of $\lambda_m^\sigma(\mu)$ in μ will be given in (5.15).

The structure of this paper is as follows.

In Section 2, we will recall from [4] some basic facts on measures, the Lebesgue-Stieltjes integral and the Riemann-Stieltjes integral. The norm $\|\cdot\|_{\mathbf{V}}$ of total variations and the weak* topology w^* in the space $\mathcal{M}_0(I,\mathbb{K})$ will be introduced in a precise way. Moreover some important results on solutions of MDE in [13] will be mentioned.

In Section 3, we will use the shooting method to establish the basic theory for eigenvalues of problem (1.4) with the Dirichlet or Neumann boundary conditions, as did in [15] for Sturm-Liouville operators with potentials. See Theorem 3.8.

In order to study the dependence of eigenvalues on measures, in Section 4, we will introduce the argument of Eq. (1.1) and Eq. (1.4). Since velocities $\overset{\bullet}{y}(t)$ are in general not continuous in t, the usual Prüfer transformation for ODE does not work. By some topological fact on liftings of homotopies, we will introduce the time-1 argument $\Theta_\mu : \mathbb{R} \to \mathbb{R}$ for Eq. (1.1). See Definition 4.2. This idea is also used in [27]. Inherited from Theorem 1.1, it will be proved that Θ_μ is continuous in μ with the weak* topology and is continuously differentiable in μ with the norm $\|\cdot\|_{\mathbf{V}}$. See Theorems 4.4 and 4.5.

In Section 5, we will use the techniques as in [15] to develop some important estimates on arguments. See Lemma 5.2. By exploiting some ideas in [27], Theorems 1.2 and 1.3 will be proved.

In Section 6, we will present some applications of the main results on eigenvalues of MDE to extremal problems of eigenvalues of Sturm-Liouville operators with potentials in $\mathcal{L}^1$ balls, studied very recently in [23, 28]. By some topological facts on Lebesgue spaces $\mathcal{L}^p(I,\mathbb{R})$ and measure space $\mathcal{M}_0(I,\mathbb{R})$, the boundedness of eigenvalues of Sturm-Liouville operators with potentials in $\mathcal{L}^1$ balls can be explained by Theorem 1.2 in a natural way. Moreover, the extremal values in [23, 28] can also be obtained from the present continuity result of eigenvalues and the limiting techniques in [23, 28], because we will show that the weak* limits of the $\mathcal{L}^p$, $1 < p < \infty$, critical potentials are some simple Dirac measures.

The ideas of [13] and of this paper will be extended to the study of the dynamics quantities of MDE in future works.

2 Basics on Measures, Weak* Topology and MDE

For general theory of the Lebesgue-Stieltjes integral and Riemann-Stieltjes integral, see, e.g., [4]. For general theory on weak topologies and weak* topologies, we refer to [6, 11]. See also a brief statement in [13, Section 2]. In the following we briefly review some basic facts to be used in this paper.

Let $I = [0,1]$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As usual, $\mathcal{C}(I,\mathbb{K})$ is the Banach space of continuous $\mathbb{K}$-valued functions of I, endowed with the supremum norm $\|\cdot\|_\infty$. By the Riesz representation theorem, the dual space of $(\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)$ is the space $\mathcal{M}_0(I,\mathbb{K})$ of $\mathbb{K}$-valued measures of I. More precisely,

$$\mathcal{M}_0(I,\mathbb{K}) := \Big\{\mu : I \to \mathbb{K} : \mu(0+) = 0,\ \mu(t+) = \mu(t) \text{ for } t \in (0,1), \text{and } \mathbf{V}(\mu, I) < +\infty\}.$$

Here, for any $t \in [0,1)$, $\mu(t+) := \lim_{s\downarrow t} \mu(s)$ is the right-limit, and $\mathbf{V}(\mu, I)$ is the total variation

$$\mathbf{V}(\mu, I) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,\ n \in \mathbb{N} \right\}.$$

For simplicity, denote

$$\|\mu\|_{\mathbf{V}} := \mathbf{V}(\mu, I), \qquad \mu \in \mathcal{M}_0(I,\mathbb{K}).$$

Then $(\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_{\mathbf{V}})$ is a Banach space. Note that, for any subinterval $I_0 \subset I$, closed, open or semi-open, the variation $\mathbf{V}(\mu, I_0)$ of μ (over I_0) is also well-defined. One has the following result for variations

$$\lim_{t\downarrow t_0} \mathbf{V}(\mu, [t_0, t]) = \lim_{t\downarrow t_0} \mathbf{V}(\mu, (t_0, t]) = 0, \qquad t_0 \in (0,1). \tag{2.1}$$

The correspondence between $\mu \in (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})$ and $\mu^* \in (\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)^*$ is given by

$$\mu^*(f) = \int_I f(t)\, \mathrm{d}\mu(t), \qquad f \in \mathcal{C}(I,\mathbb{R}), \tag{2.2}$$

which refers to the Riemann-Stieltjes integral. More general, considering $\mu \in \mathcal{M}_0(I,\mathbb{K})$ as a measure, one has the Lebesgue-Stieltjes integrals $\int_{I_0} f\, \mathrm{d}\mu$ for appropriate class of functions. For details, see [4]. For Lebesgue-Stieltjes integrals, one has the following basic estimate

$$\left| \int_{I_0} f\, \mathrm{d}\mu \right| \le \|f\|_{\infty, I_0} \cdot \mathbf{V}(f, I_0), \qquad \|f\|_{\infty, I_0} := \sup_{t\in I_0} |f(t)|, \tag{2.3}$$

where I_0 has the form (a,b), $(a,b]$, with $0 \le a < b \le 1$, or the form $[0,b)$, $[0,b]$ with $0 < b \le 1$. Due to the jump of measures $\mu(t)$ at $t = 0$, one has

$$\int_{[0,b]} f\, \mathrm{d}\mu = -f(0)\mu(0) + \int_{(0,b]} f\, \mathrm{d}\mu, \qquad b \in (0,1]. \tag{2.4}$$

The space $\mathcal{M}_0(I,\mathbb{K})$ of measures is a Banach space with the norm $\|\cdot\|_{\mathbf{V}}$. Due to the duality relation (2.2), one has also the weak* topology indicated by w^* in the space $\mathcal{M}_0(I,\mathbb{K})$. That is, $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$ iff, for each $f \in \mathcal{C}(I,\mathbb{K})$, one has

$$\lim_{n\to\infty} \int_I f \,\mathrm{d}\mu_n = \int_I f \,\mathrm{d}\mu_0.$$

By the Banach-Alaoglu Theorem [11, pp. 229-230], a subset $V \subset \mathcal{M}_0(I,\mathbb{K})$ is sequentially relatively compact in w^* iff V is bounded in the norm $\|\cdot\|_{\mathbf{V}}$.

Given a measure $\mu \in \mathcal{M}_0(I,\mathbb{K})$. We consider the scalar second-order linear MDE (1.1) with the measure μ. Given $u_0 = (y_0, z_0) \in \mathbb{K}^2$. By a solution of the initial value problem (1.1)-(1.2), it means that $y \in \mathcal{C}(I,\mathbb{K})$ and there exists a function $\overset{\bullet}{y} : I \to \mathbb{K}$ such that $(y(t), \overset{\bullet}{y}(t))$ satisfies the following integral system

$$y(t) = y_0 + \int_{[0,t]} \overset{\bullet}{y}(s)\,\mathrm{d}s, \qquad t \in [0,1], \tag{2.5}$$

$$\overset{\bullet}{y}(t) = \begin{cases} z_0, & t = 0, \\ z_0 - \int_{[0,t]} y(s)\,\mathrm{d}\mu(s), & t \in (0,1]. \end{cases} \tag{2.6}$$

It is proved in [13] that problem (1.1)-(1.2) has the unique solution $y(t) = y(t, u_0, \mu) \in \mathcal{C}(I,\mathbb{K})$. Accordingly, one has also the unique function $\overset{\bullet}{y}(t) = \overset{\bullet}{y}(t, u_0, \mu)$, $t \in I$, associated with each solution $y(t)$. Some properties are as follows.

- $y \in \mathcal{AC}(I,\mathbb{K})$ and $\overset{\bullet}{y} \in \mathcal{M}(I,\mathbb{K})$, the spaces of absolutely continuous and non-normalized $\mathbb{K}$-valued measures of I, respectively.
- There holds

$$\int_{[t_1,t_2]} \overset{\bullet}{y}(s)\,\mathrm{d}s = \int_{(t_1,t_2]} \overset{\bullet}{y}(s)\,\mathrm{d}s = y(t_2) - y(t_1), \qquad 0 \le t_1 < t_2 \le 1.$$

- At any $t \in (0,1)$, the classical right-derivative $y'_+(t) := \lim_{s\downarrow t} \frac{y(s)-y(t)}{s-t}$ exists and $\overset{\bullet}{y}(t) = y'_+(t)$. Moreover, for all $t \in (0,1)$ except in an at most countable subset of I, the classical derivative $\dot{y}(t) := \lim_{s\to t} \frac{y(s)-y(t)}{s-t}$ exists and $\overset{\bullet}{y}(t) = \dot{y}(t)$.

Due to these, $\overset{\bullet}{y}(t)$ is called the generalized right-derivative or velocity of the solution $y(t)$. Note that $\overset{\bullet}{y}(t)$ has precise meaning for all $t \in I$.

Given $\mu \in \mathcal{M}_0(I,\mathbb{K})$. The fundamental solutions of Eq. (1.1) are

$$\varphi_1(t) = \varphi_1(t,\mu) := y(t,1,0,\mu), \qquad \varphi_2(t) = \varphi_2(t,\mu) := y(t,0,1,\mu).$$

The fundamental matrix solution of Eq. (1.1) is defined as

$$N_\mu(t) := \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \overset{\bullet}{\varphi}_1(t) & \overset{\bullet}{\varphi}_2(t) \end{pmatrix}, \qquad t \in I.$$

By the linearity of Eq. (1.1) and the uniqueness of solutions, one has

$$\begin{pmatrix} y(t,y_0,z_0) \\ \overset{\bullet}{y}(t,y_0,z_0) \end{pmatrix} \equiv N_\mu(t) \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in \mathbb{K}^2. \tag{2.7}$$

Note that $N_\mu(0) = I_2$. It is proved in [13] that the Liouville law holds for MDE, i.e.,

$$\det N_\mu(t) \equiv +1, \qquad t \in I. \tag{2.8}$$

3 General Results of Eigenvalues of MDE

3.1 Definition of eigenvalues of MDE

Given a real measure $\mu \in \mathcal{M}_0(I,\mathbb{R})$. We write $\mu_\lambda(t) := \lambda t + \mu(t) \in \mathcal{M}_0(I,\mathbb{C})$ for $\lambda \in \mathbb{C}$. Eigenvalue problem (1.4) can be written as

$$\mathrm{d}\overset{\bullet}{y} + y\,\mathrm{d}\mu_\lambda(t) = 0. \tag{3.1}$$

The Dirichlet boundary condition is

$$y(0) = y(1) = 0, \tag{3.2}$$

while the Neumann boundary condition is

$$\overset{\bullet}{y}(0) = \overset{\bullet}{y}(1) = 0. \tag{3.3}$$

Definition 3.1 Given $\mu \in \mathcal{M}_0(I,\mathbb{R})$. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of the Dirichlet problem (3.1)-(3.2), if Eq. (3.1) with such a parameter λ has non-zero solutions $y(t)$ satisfying (3.2). The corresponding solutions $y(t)$ are called eigen-functions associated with λ. The eigenvalues and eigen-functions for Neumann problems (3.1)-(3.3) are defined similarly.

The following basic result from calculus is used in [13] to obtain the Liouville law and will be used frequently in this paper. For its proof, see, for example, [21].

Lemma 3.2 (Generalized Newton-Lebniz Formula) *Let $\varphi \in \mathcal{C}([a,b],\mathbb{C})$. Suppose that the right-derivative $\varphi'_+(t)$ exists for all $t \in (a,b)$ and φ'_+ is integrable on $[a,b]$ in the sense of Riemann. Then*

$$\int_{[a,b]} \varphi'_+(t)\,\mathrm{d}t = \varphi(b) - \varphi(a).$$

In particular, if $\varphi \in \mathcal{C}([a,b],\mathbb{C})$ satisfies $\varphi'_+(t) = 0$ for all $t \in (a,b)$, then $\varphi(t) \equiv \mathrm{const.}$ on $[a,b]$.

Lemma 3.3 *Given a real measure $\mu \in \mathcal{M}_0(I,\mathbb{R})$. Then all possible eigenvalues of problem (3.1)-(3.2) and problem (3.1)-(3.3) are real. Therefore eigen-functions are also real-valued.*

Proof We first consider Dirichlet eigenvalues. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of (3.1)-(3.2) with an eigen-function $y \in \mathcal{C}(I,\mathbb{C})$. Suggested by the Rayleigh form for Sturm-Liouville operators, by introducing

$$\xi(t) := \bar{y}(t)z(t) - \int_{(0,t]} z(s)\bar{z}(s)\,\mathrm{d}s + \int_{(0,t]} y(s)\bar{y}(s)\,\mathrm{d}\mu_\lambda(s), \qquad t \in I, \tag{3.4}$$

we will prove that

$$\xi(t) \equiv 0, \qquad t \in I. \tag{3.5}$$

Here $z(t) = \overset{\bullet}{y}(t)$ and the bar denotes the complex conjugate. Note that $\xi(0) = \bar{y}(0)\,\overset{\bullet}{y}(0) = 0$ by the boundary condition (3.2).

Step 1. We assert that $\xi \in \mathcal{C}(I,\mathbb{C})$.

Case 1. Since $y \in \mathcal{C}(I, \mathbb{C})$ and $y(0) = 0$, it is easy to see that each term of (3.4) is small when $0 < t \ll 1$. Hence one has $\xi(0+) = 0 = \xi(0)$.

Case 2. $0 < t_0 < t \le 1$. Then

$$
\begin{aligned}
\xi(t) - \xi(t_0) &= \bar{y}(t)z(t) - \bar{y}(t_0)z(t_0) - \int_{(t_0,t]} z(s)\bar{z}(s)\,\mathrm{d}s + \int_{(t_0,t]} y(s)\bar{y}(s)\,\mathrm{d}\mu_\lambda(s) \\
&= \left(\bar{y}(t)(z(t) - z(t_0)) + \int_{(t_0,t]} y(s)\bar{y}(s)\,\mathrm{d}\mu_\lambda(s)\right) \\
&\quad + \left(z(t_0)(\bar{y}(t) - \bar{y}(t_0)) - \int_{(t_0,t]} z(s)\bar{z}(s)\,\mathrm{d}s\right) \\
&= \int_{(t_0,t]} y(s)(\bar{y}(s) - \bar{y}(t))\,\mathrm{d}\mu_\lambda(s) \\
&\quad + \int_{(t_0,t]} (z(t_0) - z(s))\bar{z}(s)\,\mathrm{d}s \qquad \text{(by (2.5) and (2.6))} \\
&=: \psi_1(t) + \psi_2(t).
\end{aligned}
\tag{3.6}
$$

Now we have from (2.3)

$$
|\psi_1(t)| \le \|y\|_\infty \|\mu_\lambda\|_{\mathbf{V}} \|y(\cdot) - y(t)\|_{\infty,(t_0,t]}.
$$

One has $\lim_{t\downarrow t_0} \psi_1(t) = 0$, because $\lim_{t\downarrow t_0} \|y(\cdot) - y(t)\|_{\infty,(t_0,t]} = 0$ by the continuity of $y(t)$. Moreover, as $t \downarrow t_0$, $\psi_2(t) \to 0$, following simply from the property of Lebesgue integral. Thus $\xi(t_0+) = \xi(t_0)$ for all $t_0 \in (0, 1)$.

Case 3. $0 < t < t_0 \le 1$. By interchanging the position of t_0 and t in (3.6), similarly we can obtain $\xi(t_0-) = \xi(t_0)$ for all $t_0 \in (0, 1]$.

In Cases 1 and 2, we have shown that ξ is right-continuous everywhere, while in Case 3, we have obtained the left-continuity. Thus we have $\xi \in \mathcal{C}(I, \mathbb{C})$.

Step 2. We assert that $\xi'_+(t_0) = 0$ for any $t_0 \in (0, 1)$. To this end, let us go back (3.6). Since the integrand $(z(t_0) - z(s))\bar{z}(s)$ is right-continuous at $s = t_0$, one has

$$
\psi'_{2+}(t_0) = (z(t_0) - z(s))\bar{z}(s)\big|_{s=t_0} = 0.
$$

On the other hand, as $y(s)$ has the right-derivative $y'_+(t_0) =: a$ at $s = t_0$, we have $y(s) - y(t_0) = (a + o(1))(s - t_0)$ as $s \downarrow t_0$. Thus, as $t \downarrow t_0$,

$$
\begin{aligned}
\psi_1(t) &= \int_{(t_0,t]} \overline{y(s) - y(t_0)}\, y(s)\,\mathrm{d}\mu_\lambda(s) \\
&= \int_{(t_0,t]} (\bar{a} + o(1))(s - t_0) y(s)\,\mathrm{d}\mu_\lambda(s),
\end{aligned}
$$

and, by noticing $\psi_1(t_0) = 0$,

$$
\begin{aligned}
\left|\frac{\psi_1(t) - \psi_1(t_0)}{t - t_0}\right| &= \left|\int_{(t_0,t]} \frac{(\bar{a} + o(1))(s - t_0)}{t - t_0} y(s)\,\mathrm{d}\mu_\lambda(s)\right| \\
&\le (|a| + o(1)) \|y\|_\infty \mathbf{V}(\mu_\lambda, (t_0, t]) \to 0.
\end{aligned}
$$

See (2.1). Thus $\psi'_{1+}(t_0) = 0$. Hence we have $\xi'_+(t_0) = 0$ for any $t_0 \in (0, 1)$.

Step 3. We conclude (3.5) from Lemma 3.2 because $\xi(0) = 0$.

Let $t = 1$ in (3.5), where $y(1) = 0$. We can obtain

$$-\int_{(0,1]} z(s)\bar{z}(s)\,\mathrm{d}s + \lambda \int_{(0,1]} y(s)\bar{y}(s)\,\mathrm{d}s + \int_{(0,1]} y(s)\bar{y}(s)\,\mathrm{d}\mu(s) = 0.$$

As μ is a real measure, we know that

$$\lambda = \frac{\int_{(0,1]} z(s)\bar{z}(s)\,\mathrm{d}s - \int_{(0,1]} y(s)\bar{y}(s)\,\mathrm{d}\mu(s)}{\int_{(0,1]} y(s)\bar{y}(s)\,\mathrm{d}s} \in \mathbb{R}.$$

This is just the Rayleigh form for eigenvalues of (3.1).

Step 4. As for Neumann eigenvalues, one can use the ideas above to verify that the following function

$$\eta(t) := \begin{cases} 0, & t = 0, \\ \bar{y}(t)z(t) - \int_{[0,t]} z(s)\bar{z}(s)\,\mathrm{d}s + \int_{[0,t]} y(s)\bar{y}(s)\,\mathrm{d}\mu_\lambda(s), & t \in (0,1], \end{cases}$$

is identically zero, which implies that Neumann eigenvalues are also real. □

3.2 Eigenvalues by shooting method of entire functions

There are several methods to study eigenvalues of Sturm-Liouville operators with potentials or weights [25]. In this subsection we show that the shooting method [15] using the fundamental solutions can be adopted to study eigenvalues of MDE. For the argument approach to eigenvalues of (3.1)-(3.2) and (3.1)-(3.3), see the succeeding sections.

Let us recall from [13] the variant-of-constant formula for inhomogeneous MDE

$$\mathrm{d}\dot{y} + y\,\mathrm{d}\mu(t) = h(t)\,\mathrm{d}\nu(t), \tag{3.7}$$

where $\mu,\ \nu \in \mathcal{M}_0(I, \mathbb{K})$ and $h \in \mathcal{C}(I, \mathbb{K})$. For any $(y_0, z_0) \in \mathbb{K}^2$, the unique solution $(y(t), \dot{y}(t))$ of (3.7) satisfying $(y(0), \dot{y}(0)) = (y_0, z_0)$ is given by

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} & \text{for } t = 0, \\ N_\mu(t)\left(\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \int_{[0,t]} N_\mu^{-1}(s) \begin{pmatrix} 0 \\ h(s) \end{pmatrix} \mathrm{d}\nu(s)\right) & \text{for } t \in (0,1]. \end{cases} \tag{3.8}$$

Here $N_\mu^{-1}(s)$ is the inverse of $N_\mu(s)$.

Given $\mu \in \mathcal{M}_0(I, \mathbb{R})$. It is trivial that $\lambda \in \mathbb{C}$ is an eigenvalue of problem (3.1)-(3.2) iff λ satisfies

$$\varphi_2(1, \mu_\lambda) = 0. \tag{3.9}$$

The basic idea is to consider MDE (3.1) as a perturbation of the following simple ODE

$$y'' + \lambda y = 0, \tag{3.10}$$

whose fundamental solutions are

$$C_\lambda(t) = \cos\sqrt{\lambda}t = \sum_{n \in \mathbb{Z}^+} \frac{(-1)^n \lambda^n t^{2n}}{(2n)!}, \qquad S_\lambda(t) = \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} = \sum_{n \in \mathbb{Z}^+} \frac{(-1)^n \lambda^n t^{2n+1}}{(2n+1)!}.$$

Both are entire functions of $\lambda \in \mathbb{C}$. The fundamental matrix solution of Eq. (3.10) is

$$N_\lambda(t) = \exp\left(\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} t\right) = \begin{pmatrix} C_\lambda(t) & S_\lambda(t) \\ C'_\lambda(t) & S'_\lambda(t) \end{pmatrix}.$$

One has

$$C'_\lambda(t) = -\lambda S_\lambda(t), \qquad S'_\lambda(t) = C_\lambda(t), \qquad N_\lambda(t) N_\lambda^{-1}(s) = N_\lambda(t-s).$$

We aim at giving reasonable estimates for the difference between the fundamental solutions of MDE (3.1) and of ODE (3.10).

Let us write Eq. (3.1) as an inhomogeneous MDE

$$\mathrm{d}\overset{\bullet}{y} + \lambda y\, \mathrm{d}t = -y\, \mathrm{d}\mu(t).$$

From the variant-of-constant formula (3.8), $\varphi_i(t) := \varphi_i(t, \mu_\lambda)$ satisfy for $t \in (0,1]$,

$$\begin{pmatrix} \varphi_1(t) \\ \overset{\bullet}{\varphi}_1(t) \end{pmatrix} = \begin{pmatrix} C_\lambda(t) \\ C'_\lambda(t) \end{pmatrix} - \int_{[0,t]} \begin{pmatrix} S_\lambda(t-s)\varphi_1(s) \\ S'_\lambda(t-s)\varphi_1(s) \end{pmatrix} \mathrm{d}\mu(s), \tag{3.11}$$

$$\begin{pmatrix} \varphi_2(t) \\ \overset{\bullet}{\varphi}_2(t) \end{pmatrix} = \begin{pmatrix} S_\lambda(t) \\ S'_\lambda(t) \end{pmatrix} - \int_{[0,t]} \begin{pmatrix} S_\lambda(t-s)\varphi_2(s) \\ S'_\lambda(t-s)\varphi_2(s) \end{pmatrix} \mathrm{d}\mu(s). \tag{3.12}$$

Let us introduce the operator $\mathcal{Z}_{\mu_\lambda} : \mathcal{C}(I, \mathbb{C}) \to \mathcal{C}(I, \mathbb{C})$ by

$$\mathcal{Z}_{\mu_\lambda} y(t) := -\int_{[0,t]} S_\lambda(t-s) y(s)\, \mathrm{d}\mu(s), \qquad t \in I.$$

By noticing that $S_\lambda(t-s) = 0$ for $t = s$, the first columns of (3.11) and (3.12) hold at $t = 0$ as well. That is,

$$\varphi_1 = C_\lambda + \mathcal{Z}_{\mu_\lambda}\varphi_1, \qquad \varphi_2 = S_\lambda + \mathcal{Z}_{\mu_\lambda}\varphi_2. \tag{3.13}$$

For $\nu \in \mathcal{M}_0(I, \mathbb{R})$, let us introduce

$$\hat{\nu}(t) := \begin{cases} -|\nu(0)| & \text{for } t = 0, \\ \mathbf{V}(\nu, (0,t]) & \text{for } t \in (0,1]. \end{cases} \tag{3.14}$$

For example, for absolutely continuous measures with densities $\dot{\mu} = q \in \mathcal{L}^1(I, \mathbb{R})$, one has

$$\hat{\mu}(t) \equiv \int_{[0,t]} |q(s)|\, \mathrm{d}s.$$

Lemma 3.4 *We have $\hat{\nu} \in \mathcal{M}_0(I, \mathbb{R})$. Moreover, $\|\nu\|_{\mathbf{V}} = \hat{\nu}(1) - \hat{\nu}(0)$ and*

$$\left| \int_{[a,b]} f(s)\, \mathrm{d}\nu(s) \right| \le \int_{[a,b]} |f(s)|\, \mathrm{d}\hat{\nu}(s) \qquad \forall\ f \in \mathcal{C}(I, \mathbb{C}),\ [a,b] \subset I. \tag{3.15}$$

Proof Step 1. We assert that $\hat{\nu} \in \mathcal{M}_0(I, \mathbb{R})$.

By (2.1), one has $\hat{\nu}(0+) = 0$. Let us prove

$$\hat{\nu}(t) - \hat{\nu}(t_0) = \mathbf{V}(\nu, (0,t]) - \mathbf{V}(\nu, (0,t_0]) = \mathbf{V}(\nu, (t_0, t]) \qquad \forall\ 0 < t_0 < t \le 1. \tag{3.16}$$

In fact, let $\mathcal{P}' = \left\{0 < t'_0 < t'_1 < \cdots < t'_{n-1} < t'_n = t\right\}$ be a partition of $(0, t]$. By inserting t_0, we have a new partition of $(0, t]$

$$\mathcal{P}'' = \mathcal{P} \cup \{t_0\} =: \left\{0 < t''_0 < t''_1 < \cdots t''_m < \cdots < t''_n < t''_{n+1} = t\right\},$$

where $t''_m = t_0$. We have then

$$\begin{aligned}
&\sum_{i=0}^{n-1} |\nu(t'_{i+1}) - \nu(t'_i)| \le \sum_{i=0}^{n} |\nu(t''_{i+1}) - \nu(t''_i)| \\
&= \sum_{i=0}^{m-1} |\nu(t''_{i+1}) - \nu(t''_i)| + |\nu(t''_{m+1}) - \nu(t_0)| + \sum_{i=m+1}^{n} |\nu(t''_{i+1}) - \nu(t''_i)| \\
&\le \mathbf{V}(\nu, (0, t_0]) + |\nu(t''_{m+1}) - \nu(t_0)| + \mathbf{V}(\nu, (t_0, t]).
\end{aligned}$$

When considering variations, we are able to assume that $t''_{m+1} - t_0$ is small. By taking supremum over the partitions $\mathcal{P}'$ and noticing that $\nu(t_0+) = \nu(t_0)$, we can obtain

$$\mathbf{V}(\nu, (0, t]) - \mathbf{V}(\nu, (0, t_0]) \le \mathbf{V}(\nu, (t_0, t]).$$

The reversing inequality can be proved similarly. Thus we have (3.16). By (2.1) again, we have $\hat{\nu}(t_0+) = \hat{\nu}(t_0)$ from (3.16). Since $\hat{\nu}(t)$ is non-decreasing in $t \in I$, one has

$$\mathbf{V}(\hat{\nu}, I) = \hat{\nu}(1) - \hat{\nu}(0) = \|\nu\|_{\mathbf{V}} < +\infty.$$

Therefore, $\hat{\nu} \in \mathcal{M}_0(I, \mathbb{R})$.

Step 2. We assert that

$$|\nu(t) - \nu(t_0)| \le \hat{\nu}(t) - \hat{\nu}(t_0) \qquad \forall\ 0 \le t_0 < t \le 1. \tag{3.17}$$

When $t_0 > 0$, (3.17) is the same as

$$|\nu(t) - \nu(t_0)| \le \mathbf{V}(\nu, (t_0, t]) = \hat{\nu}(t) - \hat{\nu}(t_0) \qquad \forall\ 0 < t_0 < t \le 1. \tag{3.18}$$

See (3.16). To see this, notice that $\{t'_0,\, t\}$ is a partition of $(t_0, t]$ for any $t'_0 \in (t_0, t)$. From the definition of variations on subintervals, we have

$$|\nu(t) - \nu(t'_0)| \le \mathbf{V}(\nu, (t_0, t]).$$

By letting $t'_0 \downarrow t_0$ and noticing that $\nu(t_0+) = \nu(t_0)$, we can get (3.18). Hence we have proved (3.17) for the case $t_0 \in (0, 1)$.

When $t_0 = 0$, let us notice that (3.16) is not true because $\nu(0)$ may not be 0. However, by letting $t_0 \downarrow 0$ in (3.18) and noticing that $\nu(0+) = \hat{\nu}(0+) = 0$, we have $|\nu(t)| \le \hat{\nu}(t)$ for all $0 < t \le 1$. Thus

$$|\nu(t) - \nu(0)| \le |\nu(t)| + |\nu(0)| \le \hat{\nu}(t) - \hat{\nu}(0) \qquad \forall\ t \in (0, 1],$$

because we have defined $\hat{\nu}(0)$ as $-|\nu(0)|$. This shows that (3.17) is also true for $t_0 = 0$.

Step 3. Let us complete the proof of (3.15).

Case 1. Suppose that $0 \le a < b \le 1$. Since $f \in \mathcal{C}(I, \mathbb{C})$ and $|f| \in \mathcal{C}(I, \mathbb{R})$, both sides of (3.15) are Riemann-Stieltjes integrals. Let $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a partition of $[a,b]$ and $\xi_i \in [t_i, t_{i+1}]$, $0 \le i \le n-1$. We have the following estimate for the Riemann-Stieltjes sums

$$\begin{aligned}\left|\sum_{i=0}^{n-1} f(\xi_i)(\nu(t_{i+1}) - \nu(t_i))\right| &\le \sum_{i=0}^{n-1} |f(\xi_i)| \cdot |\nu(t_{i+1}) - \nu(t_i)| \\ &\le \sum_{i=0}^{n-1} |f(\xi_i)| \cdot (\hat{\nu}(t_{i+1}) - \hat{\nu}(t_i)),\end{aligned}$$

following from inequality (3.17). This proves (3.15).

Case 2. Suppose that $b = a \in [0,1]$. In case $a = 0$, both sides of (3.15) are $|f(0)||\nu(0)|$. Hence (3.15) is true. In case $a \in (0,1)$, both sides of (3.15) are 0. In case $a = 1$, (3.15) is the same as

$$|f(1)(\nu(1) - \nu(1-))| \le |f(1)|(\hat{\nu}(1) - \hat{\nu}(1-)).$$

By letting $t = 1$ and $t_0 \uparrow 1$ in (3.17), we know that this is also true. □

In order to establish existence of eigenvalues, we follow [15] to prove the following estimates on fundamental solutions of (3.1). For simplicity, let us write for $\nu \in \mathcal{M}_0(I, \mathbb{R})$

$$\check{\nu}(t) \equiv \hat{\nu}(t) - \hat{\nu}(0) = \begin{cases} 0 & \text{for } t = 0, \\ |\nu(0)| + \mathbf{V}(\nu, (0,t]) & \text{for } t \in (0,1]. \end{cases}$$

Note that $\hat{\nu}$ and $\check{\nu}$ define the same measure $\mathrm{d}\hat{\nu} = \mathrm{d}\check{\nu}$.

Lemma 3.5 *Given $\mu \in \mathcal{M}_0(I, \mathbb{R})$ and $t \in I$. Then the fundamental solution $\varphi_2(t, \mu_\lambda)$ of MDE* (3.1) *is an entire function of $\lambda \in \mathbb{C}$. Moreover, one has the following estimates*

$$|\varphi_2(t, \mu_\lambda)| \le e^{|\mathrm{Im}\sqrt{\lambda}|t + \check{\mu}(t)}, \qquad \lambda \in \mathbb{C},\ t \in I, \tag{3.19}$$

$$|\varphi_2(t, \mu_\lambda) - S_\lambda(t)| \le \frac{1}{|\lambda|} e^{|\mathrm{Im}\sqrt{\lambda}|t + \check{\mu}(t)}, \qquad \lambda \in \mathbb{C}\backslash\{0\},\ t \in I. \tag{3.20}$$

Proof Note that both (3.19) and (3.20) are trivial for $t = 0$. In the following we assume that $t \in (0,1]$.

The second equality of (3.13) is

$$\varphi_2 = S_\lambda + \mathcal{Z}_{\mu_\lambda} \varphi_2.$$

By iterating it, we have

$$\varphi_2 = \sum_{n=0}^{N} \mathcal{Z}_{\mu_\lambda}^n S_\lambda + \mathcal{Z}_{\mu_\lambda}^{N+1} \varphi_2, \qquad N \in \mathbb{N}. \tag{3.21}$$

By the definition of $\mathcal{Z}_{\mu_\lambda}$, one has for $f \in \mathcal{C}(I, \mathbb{C})$,

$$\begin{aligned}\mathcal{Z}_{\mu_\lambda}^2 f(t) &= -\int_{[0,t]} S_\lambda(t - t_2) \left(-\int_{[0,t_2]} S_\lambda(t_2 - t_1) f(t_1)\, \mathrm{d}\mu(t_1) \right) \mathrm{d}\mu(t_2) \\ &= (-1)^2 \int_{0 \le t_1 \le t_2 \le t_3 := t} f(t_1) \prod_{i=1}^{2} S_\lambda(t_{i+1} - t_i)\, \mathrm{d}\mu(t_1)\, \mathrm{d}\mu(t_2).\end{aligned}$$

Inductively,

$$\mathcal{Z}_{\mu_\lambda}^n f(t) = \int_{0\le t_1\le\cdots\le t_n\le t_{n+1}:=t} f(t_1)T_n(t_1,\cdots,t_{n+1})\,\mathrm{d}\mu(t_1)\cdots\mathrm{d}\mu(t_n), \tag{3.22}$$

where

$$T_n(t_1,\cdots,t_{n+1}) := (-1)^n\prod_{i=1}^n S_\lambda(t_{i+1}-t_i).$$

Applying (3.15) to (3.22), we have

$$\left|\mathcal{Z}_{\mu_\lambda}^n f(t)\right| \le \int_{0\le t_1\le\cdots\le t_n\le t_{n+1}:=t} |f(t_1)||T_n(t_1,\cdots,t_{n+1})|\,\mathrm{d}\hat{\mu}(t_1)\cdots\mathrm{d}\hat{\mu}(t_n).$$

One crucial observation on T_n is as follows. Since $S_\lambda(0)=0$, one has $S_\lambda(t_{i+1}-t_i)=0$ whenever $t_{i+1}=t_i$. Hence we have

$$\left|\mathcal{Z}_{\mu_\lambda}^n f(t)\right| \le \int_{0\le t_1<t_2<\cdots<t_n<t_{n+1}:=t} |f(t_1)||T_n(t_1,\cdots,t_{n+1})|\,\mathrm{d}\hat{\mu}(t_1)\cdots\mathrm{d}\hat{\mu}(t_n). \tag{3.23}$$

For simplicity, in the following we write $\omega := |\mathrm{Im}\sqrt{\lambda}|$ for $\lambda\in\mathbb{C}$. For the proof of (3.19), we use the following elementary inequality

$$|S_\lambda(t)| \le e^{\omega t}, \qquad \lambda\in\mathbb{C},\ t\in I. \tag{3.24}$$

From this, for $0\le t_1<\cdots<t_{n+1}=t\le 1$,

$$|T_n(t_1,\cdots,t_{n+1})| \le \prod_{i=1}^n e^{\omega(t_{i+1}-t_i)} = e^{\omega(t-t_1)}. \tag{3.25}$$

Let $f=S_\lambda$ in (3.23). By using (3.24) for $t=t_1$ and (3.25), we obtain

$$|\mathcal{Z}_{\mu_\lambda}^n S_\lambda(t)| \le e^{\omega t}\cdot I_n(t), \qquad I_n(t) := \int_{0\le t_1<\cdots<t_n<t} \mathrm{d}\hat{\mu}(t_1)\cdots\mathrm{d}\hat{\mu}(t_n).$$

Note that the integral $I_n(t)$ does not change under permutations of $(t_1,\cdots,t_n)$. Moreover, all permuted domains for $(t_1,t_2,\cdots,t_n)$ are disjoint and have the union being a subset of $[0,t]^n$. As $\hat{\mu}(t)$ is non-decreasing in t, we have

$$I_n(t) \le \frac{1}{n!}\int_{[0,t]^n}\mathrm{d}\hat{\mu}(t_1)\cdots\mathrm{d}\hat{\mu}(t_n) = \frac{1}{n!}\left(\int_{[0,t]}\mathrm{d}\hat{\mu}(s)\right)^n = \frac{(\check{\mu}(t))^n}{n!}. \tag{3.26}$$

In conclusion

$$|\mathcal{Z}_{\mu_\lambda}^n S_\lambda(t)| \le e^{\omega t}\frac{(\check{\mu}(t))^n}{n!}. \tag{3.27}$$

Note that this is also true for $n=0$ by (3.24).

Now let $f=\varphi_2$ in (3.22). By the arguments above, we can obtain

$$\left|\mathcal{Z}_{\mu_\lambda}^n \varphi_2(t)\right| \le \|\varphi_2\|_\infty e^{\omega t}\frac{(\check{\mu}(t))^n}{n!}. \tag{3.28}$$

Let $N \to \infty$ in (3.21). It follows from (3.28) that the second term tends to 0. By (3.27), we know that the series

$$\varphi_2(t) = \varphi_2(t, \mu_\lambda) = \sum_{n \in \mathbb{Z}^+} \mathcal{Z}^n_{\mu_\lambda} S_\lambda(t) \tag{3.29}$$

is convergent. Since each term of the right-hand side of (3.29) is an entire function of $\lambda \in \mathbb{C}$, we know that $\varphi_2(t, \mu_\lambda)$ is also an entire function of λ. Moreover, by (3.27) and (3.29), we have $|\varphi_2(t, \mu_\lambda)| \le e^{\omega t + \check{\mu}(t)}$. This is the desired estimate (3.19).

For the desired estimate (3.20), estimate (3.24) for $S_\lambda(t)$ can be replaced by the following elementary inequality

$$|S_\lambda(t)| \le \frac{e^{\omega t}}{|\sqrt{\lambda}|}, \qquad \lambda \in \mathbb{C}\backslash\{0\},\ t \in I.$$

Estimates like (3.25) can be changed accordingly. We have

$$|S_\lambda(t_1) T(t_1, t_2, \cdots, t_{n+1})| \le \frac{e^{\omega t_1}}{|\sqrt{\lambda}|} \cdot \frac{e^{\omega(t_2 - t_1)}}{|\sqrt{\lambda}|} \cdot \prod_{i=2}^{n} e^{\omega(t_{i+1} - t_i)} = \frac{e^{\omega t}}{|\lambda|}, \qquad n \ge 1.$$

Hence, by (3.26),

$$\begin{aligned} |\varphi_2(t, \mu_\lambda) - S_\lambda(t)| &= \left| \sum_{n \in \mathbb{N}} \mathcal{Z}^n_{\mu_\lambda} S_\lambda(t) \right| \le \frac{e^{\omega t}}{|\lambda|} \sum_{n \in \mathbb{N}} I_n(t) \\ &\le \frac{e^{\omega t}}{|\lambda|} \left(e^{\check{\mu}(t)} - 1 \right) \le \frac{1}{|\lambda|} e^{\omega t + \check{\mu}(t)}. \end{aligned}$$

□

Remark 3.6 (i) From the estimates of the proof of Lemma 3.5, it is easy to see that the convergence of series (3.29) is uniform in $t \in I$, in λ in any bounded set of $\mathbb{C}$, and in μ in any $\|\cdot\|_{\mathbf{V}}$-bounded set of $\mathcal{M}_0(I, \mathbb{R})$. Results (3.19) and (3.20) have been established for those measures μ with densities $q \in \mathcal{L}^2(I, \mathbb{R})$. See [15, p. 7, Theorem 1].

(ii) From the second column of (3.12), one has

$$\overset{\bullet}{\varphi}_2 = S'_\lambda + \mathcal{Z}^*_{\mu_\lambda} \varphi_2, \qquad \mathcal{Z}^*_{\mu_\lambda} y(t) := -\int_{[0,t]} S'_\lambda(t-s) y(s) \,\mathrm{d}\mu(s) \text{ for } t \in (0, 1].$$

Using some elementary estimates on $S'_\lambda(t)$ and estimates (3.19) and (3.20) for $\varphi_2(t)$, we can obtain estimates on $\overset{\bullet}{\varphi}_2(t, \mu_\lambda)$. When λ is negative, we will give in Lemma 5.1 some alternative estimates on $\varphi_2(t, \mu_\lambda)$ and $\overset{\bullet}{\varphi}_2(t, \mu_\lambda)$.

The ideas above can be applied to estimates of $\varphi_1(t, \mu_\lambda)$ and $\overset{\bullet}{\varphi}_1(t, \mu_\lambda)$. For example, using (3.11), $\varphi_1(t, \mu_\lambda)$ can be expanded as

$$\varphi_1(t, \mu_\lambda) = C_\lambda(t) + \sum_{n \in \mathbb{N}} \int_{0 \le t_1 \le \cdots \le t_{n+1} := t} C_\lambda(t_1) T(t_1, t_2, \cdots, t_{n+1}) \,\mathrm{d}\mu(t_1) \cdots \mathrm{d}\mu(t_n).$$

Instead of estimate (3.24) for $S_\lambda(t_1)$, one can use, for example,

$$|C_\lambda(t_1)| \le e^{|\mathrm{Im}\sqrt{\lambda}| t_1}, \qquad \lambda \in \mathbb{C},\ t_1 \in I.$$

The corresponding results are stated as in the following lemma, referring some detail to [15].

Lemma 3.7 *Let $\mu \in \mathcal{M}_0(I,\mathbb{R})$ and $t \in I$. There hold*

$$\begin{aligned}
|\varphi_1(t,\mu_\lambda)| &\le e^{|\mathrm{Im}\sqrt{\lambda}|t+\check{\mu}(t)},\\
|\varphi_1(t,\mu_\lambda)-C_\lambda(t)| &\le \frac{1}{|\sqrt{\lambda}|}e^{|\mathrm{Im}\sqrt{\lambda}|t+\check{\mu}(t)},\\
|\overset{\bullet}{\varphi}_1(t,\mu_\lambda)-C'_\lambda(t)| &\le \check{\mu}(t)e^{|\mathrm{Im}\sqrt{\lambda}|t+\check{\mu}(t)},\\
|\overset{\bullet}{\varphi}_2(t,\mu_\lambda)-S'_\lambda(t)| &\le \frac{1}{|\sqrt{\lambda}|}\check{\mu}(t)e^{\omega t+\check{\mu}(t)},
\end{aligned}$$

where $\lambda \in \mathbb{C}$ for the first and the third estimates, while $\lambda \in \mathbb{C}\backslash\{0\}$ for the second and the fourth estimates.

We can use the shooting method to establish the structure of eigenvalues of MDE.

Theorem 3.8 *Let $\mu \in \mathcal{M}_0(I,\mathbb{R})$. We have the following results.*

(i) *Problem* (3.1)-(3.2) *has a sequence of real (simple) eigenvalues $\lambda_m^D(\mu)$, $m \in \mathbb{N}$, increasing in $m \in \mathbb{N}$.*

(ii) *Problem* (3.1)-(3.3) *has a sequence of real (simple) eigenvalues $\lambda_m^N(\mu)$, $m \in \mathbb{Z}^+$, increasing in $m \in \mathbb{Z}^+$.*

(iii) *Moreover,* $\lim_{m\to\infty}\lambda_m^D(\mu) = \lim_{m\to\infty}\lambda_m^N(\mu) = +\infty$.

Proof Let $\mu \in \mathcal{M}_0(I,\mathbb{R})$. Eigenvalues $\lambda \in \mathbb{R}$ of (3.1)-(3.2) are zeros of equation (3.9) in the complex plane of λ. When μ has density $q \in \mathcal{L}^2(I,\mathbb{R})$, the existence of $\lambda_m^D(q)$, $m \in \mathbb{N}$, and the property $\lim_{m\to\infty}\lambda_m^D(q) = +\infty$ can be deduced simply from estimate (3.20) for $\varphi_2(1,\mu_\lambda)$, with the help of the Rouché theorem. See [15, p. 28, Counting Lemma]. The technique there can be transferred to the present case word by word.

For the Neumann eigenvalues, one can use the third estimate of Lemma 3.7 to solve the equation

$$\overset{\bullet}{\varphi}_1(1,\mu_\lambda) = 0,$$

which has a sequence of solutions, labeled as $\lambda_0^N(\mu) < \lambda_1^N(\mu) < \cdots < \lambda_m^N(\mu) < \cdots$, with the property $\lambda_m^N(\mu) \to +\infty$ as $m \to \infty$. □

Like the results for eigen-functions for ODE case, one can also obtain the results on nodes of eigen-functions of MDE (3.1).

4 Arguments and Continuity in Weak* Topology

In the next section, we will prove a deep result on eigenvalues $\lambda_m^\sigma(\mu)$ of (3.1) where $\sigma = D$ or N. That is, $\lambda_m^\sigma(\mu)$ are continuous in $\mu \in (\mathcal{M}_0(I,\mathbb{R}), w^*)$. Recall from [15] that when $\mu' = q \in \mathcal{L}^2(I,\mathbb{R})$, it is possible to use the expansions of $(\varphi_i(t,\lambda+q), \overset{\bullet}{\varphi}_i(t,\lambda+q))$ to prove that eigenvalues $\lambda_m^\sigma(q)$ of (1.6) are continuous in potentials q with respect to the weak topology w_2 of $\mathcal{L}^2(I,\mathbb{R})$. Note that the weak* topology w^* is much weaker than w_2 and therefore the desired continuity of $\lambda_m^\sigma(\mu)$ is much stronger. It seems that the approach in [15] is not easily be generalized to the present case. We will use the argument approach to eigenvalues to prove this. This idea has been recently developed by Zhang [27] to obtain continuity results for eigenvalues of Sturm-Liouville operators in potentials endowed with weak topologies.

4.1 Definition and continuity of arguments

For any non-zero solution $y(t)$ of (1.5), the argument of $y(t)$ is defined as

$$\theta(t) := \arg(y(t) - i\dot{y}(t)), \qquad t \in I, \tag{4.1}$$

which is understood as a continuous representation (in time t). See [26]. In literature like [2, 8, 9], the argument of $y(t)$ is also defined as

$$\hat{\theta}(t) := -\arg(\dot{y}(t) + iy(t)).$$

Note that $\theta(t)$ differs from $\hat{\theta}(t)$ by a constant $-\pi/2$. We will adopt definition (4.1), as used in our works [24, 26, 27]. A remarkable fact for ODE case (1.5) is that the argument $\theta(t)$ is actually absolutely continuous in $t \in I$ and is governed by the following first-order nonlinear ODE

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = q(t)\cos^2\theta + \sin^2\theta. \tag{4.2}$$

This equation is deduced from ODE (1.5) by the Prüfer transformation $y = r\cos\theta$, $-\dot{y} = r\sin\theta$. Hence the argument of (1.5) can be defined in an easy way.

For any $\vartheta \in \mathbb{R}$, let $\theta = \theta_q(t,\vartheta)$, $t \in I$, be the unique solution of equation (4.2) satisfying $\theta(0) = \vartheta$. In order to study the evolution of (1.5), for any $t \in I$, let us define the *argument* of ODE (1.5) at time t as the transformation $\Theta_q^t : \mathbb{R} \to \mathbb{R}$ defined by

$$\Theta_q^t(\vartheta) := \theta_q(t,\vartheta).$$

Obviously, $\Theta_q^0 = \mathrm{id}_{\mathbb{R}}$. As equation (4.2) is π-periodic in θ, one has

$$\Theta_q^t(\vartheta + m\pi) = \Theta_q^t(\vartheta) + m\pi, \qquad m \in \mathbb{Z}. \tag{4.3}$$

Moreover, for each $t \in \mathbb{R}$, Θ_q^t is an increasing C^∞ self-diffeomorphism of $\mathbb{R}$.

Now we are going to consider MDE (1.1). For a non-zero solution $(y(t), \overset{\bullet}{y}(t))$ of MDE (1.1), though $y(t)$ is continuous in t, the velocity $\overset{\bullet}{y}(t)$ is in general not continuous in t. Hence (4.1) fails because $\theta(t)$ cannot be continuous in t. In the following we use some idea on lifting in general topology to give a natural definition for arguments of MDE, as did in [27] for another defining way of arguments of ODE (1.5).

The topological fact we are going to use is as follows. Let

$$\mathbb{S}_1 := \left\{ u = (\cos\vartheta, \sin\vartheta)^T \in \mathbb{R}^2 : \vartheta \in \mathbb{R} \right\} \subset \mathbb{R}^2$$

be the unit circle, which has the universal covering $\sigma : \mathbb{R} \to \mathbb{S}_1$ defined by

$$\sigma(\vartheta) = (\cos\vartheta, \sin\vartheta)^T.$$

Lemma 4.1 [16] *Let (X,τ) be a topological space which is simply connected and path connected. Suppose that*

$$f : (X,\tau) \times \mathbb{S}_1 \to \mathbb{S}_1$$

is a continuous mapping, called a homotopy of maps of $\mathbb{S}_1$ with the homotopy parameter $p \in X$. Fix $p_0 \in X$. Suppose that $F_{p_0} : \mathbb{R} \to \mathbb{R}$ is a lifting of the map $f(p_0,\cdot) : \mathbb{S}_1 \to \mathbb{S}_1$, i.e., $F_{p_0} : \mathbb{R} \to \mathbb{R}$ is continuous and $\sigma \circ F_{p_0} = f(p_0,\cdot) \circ \sigma$. Then there exists a unique lifting $F : (X,\tau) \times \mathbb{R} \to \mathbb{R}$ of f such that $F(p_0,\cdot) = F_{p_0}$. More precisely,

- *$F : (X,\tau) \times \mathbb{R} \to \mathbb{R}$ is continuous,*
- *$F(p_0,\vartheta) = F_{p_0}(\vartheta)$ for $\vartheta \in \mathbb{R}$, and*
- *$\sigma(F(p,\vartheta)) = f(p,\sigma(\vartheta))$ for $(p,\vartheta) \in X \times \mathbb{R}$.*

Note that the joint continuity of $f(p,u)$ in $(p,u) \in X \times \mathbb{S}_1$ and $F(p,\vartheta)$ in $(p,\vartheta) \in X \times \mathbb{R}$ is important.

Consider MDE (1.1) with the real measures $\mu \in \mathcal{M}_0(I,\mathbb{R})$. Define

$$M_\mu(t) := \begin{pmatrix} \varphi_1(t) & -\varphi_2(t) \\ -\overset{\bullet}{\varphi}_1(t) & \overset{\bullet}{\varphi}_2(t) \end{pmatrix}, \qquad t \in I,$$

where $\varphi_i(t)$ are fundamental solutions of Eq. (1.1) as before. By the Liouville law in [13] for MDE, one has $\det M_\mu(t) \equiv +1$. Note that $M_\mu(t)$ is the fundamental matrix solution of the system

$$\overset{\bullet}{y} = -z, \qquad \mathrm{d}z = y\,\mathrm{d}\mu(t),$$

in the usual sense. This system is also equivalent to Eq. (1.1).

By Theorem 1.1, we have the following continuity results.

- Given $t \in I$. The following mapping is continuously differentiable

$$\mathbb{R}^2 \times (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \to \mathbb{R}^2, \qquad (u_0,\mu) \to M_\mu(t)u_0.$$

- For $t = 1$, the following mapping is continuous

$$\mathbb{R}^2 \times (\mathcal{M}_0(I,\mathbb{R}), w^*) \to \mathbb{R}^2, \qquad (u_0,\mu) \to M_\mu(1)u_0.$$

In order to apply Lemma 4.1, as $M_\mu(t)$ is non-singular, it induces a mapping $\hat{M}_\mu^t : \mathbb{S}_1 \to \mathbb{S}_1$ by

$$\hat{M}_\mu^t(u) := \frac{M_\mu(t)u}{|M_\mu(t)u|}, \qquad u \in \mathbb{S}_1.$$

Here $|\cdot|$ is the Euclidean norm of $\mathbb{R}^2$. Then each $\hat{M}_\mu^t$ is an orientation-preserving diffeomorphism of $\mathbb{S}_1$.

Suppose that $t \in I$ is fixed. Then $\hat{M}_\mu^t(u)$ is jointly continuous in $(\mu,u) \in (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \times \mathbb{S}_1$, which is a Banach space and therefore is simply connected and path connected. In order to apply Lemma 4.1, we need only to fix a lifting of $\hat{M}_\mu^t$ for some special μ. This can be done by choosing $\mu = 0$ as follows. When $\mu = 0$, MDE (1.1) is simply the ODE $y'' = 0$. The arguments, denoted by $\hat{\Theta}^t(\vartheta)$, are solutions of

$$\theta'(t) = \sin^2\theta(t), \qquad \theta(0) = \vartheta.$$

Explicitly, $\hat{\Theta}^t$ satisfies (4.3), and

$$\hat{\Theta}^t(\vartheta) = \begin{cases} 0 & \text{for } \vartheta = 0, \\ \cot^{-1}(\cot\vartheta - t) & \text{for } \vartheta \in (0,\pi). \end{cases}$$

Here $\cot : (0,\pi) \to \mathbb{R}$ and $\cot^{-1} : \mathbb{R} \to (0,\pi)$. See [27].

By Lemma 4.1, there exists a unique lifting

$$(\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \times \mathbb{R} \to \mathbb{R}, \qquad (\mu,\vartheta) \to \Theta_\mu^t(\vartheta) \tag{4.4}$$

of $\hat{M}_\mu^t(u)$ such that $\Theta_0^t(\vartheta) \equiv \hat{\Theta}^t(\vartheta)$. Moreover, the mapping $\Theta_\mu^t(\vartheta)$ is jointly continuous in $(\mu,\vartheta) \in (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \times \mathbb{R}$ when $t \in I$ is fixed.

Definition 4.2 The *argument* of MDE (1.1) (at time $t \in I$) is defined as the following map of $\mathbb{R}$

$$\Theta_\mu^t : \mathbb{R} \to \mathbb{R}, \qquad \vartheta \to \Theta_\mu^t(\vartheta).$$

It is obvious that Θ_μ^t is a self-homeomorphism of $\mathbb{R}$ and satisfies (4.3).

Remark 4.3 When $\mu \in \mathcal{M}_0(I,\mathbb{R})$ has density $\mu' = q \in \mathcal{L}^1(I,\mathbb{R})$, $\Theta_\mu^t(\vartheta)$ coincides with the classical argument defined from ODE (4.2). See [27].

The argument of Eq. (1.1) at time 1 is denoted simply by

$$\Theta_\mu(\vartheta) := \Theta_\mu^1(\vartheta), \qquad (\mu,\vartheta) \in \mathcal{M}_0(I,\mathbb{R}) \times \mathbb{R}.$$

With respect to the weak* topology for measures, we have the following important continuity result.

Theorem 4.4 *The following mapping is continuous*

$$(\mathcal{M}_0(I,\mathbb{R}), w^*) \times \mathbb{R} \to \mathbb{R}, \qquad (\mu,\vartheta) \to \Theta_\mu(\vartheta).$$

Proof We have known that

$$(\mathcal{M}_0(I,\mathbb{R}), w^*) \times \mathbb{S}_1 \to \mathbb{S}_1, \qquad (\mu,u) \to \hat{M}_\mu^1(u)$$

is also continuous. Applying Lemma 4.1 to the parameter space $(X,\tau) = (\mathcal{M}_0(I,\mathbb{R}), w^*)$ which is also simply connected and path connected, we have also the unique (continuous) lifting

$$(\mathcal{M}_0(I,\mathbb{R}), w^*) \times \mathbb{R} \to \mathbb{R}, \qquad (\mu,\vartheta) \to \check{\Theta}_\mu(\vartheta) \tag{4.5}$$

such that $\check{\Theta}_0 = \hat{\Theta}^1$. Since the continuity of (4.5) implies the continuity of (4.4) for $t = 1$, it follows from the uniqueness of liftings of homotopies that $\check{\Theta}_\mu(\vartheta) \equiv \Theta_\mu(\vartheta)$. Hence (4.5) shows that $\Theta_\mu(\vartheta)$ is continuous in $(\mu,\vartheta) \in (\mathcal{M}_0(I,\mathbb{R}), w^*) \times \mathbb{R}$. $\square$

Theorem 4.4 is a generalization of [27, Theorem 4.4] where the classical measures $\mu = \mu_q$ are considered.

In order to study the evolution of MDE (1.1), we introduce another concept for Eq. (1.1). Let the solution of MDE (1.1) satisfying the initial condition

$$y(0) = r_0 \cos\vartheta, \qquad \overset{\bullet}{y}(0) = -r_0 \sin\vartheta, \qquad r_0 > 0,\ \vartheta \in \mathbb{R},$$

be denoted by $y(t, r_0, \vartheta, \mu)$ with its generalized right-derivative $\overset{\bullet}{y}(t, r_0, \vartheta, \mu)$. The growth function at time t is defined by

$$R_\mu^t(\vartheta) := \sqrt{(y(t,1,\vartheta,\mu))^2 + (\overset{\bullet}{y}(t,1,\vartheta,\mu))^2} > 0. \tag{4.6}$$

Obviously,

$$R_\mu^0(\vartheta) \equiv 1 \quad \text{and} \quad R_\mu^t(\vartheta + m\pi) = R_\mu^t(\vartheta) \text{ for } m \in \mathbb{Z}.$$

Now the arguments and growths are related with solutions of Eq. (1.1) via

$$\begin{pmatrix} y(t,r_0,\vartheta,\mu) \\ -\overset{\bullet}{y}(t,r_0,\vartheta,\mu) \end{pmatrix} = \begin{pmatrix} r_0 R_\mu^t(\vartheta) \cos\Theta_\mu^t(\vartheta) \\ r_0 R_\mu^t(\vartheta) \sin\Theta_\mu^t(\vartheta) \end{pmatrix}. \tag{4.7}$$

Suggested by the classical defining ODE (4.2) for arguments, it is reasonable to introduce the following first-order nonlinear MDE

$$\mathrm{d}\theta = \sin^2\theta\,\mathrm{d}t + \cos^2\theta\,\mathrm{d}\mu(t). \tag{4.8}$$

Since $\cos^2\theta(t)$ will be discontinuous and $\mathrm{d}\mu(t)$ is not the Lebesgue measure, we are not able to find a reasonable understanding for solutions of this equation.

4.2 Properties of arguments in measures

Given $t \in I$. Theorem 1.1 (i) asserts that the fundamental matrix solution $N_\mu(t)$ is continuously differentiable in $\mu \in (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})$. Moreover, it is proved in [13] that the directional derivative of $N_\mu(t)$ at μ along the direction ν is given by

$$\partial_\mu N_\mu(t)\cdot\nu = \begin{cases} 0 & \text{for } t=0,\\ -N_\mu(t)\int_{[0,t]} N_\mu^{-1}(s)\begin{pmatrix} 0 & 0\\ \varphi_1(s,\mu) & \varphi_2(s,\mu)\end{pmatrix}\mathrm{d}\nu(s) & \text{for } t\in(0,1]. \end{cases} \tag{4.9}$$

In particular, it follows from (4.6) and (4.7) that both $R_\mu(\vartheta)$ and $\Theta_\mu(\vartheta)$ are also continuously differentiable in $\mu \in (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})$. Let us write

$$(y(t,\vartheta,\mu), \overset{\bullet}{y}(t,\vartheta,\mu)) := (y(t,1,\vartheta,\mu), \overset{\bullet}{y}(t,1,\vartheta,\mu)).$$

Theorem 4.5 *Given $\vartheta\in\mathbb{R}$. As a functional*

$$(\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \to \mathbb{R}, \qquad \mu \to \Theta_\mu(\vartheta),$$

it is continuously differentiable in μ. Moreover, the directional derivative of $\Theta_\mu(\vartheta)$ at μ along the direction $\nu \in \mathcal{M}_0(I,\mathbb{R})$ is given by

$$\partial_\mu\Theta_\mu(\vartheta)\cdot\nu = \frac{\int_I (y(s,\vartheta,\mu))^2\,\mathrm{d}\nu(s)}{(R_\mu(\vartheta))^2} = \int_I (\hat{y}(s,\vartheta,\mu))^2\,\mathrm{d}\nu(s), \tag{4.10}$$

where $R_\mu(\vartheta) := R^1_\mu(\vartheta)$ and $\hat{y}(s,\vartheta,\mu) := y(s,\vartheta,\mu)/R_\mu(\vartheta)$.

Proof Differentiating

$$y(1,\vartheta,\mu) = R_\mu(\vartheta)\cos\Theta_\mu(\vartheta), \qquad \overset{\bullet}{y}(1,\vartheta,\mu) = -R_\mu(\vartheta)\sin\Theta_\mu(\vartheta),$$

with respect to μ, we know that for any $\nu \in \mathcal{M}_0(I,\mathbb{R})$,

$$\begin{aligned} \partial_\mu y(1,\vartheta,\mu)\cdot\nu &= \alpha\cos\Theta_\mu(\vartheta) - \beta R_\mu(\vartheta)\sin\Theta_\mu(\vartheta),\\ \partial_\mu \overset{\bullet}{y}(1,\vartheta,\mu)\cdot\nu &= -\alpha\sin\Theta_\mu(\vartheta) - \beta R_\mu(\vartheta)\cos\Theta_\mu(\vartheta), \end{aligned}$$

where $\alpha := \partial_\mu R_\mu(\vartheta)\cdot\nu \in \mathbb{R}$ and $\beta := \partial_\mu\Theta_\mu(\vartheta)\cdot\nu\in\mathbb{R}$. Thus

$$\begin{aligned} \beta &= -\left((\partial_\mu \overset{\bullet}{y}(1,\vartheta,\mu)\cdot\nu)\cos\Theta_\mu(\vartheta) + (\partial_\mu y(1,\vartheta,\mu)\cdot\nu)\sin\Theta_\mu(\vartheta)\right)/R_\mu(\vartheta)\\ &= -\left((\partial_\mu \overset{\bullet}{y}(1,\vartheta,\mu)\cdot\nu)y(1,\vartheta,\mu) - (\partial_\mu y(1,\vartheta,\mu)\cdot\nu)\overset{\bullet}{y}(1,\vartheta,\mu)\right)/(R_\mu(\vartheta))^2\\ &= -U^T JV/(R_\mu(\vartheta))^2, \end{aligned}$$

where

$$U = \left(\begin{array}{c} y(1,\vartheta,\mu) \\ \overset{\bullet}{y}(1,\vartheta,\mu) \end{array} \right), \qquad J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \qquad V = \left(\begin{array}{c} \partial_\mu y(1,\vartheta,\mu)\cdot\nu \\ \partial_\mu \overset{\bullet}{y}(1,\vartheta,\mu)\cdot\nu \end{array} \right).$$

Denote $X = (\cos\vartheta, \sin\vartheta)^T$. By (2.7), we have $U = N_\mu(1)X$ and $V = (\partial_\mu N_\mu(1)\cdot\nu)X$. Now β can be rewritten as

$$\beta = -X^T QX/(R_\mu(\vartheta))^2, \qquad Q := (N_\mu(1))^T J(\partial_\mu N_\mu(1)\cdot\nu).$$

Recall that the Liouville law (2.8) is the same as

$$(N_\mu(t))^T JN_\mu(t) = J, \qquad JN_\mu^{-1}(t) = (N_\mu(t))^T J. \tag{4.11}$$

Using (4.9), we have

$$\begin{aligned} Q &= -(N_\mu(1))^T JN_\mu(1)\int_I N_\mu^{-1}(s)\left(\begin{array}{cc} 0 & 0 \\ \varphi_1(s,\mu) & \varphi_2(s,\mu) \end{array} \right) \mathrm{d}\nu(s) \\ &= -J\int_I N_\mu^{-1}(s)\left(\begin{array}{cc} 0 & 0 \\ \varphi_1(s,\mu) & \varphi_2(s,\mu) \end{array} \right) \mathrm{d}\nu(s) \qquad \text{(by (4.11))} \\ &= -\int_I (N_\mu(s))^T J\left(\begin{array}{cc} 0 & 0 \\ \varphi_1(s,\mu) & \varphi_2(s,\mu) \end{array} \right) \mathrm{d}\nu(s) \qquad \text{(by (4.11))} \\ &= -\int_I \left(\begin{array}{cc} \varphi_1^2(s,\mu) & \varphi_1(s,\mu)\varphi_2(s,\mu) \\ \varphi_1(s,\mu)\varphi_2(s,\mu) & \varphi_2^2(s,\mu) \end{array} \right) \mathrm{d}\nu(s). \end{aligned}$$

Therefore

$$\begin{aligned} (R_\mu(\vartheta))^2\cdot\beta &= \int_I X^T \left(\begin{array}{cc} \varphi_1^2(s,\mu) & \varphi_1(s,\mu)\varphi_2(s,\mu) \\ \varphi_1(s,\mu)\varphi_2(s,\mu) & \varphi_2^2(s,\mu) \end{array} \right) X\,\mathrm{d}\nu(s) \\ &= \int_I (\varphi_1(s,\mu)\cos\vartheta - \varphi_2(s,\mu)\sin\vartheta)^2\,\mathrm{d}\nu(s) \\ &= \int_I (y(s,\vartheta,\mu))^2\,\mathrm{d}\nu(s). \end{aligned}$$

This gives (4.10). □

Formula (4.10) is a generalization of the results for arguments of linear ODE which are used to deduce the formulas for differentials of eigenvalues in potentials [25]. It will be used to yield differentials of eigenvalues of MDE in measures.

Note that $\hat{y}(\cdot,\vartheta,\mu)$ in (4.10) is a non-zero solution of Eq. (1.1) and $(\hat{y}(\cdot,\vartheta,\mu))^2$ is non-negative. Formula (4.10) shows that arguments $\Theta_\mu(\vartheta)$ are monotone in μ. Precisely, let us introduce the following ordering for real measures. We say that real measures $\mu_2 \geq \mu_1$, if

$$\int_I f(t)\,\mathrm{d}\mu_2(t) \geq \int_I f(t)\,\mathrm{d}\mu_1(t) \qquad \text{for all } f\in\mathcal{C}_+ := \{f\in\mathcal{C}(I,\mathbb{R}) : f(t)\geq 0,\ t\in I\}.$$

We say that $\mu_2 > \mu_1$, if μ_i satisfy further

$$\int_I f(t)\,\mathrm{d}\mu_2(t) > \int_I f(t)\,\mathrm{d}\mu_1(t) \qquad \text{for all } f\in\mathcal{C}_+\backslash\{0\}.$$

As a consequence of Theorem 4.5, we can apply the Lagrange theorem in Calculus to obtain the following results.

Corollary 4.6 (i) *Let $\vartheta \in \mathbb{R}$. Then*

$$\mu_2 \geq \mu_1 \Longrightarrow \Theta_{\mu_2}(\vartheta) \geq \Theta_{\mu_1}(\vartheta), \qquad \mu_2 > \mu_1 \Longrightarrow \Theta_{\mu_2}(\vartheta) > \Theta_{\mu_1}(\vartheta).$$

(ii) *In particular, given $\mu \in \mathcal{M}_0(I,\mathbb{R})$ and $\vartheta \in \mathbb{R}$. Then, as a function of $\lambda \in \mathbb{R}$, $\Theta_{\lambda\mu_0+\mu}(\vartheta)$ is continuously differentiable and strictly increasing. Here $\mu_0(t) \equiv t$.*

5 Continuity and Continuous Differentiability of Eigenvalues

5.1 Continuity of eigenvalues in weak* topology

Given $\mu \in \mathcal{M}_0(I,\mathbb{R})$. We have known that eigenvalues $\lambda_m^\sigma(\mu)$ are real. By (4.7) and property (4.3) for $\Theta_{\mu_\lambda}(\vartheta)$, like eigenvalues for ODE (1.6), one sees that $\lambda \in \mathbb{R}$ is an eigenvalue of (3.1)-(3.2) iff

$$\Theta_{\lambda\mu_0+\mu}(-\pi/2) = -\pi/2 + m\pi \tag{5.1}$$

for some $m \in \mathbb{Z}$, while $\lambda \in \mathbb{R}$ is an eigenvalue of (3.1)-(3.3) iff

$$\Theta_{\lambda\mu_0+\mu}(0) = m\pi \tag{5.2}$$

for some $m \in \mathbb{Z}$. For ODE (1.6), it is well-known that (5.1) is solvable iff $m \in \mathbb{N}$, while (5.2) is solvable iff $m \in \mathbb{Z}^+$. We will show that these are also true for (3.1). In order to prove these, we need some estimates on $\Theta_{\lambda\mu_0+\mu}(\vartheta)$.

In the following we consider only $\lambda < 0$. Denote $\lambda = -\omega^2$ where $\omega > 0$.

Lemma 5.1 *We have the following estimates*

$$\left|\frac{\varphi_2(1,\mu_\lambda)}{S_\lambda(1)} - 1\right| \leq \frac{2(e^{\|\mu\|_{\mathbf{V}}/\omega} - 1)}{1 - e^{-2\omega}}, \tag{5.3}$$

$$\left|\frac{\overset{\bullet}{\varphi}_2(1,\mu_\lambda)}{C_\lambda(1)} - 1\right| \leq \frac{2\|\mu\|_{\mathbf{V}}}{\omega} e^{\|\mu\|_{\mathbf{V}}/\omega}. \tag{5.4}$$

Proof We still use the expansion (3.29) for $\varphi_2(t) := \varphi_2(t,\mu_\lambda)$. That is,

$$\varphi_2(t) = S_\lambda(t) + \sum_{n=1}^{\infty} \int_{0\leq t_1 \leq \cdots \leq t_{n+1} := t} S_\lambda(t_1) T_n(t_1,\cdots,t_{n+1})\, \mathrm{d}\mu(t_1)\cdots \mathrm{d}\mu(t_n).$$

In (3.25), we have used (3.24) for $S_\lambda(t)$ which is true for all $\lambda \in \mathbb{C}$. When $\lambda = -\omega^2 < 0$, we can give another estimate

$$0 \leq S_\lambda(t) = (\sinh \omega t)/\omega = (e^{\omega t} - e^{-\omega t})/(2\omega) < e^{\omega t}/\omega, \qquad t \in I.$$

Thus

$$|T_n(t_1,\cdots,t_{n+1})| \leq e^{\omega t}/\omega^{n+1}.$$

Hence

$$|\varphi_2(t) - S_\lambda(t)| \leq \sum_{n=1}^{\infty} \frac{e^{\omega t}}{\omega^{n+1}} I_n(t) \leq \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{e^{\omega t}}{\omega^n} \frac{(\check{\mu}(t))^n}{n!} = \frac{e^{\omega t}(e^{\check{\mu}(t)/\omega} - 1)}{\omega}. \tag{5.5}$$

Letting $t = 1$, we get

$$|\varphi_2(1) - S_\lambda(1)| \le \frac{e^{\omega}(e^{\|\mu\|_{\mathbf{V}}/\omega} - 1)}{\omega}.$$

Thus

$$\left|\frac{\varphi_2(1,\mu_\lambda)}{S_\lambda(1)} - 1\right| \le \frac{e^{\omega}(e^{\|\mu\|_{\mathbf{V}}/\omega} - 1)}{\omega S_\lambda(1)} = \frac{2(e^{\|\mu\|_{\mathbf{V}}/\omega} - 1)}{1 - e^{-2\omega}},$$

obtaining estimate (5.3).

Note that $S'_\lambda(t) \equiv C_\lambda(t)$. From (3.12), we have, for $t \in I$,

$$\begin{aligned}\overset{\bullet}{\varphi}_2(t) - C_\lambda(t) &= -\int_{[0,t]} C_\lambda(t-s)\varphi_2(s)\,\mathrm{d}\mu(s)\\ &= -\int_{[0,t]} C_\lambda(t-s)S_\lambda(s)\,\mathrm{d}\mu(s) - \int_{[0,t]} J_\lambda(t,s)\,\mathrm{d}\mu(s),\end{aligned} \tag{5.6}$$

where

$$J_\lambda(t,s) = C_\lambda(t-s)(\varphi_2(s) - S_\lambda(s)).$$

Note that

$$0 \le C_\lambda(t-s)S_\lambda(s) = \cosh(\omega(t-s))\frac{\sinh(\omega s)}{\omega} < \frac{e^{\omega t}}{\omega},$$

and, by (5.5),

$$|J_\lambda(t,s)| \le \cosh(\omega(t-s))\frac{e^{\omega s}(e^{\check{\mu}(s)/\omega} - 1)}{\omega} \le \frac{e^{\omega t}(e^{\check{\mu}(t)/\omega} - 1)}{\omega}.$$

Now (5.6) can yield

$$|\overset{\bullet}{\varphi}_2(1,\mu_\lambda) - C_\lambda(1)| \le \frac{e^{\omega}}{\omega}e^{\|\mu\|_{\mathbf{V}}/\omega}\|\mu\|_{\mathbf{V}}.$$

Note that $C_\lambda(1) = \cosh\omega > e^{\omega}/2$. Thus

$$\left|\frac{\overset{\bullet}{\varphi}_2(1,\mu_\lambda)}{C_\lambda(1)} - 1\right| \le \frac{2}{\omega}e^{\|\mu\|_{\mathbf{V}}/\omega}\|\mu\|_{\mathbf{V}}.$$

This proves (5.4). □

Note that when $\lambda \to -\infty$, $C_\lambda(1) = \cosh\omega \to +\infty$ and $S_\lambda(1) = (\sinh\omega)/\omega \to +\infty$. Meanwhile, the errors in (5.3) and (5.4) can be well-controlled by the norm $\|\mu\|_{\mathbf{V}}$ and have the order $O(1/\omega)$.

Lemma 5.2 *There hold the following asymptotical estimates*

$$\lim_{\lambda\to-\infty} \Theta_\mu(-\pi/2,\lambda) = -\pi/2, \tag{5.7}$$

$$\lim_{\lambda\to+\infty} \Theta_\mu(-\pi/2,\lambda) = +\infty. \tag{5.8}$$

Proof For the ODE case, both of these can be estimated from first-order ODE (4.2) for arguments. For the MDE case, since we are not able to understand first-order nonlinear MDE like (4.8), the proof is much complicated.

Let $\lambda \ll -1$. The initial argument of $(\varphi_2(t,\mu_\lambda), -\overset{\bullet}{\varphi}_2(t,\mu_\lambda))$ at $t=0$ is taken as $\vartheta = -\pi/2$. Let $(r_0,\vartheta) = (1,-\pi/2)$ in (4.7). We get

$$\cot \Theta_\mu(-\pi/2,\lambda) = \frac{\varphi_2(1,\mu_\lambda)}{\overset{\bullet}{\varphi}_2(1,\mu_\lambda)}. \tag{5.9}$$

We argue by connecting μ to the measure 0 using the homotopy $\tau\mu$, $\tau \in [0,1]$. Thus (5.9) is also true for $\tau\mu$. Now the estimates (5.3) and (5.4) are true uniformly in $\tau \in [0,1]$. Thus, if $\lambda \ll -1$, one has uniformly (in $\tau \in [0,1]$)

$$\begin{aligned}\cot \Theta_\mu(-\pi/2,\tau\lambda) &= \frac{\varphi_2(1,\lambda,\tau\mu)}{\overset{\bullet}{\varphi}_2(1,\lambda,\tau\mu)} = \frac{\varphi_2(1,\lambda,\tau\mu)/S_\lambda(1)}{\overset{\bullet}{\varphi}_2(1,\lambda,\tau\mu)/C_\lambda(1)} \cdot \frac{S_\lambda(1)}{C_\lambda(1)} \\ &= \frac{1+o(1/\omega)}{1+o(1/\omega)} \frac{\tanh\omega}{\omega} = O\left(\frac{1}{\omega}\right).\end{aligned}$$

Thus, for any $\varepsilon \in (0,1)$, there exists a constant $\Lambda_0 = \Lambda_0(\varepsilon, \|\mu\|_{\mathbf{V}}) < 0$ such that

$$\lambda \le \Lambda_0 \Longrightarrow |\cot \Theta_{\tau\mu}(-\pi/2,\lambda)| < \varepsilon \qquad \text{for all } \tau \in [0,1]. \tag{5.10}$$

Note that $\cot : (0,\pi) \to \mathbb{R}$ is a decreasing homeomorphism. Denote

$$\theta_\varepsilon := \max\left(\pi/2 - \cot^{-1}\varepsilon,\ \cot^{-1}(-\varepsilon) - \pi/2\right).$$

Then $0 < \theta_\varepsilon < \pi/4$ because $\varepsilon \in (0,1)$. Moreover,

$$\lim_{\varepsilon \downarrow 0} \theta_\varepsilon = 0. \tag{5.11}$$

Let $\lambda \le \Lambda_0(\varepsilon, \|\mu\|_{\mathbf{V}})$. For any $\tau \in [0,1]$, (5.10) implies that there exists $k_{\lambda,\tau} \in \mathbb{Z}$ such that

$$|\Theta_{\tau\mu}(-\pi/2,\lambda) + \pi/2 + k_{\lambda,\tau}\pi| < \theta_\varepsilon. \tag{5.12}$$

As $\theta_\varepsilon < \pi/4$ and $\Theta_{\tau\mu}(-\pi/2,\lambda)$ is continuous in $\tau \in [0,1]$, one sees from (5.12) that $k_{\lambda,\tau} = k_{\lambda,0}$ is necessarily independent of $\tau \in [0,1]$. Now we can use the continuity of $\Theta_{\tau\mu}(-\pi/2,\lambda)$ is continuous in $\lambda \in (-\infty,\Lambda_0]$ to obtain from (5.12) that $k_{\lambda,0}$ is actually independent of $(-\infty,\Lambda_0]$. That is, there exists some $k_0 \in \mathbb{Z}$ such that

$$|\Theta_{\tau\mu}(-\pi/2,\lambda) + \pi/2 + k_0\pi| < \theta_\varepsilon \qquad \text{for all } \tau \in [0,1],\ \lambda \in (-\infty,\Lambda_0]. \tag{5.13}$$

When $\tau = 0$, it is well-known that $\lim_{\lambda \to -\infty} \Theta_{0\cdot\mu}(-\pi/2,\lambda) = -\pi/2$. This implies that $k_0 = 0$. Let $\tau = 1$ in (5.13). We get

$$|\Theta_\mu(-\pi/2,\lambda) + \pi/2| < \theta_\varepsilon \qquad \text{for all } \lambda \in (-\infty,\Lambda_0].$$

By noticing the fact (5.11), we have proved (5.7).

Result (5.8) can be proved by developing estimates on $\Theta_\mu(-\pi/2,\lambda)$ for $\lambda \gg 1$. Since we have the existence of eigenvalues in Theorem 3.8, (5.8) can be obtained in a simple way. From Corollary 4.6 (ii), $\lambda \to \Theta_\mu(-\pi/2,\lambda)$ is strictly increasing in $\lambda \in \mathbb{R}$. Note that (5.1) is solvable only when $m \in \mathbb{N}$. If (5.8) fails, (5.1) is solvable only for finitely many $m \in \mathbb{N}$. Hence problem (3.1)-(3.2) would have only finitely many eigenvalues, a contradiction to Theorem 3.8. □

By Theorem 3.8, the Dirichlet eigenvalues $\lambda_m^D(\mu)$, $m \in \mathbb{N}$, are zeros of the entire function $\varphi_2(1,\mu_\lambda)$. See (3.9). This is not convenient for us to study the dependence of $\lambda_m^D(\mu)$ on μ. Using the arguments, we have another characterization.

Theorem 5.3 *Let $\mu \in \mathcal{M}_0(I, \mathbb{R})$. Then $\lambda = \lambda_m^D(\mu)$ for $m \in \mathbb{N}$ iff λ satisfies* (5.1) *with the same m. Similarly, $\lambda = \lambda_m^N(\mu)$ for $m \in \mathbb{Z}^+$ iff λ satisfies* (5.2) *with the same m.*

Proof By Corollary 4.6 (ii) and asymptotical results (5.7) and (5.8), the range of the function $\Theta_\mu(-\pi/2, \cdot)$ is $(-\pi/2, \infty)$. Hence (5.1) is solvable iff $m \in \mathbb{N}$.

For the Neumann eigenvalues, it follows from (5.7) and (5.8) that

$$\lim_{\lambda \to -\infty} \Theta_\mu(0, \lambda) = -\pi/2, \qquad \lim_{\lambda \to +\infty} \Theta_\mu(0, \lambda) = +\infty.$$

This shows that (5.2) is solvable iff $m \in \mathbb{Z}^+$. $\square$

Now we can give the continuity of eigenvalues in measures with the weak* topology.

Proof of Theorem 1.2. We only give the proof for $\sigma = D$. Let $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I, \mathbb{R}), w^*)$. Denote $\xi_n := \lambda_m^D(\mu_n)$, $n \geq 0$. By Theorem 5.3, we have

$$\Theta_{\mu_n}(-\pi/2, \xi_n) \equiv -\pi/2 + m\pi, \qquad \forall\, n \geq 0. \tag{5.14}$$

We need to prove $\lim_{n\to\infty} \xi_n = \xi_0$. Otherwise, without loss of generality, assume that there exists $\varepsilon_0 > 0$ such that $|\xi_n - \xi_0| \geq \varepsilon_0$ for all $n \geq 1$. Furthermore, we can assume $\xi_n - \xi_0 \geq \varepsilon_0$ for all $n \geq 1$. By Corollary 4.6 (ii), we have

$$-\pi/2 + m\pi = \Theta_{\mu_n}(-\pi/2, \xi_n) \geq \Theta_{\mu_n}(-\pi/2, \xi_0 + \varepsilon_0), \qquad n \geq 1.$$

Since $\mu_n \to \mu_0$, by letting $n \to \infty$, we know from Theorem 4.4 that

$$-\pi/2 + m\pi \geq \Theta_{\mu_0}(-\pi/2, \xi_0 + \varepsilon_0).$$

By Corollary 4.6 (ii) again, we get

$$-\pi/2 + m\pi \geq \Theta_{\mu_0}(-\pi/2, \xi_0 + \varepsilon_0) > \Theta_{\mu_0}(-\pi/2, \xi_0).$$

From (5.14) with $n = 0$, this is impossible. $\square$

5.2 Continuous differentiability of eigenvalues in measures

Proof of Theorem 1.3. Since $\Theta_\mu(\vartheta)$ is continuously differentiable in $\mu \in (\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_{\mathbf{V}})$, see Theorem 4.5, it follows from the implicit function expressions (5.1) and (5.2) that each $\lambda_m^\sigma(\mu)$ is continuously differentiable in μ. $\square$

Denote

$$\hat{y}_m^D(t) := \hat{y}(t, -\pi/2, \lambda_m^D(\mu)\mu_0 + \mu), \qquad E_m^D(t, \mu) := \frac{\hat{y}_m^D(t)}{\|\hat{y}_m^D\|_2}.$$

Then both $\hat{y}_m^D(t)$ and $E_m^D(t, \mu)$ are eigen-functions associated with $\lambda_m^D(\mu)$. For any $\nu \in \mathcal{M}_0(I, \mathbb{R})$, the directional derivative of $\lambda_m^D(\mu)$ along the direction ν is denoted by $\ell(\nu) \in \mathbb{R}$. That is, for $\tau \in \mathbb{R}$ with $|\tau| \ll 1$,

$$\lambda_m^D(\mu + \tau\nu) = \lambda_m^D(\mu) + \tau\ell(\nu) + o(\tau).$$

Hence

$$\lambda_m^D(\mu + \tau\nu)\mu_0 + \mu + \tau\nu = \big(\lambda_m^D(\mu)\mu_0 + \mu\big) + \tau(\ell(\nu)\mu_0 + \nu) + o(\tau).$$

By (5.1), we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial \tau} \Theta_{\lambda_m^D(\mu+\tau\nu)\mu_0+\mu+\tau\nu}(-\pi/2)\bigg|_{\tau=0} \\ &= \int_I \left(\hat{y}_m^D(s)\right)^2 \mathrm{d}\left(\ell(\nu)\mu_0(s)+\nu(s)\right) \qquad \text{(by (4.10))} \\ &= \ell(\nu)\int_I \left(\hat{y}_m^D(s)\right)^2 \mathrm{d}s + \int_I \left(\hat{y}_m^D(s)\right)^2 \mathrm{d}\nu(s). \end{aligned}$$

Thus

$$\ell(\nu) = -\frac{\int_I \left(\hat{y}_m^D(s)\right)^2 \mathrm{d}\nu(s)}{\int_I \left(\hat{y}_m^D(s)\right)^2 \mathrm{d}s} = -\int_I (E_m^D(s,\mu))^2 \,\mathrm{d}\nu(s),$$

which is bounded linear functional of $(\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})$ because $-(E_m^D(\cdot,\mu))^2 \in \mathcal{C}(I,\mathbb{R})$.

The directional derivatives of the Neumann eigenvalues $\lambda_m^N(\mu)$ can be found in a similar way. We write down these as the following results.

Corollary 5.4 *For eigenvalue $\lambda_m^\sigma(\mu)$, $\mu \in \mathcal{M}_0(I,\mathbb{R})$, let $E_m^\sigma(\cdot,\mu)$ be an eigen-function associated with $\lambda_m^\sigma(\mu)$ and satisfying the following normalization condition*

$$\|E_m^\sigma(\cdot,\mu)\|_2 = \left(\int_I (E_m^\sigma(t,\mu))^2 \,\mathrm{d}t\right)^{1/2} = 1.$$

The the Fréchet derivative of $\lambda_m^\sigma(\mu)$ is given by

$$\begin{aligned} \partial_\mu \lambda_m^\sigma(\mu) &= -(E_m^\sigma(\cdot,\mu))^2 \in (\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty) \\ &\hookrightarrow (\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)^{**} \cong (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})^*. \end{aligned} \tag{5.15}$$

Remark 5.5 For a general differentiable functional $F : (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \to \mathbb{R}$, Fréchet derivatives $\partial_\mu F(\mu)$ are in the double-dual space $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)^{**} \cong (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})^*$. Since $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)$ is not reflexive, the double-dual space $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)^{**}$ is unclear. However, for eigenvalues $\lambda_m^\sigma(\mu)$, (5.15) shows that Fréchet derivatives are of eigenvalues are actually in $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)$ which is a subspace of $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)^{**}$. This is important when one considers extremal problems of eigenvalues of MDE.

6 Extremal Eigenvalues of ODE with Potentials in $\mathcal{L}^1$ Balls

6.1 More facts on weak topologies

In the Lebesgue spaces $\mathcal{L}^p(I,\mathbb{R})$, $p \in [1,\infty]$, besides the norms $\|\cdot\|_p$, one has the following weak topologies [6, 11]. For $p \in [1,\infty)$, we use w_p to indicate the topology of weak convergence in $\mathcal{L}^p(I,\mathbb{R})$, and for $p=\infty$, by considering $\mathcal{L}^\infty(I,\mathbb{R})$ as the dual space of $(\mathcal{L}^1(I,\mathbb{R}), \|\cdot\|_1)$, we have the topology w_∞ of the weak* convergence. In a unified way, $q_n \to q_0$ in $(\mathcal{L}^p(I,\mathbb{R}), w_p)$ iff

$$\int_I f(t)q_n(t)\,\mathrm{d}t \to \int_I f(t)q_0(t)\,\mathrm{d}t \qquad \forall f \in \mathcal{L}^{p^*}(I,\mathbb{R}).$$

Here $p^* := p/(p-1) \in [1,\infty]$ is the conjugate exponent of p.

Considering $q \in \mathcal{L}^p(I, \mathbb{R})$ as a density, one has the measure or distribution

$$\mu_q(t) := \int_{[0,t]} q(s)\, \mathrm{d}s, \qquad t \in I. \tag{6.1}$$

Then $\mu_q \in \mathcal{M}_0(I, \mathbb{R})$. Moreover, $\|\mu_q\|_{\mathbf{V}} = \|q\|_1$. That is,

$$(\mathcal{L}^1(I, \mathbb{R}), \|\cdot\|_1) \hookrightarrow (\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_{\mathbf{V}}) \tag{6.2}$$

is an isometric embedding.

The following is a direct consequence of definition of weak topologies.

Lemma 6.1 *The following embeddings are continuous*

$$(\mathcal{L}^{p'}(I, \mathbb{R}), w_{p'}) \hookrightarrow (\mathcal{L}^p(I, \mathbb{R}), w_p) \hookrightarrow (\mathcal{M}_0(I, \mathbb{R}), w^*) \qquad \forall\ \infty \ge p' > p \ge 1.$$

For $r \in [0, \infty)$, define balls

$$\begin{aligned} B_p[r] &:= \{q \in \mathcal{L}^p(I, \mathbb{R}) : \|q\|_p \le r\}, \qquad p \in [1, \infty], \\ B_0[r] &:= \{\mu \in \mathcal{M}_0(I, \mathbb{R}) : \|\mu\|_{\mathbf{V}} \le r\}. \end{aligned}$$

Via (6.1), by the Hölder inequality and the isometrical embedding of (6.2), one has the following results on these balls.

Lemma 6.2 *Let $r > 0$. The following inclusions are proper*

$$B_{p'}[r] \subset B_p[r] \subset B_0[r] \qquad \forall\ \infty \ge p' > p \ge 1.$$

As for the compactness of these balls in weak topologies, we have the following results.

Lemma 6.3 *Let $r > 0$. Then $B_p[r] \subset (\mathcal{L}^p(I, \mathbb{R}), w_p)$ with $1 < p \le \infty$, and $B_0[r] \subset (\mathcal{M}_0(I, \mathbb{R}), w^*)$ are sequentially compact, while $B_1[r] \subset (\mathcal{L}^1(I, \mathbb{R}), w_1)$ is not sequentially compact.*

For $\mathcal{L}^p$ balls $B_p[r]$, we have the following result.

Lemma 6.4 [28] *Given $r \in [0, \infty)$. There holds*

$$\overline{\bigcup_{p \in (1,\infty)} B_p[r]} = B_1[r],$$

where the closure is taken in the $\mathcal{L}^1$ space $(\mathcal{L}^1(I, \mathbb{R}), \|\cdot\|_1)$.

6.2 Extremal problems of eigenvalues in $\mathcal{L}^p$ balls

For a potential $q \in \mathcal{L}^p(I, \mathbb{R})$ where $p \in [1, \infty]$, we use $\{\lambda_m^D(q)\}_{m \in \mathbb{N}}$ or $\{\lambda_m^N(q)\}_{m \in \mathbb{Z}^+}$ to denote eigenvalues of (1.6), with the Dirichlet boundary condition (3.2) or with the Neumann boundary condition (3.3). In [23, 28], the authors have studied the following extremal values

$$\mathbf{L}_{m,p}^\sigma(r) := \inf \{\lambda_m^\sigma(q) : q \in B_p[r]\}, \qquad r \in [0, \infty).$$

The most interesting case is $p = 1$.

It is proved in [27] that $\lambda_m^\sigma(q)$ are continuous in $q \in (\mathcal{L}^p(I,\mathbb{R}), w_p)$ for all $p \in [1,\infty]$. When $p \in (1,\infty]$, by the compactness of Lemma 6.3, one has $\mathbf{L}_{m,p}^\sigma(r) \in \mathbb{R}$ are finite. Moreover, $\mathbf{L}_{m,p}^\sigma(r)$ can be attained by some potentials $q_{m,p,r}^\sigma \in B_p[r]$. When $p \in (1,\infty)$, the authors have derived in [23, 28] the critical equations for $q_{m,p,r}^\sigma$ and given a characterization on $\mathbf{L}_{m,p}^\sigma(r)$. For the most interesting case $p = 1$, by the non-compactness of $\mathcal{L}^1$ balls $B_1[r]$, it is a question whether $\mathbf{L}_{m,1}^\sigma(r)$ are finite for all $r \in [0,\infty)$. However, due to the denseness of Lemma 6.4, one has

$$\mathbf{L}_{m,1}^\sigma(r) = \lim_{p\downarrow 1} \mathbf{L}_{m,p}^\sigma(r) \qquad \forall\, r \in [0,\infty). \tag{6.3}$$

Based on the characterization for $\mathbf{L}_{m,p}^\sigma(r)$ for $p \in (1,\infty)$ and equality (6.3), the authors of [23, 28] have used complicated limiting techniques to prove that all $\mathbf{L}_{m,1}^\sigma(r)$ are finite. Moreover, these important extremal values $\mathbf{L}_{m,1}^\sigma(r)$ are elementary functions of r which have been constructed explicitly.

For example, let us consider the zeroth Neumann eigenvalues $\lambda_0^N(q)$ of (1.6). By introducing the following elementary function

$$\hat{\mathbf{Z}}_0(x) = \sqrt{-x}\tanh\sqrt{-x}, \qquad x \in (-\infty, 0], \tag{6.4}$$

one has actually

$$\mathbf{L}_{0,1}^N(r) \equiv \hat{\mathbf{Z}}_0^{-1}(r) \qquad \forall\, r \in [0,\infty). \tag{6.5}$$

Now we apply the results of this paper on MDE to give a quite natural explanation to the finiteness of extremal values in [23, 28]. Moreover, the extremal values there can be obtained by finding the weak* limits of the critical potentials $q_{m,p,r}^\sigma \in B_p[r] \subset \mathcal{M}_0(I,\mathbb{R})$.

Since $\mathcal{M}_0(I,\mathbb{R})$ is the dual space of the (separable) Banach space $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)$, it is well-known that any $\|\cdot\|_{\mathbf{V}}$-bounded subset $S \subset \mathcal{M}_0(I,\mathbb{R})$ is relatively sequentially compact in the weak* topology w^*. As an immediate consequence of Theorem 1.2, we have the following results.

Corollary 6.5 (i) *Let $S \subset (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}})$ be a bounded set. Then, for each m, both $\inf_{\mu\in S}\lambda_m^\sigma(\mu)$ and $\sup_{\mu\in S}\lambda_m^\sigma(\mu)$ are finite.*

(ii) *In particular, let m and r be given. Then there exist $\underline{\mu}_{m,r}^\sigma$, $\overline{\mu}_{m,r}^\sigma \in B_0[r]$ such that*

$$\mathbf{L}_m^\sigma(r) := \inf_{\mu\in B_0[r]} \lambda_m^\sigma(\mu) = \lambda_m^\sigma(\underline{\mu}_{m,r}^\sigma) > -\infty,$$
$$\mathbf{M}_m^\sigma(r) := \sup_{\mu\in B_0[r]} \lambda_m^\sigma(\mu) = \lambda_m^\sigma(\overline{\mu}_{m,r}^\sigma) < +\infty.$$

For eigenvalues of Sturm-Liouville operators with $\mathcal{L}^p$ potentials, the point is as follows. Via (6.1), any $q \in \mathcal{L}^p(I,\mathbb{R})$ induces a measure $\mu_q \in \mathcal{M}_0(I,\mathbb{R})$. Using eigenvalues of MDE of this paper, one has $\lambda_m^\sigma(q) = \lambda_m^\sigma(\mu_q)$. By the inclusions of Lemma 6.2 and the finiteness results of Corollary 6.5, we can obtain the following results immediately.

Corollary 6.6 *There hold the following finiteness results*

$$-\infty < \mathbf{L}_m^\sigma(r) \le \mathbf{L}_{m,p}^\sigma(r) \le \mathbf{M}_{m,p}^\sigma(r) \le \mathbf{M}_m^\sigma(r) < +\infty$$

for all $p \in [1,\infty]$ and all $r \in [0,\infty)$.

6.3 Weak* limits of minimal potentials in $\mathcal{L}^p$ balls

As mentioned before, the extremal values $\mathbf{L}^\sigma_{m,1}(r)$ in $\mathcal{L}^1$ balls, like (6.4) and (6.5), are obtained by complicated limiting analysis. In this subsection, let us see how we can use MDE to obtain them in a simple way. As a concise example, we only consider the minimal values $\mathbf{L}^N_{0,1}(r)$ of the zeroth Neumann eigenvalues $\lambda_0^N(q)$. This is based on (6.3). For the zeroth Neumann eigenvalues, it is

$$\mathbf{L}^N_{0,1}(r) = \lim_{p\downarrow 1} \mathbf{L}^N_{0,p}(r). \tag{6.6}$$

In order to consider (6.6), let us introduce the following Dirac measures in $\mathcal{M}_0(I,\mathbb{R})$. Given $a \in (0,1]$ and $r \in \mathbb{R}$. One has the following Dirac measure at $t = a$ of the total mass r

$$\Delta_{a,r}(t) = \begin{cases} 0 & \text{for } t \in [0,a), \\ r & \text{for } t \in [a,1]. \end{cases} \tag{6.7}$$

Let $a = 0$ and $r \in \mathbb{R}$. One has the following Dirac measure at $t = 0$ of the total mass r

$$\Delta_{0,r}(t) = \begin{cases} -r & \text{for } t = 0, \\ 0 & \text{for } t \in (0,1]. \end{cases} \tag{6.8}$$

Note that both (6.7) and (6.8) satisfy the normalization condition $\Delta_{a,r}(0+) = 0$.

In the following, let $r > 0$ be fixed. For any $p \in (1,\infty)$, there exist some potentials $q_p \in B_p[r]$ such that $\mathbf{L}^N_{0,p}(r) = \lambda_0^N(q_p)$. These q_p are called minimizers.

Let us recall from [28] some facts on minimizers. Basically, as we are considering the minimal values, $q_p(t) > 0$ for all $t \in I$. Moreover, they are on the $\mathcal{L}^p$ sphere

$$\|q_p\|_p = r. \tag{6.9}$$

Consequently,

$$\mathbf{L}^N_{0,p}(r) = \min\left\{\lambda_0^N(q) : \|q\|_p = r\right\}.$$

Due to the continuous differentiability of $\lambda_0^N(q)$ and the $\mathcal{L}^p$ sphere, the minimizers q_p can be computed using the variational method. In [28], detailed computation has been undertaken for the zeroth periodic eigenvalues. However, due to the relation between the zeroth Neumann eigenvalues and the zeroth periodic eigenvalues, computation there can be transformed to that for $\mathbf{L}^N_{0,p}(r)$. In the following we recall some results on $\mathbf{L}^N_{0,p}(r)$, with a minor change of statements. For details, see [28, Section 6].

Let us introduce

$$\xi_p := -\lambda_0^N(q_p) = -\mathbf{L}^N_{0,p}(r) > 0, \qquad y_p(t) := (q_p(t))^{(p-1)/2} = (q_p(t))^{2p^*-2}. \tag{6.10}$$

We know that $y_p(t)$ satisfies the following critical equation

$$\ddot{y}_p - \xi_p y_p + y_p^{2p^*-1} = 0, \qquad t \in I. \tag{6.11}$$

Moreover, $y_p(t)$ satisfies the Neumann boundary condition

$$\dot{y}_p(0) = \dot{y}_p(1) = 0, \tag{6.12}$$

and the following condition, deduced from the constraint (6.9),

$$\|y_p\|_{2p^*}^{2p^*/p} = r. \tag{6.13}$$

By (6.10), critical equation (6.11) can be written as

$$\ddot{y}_p - \xi_p y_p + q_p(t) y_p(t) = 0.$$

Hence $y_p(t)$ is actually an eigen-function associated with $\lambda_0^N(q_p) = -\xi_p$. Note all of these objects are dependent on r as well.

For the phase portrait of critical equation (6.11), see [28, Figure 1]. The trick there is to consider ξ_p as a parameter and all solutions of the autonomous equation (6.11) satisfying (6.12) can be analyzed in details. Finally, $\mathbf{L}_{0,p}^N(r) = -\xi_p$ is determined by the constraint (6.13).

Note that for the Neumann eigenvalues $\lambda_m^N(q)$ of a potential q, by introducing the reflection of q as $\hat{q}(t) := q(1-t)$, we have $\|\hat{q}\|_p = \|q\|_p$ and $\lambda_m^N(\hat{q}) = \lambda_m^N(q)$.

Lemma 6.7 [28] *Given $r \in (0,\infty)$. We have the following results.*

- *There exists some $P_r > 1$ such that if $p \in [P_r, \infty)$, the only solution $y_p(t)$ of (6.11)–(6.13) is constant. In this case, one has $q_p(t) \equiv r$ and $\mathbf{L}_{0,p}^N(r) = -r$.*
- *On the other hand, there exists some $1 < P'_r \le P_r$ such that if $p \in (1, P'_r)$, problem (6.11)–(6.13) has two non-constant solutions. If one is $y_p(t)$, another is the reflection $\hat{y}_p(t) := y_p(1-t)$. In this case, by (6.10), $\mathbf{L}_{0,p}^N(r)$ has two non-constant minimizers, correlated by the reflection. Moreover, $\mathbf{L}_{0,p}^N(r) < -r$.*

In order to study $\mathbf{L}_{0,1}^N(r)$ using (6.6), we may assume that p is sufficiently near 1. Hence we need only to consider the second case of Lemma 6.7. For definite, let us choose q_p and y_p such that

$$a_p := \min_{t\in I} y_p(t) = y_p(0). \tag{6.14}$$

From the phase portrait of (6.11), $y_p(t)$ is strictly increasing in $t \in I$. Therefore

$$b_p := \max_{t\in I} y_p(t) = y_p(1). \tag{6.15}$$

In fact, as (6.11) is autonomous and $y_p(t)$ satisfies (6.12), $y_p(t)$ can be considered as a (positive) periodic solution of (6.11) of the minimal period 2.

Some estimates on $y_p(t)$ are as follows. Note that (6.11) has the unique (positive) equilibrium $y = \xi_p^{1/(2p^*-2)} = \xi_p^{(p-1)/2}$, we have

$$0 < a_p < \xi_p^{(p-1)/2} < b_p. \tag{6.16}$$

From the conservative law for (6.11), one has also

$$-\xi_p a_p^2 + a_p^{2p^*}/p^* = -\xi_p b_p^2 + b_p^{2p^*}/p^*.$$

This implies that b_p actually satisfies

$$\xi_p^{(p-1)/2} < b_p < \xi_p^{(p-1)/2}\mathbf{b}_p, \qquad \mathbf{b}_p := (p^*)^{(p-1)/2}. \tag{6.17}$$

Lemma 6.8 *We assert that*

$$\lim_{p\downarrow 1} \|q_p\|_1 = r. \tag{6.18}$$

Proof By the Hölder inequality, one has $\|q_p\|_1 \leq \|q_p\|_p = r$. Hence $\limsup_{p\downarrow 1} \|q_p\|_1 \leq r$. On the other hand, condition (6.13) is

$$\int_I (y_p(t))^{2p^*}\, \mathrm{d}t = \|y_p\|_{2p^*}^{2p^*} = r^p.$$

Thus

$$\begin{aligned} \|q_p\|_1 &= \int_I (y_p(t))^{2p^*-2}\, \mathrm{d}t = \int_I \frac{(y_p(t))^{2p^*}}{(y_p(t))^2}\, \mathrm{d}t \\ &\geq \frac{1}{b_p^2} \int_I (y_p(t))^{2p^*}\, \mathrm{d}t = \frac{r^p}{b_p^2}. \end{aligned} \tag{6.19}$$

Since $\lim_{p\downarrow 1} \xi_p = -\mathbf{L}_{0,1}^N(r) \in (0,\infty)$ and $\lim_{p\downarrow 1} \mathbf{b}_p = 1$, we have from (6.17) $\lim_{p\downarrow 1} b_p = 1$. Now (6.13) shows that $\liminf_{p\downarrow 1} \|q_p\|_1 \geq 1$. Hence we have (6.18). □

Via (6.1), q_p induces a measure $Q_p = \mu_{q_p} \in B_0[r]$. Note that $\|Q_p\|_{\mathbf{V}} = \|q_p\|_1$. By (6.18), one has $\lim_{p\downarrow 1} \|Q_p\|_{\mathbf{V}} = r$. We will find the weak* limits of $\lim_{p\downarrow 1} Q_p$ in Lemma 6.10 below. To this end, let us recall the Alexandroff Theorem [6, p. 316] for weak* convergence.

Lemma 6.9 *Let $\mu_n,\ \mu_0 \in \mathcal{M}_0(I,\mathbb{K})$. Suppose that*

- *$\{\|\mu_n\|_{\mathbf{V}}\}_{n\in\mathbb{N}}$ is bounded, and*
- *for any open subset $B \subset I$ satisfying $\mu_0(B) = \mu_0(\bar{B})$, one has $\mu_n(B) \to \mu_0(B)$.*

Then one has $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$.

Lemma 6.10 *Let $r > 0$ be fixed. As $p \downarrow 1$, $Q_p \to \Delta_{1,r}$ in the space $(\mathcal{M}_0(I,\mathbb{R}), w^*)$, where $\Delta_{1,r}$ is the Dirac measure given by (6.7).*

Proof Since $\{Q_p : p \in (1, P_r')\} \subset B_0[r]$, for any sequence $\{p_n\} \subset (1, P_r')$ such that $p_n \downarrow 1$, by the compactness of Lemma 6.3, one has a sub-sequence $\{p_n'\}$ such that $Q_{p_n',r} \to \mu_* \in B_0[r]$ in the weak* topology w^*. We will prove

$$\mu_* = \Delta_{1,r}. \tag{6.20}$$

The only estimate for $y_p(t)$ in [28] to be used is as follows. For any $\beta \in (0,1)$, there exists some $p_\beta \in (1, P_r')$ such that

$$\sup_{t\in[0,\beta],\ p\in(1,p_\beta)} y_p(t) < 1. \tag{6.21}$$

This can be obtained from the qualitative analysis for critical equation (6.11). By (6.10), $q_p(t) = (y_p(t))^{2p^*-2}$. As $p \downarrow 1$, we have $p^* \uparrow \infty$ and (6.21) implies

$$q_p \to 0 \ \text{ in } (\mathcal{C}([0,\beta],\mathbb{R}), \|\cdot\|_\infty) \qquad \forall\ \beta \in (0,1). \tag{6.22}$$

Now we are going to apply Lemma 6.9 to $\mu_{n'} := Q_{p_n',r}$ and $\mu_0 := \Delta_{1,r}$. Let B be an open interval of the closed interval I. We distinguish B in several cases.

Case 1. $B = [0,b)$ for some $b \in (0,1)$. In this case $\Delta_{1,r}(B) = 0 = \Delta_{1,r}(\bar{B})$, while (6.22) shows that

$$Q_{p_n',r}(B) = \int_B q_{p_n',r}(t)\, \mathrm{d}t \to 0. \tag{6.23}$$

Case 2. $B = [0, 1)$. In this case, one has $\Delta_{1,r}(B) = 0 \neq r = \Delta_{1,r}(\bar{B})$. We need not to consider.

Case 3. $B = (a, b)$ for some $0 \leq a < b < 1$. In this case $\Delta_{1,r}(B) = 0 = \Delta_{1,r}(\bar{B})$. One has also (6.23), following from (6.22).

Case 4. $B = (a, 1)$ for some $0 \leq a < 1$. In this case, one has $\Delta_{1,r}(B) = 0 \neq r = \Delta_{1,r}(\bar{B})$. We need not to consider.

Case 5. Finally, $B = (a, 1]$ is also an open interval of I where $0 \leq a < 1$. In this case, one has $\Delta_{1,r}(B) = r = \Delta_{1,r}(\bar{B})$. We have

$$Q_{p'_n,r}(B) = \int_{(a,1]} q_{p'_n,r}(t)\, \mathrm{d}t = \|q_{p'_n,r}\|_1 - \int_{[0,a)} q_{p'_n,r}(t)\, \mathrm{d}t \to r,$$

following from (6.18) and (6.23).

In conclusion, Lemma 6.9 shows that (6.20) holds. Since the limit $\Delta_{1,r}$ of (6.20) is independent of the choice of subsequences p_n and p'_n, we have proved that $Q_p \to \Delta_{1,r}$ in $(\mathcal{M}_0(I, \mathbb{R}), w^*)$ as $p \downarrow 1$. $\square$

Note that, as $p \downarrow 1$,

$$y_p(1) = \max_{t \in I} y_p(t) = b_p \to 1,$$

while

$$\max_{t \in I} q_p(t) = q_p(1) = b_p^{2p^*-2} \to +\infty.$$

See Figure ???

As a consequence of Theorem 1.2 and Lemma 6.10, we conclude from (6.6) the following explanation to the extremal values $\mathbf{L}_{0,1}^N(r)$.

Corollary 6.11 *There holds the following equality*

$$\mathbf{L}_{0,1}^N(r) = \inf\{\lambda_0^N(q) : q \in B_1[r]\} = \lambda_0^N(\Delta_{1,r}), \qquad r \geq 0. \tag{6.24}$$

Due to the reflections, one has also

$$\mathbf{L}_{0,1}^N(r) = \inf\{\lambda_0^N(q) : q \in B_1[r]\} = \lambda_0^N(\Delta_{0,r}), \qquad r \geq 0, \tag{6.25}$$

where $\Delta_{0,r}$ is given by (6.8). The MDE (1.1) with the Dirac measures $\mu = \Delta_{a,r}$ with $a = 0$ or 1 have be analyzed explicitly in [13]. The result is

$$\lambda_0^N(\Delta_{a,r}) = \hat{\mathbf{Z}}_0^{-1}(r), \qquad a = 0,\ 1, \tag{6.26}$$

where the homeomorphism $\hat{\mathbf{Z}}_0$ is as in (6.4). Now results (6.24)–(6.26) have given a clear explanation to (6.4)-(6.5).

This idea also applies to minimal values in [23] of higher-order eigenvalues of Sturm-Liouville operators with potentials in $\mathcal{L}^1$ balls.

Remark 6.12 The minimal problem $\mathbf{L}_{0,p}^N(r)$ of the zeroth Neumann eigenvalues has minimal potentials $q_p \in B_p[r]$ when $1 < p \leq \infty$. However, in case $p = 1$, it has no any minimal potential in $B_1[r]$, due to the non-compactness of $B_1[r]$. In the ball $B_0[r]$ of measures, $\mathbf{L}_0^N(r)$ has also the minimal measures. See Corollary 6.5. These observations on minimizers also apply to minimal values $\mathbf{L}_{m,p}^\sigma(r)$ and $\mathbf{L}_m^\sigma(r)$ for all $m \in \mathbb{N}$. Different from the minimal problems, the maximal problems $\mathbf{M}_{m,p}^\sigma(r)$ always have maximizers in the $\mathcal{L}^p$ balls, including the case $p = 1$. See [23, 28]. In this sense, the maximal problems in $\mathcal{L}^p$ balls are relatively simple than the minimal problems.

Since we have extended Sturm-Liouville theory to MDE, the following extremal values for eigenvalues of MDE

$$\mathbf{L}_0^N(r) := \min\left\{\lambda_0^N(\mu) : \mu \in B_0[r]\right\}$$

are well-defined and can be attained by some measures in $B_0[r]$. See Corollary 6.5. As $B_1[r] \subset B_0[r]$, one has

$$\mathbf{L}_0^N(r) \le \mathbf{L}_{0,1}^N(r).$$

We conjecture that

$$\mathbf{L}_0^N(r) = \lambda_0^N(\Delta_{0,r}) = \lambda_0^N(\Delta_{1,r}) = \hat{\mathbf{Z}}_0^{-1}(r) = \mathbf{L}_{0,1}^N(r).$$

That is, the minimal values of $\lambda_0^N(q)$ in $\mathcal{L}^1(I,\mathbb{R})$ balls coincide with the minimal values of $\lambda_0^N(\mu)$ in $\mathcal{M}_0(I,\mathbb{R})$ balls. For the minimal values of higher order eigenvalues, we conjecture that the analogous results hold as well.

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