Measure Differential Equations, I. Continuity of Solutions in Measures with Weak* Topology

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Abstract

Measure differential equations (MDE) are frequently used to model non-classical problems like the quantum effects. In Part I of these serial papers, we will first use the Riemann-Stieltjes integrals to give an explanation to solutions of initial value problems of second-order linear MDE. Then we will present some deep results on dependence of solutions on measures. That is, solutions and some of their derivatives of MDE are continuously dependent on measures, considered in the weak* topology. Examples show that these continuity results are optimal. In Part II, these results will be used to prove the continuity of eigenvalues of MDE in measures with weak* topology.

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1 Introduction

Classical problems are usually modeled using ordinary or partial differential equations. For example, the oscillation of a string is modeled as

$$\ddot{y} + q(t)y = 0, \tag{1.1}$$

where the potential $q(t)$ is continuous or, more generally, locally (Lebesgue) integrable. When quantum effect is considered, many problems can be modeled using differential equations with non-absolutely continuous measures. For example, in 1931, Kronig and Penney studied in [9] the quantum mechanics in crystal lattices and derived the following differential equation

$$-\ddot{y} + rH'(t)y = \lambda y. \tag{1.2}$$

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Here $r \in \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(t) := n \qquad \text{for } t \in [2n\pi, 2(n+1)\pi),\ n \in \mathbb{Z}.$$

Note that $rH'(t)$ defines the Dirac measure of mass r at points $2n\pi$, $n \in \mathbb{Z}$. Solutions $y(t)$ of Eq. (1.2) are well-defined by allowing jumps of the derivatives $\dot{y}(t)$. Eq. (1.2) is called the Kronig-Penney Hamiltonian in some literature. For some recent studies on the dynamics and spectrum of generalized Kronig-Penney Hamiltonians, see Niikuni [16].

Mathematically, one can use different integrals to explain solutions of different generalizations of ordinary differential equations (ODE), including stochastic differential equations [17]. Some important generalizations of ODE, usually called *generalized ordinary differential equations*, are initially studied in 1950s by Kurzweil [10, 11], whih are followed by Jarník [8] and summarized in more recent books like [20, 21]. However, following [22], we call equations with measures like Eq. (1.2) *measure differential equations* (MDE). Here, by a measure, it means that it is in the dual space of the Banach space of continuous functions. More recently, Tvrdý et al. [7, 23] have introduced the so-called regulated functions and used the Perron-Stieltjes integrals or the Kurzweil-Stieltjes integrals to study solutions in the class of regulated functions for general first-order linear systems of equations with measures.

In this paper, we are concerned with the dependence of solutions of MDE on measures. Note that in works mentioned above, they have obtained some interesting continuous dependence results of solutions on measures, with a choice of topologies for measures different from ours. Our consideration for MDE is also motivated by recent works [25, 26, 30, 31] on dependence of solutions and eigenvalues of ODE on potentials. Throughout this paper let us use I to denote the unit interval $[0, 1]$. For $p \in [1, \infty]$, let $\mathcal{L}^p(I, \mathbb{R})$ be the Lebesgue space with the L^p norm $\|\cdot\|_p$. For $q \in \mathcal{L}^p(I, \mathbb{R})$ and $u_0 = (y_0, z_0) \in \mathbb{R}^2$, we use $y(t, u_0, q)$, $t \in I$, to denote the solution of ODE (1.1) satisfying $(y(0), \dot{y}(0)) = (y_0, z_0)$. It is well-known that $y(\cdot, u_0, q)$, together with its derivative $\dot{y}(\cdot, u_0, q)$, is continuously dependent on potentials $q \in (\mathcal{L}^p(I, \mathbb{R}), \|\cdot\|_p)$. However, in the Lebesgue spaces $\mathcal{L}^p(I, \mathbb{R})$, besides the usual topologies induced by $\|\cdot\|_p$, one has also the weak topologies [6, 12]. More precisely, for $\mathcal{L}^p(I, \mathbb{R})$ with $p \in [1, \infty)$, we use w_p to indicate the topology of weak convergence, and for $\mathcal{L}^\infty(I, \mathbb{R})$ which is the dual space of $(\mathcal{L}^1(I, \mathbb{R}), \|\cdot\|_1)$, we use w_∞ to indicate the topology of weak* convergence. In a unified way, $q_n \to q_0$ in $(\mathcal{L}^p(I, \mathbb{R}), w_p)$, $1 \le p \le \infty$, iff

$$\int_I f(t) q_n(t)\, \mathrm{d}t \to \int_I f(t) q_0(t)\, \mathrm{d}t \qquad \forall f \in \mathcal{L}^{p^*}(I, \mathbb{R}).$$

Here $p^* := p/(p-1) \in [1, \infty]$. One of the main results of [30] asserts that the solution mapping of Eq. (1.1)

$$(\mathcal{L}^p(I, \mathbb{R}), w_p) \to (\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty), \qquad q \to y(\cdot, u_0, q)$$

is continuous. By the results in [26], the derivative mapping

$$(\mathcal{L}^p(I, \mathbb{R}), w_p) \to (\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty), \qquad q \to \dot{y}(\cdot, u_0, q)$$

is also continuous. Here $(\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty)$ is the Banach space of continuous real functions of I with the supremum norm $\|\cdot\|_\infty$. Furthermore, for any potential $q \in \mathcal{L}^p(I, \mathbb{R})$, let us use $\lambda_m(q)$ to denote eigenvalues of

$$\ddot{y} + (\lambda + q(t))y = 0, \qquad t \in I, \tag{1.3}$$

with a separated boundary condition such as the Dirichlet or the Neumann boundary condition. It is proved in [19, 30] that eigenvalues $\lambda_m(q)$ are actually continuous in $q \in (\mathcal{L}^p(I,\mathbb{R}), w_p)$. These continuity results have been extended to the one-dimensional Dirac operators by Meng and Zhang [14] and to the so-called one-dimensional p-Laplacian by Yan and Zhang[26]. For the corresponding results on the principal eigenvalues of elliptic operators, see a recent paper by Cuesta and Ramos Quoirin [5]. Based on these properties, Wei, Meng and Zhang [25, 31] have studied several extremal problems on eigenvalues of (1.3). For example, let us use $\lambda_0(q)$ to denote the zeroth Neumann eigenvalue of (1.3). Define

$$\hat{\mathbf{L}}_p(r) := \inf\{\lambda_0(q) : q \in \mathcal{L}^p(I,\mathbb{R}),\ \|q\|_p \le r\}, \qquad r \in [0,\infty).$$

When $p \in (1,\infty]$, by the continuity of $\lambda_0(q)$ in $q \in (\mathcal{L}^p(I,\mathbb{R}), w_p)$ and the sequential compactness of balls of $(\mathcal{L}^p(I,\mathbb{R}), \|\cdot\|_p)$ in weak topologies w_p , one sees that $-\infty < \hat{\mathbf{L}}_p(r) \le 0$ for all $r \in [0,\infty)$. However, since L^1 balls are non-compact even in the weak topology w_1, it is not easy to see whether $\hat{\mathbf{L}}_1(r)$ is finite for all r. In [31], Zhang has used complicated limiting techniques to obtain the following results.

There holds

$$-\infty < \hat{\mathbf{L}}_1(r) \le 0 \qquad \forall r \in [0,\infty). \tag{1.4}$$

In fact, by letting

$$\hat{\mathbf{Z}}_0(x) := \sqrt{-x}\tanh\sqrt{-x} \qquad \textit{for } x \in (-\infty, 0], \tag{1.5}$$

one has

$$\hat{\mathbf{L}}_1(r) \equiv \hat{\mathbf{Z}}_0^{-1}(r) \qquad \forall r \in [0,\infty). \tag{1.6}$$

In our opinion, the most difficult part is the finiteness result (1.4) due to non-compactness of L^1 balls. In these serial papers, we will apply the weak* compactness of measures to give a natural explanation to the finiteness results like (1.4) by extending Sturm-Liouville theory to MDE. For the ease of notation, let us consider a potential $q \in \mathcal{L}^p(I,\mathbb{R})$ as a density. The corresponding measure (or distribution) is

$$\mu_q(t) := \int_{[0,t]} q(s)\,\mathrm{d}s, \qquad t \in I. \tag{1.7}$$

Then $\mu_q \in \mathcal{M}_0(I,\mathbb{R})$, the space of real-valued measures on I. By the Riesz representation theorem, one has $\mathcal{M}_0(I,\mathbb{R}) \cong (\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)^*$, the dual space of continuous linear functionals of $(\mathcal{C}(I,\mathbb{R}), \|\cdot\|_\infty)$. Recall that for ODE (1.1), solutions $y(t)$ are continuously differentiable and $\dot{y}(t)$ are absolutely continuous in $t \in I$. Using the measure μ_q, Eq. (1.1) can be written as

$$\mathrm{d}\dot{y} + y\,\mathrm{d}\mu_q(t) = 0.$$

In general, let $\mu \in \mathcal{M}_0(I,\mathbb{K}) \cong (\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)^*$ be a $\mathbb{K}$-valued measure, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In these papers, we will write the second-order linear MDE with the measure μ as

$$\mathrm{d}\overset{\bullet}{y} + y\,\mathrm{d}\mu(t) = 0, \qquad t \in I. \tag{1.8}$$

By letting $z(t) := \overset{\bullet}{y}(t)$, Eq. (1.8) is equivalent to the following system for $(y(t), z(t))$

$$\mathrm{d}y(t) = z(t)\,\mathrm{d}t, \qquad \mathrm{d}z(t) = -y(t)\,\mathrm{d}\mu(t), \tag{1.9}$$

which can be explained using the Lebesgue-Stieltjes integral because $\mu(t)$ defines a measure. See Definition 3.1. We give in Theorem 3.5 a simpler proof for the existence and uniqueness of solutions of initial value problems of Eq. (1.8). Consequently, any solution $y(t)$ of Eq. (1.8) is always absolutely continuous on I, while $\overset{\bullet}{y}(t)$ is some generalization of the classical derivative $\dot{y}(t)$. We call $\overset{\bullet}{y}(t)$ the *generalized right-derivative* or the *velocity* of the solution $y(t)$. Note that velocities $\overset{\bullet}{y}(t)$ have, in general, jumps because $\mu(t)$ may be discontinuous and has no density. However, it will be proved that $\overset{\bullet}{y}(t)$ is also some non-normalized measure of I. When the measure $\mu \in \mathcal{M}_0(I, \mathbb{K})$ is absolutely continuous with the density $q(t) = \dot{\mu}(t) \in \mathcal{L}^1(I, \mathbb{K})$, MDE (1.8) returns to ODE (1.1), while $\overset{\bullet}{y}(t)$ is simply the classical derivative $\dot{y}(t)$.

This paper is organized as follows.

In Section 2, we will recall basic facts on measures, the Lebesgue-Stieltjes integral and the Riemann-Stieltjes integral [4]. Since $\mathcal{M}_0(I, \mathbb{K}) = (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)^*$, besides the norm $\|\cdot\|_{\mathbf{V}}$ of total variations, one has also in the space $\mathcal{M}_0(I, \mathbb{K})$ the weak* topology which is indicated by w^*. Some facts on $\|\cdot\|_{\mathbf{V}}$ and w^* will be also mentioned in this section.

In Section 3, for any $u_0 = (y_0, z_0) \in \mathbb{K}^2$, we will use the Riemann-Stieltjes integral only to give a simple explanation to the initial value problem of Eq. (1.8) with initial value

$$y(0) = y_0, \qquad \overset{\bullet}{y}(0) = z_0. \tag{1.10}$$

See Theorem 3.5. For the purpose of the study for eigenvalues of MDE in [15], based on the Generalized Newton-Leibnitz Formula concerning right-derivatives [24], we will show that the Liouville law is also true for MDE (1.8). See Theorem 3.10. The Variant-of-Constant Formula for inhomogeneous MDE will be obtained using this idea. See Theorem 3.11.

Let us use $y(t, u_0, \mu)$ to denote the solution of problem (1.8)-(1.10) with the velocity $\overset{\bullet}{y}(t, u_0, \mu)$, $t \in I$. In the usual topology on measures, we will obtain the following results.

Theorem 1.1 *Let $u_0 \in \mathbb{K}^2$ and $t \in [0,1]$ be fixed. Then solution functionals $y(t, u_0, \mu)$ and $\overset{\bullet}{y}(t, u_0, \mu)$ are continuously differentiable in $\mu \in (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}})$.*

The proof is given in Section 3.3. Moreover, the Fréchet derivatives of $y(t, u_0, \mu)$ and $\overset{\bullet}{y}(t, u_0, \mu)$ in μ are given in (3.28) and (3.29).

In Section 4, we will give explicit solutions for Eq. (1.8) with the Dirac measures $\Delta_{a,r}$. See (2.6) and (2.7). Correspondingly, the Dirichlet and Neumann eigenvalues of these measures will be found explicitly. These examples are very useful in [15].

In Section 5, suggested by the examples in Section 4 and the results of [7, 26, 30] on the dependence of solutions of ODE on potentials, we will prove the following main result of this paper, which provides a deep understanding on dependence of solutions of MDE on measures.

Theorem 1.2 *Let $u_0 \in \mathbb{K}^2$ be fixed. Then the solution mapping*

$$(\mathcal{M}_0(I, \mathbb{K}), w^*) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty), \qquad \mu \to y(\cdot, u_0, \mu) \tag{1.11}$$

is continuous. Moreover, the following ending velocity functional is also continuous

$$(\mathcal{M}_0(I, \mathbb{K}), w^*) \to \mathbb{R}, \qquad \mu \to \overset{\bullet}{y}(1, u_0, \mu). \tag{1.12}$$

For the proof of (1.11) and (1.12), uniformly completely continuous operators introduced by Zhang [28] will play an important role. Since we are using the weak* topology, Theorem 1.2 cannot be deduced from the known works. Actually the present continuity results have generalized those in [7, 18, 19, 26, 30] with various weak topologies. It is remarkable that Example 5.4 shows that continuity (1.12) for velocities cannot be improved to any time $t \in (0,1)$. This fact is consistent with a basic fact for the weak* topology, i.e., global weak* convergence does not imply local weak* convergence. See Remark 5.5.

As mentioned before, the present results and approaches for solutions will be used in [15] to yield the following deep results on eigenvalues of MDE, i.e., as nonlinear functionals,

$$\lambda_m : (\mathcal{M}_0(I,\mathbb{R}), w^*) \to \mathbb{R}$$

are continuous. Here $\lambda_m(\mu)$ are the Dirichlet or Neumann eigenvalues of MDE with the measure μ. Consequently, the finiteness result (1.4) is natural, because the L^1 balls are embedded into the $\|\cdot\|_{\mathbf{V}}$ balls of $\mathcal{M}_0(I,\mathbb{R})$ which are sequentially compact in the weak* topology. The results for extremal values like (1.6) can then be obtained by finding the limiting measures of the critical potentials in L^p balls, $1 < p < \infty$, of [25, 31]. These extremal values have given not only a deep understanding on eigenvalues, but also have some important applications [3, 29].

2 Measures, Lebesgue-Stieltjes Integral and Weak* Topology

For general theory of the Lebesgue-Stieltjes integral and the Riemann-Stieltjes integral, see, e.g., [4]. In most cases, we will use the Riemann-Stieltjes integral.

Let $I = [0,1]$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For a function $\mu : I \to \mathbb{K}$, the *total variation* of μ (over I) is defined as

$$\mathbf{V}(\mu, I) := \sup\left\{\sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,\ n \in \mathbb{N}\right\}.$$

For any subinterval $I_0 \subset I$, closed, open or semi-open, the total variation of μ over I_0 is also well-defined. For example, if $I_0 = (a,b] \subset I$, the total variation is

$$\mathbf{V}(\mu, I_0) := \sup\left\{\sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : a < t_0 < t_1 < \cdots < t_{n-1} < t_n = b,\ n \in \mathbb{N}\right\}.$$

Let

$$\mathcal{M}(I,\mathbb{K}) := \{\mu : I \to \mathbb{K} : \mu(0+)\ \exists,\ \mu(t+) = \mu(t)\ \forall t \in (0,1),\ \mathbf{V}(\mu, I) < \infty\}$$

be the space of non-normalized $\mathbb{K}$-value measures of I. Here, for any $t \in [0,1)$, $\mu(t+) := \lim_{s\downarrow t} \mu(s)$ is the right-limit. The space of (normalized) $\mathbb{K}$-valued measures is

$$\mathcal{M}_0(I,\mathbb{K}) := \{\mu \in \mathcal{M}(I,\mathbb{K}) : \mu(0+) = 0\}.$$

Note that the normalization condition for $\mu \in \mathcal{M}_0(I,\mathbb{K})$ is $\mu(0+) = 0$. Hence $\mu(0) \neq 0$ is possible. Due to the right-continuity of $\mu \in \mathcal{M}_0(I,\mathbb{K})$ on $(0,1)$, one has the following result for variations

$$\lim_{t\downarrow t_0} \mathbf{V}(\mu, [t_0,t]) = \lim_{t\downarrow t_0} \mathbf{V}(\mu, (t_0,t]) = 0, \qquad t_0 \in (0,1). \tag{2.1}$$

For simplicity, we write $\mathbf{V}(\mu, I)$ as $\|\mu\|_{\mathbf{V}}$. By the Riesz representation theorem [12], $(\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_{\mathbf{V}})$ is the same as the dual space of the Banach space $(\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)$ of continuous $\mathbb{K}$-valued functions of I. In fact, $\mu \in (\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_{\mathbf{V}})$ defines $\mu^* \in (\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)^*$ by

$$\mu^*(f) = \int_I f(t)\,\mathrm{d}\mu(t), \qquad f \in \mathcal{C}(I,\mathbb{K}), \tag{2.2}$$

which refers to the Riemann-Stieltjes integral. Moreover, one has

$$\|\mu\|_{\mathbf{V}} = \mathbf{V}(\mu, I) = \sup\left\{ \textstyle\int_I f\,\mathrm{d}\mu : f \in \mathcal{C}(I,\mathbb{K}),\ \|f\|_\infty = 1 \right\}.$$

Given $\mu \in \mathcal{M}_0(I,\mathbb{K})$ and $f \in \mathcal{C}(I,\mathbb{K})$. For any subinterval $I_0 \subset I$, closed, open or semi-open, the Lebesgue-Stieltjes integral $\int_{I_0} f\,\mathrm{d}\mu$ is also defined. Due to the jump of a measure $\mu(t)$ at $t = 0$, one has

$$\int_{[0,b]} f\,\mathrm{d}\mu = -f(0)\mu(0) + \int_{(0,b]} f\,\mathrm{d}\mu, \qquad b \in (0,1]. \tag{2.3}$$

That is, $\int_{[0,b]} f\,\mathrm{d}\mu$ and $\int_{(0,b]} f\,\mathrm{d}\mu$ may differ.

In case I_0 has the form (a,b), $(a,b]$, where $0 \le a < b \le 1$, or the form $[0,b)$, $[0,b]$ where $0 < b \le 1$, one has the following basic inequality

$$\left|\int_{I_0} f\,\mathrm{d}\mu\right| \le \|f\|_{\infty,I_0} \cdot \mathbf{V}(f, I_0), \qquad \|f\|_{\infty,I_0} := \sup_{t\in I_0} |f(t)|. \tag{2.4}$$

Typical examples are as follows.

• Let $\mu_0 : I \to \mathbb{K}$ be $\mu_0(t) \equiv t$. Then μ_0 yields the Lebesgue measure of I and the Lebesgue integral. More generally, any $q \in \mathcal{L}^1(I,\mathbb{K})$ induces a measure by (1.7). In this case, one has

$$\|\mu_q\|_{\mathbf{V}} = \|q\|_1 = \|q\|_{\mathcal{L}^1(I,\mathbb{K})}, \tag{2.5}$$

$$\int_{I_0} f(t)\,\mathrm{d}\mu_q(t) = \int_{I_0} f(t)q(t)\,\mathrm{d}t = \int_{\bar{I}_0} f(t)\,\mathrm{d}\mu_q(t).$$

• Given $a \in (0,1]$ and $r \in \mathbb{K}$. One has the Dirac measure at $t = a$ of the total mass r

$$\Delta_{a,r}(t) = \begin{cases} 0 & \text{for } t \in [0,a), \\ r & \text{for } t \in [a,1]. \end{cases} \tag{2.6}$$

• Let $a = 0$ and $r \in \mathbb{K}$. One has the Dirac measure at $t = 0$ of the total mass r

$$\Delta_{0,r}(t) = \begin{cases} -r & \text{for } t = 0, \\ 0 & \text{for } t \in (0,1]. \end{cases} \tag{2.7}$$

Since we will deal with several topologies, including weak topologies [6, 12], we give some basic relations between these topologies.

In the Lebesgue space $\mathcal{L}^p(I,\mathbb{K})$, $p \in [1,\infty]$, besides the usual topology induced by the L^p norm $\|\cdot\|_p$, one has also the weak topology w_p as mentioned in the Introduction. Let us recall the characterization for relatively sequentially weakly compact subsets in the space $(\mathcal{L}^1(I,\mathbb{K}), w_1)$, see, for example, [6, p. 294].

Lemma 2.1 *Let $V \subset \mathcal{L}^1(I,\mathbb{K})$. Then V is relatively sequentially weakly compact iff*

- *the set V is bounded in $(\mathcal{L}^1(I,\mathbb{K}), \|\cdot\|_1)$, and*
- *for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|\int_B q(t)\,\mathrm{d}t\right| < \varepsilon$ for any $q \in V$ and any measurable set $B \subset [0,1]$ of measure less than δ.*

From this characterization, it is easy to prove the following results. See, for example, [26].

Lemma 2.2 *Suppose that $q_n \to q_0$ in $(\mathcal{L}^1(I,\mathbb{K}), w_1)$ and $f_n \to f_0$ in $(\mathcal{L}^\infty(I,\mathbb{K}), \|\cdot\|_\infty)$. Then $h_n \to h_0$ in $(\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty)$, where*

$$h_n(t) := \int_0^t f_n(s) q_n(s)\,\mathrm{d}s, \qquad t \in I,\ n \geq 0.$$

In the space $\mathcal{M}_0(I,\mathbb{K})$ of measures, one has the usual topology induced by the norm $\|\cdot\|_{\mathbf{V}}$. Due to duality relation (2.2), one has the following weak* topology w^*.

Definition 2.3 Let $\mu_0,\ \mu_n \in \mathcal{M}_0(I,\mathbb{K})$, $n \in \mathbb{N}$. We say that μ_n is weakly* convergent to μ_0 iff, for each $f \in \mathcal{C}(I,\mathbb{K})$, one has

$$\lim_{n\to\infty} \int_I f\,\mathrm{d}\mu_n = \int_I f\,\mathrm{d}\mu_0.$$

For example, as $a \downarrow 0$, one has

$$\int_I f\,\mathrm{d}\Delta_{a,r} = rf(a) \to rf(0) = \int_I f\,\mathrm{d}\Delta_{0,r}$$

for each $f \in \mathcal{C}(I,\mathbb{K})$. Thus $\Delta_{a,r} \to \Delta_{0,r}$ in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$.

We remark that in some literature, this topology just called the weak topology for measures.

By the Banach-Alaoglu Theorem [12, pp. 229-230], we have the following simple characterization on relatively sequentially compact subsets of $(\mathcal{M}_0(I,\mathbb{K}), w^*)$.

Lemma 2.4 *A subset $V \subset (\mathcal{M}_0(I,\mathbb{K}), w^*)$ is relatively sequentially compact iff V is bounded in the $\|\cdot\|_{\mathbf{V}}$ norm. That is, any sequence $\{\mu_n\} \subset \mathcal{M}_0(I,\mathbb{K})$ with $\sup_n \|\mu_n\|_{\mathbf{V}} < \infty$ has a subsequence converging to some μ_0 in the space $(\mathcal{M}_0(I,\mathbb{K}), w^*)$.*

In the space $\mathcal{M}_0(I,\mathbb{K})$ of measures, besides the topologies $\|\cdot\|_{\mathbf{V}}$ and w^*, sometimes the topology induced by the supremum norm $\|\cdot\|_\infty$ is also used in [7, 23]. As $\|\mu\|_\infty \leq \|\mu\|_{\mathbf{V}}$ for all $\mu \in \mathcal{M}_0(I,\mathbb{K})$, one sees that $\|\cdot\|_\infty$ is also weaker than $\|\cdot\|_{\mathbf{V}}$.

We have the following relations for these weak topologies.

Lemma 2.5 *All of the following embeddings are continuous*

$$(\mathcal{L}^{p'}(I,\mathbb{K}), w_{p'}) \hookrightarrow (\mathcal{L}^p(I,\mathbb{K}), w_p) \hookrightarrow (\mathcal{L}^1(I,\mathbb{K}), w_1), \qquad \infty \geq p' > p > 1,$$

$$(\mathcal{L}^1(I,\mathbb{K}), w_1) \hookrightarrow (\mathcal{M}_0(I,\mathbb{K}), w^*), \qquad (\mathcal{L}^1(I,\mathbb{K}), w_1) \hookrightarrow (\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_\infty). \tag{2.8}$$

However,

$$\mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_\infty) \not\Longrightarrow \mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I,\mathbb{K}), w^*), \tag{2.9}$$

$$\mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I,\mathbb{K}), w^*) \not\Longrightarrow \mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I,\mathbb{K}), \|\cdot\|_\infty). \tag{2.10}$$

Proof The embeddings in (2.8) are given by (1.7). The continuity of the latter embedding of (2.8) follows immediately from Lemma 2.2.

For (2.9), consider

$$\mu_n(t) = (1/n)\sin(2\pi n^2 t) \to 0$$

in $(\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_\infty)$. By (2.5), one has

$$\|\mu_n\|_{\mathbf{V}} = \|\dot{\mu}_n\|_1 = 4n \to \infty.$$

Hence $\mu_n \not\to 0$ in $(\mathcal{M}_0(I,\mathbb{R}), w^*)$.

For (2.10), let $r \in \mathbb{K}\backslash\{0\}$. One has

$$\Delta_{a,r} \to \Delta_{0,r}$$

in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$ as $a \downarrow 0$, while

$$\|\Delta_{a,r} - \Delta_{0,r}\|_\infty = |r| \neq 0$$

for all $a \in (0,1]$. □

Example 2.6 Given $r \in \mathbb{K}\backslash\{0\}$. Consider $\Delta_{a,r}$, $a \in [0,1]$, as a path in $\mathcal{M}_0(I,\mathbb{K})$. One has

$$\|\Delta_{a,r} - \Delta_{a',r}\|_\infty = |r|, \qquad \|\Delta_{a,r} - \Delta_{a',r}\|_{\mathbf{V}} = 2|r|,$$

for all $a,\ a' \in I$ and $a \neq a'$. Consequently, when $r \neq 0$, the path $a \to \Delta_{a,r}$ is nowhere continuous with respect to the topologies $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathbf{V}}$. However, with respect to the weak* topology w^*, the path $I \ni a \to \Delta_{a,r} \in (\mathcal{M}_0(I,\mathbb{K}), w^*)$ is continuous at any $a \in I$.

3 Solutions of Initial Value Problems of MDE

3.1 Notion of solutions and existence and uniqueness

The scalar second-order linear MDE with the measure $\mu \in \mathcal{M}_0(I,\mathbb{K})$ is written as in the form (1.8). By introducing the function $z(t) = \overset{\bullet}{y}(t) : I \to \mathbb{K}$, Eq. (1.8) is understood as the system (1.9), while the initial condition (1.10) is understood as $(y(0), z(0)) = (y_0, z_0)$. We can introduce solutions of problem (1.8)-(1.10) as follows.

Definition 3.1 By a *solution* $y(t)$ of initial value problem (1.8)-(1.10), it means that

- $y \in \mathcal{C}(I,\mathbb{K})$, and
- there exists a function $z : I \to \mathbb{K}$ such that the following are satisfied

$$y(t) = y_0 + \int_{[0,t]} z(s)\,\mathrm{d}s, \qquad t \in [0,1], \tag{3.1}$$

$$z(t) = \begin{cases} z_0, & t = 0, \\ z_0 - \int_{[0,t]} y(s)\,\mathrm{d}\mu(s), & t \in (0,1]. \end{cases} \tag{3.2}$$

Since we have assumed that $y \in \mathcal{C}(I,\mathbb{K})$, the right-hand sides of (3.1) and (3.2) are the Lebesgue integral and Riemann-Stieltjes integral respectively.

Remark 3.2 Note that such a definition for solutions of Eq. (1.8) is the same as that for stochastic differential equations [17]. Let us mention that Eq. (3.2) is different for $t = 0$ and for $t \in (0, 1]$. However, if (3.2) is replaced by

$$z(t) = z_0 - \int_{(0,t]} y(s)\,\mathrm{d}\mu(s), \qquad t \in [0, 1], \tag{3.3}$$

integral system (3.1)-(3.3) is another explanation to solutions of (1.8)-(1.10). See the difference (2.3) for Lebesgue-Stieltjes integrals. Due to jumps of measures at $t = 0$, solutions of (3.1)-(3.3) will have no longer the continuity (1.12). See Section 5. That is why we use (3.2), instead of (3.3). Another reason is that if $\mu = \Delta_{0,r}$, system (3.1)-(3.3) is the same as $\ddot{y} = 0$ which is not the MDE we want to study.

Note in the definition of solutions we have not appointed any space for $z(t)$. This will be convenient to prove the existence and uniqueness theorem. Since solutions of (1.8)-(1.10) are defined via fixed point equations (3.1)-(3.2) for (y, z), there are many methods to prove this. For example, one can find a proof from [7, 23] based on the Kurzweil-Stieltjes integral, which applies also to the first-order linear MDE. As we are only interesting in second-order MDE, for our later purpose, we will use the ideas in [30] to give a proof, which is very insight for further properties of solutions of MDE (1.8).

Lemma 3.3 *For any $z_0 \in \mathbb{K}$ and $y \in \mathcal{C}(I, \mathbb{K})$, let us use $\hat{y}$ to denote the function defined by the right-hand side of Eq.* (3.2)*. Then $\hat{y} \in \mathcal{M}(I, \mathbb{K})$. In particular, $\hat{y}$ is Lebesgue integrable on I.*

Proof From (2.1) and (2.4), $\hat{y}(0+) = z_0 + y(0)\mu(0)$ exists. Let $0 < t_0 < t \le 1$. One has

$$|\hat{y}(t) - \hat{y}(t_0)| = \left| \int_{(t_0,t]} y(s)\,\mathrm{d}\mu(s) \right| \le \|y\|_\infty \mathbf{V}(\mu, (t_0, t]).$$

By (2.1), $\hat{y}(t_0+) = \hat{y}(t_0)$ for all $t_0 \in (0, 1)$. Finally, for a partition $0 = t_0 < t_1 < \cdots < t_n = 1$, one has

$$\begin{aligned}
\sum_{i=0}^{n-1} |\hat{y}(t_{i+1}) - \hat{y}(t_i)| &= \left| \int_{[0,t_1]} y(s)\,\mathrm{d}\mu(s) \right| + \sum_{i=1}^{n-1} \left| \int_{(t_i,t_{i+1}]} y(s)\,\mathrm{d}\mu(s) \right| \\
&\le |y(0)||\mu(0)| + \|y\|_\infty \mathbf{V}(\mu, (t_0, t_1]) + \|y\|_\infty \sum_{i=1}^{n-1} \mathbf{V}(\mu, (t_i, t_{i+1}]) \\
&\le |y(0)||\mu(0)| + \|y\|_\infty \mathbf{V}(\mu, I) < \infty,
\end{aligned}$$

where (2.4) and (2.3) are used. Hence $\hat{y} \in \mathcal{M}(I, \mathbb{K})$. □

Let us introduce $G : [0, 1]^2 \to \mathbb{R}$ by

$$G(t, s) := \begin{cases} t - s & \text{for } 0 \le s \le t \le 1, \\ 0 & \text{for } 0 \le t < s \le 1. \end{cases}$$

It is easy to see that $G \in \mathcal{C}(I^2, \mathbb{R})$. For $y \in \mathcal{C}(I, \mathbb{K})$, the following is well-defined

$$\mathcal{Z}y(t) := \int_I G(t, s)y(s)\,\mathrm{d}\mu(s), \qquad t \in I, \tag{3.4}$$

and, in fact, $\mathcal{Z} : (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$ is a bounded linear operator. For second-order MDE, we have the following explanation to solutions.

Lemma 3.4 *Denote*

$$\hat{y}_0(t) := y_0 + z_0 t \in \mathcal{C}(I, \mathbb{K}).$$

A function $y \in \mathcal{C}(I, \mathbb{K})$ *is a solution of problem* (1.8)-(1.10) *iff it satisfies*

$$y = \hat{y}_0 - \mathcal{Z} y. \tag{3.5}$$

Proof For the necessity, let us assume that $y \in \mathcal{C}(I, \mathbb{K})$ is a solution of (1.8)-(1.10). Given $t \in (0, 1]$. Substituting (3.2) into (3.1), we have

$$\begin{aligned} y(t) &= y_0 + \int_{[0,t]} \left(z_0 - \int_{[0,u]} y(s) \, \mathrm{d}\mu(s) \right) \mathrm{d}u \\ &= y_0 + z_0 t - \int_{[0,t]} \left(\int_{[0,u]} y(s) \, \mathrm{d}\mu(s) \right) \mathrm{d}u \\ &= \hat{y}_0(t) - \int_{[0,t]} \left(\int_{[s,t]} y(s) \, \mathrm{d}u \right) \mathrm{d}\mu(s) \qquad \text{(by the Fubini theorem)} \\ &= \hat{y}_0(t) - \int_{[0,t]} (t-s) y(s) \, \mathrm{d}\mu(s) \\ &= \hat{y}_0(t) - \int_{[0,1]} G(t,s) y(s) \, \mathrm{d}\mu(s), \end{aligned}$$

because $G(t, s) = 0$ for $s > t$. This shows (3.5) is true for $t \in (0, 1]$. When $t = 0$, by noticing that $G(0, s)y(s) \equiv 0$, we know that (3.5) is also true.

For the sufficiency, if $y \in \mathcal{C}(I, \mathbb{K})$ satisfies (3.5), one can use (3.2) to introduce $z(t)$ and verify that $(y(t), z(t))$ satisfies (3.1)-(3.2). □

Theorem 3.5 *For each* $(y_0, z_0) \in \mathbb{K}^2$*, problem* (1.8)-(1.10) *has the unique solution* $y(t)$ *defined on* I*.*

Proof Step 1. We first prove the existence and uniqueness of solutions of (1.8)-(1.10) for $t \in [0, T]$, where $0 < T \leq 1$ is determined later. As in Lemma 3.4, $y : [0, T] \to \mathbb{K}$ is a solution of (1.8)-(1.10) iff $y \in \mathcal{C}_T := \mathcal{C}([0, T], \mathbb{K})$ satisfies

$$y = \hat{y}_0 - \mathcal{Z}_T y, \tag{3.6}$$

where $\mathcal{Z}_T : \mathcal{C}_T \to \mathcal{C}_T$ is

$$\mathcal{Z}_T y(t) = \int_{[0,T]} G_T(t,s) y(s) \, \mathrm{d}\mu(s), \qquad t \in [0, T],$$

with

$$G_T = G|_{[0,T]^2} \in \mathcal{C}([0, T]^2, \mathbb{R}).$$

One has $\max_{(t,s)\in[0,T]^2} |G_T(t, s)| = T$. As an operator from the Banach space $(\mathcal{C}_T, \|\cdot\|_\infty)$ to itself, one has

$$\begin{aligned} |(\mathcal{Z}_T y_1)(t) - (\mathcal{Z}_T y_2)(t)| &= \left| \int_{[0,T]} G_T(t,s)(y_1(s) - y_2(s)) \, \mathrm{d}\mu(s) \right| \\ &\leq \|G_T(t,\cdot)(y_1(\cdot) - y_2(\cdot))\|_\infty \cdot \|\mu\|_{\mathbf{V}} \\ &\leq T \|\mu\|_{\mathbf{V}} \cdot \|y_1 - y_2\|_\infty. \end{aligned}$$

See (2.4). Suppose that T satisfies

$$0 < T \leq 1 \quad \text{and} \quad T < 1/\|\mu\|_{\mathbf{V}}. \tag{3.7}$$

Then Eq. (3.6) has the unique fixed point $y_T \in \mathcal{C}_T$, which is the unique solution of (1.8)-(1.10) on $[0, T]$.

Step 2. By noticing the restriction (3.7) on the length T on the existence interval $[0, T]$ is determined by $\|\mu\|_{\mathbf{V}}$ only, we can use the continuation of solutions, as did for ODE, in finitely many steps, to obtain the unique global solution y of problem (1.8)-(1.10). □

Remark 3.6 Under condition (3.7) on T, $\mathcal{Z}_T : \mathcal{C}_T \to \mathcal{C}_T$ is a contraction. It is possible to prove that there exists some $k \in \mathbb{N}$, depending on $\|\mu\|_{\mathbf{V}}$ only, such that the kth iteration $\mathcal{Z}^k : (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$ is a contraction. See the estimates in [15]. Hence the global existence and uniqueness of solutions on I can be deduced from the Banach Contraction Principle in a direct way.

Since problem (1.8)-(1.10) has the unique solution $y \in \mathcal{C}(I, \mathbb{K})$, by Eq. (3.2), we have the unique function $z : I \to \mathbb{K}$ associated with the solution y. We use also $\overset{\bullet}{y}$ to denote this function z. That is,

$$\overset{\bullet}{y}(t) = \begin{cases} z_0 & \text{for } t = 0, \\ z_0 - \int_{[0,t]} y(s)\, \mathrm{d}\mu(s) & \text{for } t \in (0, 1]. \end{cases} \tag{3.8}$$

In general, $\overset{\bullet}{y}(t)$ are not continuous in time t. This is quite reasonable because we are considering equations with measures μ which represent bumps in physics. Note that $y \in \mathcal{C}(I, \mathbb{K})$ and, by Lemma 3.3, $\overset{\bullet}{y} \in \mathcal{M}(I, \mathbb{K}) \subset \mathcal{L}^1(I, \mathbb{K})$. Some further properties for solutions $y(t)$ and the associated functions $\overset{\bullet}{y}(t)$ are as follows.

Corollary 3.7 (i) *There holds*

$$\int_{[t_1,t_2]} \overset{\bullet}{y}(s)\, \mathrm{d}s = \int_{(t_1,t_2]} \overset{\bullet}{y}(s)\, \mathrm{d}s = y(t_2) - y(t_1), \qquad 0 \leq t_1 \leq t_2 \leq 1. \tag{3.9}$$

(ii) *At any point $t_0 \in (0, 1)$, classical right-derivative*

$$y'_+(t_0) := \lim_{t \downarrow t_0} \frac{y(t) - y(t_0)}{t - t_0}$$

exists, and

$$\overset{\bullet}{y}(t_0) = y'_+(t_0). \tag{3.10}$$

(iii) *The solution $y : I \to \mathbb{K}$ is actually absolutely continuous, and*

$$\overset{\bullet}{y}(t_0) = \dot{y}(t_0) := \lim_{t \to t_0} \frac{y(t) - y(t_0)}{t - t_0} \tag{3.11}$$

for Lebesgue-a.e. $t_0 \in I$.

Proof (i) Since $\overset{\bullet}{y} \in \mathcal{L}^1(I, \mathbb{K})$, the two integrals of (3.9) are equal. The last equality of (3.9) can be simply deduced from Eq. (3.1).

(ii) For $t_0 \in (0,1)$, it follows from the right-continuity of $\overset{\bullet}{y}$ at t_0 and equality (3.9) that

$$y'_+(t_0) = \lim_{t\downarrow t_0} \frac{y(t)-y(t_0)}{t-t_0} = \lim_{t\downarrow t_0} \frac{1}{t-t_0}\int_{[t_0,t]} \overset{\bullet}{y}(s)\,\mathrm{d}s = \overset{\bullet}{y}(t_0).$$

Hence we have (3.10).

(iii) The absolute continuity of $y(t)$ is obvious from equality (3.9). As $\mu(t)$ is a measure, $\mu(t)$ has only at most countable many discontinuous points. In particular, the set

$$I_0 := \{t_0 \in (0,1) : \mu(t) \text{ is continuous at } t_0\}$$

has the full Lebesgue measure. For any $t_0 \in I_0$, one has $\lim_{t\uparrow t_0} \mathbf{V}(\mu, (t, t_0]) = 0$ and, by (3.2),

$$\lim_{t\uparrow t_0} \left|\overset{\bullet}{y}(t) - \overset{\bullet}{y}(t_0)\right| = \lim_{t\uparrow t_0} \left|\int_{(t,t_0]} y(s)\,\mathrm{d}\mu(s)\right| \le \lim_{t\uparrow t_0} \|y\|_\infty \mathbf{V}(\mu, (t, t_0]) = 0.$$

That is, $\overset{\bullet}{y}$ is left-continuous at any $t_0 \in I_0$. As $\overset{\bullet}{y} \in \mathcal{M}(I, \mathbb{K})$, we conclude that $\overset{\bullet}{y}$ is continuous at any $t_0 \in I_0$. From equality (3.9) again, we know that at $t_0 \in I_0$, y has the classical derivative $\dot{y}(t_0)$ and (3.11) holds for all $t_0 \in I_0$. Since I_0 has the full Lebesgue measure, the proof is complete. □

Due to (3.10) and (3.11), the function $\overset{\bullet}{y}$ associated with a solution y is called the *generalized right-derivative* or the *velocity* of the solution y. One has then $(y(0), \overset{\bullet}{y}(0)) = (y_0, z_0)$. This gives an explanation to initial condition (1.10).

The ideas above can be extended to the existence and uniqueness of (local) solutions to initial value problems of the following second-order nonlinear MDE

$$\mathrm{d}\overset{\bullet}{y} + f(t,y)\,\mathrm{d}t + g(t,y)\,\mathrm{d}\mu(t) = 0. \tag{3.12}$$

Theorem 3.8 *Suppose that $f(t,y)$ and $g(t,y)$ in Eq. (3.12) are continuous in (t,y) and are locally Lipschitz in y. Then, for any $t_0 \in \mathbb{R}$ and $(y_0, z_0) \in \mathbb{K}^2$, Eq. (3.12) has a unique (local) solution $y(t)$ satisfying the initial condition $(y(t_0), \overset{\bullet}{y}(t_0)) = (y_0, z_0)$. Precisely, there exists some interval $I_0 = [t_0, t_0 + \delta)$, $\delta > 0$, and a unique continuous function $y : I_0 \to \mathbb{K}$, and a unique function $\overset{\bullet}{y} : I_0 \to \mathbb{K}$ such that $(y(t_0), \overset{\bullet}{y}(t_0)) = (y_0, z_0)$, and for $t \in (t_0, t_0 + \delta)$,*

$$\begin{cases} y(t) = y_0 + \int_{[t_0,t]} \overset{\bullet}{y}(s)\,\mathrm{d}s, \\ \overset{\bullet}{y}(t) = z_0 - \int_{[t_0,t]} f(s, y(s))\,\mathrm{d}s - \int_{[t_0,t]} g(s, y(s))\,\mathrm{d}\mu(s). \end{cases}$$

Theorem 3.5 and Theorem 3.8 can be extended to kth, $k \ge 3$, order measure differential equations. However, the approach here does not apply to first order (linear) measure differential equations.

3.2 Fundamental matrix solutions and the Liouville law

The following basic results from calculus will be used frequently in this paper. For their proofs, see, for example, [24].

Lemma 3.9 (Generalized Newton-Leibnitz Formula) *Let* $\varphi \in \mathcal{C}([a,b],\mathbb{C})$. *Suppose that the right-derivative* $\varphi'_+(t)$ *exists for all* $t \in (a,b)$ *and* φ'_+ *is Riemann integrable on* $[a,b]$. *Then*

$$\int_{[a,b]} \varphi'_+(t)\,\mathrm{d}t = \varphi(b) - \varphi(a).$$

In particular, if $\varphi \in \mathcal{C}([a,b],\mathbb{C})$ *satisfies* $\varphi'_+(t) = 0$ *for all* $t \in (a,b)$, *then* $\varphi(t) \equiv \mathrm{const.}$ *on* $[a,b]$.

For linear ODE (1.1), an important property for solutions is the so-called the Liouville law. We will prove that this is also true for MDE (1.8). Given $\mu \in \mathcal{M}_0(I,\mathbb{K})$ and $(y_0,z_0) \in \mathbb{K}^2$. We use $y(t,y_0,z_0)$ to denote the unique solution of (1.8)-(1.10) with the velocity $\overset{\bullet}{y}(t,y_0,z_0)$. Let

$$\varphi_1(t) := y(t,1,0), \qquad \varphi_2(t) := y(t,0,1),$$

called the fundamental solutions of (1.8). By the linearity of (1.8) and the uniqueness of solutions, one has

$$y(t,y_0,z_0) \equiv y_0\varphi_1(t) + z_0\varphi_2(t), \qquad \overset{\bullet}{y}(t,y_0,z_0) \equiv y_0\,\overset{\bullet}{\varphi}_1(t) + z_0\,\overset{\bullet}{\varphi}_2(t).$$

The fundamental matrix solution of (1.8) is defined as

$$N_\mu(t) := \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \overset{\bullet}{\varphi}_1(t) & \overset{\bullet}{\varphi}_2(t) \end{pmatrix}, \qquad t \in I. \tag{3.13}$$

Then one has

$$\begin{pmatrix} y(t,y_0,z_0) \\ \overset{\bullet}{y}(t,y_0,z_0) \end{pmatrix} \equiv N_\mu(t) \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in \mathbb{K}^2.$$

Note that $N_\mu(0) = I_2$, the identity matrix. Now the Liouville law for MDE (1.8) is as follows.

Theorem 3.10 *There holds the following equality*

$$\det N_\mu(t) \equiv +1, \qquad t \in I. \tag{3.14}$$

Proof Note that $N_\mu(t)$ is only right-continuous on $(0,1)$. The proof is quite different from that for ODE. For the present case, we need to apply Lemma 3.9 to prove (3.14). Define

$$m_\mu(t) := \det N_\mu(t) = \varphi_1(t)\,\overset{\bullet}{\varphi}_2(t) - \varphi_2(t)\,\overset{\bullet}{\varphi}_1(t), \qquad t \in I.$$

Step 1. We assert that $m_\mu \in \mathcal{C}(I,\mathbb{K})$. For any $t_0,\ t \in I$, let us introduce

$$\begin{aligned} \psi_1(t) &= (\varphi_1(t) - \varphi_1(t_0))\,\overset{\bullet}{\varphi}_2(t), \\ \psi_2(t) &= (\varphi_2(t) - \varphi_2(t_0))\,\overset{\bullet}{\varphi}_1(t), \\ \psi_3(t) &= \varphi_1(t_0)(\overset{\bullet}{\varphi}_2(t) - \overset{\bullet}{\varphi}_2(t_0)) - \varphi_2(t_0)(\overset{\bullet}{\varphi}_1(t) - \overset{\bullet}{\varphi}_1(t_0)). \end{aligned}$$

It is easy to verify that

$$m_\mu(t) - m_\mu(t_0) \equiv \psi_1(t) - \psi_2(t) + \psi_3(t). \tag{3.15}$$

Note that $\psi_i(t_0) = 0$, $1 \le i \le 3$. Since $\varphi_i \in \mathcal{C}(I, \mathbb{K})$ and $\| \overset{\bullet}{\varphi}_i \|_\infty < \infty$, one has

$$\lim_{t \to t_0} \psi_i(t) = 0 = \psi_i(t_0), \qquad i = 1,\ 2. \tag{3.16}$$

For the term $\psi_3(t)$, let us introduce

$$\eta_{t_0}(t) := \varphi_1(t_0)\varphi_2(t) - \varphi_2(t_0)\varphi_1(t), \qquad t \in I, \tag{3.17}$$

which is also a solution of (1.8). Moreover, $\eta_{t_0}(t_0) = 0$.

We need to distinguish $t_0 \in [0, 1]$ in several cases.

Case 1. $t_0 \in (0, 1]$. For any $t \in (0, t_0)$, we use Eq. (3.2) to obtain

$$\begin{aligned} \psi_3(t) &= -\varphi_1(t_0)(\overset{\bullet}{\varphi}_2(t_0) - \overset{\bullet}{\varphi}_2(t)) + \varphi_2(t_0)(\overset{\bullet}{\varphi}_1(t_0) - \overset{\bullet}{\varphi}_1(t)) \\ &= \varphi_1(t_0) \int_{(t, t_0]} \varphi_2(s)\, \mathrm{d}\mu(s) - \varphi_2(t_0) \int_{(t, t_0]} \varphi_1(s)\, \mathrm{d}\mu(s) \\ &= \int_{(t, t_0]} \eta_{t_0}(s)\, \mathrm{d}\mu(s), \end{aligned} \tag{3.18}$$

where $\eta_{t_0}(s)$ is as in (3.17). By (2.4), we have, as $t \uparrow t_0$,

$$|\psi_3(t)| \le \|\eta_{t_0}\|_{\infty, (t, t_0]} \|\mu\|_{\mathbf{V}} \to 0,$$

because $\eta_{t_0}(s) \in \mathcal{C}(I, \mathbb{K})$ and $\eta_{t_0}(t_0) = 0$. Thus $\psi_3(t_0-) = 0 = \psi_3(t_0)$ for all $t_0 \in (0, 1]$.

Case 2. $t_0 \in (0, 1)$. For any $t \in (t_0, 1)$, we have from (3.18)

$$\psi_3(t) = -\int_{(t_0, t]} \eta_{t_0}(s)\, \mathrm{d}\mu(s). \tag{3.19}$$

Thus, as $t \downarrow t_0$,

$$|\psi_3(t)| \le \|\eta_{t_0}\|_{\infty, (t_0, t]} \|\mu\|_{\mathbf{V}} \to 0$$

because of the continuity of $\eta_{t_0}(s)$ and $\eta_{t_0}(t_0) = 0$. Therefore $\psi_3(t_0+) = 0 = \psi_3(t_0)$ for all $t_0 \in (0, 1)$.

Case 3. $t_0 = 0$. For any $t \in (0, 1)$, we have from (3.2) that equality (3.19) shall be

$$\psi_3(t) = -\int_{[0, t]} \eta_0(s)\, \mathrm{d}\mu(s).$$

One has also $\psi_3(0+) = 0 = \psi_3(0)$ because of $\eta_0(0+) = \eta_0(0) = 0$.

We have thus proved that $\psi_3 \in \mathcal{C}(I, \mathbb{K})$ and therefore $m_\mu \in \mathcal{C}(I, \mathbb{K})$, following from (3.15) and (3.16).

Step 2. We assert that $m_\mu(t)$ has right-derivative 0 at any $t_0 \in (0, 1)$. By Corollary 3.7, one has $\overset{\bullet}{\varphi}_2(t_0+) = \overset{\bullet}{\varphi}_2(t_0)$ and $\varphi'_{1+}(t_0) = \overset{\bullet}{\varphi}_1(t_0)$. See (3.10). Thus

$$\begin{aligned} \psi'_{1+}(t_0) &= \lim_{t \downarrow t_0} \frac{\psi_1(t) - \psi_1(t_0)}{t - t_0} = \lim_{t \downarrow t_0} \left(\frac{\varphi_1(t) - \varphi_1(t_0)}{t - t_0} \cdot \overset{\bullet}{\varphi}_2(t) \right) \\ &= \varphi'_{1+}(t_0)\, \overset{\bullet}{\varphi}_2(t_0+) = \overset{\bullet}{\varphi}_1(t_0)\, \overset{\bullet}{\varphi}_2(t_0). \end{aligned}$$

Similarly, one has also $\psi'_{2+}(t_0) = \overset{\bullet}{\varphi}_1(t_0)\, \overset{\bullet}{\varphi}_2(t_0)$. For $\psi_3(t)$, let $0 < t_0 < s \le t < 1$. Note that $\eta_{t_0}(\cdot)$ in (3.17) is a solution of (1.8). It then follows from (3.10) that

$$\eta_{t_0}(s) = \big(a + o(1)\big)(s - t_0), \qquad a := \overset{\bullet}{\eta}_{t_0}(t_0).$$

By (3.19), as $t \downarrow t_0$,

$$\left|\frac{\psi_3(t)-\psi_3(t_0)}{t-t_0}\right| = \left|-\int_{(t_0,t]} \big(a+o(1)\big)\frac{s-t_0}{t-t_0}\,\mathrm{d}\mu(s)\right|$$
$$\leq (|a|+o(1))\mathbf{V}(\mu,(t_0,t]) \to 0.$$

See (2.1). Thus $\psi'_{3+}(t_0) = 0$. By the decomposition (3.15), we have $m'_{\mu+}(t_0) = 0$ where $t_0 \in (0,1)$ is arbitrary.

Step 3. Finally, we have $m_\mu(0) = +1$ because $N_\mu(0) = I_2$. It follows from Lemma 3.9 that $m_\mu(t) \equiv m_\mu(0) = +1$ for all $t \in I$. □

We remark that such a proof will be used later in studying measure differential equations.

3.3 Variant-of-constant formula for inhomogeneous MDE

Now we consider second-order inhomogeneous MDE

$$\mathrm{d}\overset{\bullet}{y} + y\,\mathrm{d}\mu(t) = h(t)\,\mathrm{d}\nu(t), \tag{3.20}$$

where $\mu,\ \nu \in \mathcal{M}_0(I,\mathbb{K})$ and $h \in \mathcal{C}(I,\mathbb{K})$. It can be proved that solutions of (3.20) are well-defined. With the fundamental matrix solution $N_\mu(t)$ defined by (3.13), the variant-of-constant formula for inhomogeneous equation (3.20) is as follows.

Theorem 3.11 *Let* $(y_0,z_0) \in \mathbb{K}^2$. *Then the unique solution* $(y(t),\overset{\bullet}{y}(t))$ *of inhomogeneous equation* (3.20) *satisfying* $(y(0),\overset{\bullet}{y}(0)) = (y_0,z_0)$ *is given by the variant-of-constant formula*

$$\begin{pmatrix} y(t) \\ \overset{\bullet}{y}(t) \end{pmatrix} = N_\mu(t)\left(\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \int_{[0,t]} N_\mu^{-1}(s)\begin{pmatrix} 0 \\ h(s) \end{pmatrix}\mathrm{d}\nu(s)\right), \qquad t \in (0,1]. \tag{3.21}$$

Here $N_\mu^{-1}(s)$ *is the inverse of* $N_\mu(s)$.

Proof Due to the relation between Eq. (1.8) and Eq. (3.20), in order to prove (3.21) for general $(y_0,z_0) \in \mathbb{K}^2$, it is necessary and sufficient to show that (3.21) is true for the solution of Eq. (3.20) with $(y_0,z_0) = (0,0)$. To this end, define $(y^*0), z^*(0)) = (0,0)$ and, for $t \in (0,1]$,

$$\begin{pmatrix} y^*(t) \\ z^*(t) \end{pmatrix} := N_\mu(t)\int_{[0,t]} N_\mu^{-1}(s)\begin{pmatrix} 0 \\ h(s) \end{pmatrix}\mathrm{d}\nu(s)$$
$$= \begin{pmatrix} -\int_{[0,t]} (\varphi_1(t)\varphi_2(s)-\varphi_2(t)\varphi_1(s))\,h(s)\,\mathrm{d}\nu(s) \\ -\int_{[0,t]}(\overset{\bullet}{\varphi}_1(t)\varphi_2(s)-\overset{\bullet}{\varphi}_2(t)\varphi_1(s))h(s)\,\mathrm{d}\nu(s) \end{pmatrix}. \tag{3.22}$$

We need to verify that $(y^*(t),z^*(t))$ is the solution of Eq. (3.20) satisfying $(y_0,z_0) = (0,0)$. That is, for all $u \in (0,1]$, one will have the following two equalities

$$y^*(u) = \int_{[0,u]} z^*(t)\,\mathrm{d}t, \tag{3.23}$$

$$z^*(u) = -\int_{[0,u]} y^*(t)\,\mathrm{d}\mu(t) + \int_{[0,u]} h(t)\,\mathrm{d}\nu(t). \tag{3.24}$$

By (3.22), we have

$$\begin{aligned}
I(u) &:= y^*(u) - \int_{[0,u]} z^*(t)\,\mathrm{d}t \\
&= -\int_{[0,u]} (\varphi_1(u)\varphi_2(s) - \varphi_2(u)\varphi_1(s))\, h(s)\,\mathrm{d}\nu(s) \\
&\quad + \int_{[0,u]} \left(\int_{[0,t]} (\overset{\bullet}{\varphi}_1(t)\varphi_2(s) - \overset{\bullet}{\varphi}_2(t)\varphi_1(s)) h(s)\,\mathrm{d}\nu(s) \right) \mathrm{d}t \\
&= -\int_{[0,u]} (\varphi_1(u)\varphi_2(s) - \varphi_2(u)\varphi_1(s))\, h(s)\,\mathrm{d}\nu(s) \\
&\quad + \int_{[0,u]} \left(\int_{[s,u]} (\overset{\bullet}{\varphi}_1(t)\varphi_2(s) - \overset{\bullet}{\varphi}_2(t)\varphi_1(s))\,\mathrm{d}t \right) h(s)\,\mathrm{d}\nu(s),
\end{aligned}$$

following from the Fubini theorem. Now

$$\begin{aligned}
&\int_{[s,u]} (\overset{\bullet}{\varphi}_1(t)\varphi_2(s) - \overset{\bullet}{\varphi}_2(t)\varphi_1(s))\,\mathrm{d}t \\
&= \varphi_2(s) \int_{[s,u]} \overset{\bullet}{\varphi}_1(t)\,\mathrm{d}t - \varphi_1(s) \int_{[s,u]} \overset{\bullet}{\varphi}_2(t)\,\mathrm{d}t \\
&= \varphi_2(s)(\varphi_1(u) - \varphi_1(s)) - \varphi_1(s)(\varphi_2(u) - \varphi_2(s)) \\
&= \varphi_1(u)\varphi_2(s) - \varphi_2(u)\varphi_1(s),
\end{aligned}$$

where formula (3.9) is used. Hence $I(u) = 0$ and we obtain (3.23). Equality (3.24) can be proved in a similar way. □

Let us write the solution of problem (1.8)-(1.10) as $y(t, u_0, \mu)$ with the velocity $\overset{\bullet}{y}(t, u_0, \mu)$. We consider the dependence of $(y(t, u_0, \mu), \overset{\bullet}{y}(t, u_0, \mu))$ on measures $\mu \in \mathcal{M}_0(I, \mathbb{K})$ with the usual topology $\|\cdot\|_{\mathbf{V}}$. A preliminary result is as follows.

Theorem 3.12 *Let $u_0 \in \mathbb{K}^2$. Suppose that*

$$\mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}}). \tag{3.25}$$

Then one has

$$\lim_{n\to\infty} \|y(\cdot, u_0, \mu_n) - y(\cdot, u_0, \mu_0)\|_\infty = 0, \tag{3.26}$$

$$\lim_{n\to\infty} \| \overset{\bullet}{y}(\cdot, u_0, \mu_n) - \overset{\bullet}{y}(\cdot, u_0, \mu_0)\|_\infty = 0. \tag{3.27}$$

Proof In order to emphasize the dependence on μ, we write the operator $\mathcal{Z}$ of (3.4) as $\mathcal{Z}_\mu$. It is easy to see that $\mathcal{Z}_\mu(y)$ is jointly continuous in $(\mu, y) \in (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}}) \times (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$.

Given $\mu_0 \in \mathcal{M}_0(I, \mathbb{K})$. There is some $\delta_0 > 0$ and $k_0 \in \mathbb{N}$ such that $\mathcal{Z}_\mu^{k_0} : (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$ is contracting uniformly in

$$\mu \in B_{\delta_0}(\mu_0) = \{\mu \in \mathcal{M}_0(I, \mathbb{K}) : \|\mu - \mu_0\|_{\mathbf{V}} < \delta_0\}.$$

See Remark 3.6. Now continuity result (3.26) can be obtained by applying the uniform contraction principle of Banach to the fixed point equation (3.5).

For result (3.27), we have from (3.2)

$$\overset{\bullet}{y}(t, u_0, \mu_n) = \begin{cases} z_0, & t = 0, \\ z_0 - \int_{[0,t]} y(s, u_0, \mu_n)\, \mathrm{d}\mu_n(s), & t \in (0, 1], \end{cases}$$

Thus

$$\begin{aligned} &\| \overset{\bullet}{y}(\cdot, u_0, \mu_n) - \overset{\bullet}{y}(\cdot, u_0, \mu_0)\|_\infty \\ &= \sup_{t\in(0,1]} \left| \int_{[0,t]} y(s, u_0, \mu_n)\, \mathrm{d}\mu_n(s) - \int_{[0,t]} y(s, u_0, \mu_0)\, \mathrm{d}\mu_0(s) \right| \\ &\le \sup_{t\in(0,1]} \left| \int_{[0,t]} y(s, u_0, \mu_n)\, \mathrm{d}(\mu_n - \mu_0)(s) \right| \\ &\quad + \sup_{t\in(0,1]} \left| \int_{[0,t]} (y(s, u_0, \mu_n) - y(s, u_0, \mu_0))\, \mathrm{d}\mu_0(s) \right| \\ &\le \|y(\cdot, u_0, \mu_n)\|_\infty \|\mu_n - \mu_0\|_{\mathbf{V}} + \|y(\cdot, u_0, \mu_n) - y(\cdot, u_0, \mu_0)\|_\infty \|\mu_0\|_{\mathbf{V}}. \end{aligned}$$

Let $n \to \infty$. We know from condition (3.25) and result (3.26) that $\|\mu_n - \mu_0\|_{\mathbf{V}} \to 0$, $\|y(\cdot, u_0, \mu_n) - y(\cdot, u_0, \mu_0)\|_\infty \to 0$ and $\|y(\cdot, u_0, \mu_n)\|_\infty$ is bounded. Hence we have (3.27). □

Proof of Theorem 1.1. Since $\mathcal{Z}_\mu(y) : (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}}) \times (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty) \to (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$ is a bounded bilinear operator in (μ, y), the Banach contraction principle shows that $(y(t, u_0, \mu), \overset{\bullet}{y}(t, u_0, \mu))$ is actually continuously differentiable in $\mu \in (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}})$. □

In the following, let us work out the (Fréchet) derivatives with respect to μ. By the linearity, we need only to compute $\partial_\mu N_\mu(t)$. Since $N_\mu(0) = I_2$, we have

$$\partial_\mu N_\mu(0) = 0. \tag{3.28}$$

Given $\mu \in \mathcal{M}_0(I, \mathbb{K})$. Let us compute the directional derivative $\partial_\mu N_\mu(t) \cdot \nu$ along the direction $\nu \in \mathcal{M}_0(I, \mathbb{K})$. For $\tau \in \mathbb{R}$, $\varphi_i(t, \mu + \tau\nu)$, $i = 1, 2$, satisfy

$$\mathrm{d}\overset{\bullet}{\varphi}_i(t, \mu + \tau\nu) + \varphi_i(t, \mu + \tau\nu)\, \mathrm{d}\mu(t) + \tau\varphi_i(t, \mu + \tau\nu)\, \mathrm{d}\nu(t) = 0.$$

Differentiating this equation with respect to τ, we know that

$$\Phi_i(t) := \left. \frac{\mathrm{d}\varphi_i(t, \mu + \tau\nu)}{\mathrm{d}\tau} \right|_{\tau=0}$$

satisfy the following inhomogeneous MDE

$$\mathrm{d}\overset{\bullet}{\Phi}_i(t) + \Phi_i(t)\, \mathrm{d}\mu(t) = -\varphi_i(t, \mu)\, \mathrm{d}\nu(t).$$

Moreover, one has $(\Phi_i(0), \overset{\bullet}{\Phi}_i(0)) = (0, 0)$. See (3.28). By the variant-of-constant formula (3.21), we can obtain $\Phi_i(t)$ and $\overset{\bullet}{\Phi}_i(t)$. The final result can be stated in the matrix form in the following way.

Corollary 3.13 *In matrix form, the derivative of $N_\mu(t)$ in $\mu \in (\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_{\mathbf{V}})$ is*

$$\partial_\mu N_\mu(t) \cdot \nu = -N_\mu(t) \int_{[0,t]} N_\mu^{-1}(s) \begin{pmatrix} 0 & 0 \\ \varphi_1(s, \mu) & \varphi_2(s, \mu) \end{pmatrix} \mathrm{d}\nu(s), \qquad t \in (0, 1]. \tag{3.29}$$

We remark that for ODE, formulas like (3.29) can be found in [19, 27].

4 A Basic Example of Linear MDE

Let $r \in \mathbb{R}$ and $a \in [0,1]$. We have the Dirac measures $\Delta_{a,r} \in \mathcal{M}_0(I,\mathbb{R})$. See (2.6) and (2.7). With a complex parameter $\lambda \in \mathbb{C}$, we consider the following second-order MDE

$$\mathrm{d}\overset{\bullet}{y} + \lambda y \,\mathrm{d}t + y\,\mathrm{d}\Delta_{a,r}(t) = 0, \qquad t \in I = [0,1]. \tag{4.1}$$

4.1 Eigenvalues of MDE with Dirac measures

For $u_0 = (y_0, z_0) \in \mathbb{C}^2$, the solution $(y(t), \overset{\bullet}{y}(t))$ of problem (4.1)-(1.10) can be computed explicitly. Denote

$$\omega = \sqrt{\lambda} \in \mathbb{C}.$$

Note that

$$\cos\omega t = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n t^{2n}}{(2n)!}, \qquad \frac{\sin\omega t}{\omega} = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n t^{2n+1}}{(2n+1)!},$$

are well-defined for all $\lambda \in \mathbb{C}$ and are analytic in λ.

Case $a = 0$. The solution of problem (4.1)-(1.10) is

$$\begin{cases} y(t) = y_0 \cos\omega t + (z_0 - r y_0)\frac{\sin\omega t}{\omega}, & t \in [0,1], \\ \overset{\bullet}{y}(t) = \begin{cases} z_0 & \text{for } t = 0, \\ -\omega y_0 \sin\omega t + (z_0 - r y_0)\cos\omega t & \text{for } t \in (0,1]. \end{cases} \end{cases}$$

Case $a \in (0,1]$. We have

$$\begin{pmatrix} y(t) \\ \overset{\bullet}{y}(t) \end{pmatrix} = \begin{cases} \begin{pmatrix} y_0 \cos\omega t + z_0 \frac{\sin\omega t}{\omega} \\ -\omega y_0 \sin\omega t + z_0 \cos\omega t \end{pmatrix} & \text{for } t \in [0,a), \\ \begin{pmatrix} \hat{y}_0 \cos\omega(t-a) + \hat{z}_0 \frac{\sin\omega(t-a)}{\omega} \\ -\omega\hat{y}_0 \sin\omega(t-a) + \hat{z}_0 \cos\omega(t-a) \end{pmatrix} & \text{for } t \in [a,1], \end{cases} \tag{4.2}$$

where

$$(\hat{y}_0, \hat{z}_0) = \left(y_0 \cos\omega a + z_0 \frac{\sin\omega a}{\omega},\ -\omega y_0 \sin\omega a + z_0 \cos\omega a - r\hat{y}_0 \right). \tag{4.3}$$

It is easy to see that $y \in \mathcal{C}(I,\mathbb{C})$, while $\overset{\bullet}{y}(t)$ has jump at $t = a$, depending on the initial values (y_0, z_0) and the parameters $a,\ r,\ \lambda$.

Let us pay special attention to the ending velocity $\overset{\bullet}{y}(1)$. From formulas above, we have

$$\overset{\bullet}{y}(1) = \begin{cases} -\omega y_0 \sin\omega + (z_0 - r y_0)\cos\omega & \text{for } a = 0, \\ -\omega\hat{y}_0 \sin\omega(1-a) + \hat{z}_0 \cos\omega(1-a) & \text{for } a \in (0,1]. \end{cases} \tag{4.4}$$

Lemma 4.1 *Considering $\overset{\bullet}{y}(1)$ of (4.4) as a function of the parameter $a \in [0,1]$, it is continuous.*

Proof From formula (4.3), it is easy to see that $(\hat{y}_0, \hat{z}_0)$ is continuous in $a \in (0,1]$. Moreover, one has $\lim_{a\downarrow 0} \hat{y}_0 = y_0$ and $\lim_{a\downarrow 0} \hat{z}_0 = z_0 - r y_0$. Now the desired continuity follows from formula (4.4). □

Remark 4.2 The continuity result in Lemma 4.1 is a remarkable fact, because Example 2.6 shows that the path $a \to \Delta_{a,r}$ is not continuous in the topologies $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathbf{V}}$. However, as observed in Example 2.6, this path is continuous in the topology w^*. Theorem 1.2 we will prove in this paper has given a natural explanation to this continuity result.

For the real measure $\Delta_{a,r}$, $\lambda \in \mathbb{C}$ is called an *eigenvalue* of (4.1) with the Dirichlet boundary condition

$$y(0) = y(1) = 0 \tag{4.5}$$

if MDE (4.1) has non-zero solutions $y(t)$ satisfying (4.5). In this case, these solutions $y(t)$ are called *eigen-functions* associated with λ.

In order to study eigenvalues of problem (4.1)-(4.5), we need only to consider the solution of Eq. (4.1) with the initial value $(y_0, z_0) = (0, 1)$. In this case,

$$(\hat{y}_0, \hat{z}_0) = \left(\frac{\sin \omega a}{\omega}, \cos \omega a - r \frac{\sin \omega a}{\omega} \right) \qquad \text{for } a \in (0, 1].$$

Denote

$$\Phi_2(a, \lambda) := y(1).$$

From the formulas above, we can obtain

$$\Phi_2(a, \lambda) \equiv \frac{\sin \omega}{\omega} - r \frac{\sin \omega a}{\omega} \frac{\sin \omega (1-a)}{\omega}, \qquad a \in [0, 1], \ \lambda \in \mathbb{C}. \tag{4.6}$$

Note that $\Phi_2(1-a, \lambda) \equiv \Phi_2(a, \lambda)$. Now $\lambda \in \mathbb{C}$ is an eigenvalue of (4.1)-(4.5) iff λ satisfies

$$\Phi_2(a, \lambda) = 0. \tag{4.7}$$

By using (4.6), Eq. (4.7) can be analyzed for all cases $(a, r) \in [0, 1] \times \mathbb{R}$. In particular, we have the following results.

Lemma 4.3 *Given $(a, r) \in [0, 1] \times \mathbb{R}$. The Dirichlet eigenvalues of $\Delta_{a,r}$ are a real sequence $\{\lambda_m^D(\Delta_{a,r})\}_{m \in \mathbb{N}}$ such that $\lim_{m \to \infty} \lambda_m^D(\Delta_{a,r}) = +\infty$.*

Since we will use the results only for some special a, we will not give a detailed proof of Lemma 4.3 in all cases.

The Neumann boundary condition for ODE (1.3) is $\dot{y}(0) = \dot{y}(1) = 0$. However, for MDE (4.1), the classical derivatives are, in general, meaningless. Using the velocities of solutions of MDE, the Neumann boundary condition is defined as

$$\overset{\bullet}{y}(0) = \overset{\bullet}{y}(1) = 0. \tag{4.8}$$

To study eigenvalue problem (4.1)-(4.8), let us consider the solution $(y(t), \overset{\bullet}{y}(t))$ of (4.1) with the initial value $(y_0, z_0) = (1, 0)$. Define

$$\Phi_1(a, \lambda) := \overset{\bullet}{y}(1).$$

From formula (4.4), one has

$$\Phi_1(a, \lambda) \equiv -\left(\omega \sin \omega + r \cos \omega a \cos \omega (1-a) \right), \qquad a \in [0, 1], \ \lambda \in \mathbb{C}. \tag{4.9}$$

Note that $\Phi_1(1-a, \lambda) \equiv \Phi_1(a, \lambda)$. Now $\lambda \in \mathbb{C}$ is an eigenvalue of (4.1)-(4.8) iff λ satisfies

$$\Phi_1(a, \lambda) = 0. \tag{4.10}$$

The corresponding results are as follows.

Lemma 4.4 *Given* $(a,r)\in[0,1]\times\mathbb{R}$*. The Neumann eigenvalues of* $\Delta_{a,r}$ *are a real sequence* $\{\lambda_m^N(\Delta_{a,r})\}_{m\in\mathbb{Z}^+}$ *such that* $\lim_{m\to\infty}\lambda_m^N(\Delta_{a,r})=+\infty$*.*

Let us write

$$\Lambda^D_{m,a,r}:=\lambda^D_m(\Delta_{a,r})\text{ for }m\in\mathbb{N},\qquad \Lambda^N_{m,a,r}:=\lambda^N_m(\Delta_{a,r})\text{ for }m\in\mathbb{Z}^+.$$

Note that $\Phi_i(1-a,\lambda)\equiv\Phi_i(a,\lambda)$. One has $\lambda_m^\sigma(\Delta_{1-a,r})=\lambda_m^\sigma(\Delta_{a,r})$, where σ denotes D or N. Thus one may restrict a in $[1/2,1]$. Some special cases for a are as follows.

4.2 Eigenvalues and eigen-functions of MDE with Dirac measures $\Delta_{1,r}$

In this subsection we consider the case $a=1$. This example will be used in [15] to explain extremal problems of Neumann eigenvalues in [31]. In this case, it follows from (4.6) that

$$\Phi_2(1,\lambda)\equiv(\sin\omega)/\omega.$$

Lemma 4.5 *For any* $r\in\mathbb{R}$*, the Dirichlet eigenvalues of* $\Delta_{1,r}$ *are*

$$\Lambda^D_{m,1,r}=(m\pi)^2,\qquad m\in\mathbb{N},\tag{4.11}$$

while the corresponding Dirichlet eigen-functions can be taken as

$$E^D_{m,1,r}(t)\equiv\sin(m\pi t).\tag{4.12}$$

Here (4.12) follows simply from (4.2) where $a=1$ and $(y_0,z_0)=(0,1)$.

Remark 4.6 Note that eigenvalues (4.11) and eigen-functions (4.12) of the measure $\Delta_{1,r}$ are the same as the classical Dirichlet objects of (1.3) with the potential $q(t)=0$. The reason is as follows. Since we are considering the solution $y(t)$ of Eq. (4.1) so that $y(0)=0$, the bump $\Delta_{1,r}$ has no effect on the solution $y(t)$ with the initial value $(y_0,z_0)=(0,1)$. That is, $y(t)\equiv\frac{\sin\omega t}{\omega}$ on I. However, the bump $\Delta_{1,r}$ has influence on the velocity $\overset{\bullet}{y}(1)$ at time $t=1$. Accordingly, the Neumann objects of $\Delta_{1,r}$ will be different from the classical ones. See (4.14)–(4.16) below.

Now we consider the Neumann eigenvalues and eigen-functions of $\Delta_{1,r}$. The solution of Eq. (4.1) with $(y_0,z_0)=(1,0)$ is still $y(t)=\cos\omega t$ on I. The velocity $\overset{\bullet}{y}(t)=\dot{y}(t)=-\omega\sin\omega t$ coincides with the classical derivative for $t\in[0,1)$. However, at time $t=1$, due to the effect of $\Delta_{1,r}$, one has

$$\overset{\bullet}{y}(1)=\dot{y}(1)-ry(1)=-\omega\sin\omega-r\cos\omega.$$

Now $\lambda\in\mathbb{C}$ is a Neumann eigenvalue of (4.1) iff

$$\omega\sin\omega+r\cos\omega=0.\tag{4.13}$$

See also (4.9) and (4.10) for general a. Note that

$$\cos\sqrt{\lambda}=\cosh\sqrt{-\lambda},\qquad \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}=\frac{\sinh\sqrt{-\lambda}}{\sqrt{-\lambda}},\qquad \sqrt{\lambda}\sin\sqrt{\lambda}=-\sqrt{-\lambda}\sinh\sqrt{-\lambda},$$

for $\lambda \in \mathbb{R}$ with $\lambda \le 0$. Since r is real, we know that all solutions λ of (4.13) must be real. There are infinitely many branches for solutions of (4.13). In details, let us introduce intervals

$$\hat{J}_0 = \left(-\infty,\, (\pi/2)^2\right), \\ \hat{J}_m = \left(((m-1/2)\pi)^2,\, ((m+1/2)\pi)^2\right), \quad m \in \mathbb{N}.$$

Define the functions $\hat{\mathbf{Z}}_m : \hat{J}_m \to \mathbb{R}$ by

$$\hat{\mathbf{Z}}_0(\lambda) = \begin{cases} \sqrt{-\lambda}\tanh\sqrt{-\lambda} & \text{for } \lambda \in (-\infty, 0], \\ -\sqrt{\lambda}\tan\sqrt{\lambda} & \text{for } \lambda \in [0, (\pi/2)^2), \end{cases}$$
$$\hat{\mathbf{Z}}_m(\lambda) = -\sqrt{\lambda}\tan\sqrt{\lambda} \qquad \text{for } \lambda \in \hat{J}_m,\ m \in \mathbb{N}.$$

It is easy to verify that each $\hat{\mathbf{Z}}_m : \hat{J}_m \to \mathbb{R}$ is a diffeomorphism. Note that the function of (1.5) for extremal values of the zeroth Neumann eigenvalues with potentials in L^1 balls is just the restriction of the present function $\hat{\mathbf{Z}}_0$ to $(-\infty, 0]$. By Eq. (4.13), we can obtain the following conclusions.

Lemma 4.7 *For any $r \in \mathbb{R}$, the Neumann eigenvalues of $\Delta_{1,r}$ are given by*

$$\Lambda^N_{m,1,r} = \hat{\mathbf{Z}}_m^{-1}(r) \in \hat{J}_m, \qquad m \in \mathbb{Z}^+. \tag{4.14}$$

The Neumann eigen-functions associated with $\Lambda^N_{m,1,r}$ can be taken as

$$E^N_{m,1,r}(t) = \cos\sqrt{\Lambda^N_{m,1,r}}\, t. \tag{4.15}$$

Note that $\Lambda^N_{0,1,r} < 0$ for $m = 0$ and $r > 0$, and $\Lambda^N_{m,1,r} \ge 0$ for other case. Hence, if $m = 0$ and $r > 0$, the real form of (4.15) reads as

$$E^N_{0,1,r}(t) = \cosh\sqrt{-\Lambda^N_{0,1,r}}\, t. \tag{4.16}$$

We remark that when $r = 0$, no bump is present and (4.14)–(4.16) return to the classical Neumann objects

$$\Lambda^N_{m,1,0} = (m\pi)^2, \qquad E^N_{m,1,0}(t) = \cos m\pi t.$$

4.3 Eigenvalues and eigen-functions of MDE with Dirac measures $\Delta_{1/2,r}$

Let us consider another special case for a: $a = 1/2$. This example will be used in [15] to explain the extremal problems of periodic eigenvalues in [31].

At first we consider the Dirichlet problem for $\Delta_{1/2,r}$. By (4.6),

$$\Phi_2(1/2, \lambda) = \frac{\sin(\omega/2)}{\omega}\left(2\cos(\omega/2) - r\frac{\sin(\omega/2)}{\omega}\right).$$

Now Eq. (4.7) for the Dirichlet eigenvalues is decomposed into the following two classes

$$\sin(\sqrt{\lambda}/2) = 0, \tag{4.17}$$
$$2\sqrt{\lambda}\cot(\sqrt{\lambda}/2) = r. \tag{4.18}$$

Eq. (4.17) yields the following Dirichlet eigenvalues of $\Delta_{1/2,r}$

$$\lambda^D_{2m}(\Delta_{1/2,r}) = \Lambda^D_{2m,1/2,r} := (2m\pi)^2, \qquad m \in \mathbb{N}. \tag{4.19}$$

They are independent of the bump $r \in \mathbb{R}$. Solutions of Eq. (4.18) can be determined as follows. Let us introduce intervals

$$\begin{aligned} J_1 &= \left(-\infty,\, (2\pi)^2\right), \\ J_{2m-1} &= \left((2(m-1)\pi)^2,\, (2m\pi)^2\right), \quad m \geq 2. \end{aligned}$$

Define the functions $\mathbf{Z}_{2m-1} : J_{2m-1} \to \mathbb{R}$ by

$$\mathbf{Z}_1(\lambda) = \begin{cases} 2\sqrt{-\lambda}\coth(\sqrt{-\lambda}/2) & \text{for } \lambda \in (-\infty, 0), \\ 4 & \text{for } \lambda = 0, \\ 2\sqrt{\lambda}\cot(\sqrt{\lambda}/2) & \text{for } \lambda \in (0, (2\pi)^2). \end{cases} \tag{4.20}$$

$$\mathbf{Z}_{2m-1}(\lambda) = 2\sqrt{\lambda}\cot(\sqrt{\lambda}/2) \qquad \text{for } \lambda \in J_{2m-1},\ m \geq 2. \tag{4.21}$$

Then $\mathbf{Z}_{2m-1} : J_{2m-1} \to \mathbb{R}$ is a decreasing diffeomorphism for each $m \in \mathbb{N}$. Now Eq. (4.18) gives the following eigenvalues

$$\lambda^D_{2m-1}(\Delta_{1/2,r}) = \Lambda^D_{2m-1,1/2,r} := \mathbf{Z}^{-1}_{2m-1}(r) \in J_{2m-1}, \qquad m \in \mathbb{N}. \tag{4.22}$$

Lemma 4.8 *Let $r \in \mathbb{R}$. Then all Dirichlet eigenvalues of $\Delta_{1/2,r}$ are precisely $\{\Lambda^D_{m,1/2,r}\}_{m\in\mathbb{N}}$ given by* (4.19) *and* (4.20)*–*(4.22)*. These eigenvalues are ordered as*

$$\Lambda^D_{1,1/2,r} < \Lambda^D_{2,1/2,r} < \cdots < \Lambda^D_{m,1/2,r} < \cdots, \qquad \Lambda^D_{m,1/2,r} \to +\infty.$$

The corresponding Dirichlet eigen-functions are as follows. For even-order eigenvalues $\Lambda^D_{2m,1/2,r} = (2m\pi)^2$, $m \in \mathbb{N}$, we can take the classical eigen-functions as

$$E^D_{2m,1/2,r}(t) = \sin(2m\pi t), \qquad t \in I.$$

For odd-order eigenvalues $\Lambda^D_{2m-1,1/2,r} = \mathbf{Z}^{-1}_{2m-1}(r)$, $m \in \mathbb{N}$, let

$$c_{2m-1,r} := \sqrt{|\Lambda^D_{2m-1,1/2,r}|}.$$

One sees that $c_{2m-1,r} > 0$ for all $(m,r) \in \mathbb{N}\times\mathbb{R}$ except $(m,r) = (1,4)$. Then one can take an eigen-function associated with $\Lambda^D_{2m-1,1/2,r}$ as

$$E^D_{2m-1,1/2,r}(t) = \begin{cases} c_{2m-1,r}\cdot y(t;0,1) & \text{if } (m,r) \neq (1,4), \\ \lim_{r\to 4} E^D_{1,1/2,r}(t) & \text{if } (m,r) = (1,4). \end{cases}$$

Here $y(t;0,1)$ is the solution of MDE (4.1) with $\lambda = \Lambda^D_{2m-1,1/2,r}$ and $(y_0, z_0) = (0,1)$. Explicitly, for $m = 1$,

$$E^D_{1,1/2,r}(t) \equiv \begin{cases} \sin\sqrt{\Lambda^D_{1,1/2,r}}\,\hat{t} & \text{for } r \in (-\infty, 4), \\ \hat{t} & \text{for } r = 4, \\ \sinh\sqrt{-\Lambda^D_{1,1/2,r}}\,\hat{t} & \text{for } r \in (4, \infty), \end{cases} \tag{4.23}$$

and, for $m \geq 2$,

$$E^D_{2m-1,1/2,r}(t) \equiv \sin \sqrt{\Lambda^D_{2m-1,1/2,r}}\, \hat{t}. \tag{4.24}$$

Here

$$\hat{t} := \min(t, 1-t) \qquad \text{for } t \in I.$$

We remark that when $r = 0$, odd-order Dirichlet objects (4.22), (4.23) and (4.24) return to the classical ones

$$\Lambda^D_{2m-1,1/2,0} = ((2m-1)\pi)^2, \qquad E^D_{2m-1,1/2,0}(t) = \sin(2m-1)\pi t.$$

Next we consider the Neumann eigenvalues of $\Delta_{1/2,r}$ and their eigen-functions. We state the results only. In this case, the odd-order Neumann eigenvalues and eigen-functions are the same as the classical ones

$$\lambda^N_{2m-1}(\Delta_{1/2,r}) = \Lambda^N_{2m-1,1/2,r} := ((2m-1)\pi)^2, \qquad m \in \mathbb{N}. \tag{4.25}$$

$$E^N_{2m-1,1/2,r}(t) \equiv \cos((2m-1)\pi t), \qquad m \in \mathbb{N}. \tag{4.26}$$

They are independent of r. For even-order Neumann objects, let us introduce intervals

$$\begin{aligned} J_0 &= \left(-\infty,\, \pi^2\right), \\ J_{2m} &= \left(((2m-1)\pi)^2,\, ((2m+1)\pi)^2\right), \quad m \in \mathbb{N}. \end{aligned}$$

Define the functions $\mathbf{Z}_{2m} : J_{2m} \to \mathbb{R}$ by

$$\mathbf{Z}_0(\lambda) = \begin{cases} 2\sqrt{-\lambda}\tanh(\sqrt{-\lambda}/2) & \text{for } \lambda \in (-\infty, 0], \\ -2\sqrt{\lambda}\tan(\sqrt{\lambda}/2) & \text{for } \lambda \in [0, \pi^2), \end{cases} \tag{4.27}$$

$$\mathbf{Z}_{2m}(\lambda) = -2\sqrt{\lambda}\tan(\sqrt{\lambda}/2) \qquad \text{for } \lambda \in J_{2m},\ m \in \mathbb{N}. \tag{4.28}$$

Then $\mathbf{Z}_{2m} : J_{2m} \to \mathbb{R}$ is a diffeomorphism for each $m \in \mathbb{Z}^+$. Now the even-order Neumann eigenvalues of $\Delta_{1/2,r}$ are

$$\lambda^N_{2m}(\Delta_{1/2,r}) = \Lambda^N_{2m,1/2,r} := \mathbf{Z}^{-1}_{2m}(r) \in J_{2m}, \qquad m \in \mathbb{Z}^+. \tag{4.29}$$

The corresponding Neumann eigen-functions are given by

$$E^N_{0,1/2,r}(t) = \begin{cases} \cosh \sqrt{-\Lambda^N_{0,1/2,r}}\, \hat{t} & \text{for } r > 0, \\ 1 & \text{for } r = 0, \\ \cos \sqrt{\Lambda^N_{0,1/2,r}}\, \hat{t} & \text{for } r < 0, \end{cases} \tag{4.30}$$

$$E^N_{2m,1/2,r}(t) = \cos \sqrt{\Lambda^N_{m,1/2,r}}\, \hat{t}, \qquad m \in \mathbb{N}. \tag{4.31}$$

Lemma 4.9 *Let $r \in \mathbb{R}$. Then all Neumann eigenvalues of $\Delta_{1/2,r}$ are precisely $\{\Lambda^N_{m,1/2,r}\}_{m\in\mathbb{Z}^+}$ given by* (4.25) *and* (4.27)*–*(4.29)*. These eigenvalues are ordered as*

$$\Lambda^N_{0,1/2,r} < \Lambda^N_{1,1/2,r} < \cdots < \Lambda^N_{m,1/2,r} < \cdots, \qquad \Lambda^N_{m,1/2,r} \to +\infty.$$

The corresponding Neumann eigen-functions are given by (4.26) *and* (4.30)*–*(4.31)*.*

The Dirichlet and Neumann eigenvalues of $\Delta_{1,r}$ and $\Delta_{1/2,r}$ are plotted in Figure 1.

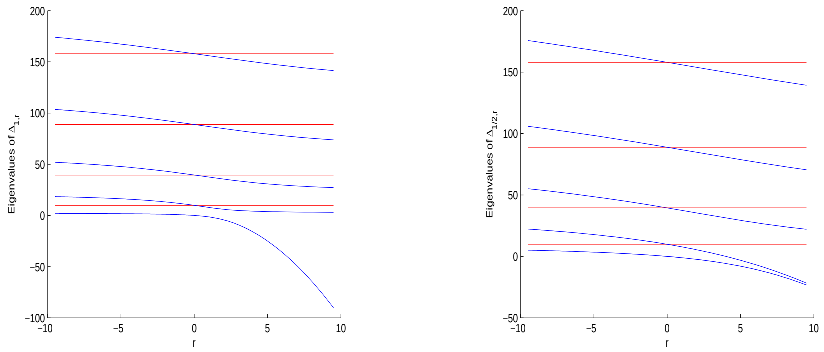

Figure 1: Eigenvalues $\{\Lambda^D_{m,a,r} : 1 \le m \le 4\} \cup \{\Lambda^N_{m,a,r} : 0 \le m \le 4\}$ of $\Delta_{a,r}$. Left: $a = 1$. Right: $a = 1/2$.

Some observations are as follows.

• Eigen-functions $E^D_{m,1/2,r}(t)$ and $E^N_{m,1/2,r}(t)$ are symmetric with respect to $t = 1/2$.

• When $r \neq 0$, $E^D_{2m-1,1/2,r}(t)$ and $E^N_{2m,1/2,r}(t)$ have $t = 1/2$ as a turning point.

• When $r = 0$, even-order Neumann objects (4.29)–(4.31) will return to the classical ones as well

$$\Lambda^N_{2m,1/2,0} = (2m\pi)^2, \qquad E^N_{2m,1/2,0}(t) = \cos 2m\pi t.$$

That is, all objects return to the classical ones.

The odd-order Dirichlet eigen-functions $E^D_{2m-1,1/2,r}(t)$ and even-order Neumann eigen-functions $E^N_{2m,1/2,r}(t)$ are plotted in Figure 2 and Figure 3 respectively, with the choice of some typical bumps r.

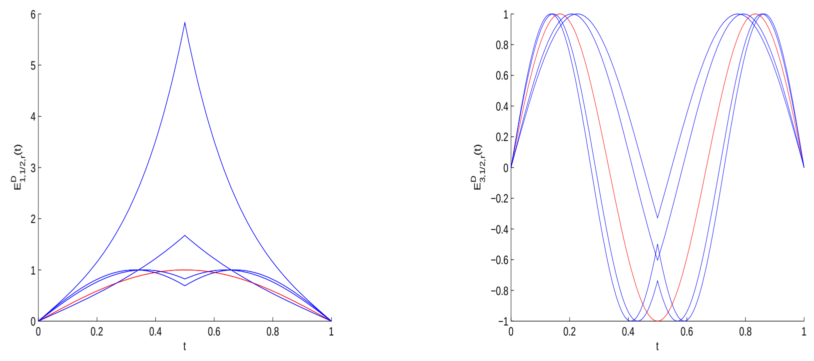

Figure 2: Dirichlet eigen-functions $E^D_{2m-1,1/2,r}(t)$. Left: $m = 1$, $r = 0$, ± 6, ± 10. Right: $m = 2$, $r = 0$, ± 20, ± 40.

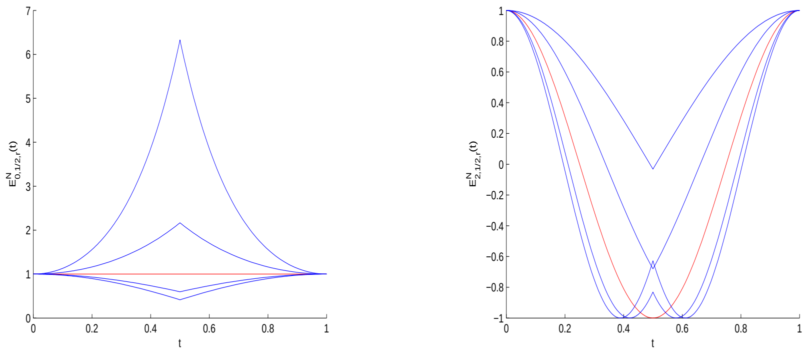

Figure 3: Neumann eigen-functions $E^N_{2m,1/2,r}(t)$. Left: $m = 0$, $r = 0, \pm 5, \pm 10$. Right: $m = 1$, $r = 0, \pm 10, \pm 20$.

5 Continuity of Solutions in Measures with Weak* Topology

In this section we will complete the proof of Theorem 1.2 on continuous dependence of solutions of MDE (1.8) on measures μ.

Given $u_0 \in \mathbb{K}^2$ and $\mu \in \mathcal{M}_0(I, \mathbb{K})$. We use $y(t, u_0, \mu)$ and $\dot{y}(t, u_0, \mu)$ to denote the solution and its velocity of MDE (1.8) satisfying $(y(0), \dot{y}(0)) = u_0$. The main result of [7], applied to MDE (1.8), can be stated as the following continuity result.

Theorem 5.1 *Given $u_0 \in \mathbb{K}^2$. Suppose that $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_\infty)$. Then both* (3.26) *and* (3.27) *hold.*

By the continuous embedding of (2.8), Theorem 3.12 can be also deduced from Theorem 5.1. However, the topology $\|\cdot\|_\infty$ for $\mathcal{M}_0(I, \mathbb{K})$ is not compatible with the nature of $\mathcal{M}_0(I, \mathbb{K})$. For example, $(\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_\infty)$ is not a Banach space. Moreover, one has no natural explanation to compact subsets of $(\mathcal{M}_0(I, \mathbb{K}), \|\cdot\|_\infty)$. Since $\mathcal{M}_0(I, \mathbb{K})$ is the dual space of $(\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$, it is quite natural to consider the weak* topology w^* in the space $\mathcal{M}_0(I, \mathbb{K})$. This is actually the main motivation of this paper. From fact (2.10) on topologies w^* and $\|\cdot\|_\infty$, Theorem 1.2 cannot be deduced simply from Theorem 5.1.

The basic idea of the proof of Theorem 1.2 is the same as [30, Sections 2 and 3], in which the concept of uniformly completely continuous operators introduced in [28] and used in [13, 30] plays an important role. Here we adopt some variant of the proof in [30].

Let $u_0 \in \mathbb{K}^2$ be fixed. In the following we always assume that

$$\mu_n \to \mu_0 \qquad \text{in } (\mathcal{M}_0(I, \mathbb{K}), w^*). \tag{5.1}$$

For simplicity, denote

$$Y_n(t) = y(t, u_0, \mu_n), \qquad Z_n(t) := \dot{Y}_n(t) = \dot{y}(t, u_0, \mu_n), \quad n \in \mathbb{Z}^+.$$

Lemma 5.2 *The sequence $\{Y_n\}_{n \in \mathbb{Z}^+}$ is bounded in $(\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty)$. That is, there exists $C > 0$ such that $\|Y_n\|_\infty \le C$ for all $n \in \mathbb{Z}^+$.*

Proof From the integral equations (3.1) and (3.2) for $Y_n(t)$ and $Z_n(t)$, one has

$$\begin{cases} |Y_n(t)| \le |y_0| + \int_{[0,t]} |Z_n(s)| \, \mathrm{d}s, \qquad t \in I, \\ \\ |Z_n(u)| \le |z_0| + \hat{Y}_n(u) \cdot \|\mu_n\|_{\mathbf{V}} \le |z_0| + c \cdot \hat{Y}_n(u), \qquad u \in I, \end{cases} \tag{5.2}$$

where $c = \sup_{n \in \mathbb{Z}^+} \|\mu_n\|_{\mathbf{V}} < \infty$, and

$$\hat{Y}_n(u) := \max_{s \in [0,u]} |Y_n(s)|.$$

Obviously, $\hat{Y}_n \in \mathcal{C}(I, \mathbb{R})$ and $\hat{Y}_n(u)$ is non-decreasing in $u \in I$. By inequalities (5.2), we have, for any $v \in [0, t] \subset I$,

$$\begin{aligned} |Y_n(v)| &\le |y_0| + \int_{[0,v]} \Big(|z_0| + c \cdot \hat{Y}_n(s) \Big) \, \mathrm{d}s \\ &\le |y_0| + c|z_0| + c \int_{[0,v]} \hat{Y}_n(s) \, \mathrm{d}s \\ &\le c_1 + c \int_{[0,t]} \hat{Y}_n(s) \, \mathrm{d}s, \qquad c_1 := |y_0| + c|z_0|. \end{aligned}$$

Thus

$$\hat{Y}_n(t) = \max_{v \in [0,t]} |Y_n(v)| \le c_1 + c \int_{[0,t]} \hat{Y}_n(s) \, \mathrm{d}s, \qquad t \in I.$$

Note that $\hat{Y}_n(0) = |Y_n(0)| = |y_0|$. Now the Gronwall inequality shows that $\sup_{n \in \mathbb{Z}^+} \|\hat{Y}_n\|_\infty \le C$ for some $C > 0$. Consequently, $\|Y_n\|_\infty \le \|\hat{Y}_n\|_\infty \le C$. □

Lemma 5.3 *The sequence $\{Y_n\}_{n \in \mathbb{Z}^+}$ is relatively compact in $(\mathcal{C}(I, \mathbb{K}), \| \cdot \|_\infty)$.*

Proof Let us keep the notation in the proof of Theorems 3.5 and 3.12. By Lemma 3.4,

$$Y_n = \hat{y}_0 - \mathcal{Z}_{\mu_n}(Y_n), \qquad n \in \mathbb{Z}^+. \tag{5.3}$$

See (3.5). For any $0 \le t_1 \le t_2 \le 1$, one has from (5.3)

$$\begin{aligned} |Y_n(t_2) - Y_n(t_1)| &= |\mathcal{Z}_{\mu_n}(Y_n)(t_2) - \mathcal{Z}_{\mu_n}(Y_n)(t_1)| \\ &= \left| \int_I (G(t_2, s) - G(t_1, s)) Y_n(s) \, \mathrm{d}\mu_n(s) \right| \\ &\le \max_{s \in I} |G(t_2, s) - G(t_1, s)| \|Y_n\|_\infty \|\mu_n\|_{\mathbf{V}} \\ &\le cC \max_{s \in I} |G(t_2, s) - G(t_1, s)|. \end{aligned}$$

Since $G \in \mathcal{C}(I^2, \mathbb{R})$, $\{Y_n\}$ is equi-continuous. Combining Lemma 5.2, the Ascoli-Arzela theorem implies the compactness of $\{Y_n\}$. □

Proof of Theorem 1.2. The proof is analogous to that of [30, Theorem 3.5].

For any subsequence $\{Y_{n'}\}$ of $\{Y_n\}$, by Lemma 5.3, one has some sub-subsequence

$$Y_{n''} \to Y_* \text{ in } (\mathcal{C}(I, \mathbb{K}), \| \cdot \|_\infty) \tag{5.4}$$

for some $Y_* \in \mathcal{C}(I, \mathbb{K})$. Now

$$\begin{aligned}
&\mathcal{Z}_{\mu''}(Y_{n''})(t) - \mathcal{Z}_{\mu_0}(Y_*)(t) \\
&= \int_I G(t,s) Y_{n''}(s)\,\mathrm{d}\mu_{n''}(s) - \int_I G(t,s) Y_*(s)\,\mathrm{d}\mu_0(s) \\
&= \int_I G(t,s)(Y_{n''}(s) - Y_*(s))\,\mathrm{d}\mu_{n''}(s) \\
&\quad + \left(\int_I G(t,s) Y_*(s)\,\mathrm{d}\mu_{n''}(s) - \int_I G(t,s) Y_*(s)\,\mathrm{d}\mu_0(s) \right) \\
&=: J_{n''}(t) + K_{n''}(t).
\end{aligned}$$

As $n'' \to \infty$, we have

$$|J_{n''}(t)| \le \|\mu_{n''}\|_{\mathbf{V}} \cdot \|Y_{n''} - Y_*\|_\infty \le c\|Y_{n''} - Y_*\|_\infty \to 0.$$

See (5.4). For the term $K_{n''}(t)$, let us notice that $G(t,\cdot) \in \mathcal{C}(I, \mathbb{R})$ and $Y_* \in \mathcal{C}(I, \mathbb{K})$. Thus $G(t,\cdot)Y_*(\cdot) \in \mathcal{C}(I, \mathbb{K})$. By the assumption (5.1) and definition of weak* convergence, we know that $K_{n''}(t) \to 0$ for each $t \in I$. In particular,

$$\lim_{n'' \to \infty} \mathcal{Z}_{\mu_{n''}}(Y_{n''})(t) = \mathcal{Z}_{\mu_0}(Y_*)(t) \qquad \text{for each } t \in I. \tag{5.5}$$

From the uniform convergence (5.4) and the pointwise convergence (5.5), the uniqueness of limits of equalities (5.3) for $n = n'' \to \infty$ yields

$$Y_*(t) = \hat{y}_0(t) - \mathcal{Z}_{\mu_0}(Y_*)(t), \qquad t \in I.$$

By Lemma 3.4, this means that $Y_*(t)$ is necessarily the unique solution $Y_0(t) = y(t, u_0, \mu_0)$ of problem (1.8)-(1.10) with $\mu = \mu_0$. As the limit $Y_* = Y_0$ of (5.4) is independent of the choice of subsequences n' and n'', we conclude from (5.4)

$$Y_n \to Y_0 \text{ in } (\mathcal{C}(I, \mathbb{K}), \|\cdot\|_\infty). \tag{5.6}$$

This is the precise meaning of the continuity of (1.11).

Finally the continuity (1.12) can be obtained as follows. By the integral equations for $Z_n(t) = \dot{Y}_n(t)$, we have

$$\begin{aligned}
\dot{Y}_n(1) &= z_0 - \int_{[0,1]} Y_n(s)\,\mathrm{d}\mu_n(s) \\
&= z_0 - \int_{[0,1]} (Y_n(s) - Y_0(s))\,\mathrm{d}\mu_n(s) - \int_{[0,1]} Y_0(s)\,\mathrm{d}\mu_n(s) \\
&\to z_0 - 0 - \int_{[0,1]} Y_0(s)\,\mathrm{d}\mu_0(s) = \dot{Y}_0(1).
\end{aligned} \tag{5.7}$$

Here the first limit can be obtained simply from (5.6) we have proved, while the second limit follows from the definition of weak* convergence. □

In the following we give an example to show that continuity result (1.12) for velocities cannot be improved.

Example 5.4 Given $t_0 \in (0,1)$. We assert that there exist $u_0 \in \mathbb{R}^2$ and $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I,\mathbb{R}), w^*)$ such that $\dot{y}(t_0,u_0,\mu_n) \not\to \dot{y}(t_0,u_0,\mu_0)$.

We only construct examples for $t_0 = 1/2$. Define measures

$$\mu_n(t) = \begin{cases} 0 & \text{for } t \in [0,1/2) =: I^1, \\ n(t-1/2) & \text{for } t \in [1/2, 1/2+1/n) =: I_n^2, \\ 1 & \text{for } t \in [1/2+1/n, 1] =: I_n^3, \end{cases}$$

where $n \in \mathbb{N}$, and

$$\mu_0(t) = \begin{cases} 0 & \text{for } t \in [0,1/2), \\ 1 & \text{for } t \in [1/2,1]. \end{cases}$$

Then, for any given $f \in \mathcal{C}(I,\mathbb{R})$, one has

$$\begin{aligned} \int_I f \,\mathrm{d}\mu_n &= \int_{[1/2,1/2+1/n]} f(t)\,\mathrm{d}(n(t-1/2)) \\ &= \int_{[0,1]} f(1/2+s/n)\,\mathrm{d}s \\ &= f(1/2+s_n) \qquad \text{for some } s_n \in [0,1/n] \\ &\to f(1/2) = \int_I f\,\mathrm{d}\mu_0. \end{aligned}$$

That is, $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I,\mathbb{R}), w^*)$. The solutions, denoted by $y(t,\mu_n)$, of the corresponding MDE with initial value $u_0 = (1,0)$ are

$$\begin{pmatrix} y(t,\mu_n) \\ \dot{y}(t,\mu_n) \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } t \in I^1, \\ \begin{pmatrix} \cos\sqrt{n}(t-1/2) \\ -\sqrt{n}\sin\sqrt{n}(t-1/2) \end{pmatrix} & \text{for } t \in I_n^2, \\ \begin{pmatrix} -\sqrt{n}\left(t-\frac{1}{2}-\frac{1}{n}\right)\sin\frac{1}{\sqrt{n}} + \cos\frac{1}{\sqrt{n}} \\ -\sqrt{n}\sin\frac{1}{\sqrt{n}} \end{pmatrix} & \text{for } t \in I_n^3, \end{cases}$$

and

$$\begin{pmatrix} y(t,\mu_0) \\ \dot{y}(t,\mu_0) \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } t \in [0,1/2), \\ \begin{pmatrix} -t+3/2 \\ -1 \end{pmatrix} & \text{for } t \in [1/2,1]. \end{cases}$$

From these one sees $\lim_{n\to\infty} \dot{y}(1,\mu_n) = -1 = \dot{y}(1,\mu_0)$, verifying (1.12) for this example. However, at $t = 1/2$, one has

$$\lim_{n\to\infty} \dot{y}(1/2,\mu_n) = 0 \neq -1 = \dot{y}(1/2,\mu_0).$$

Remark 5.5 Example (5.4) shows that the velocity continuity result (1.12) in weak* topology is optimal. This is quite reasonable, because, in general, $\mu_n \to \mu_0$ in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$ cannot imply that $\mu_n|_{[0,t]} \to \mu_0|_{[0,t]}$ in $(\mathcal{M}_0([0,t],\mathbb{K}), w^*)$ where $t \in (0,1)$. One can also find the difference between $\dot{Y}_n(t)$ and $\dot{Y}_n(1)$ from equality (5.7). However, global weak convergence in $(\mathcal{L}^p(I,\mathbb{K}), w_p)$ can yield local weak convergence in $(\mathcal{L}^p([0,t],\mathbb{K}), w_p)$ for all $t \in (0,1)$. This is why the continuity results of solutions and velocities in w_p topologies in [26, 30] are stronger than (1.12).

Arguing as in (5.7), we can obtain the following continuity on velocities.

Corollary 5.6 *Let μ_0, $\mu_n \in \mathcal{M}_0(I,\mathbb{K})$, $n \in \mathbb{N}$. Suppose that $t_0 \in (0,1]$ possesses the following convergence*

$$\mu_n|_{[0,t_0]} \to \mu_0|_{[0,t_0]} \qquad \text{in } (\mathcal{M}_0([0,t_0],\mathbb{K}), w^*).$$

Then one has

$$\overset{\bullet}{y}(t_0,u_0,\mu_n) \to \overset{\bullet}{y}(t_0,u_0,\mu_0).$$

We have now a clear description for continuity of solutions in weak and weak* topologies. For linear ODE (1.1), both solutions and their derivatives are continuous in the sense of (3.26)–(3.27) in potentials with weak topologies w_p, $1 \le p \le \infty$. See [19, 26, 30]. For second-order linear MDE (1.8), the continuity results (3.26)–(3.27) are also true in the $\|\cdot\|_\infty$ topology of measures, while in the weak* topology of measures, we have the optimal continuity results (1.11)–(1.12).

The proof of Theorem 1.2 can be generalized to inhomogeneous MDE.

Corollary 5.7 *Solutions $(y(t,u_0,\mu,\nu), \overset{\bullet}{y}(t,u_0,\mu,\nu))$ of inhomogeneous MDE* (3.20)-(1.10) *are jointly continuous in $(\mu,\nu) \in (\mathcal{M}_0(I,\mathbb{K}), w^*)^2$ in the sense that $\mu_n \to \mu_0$ and $\nu_n \to \nu_0$ in $(\mathcal{M}_0(I,\mathbb{K}), w^*)$ will imply*

$$\lim_{n\to\infty} \|y(\cdot,u_0,\mu_n,\nu_n) - y(\cdot,u_0,\mu_0,\nu_0)\|_\infty = 0,$$

$$\lim_{n\to\infty} |\overset{\bullet}{y}(1,u_0,\mu_n,\nu_n) - \overset{\bullet}{y}(1,u_0,\mu_0,\nu_0)| = 0.$$

Due to the linearity of Eq. (1.8) *and Eq.* (3.20)*, in fact, the following mappings are continuous*

$$\mathbb{K}^2 \times (\mathcal{M}_0(I,\mathbb{K}), w^*)^2 \to (\mathcal{C}(I,\mathbb{K}), \|\cdot\|_\infty), \qquad (u_0,\mu,\nu) \to y(\cdot,u_0,\mu,\nu),$$

$$\mathbb{K}^2 \times (\mathcal{M}_0(I,\mathbb{K}), w^*)^2 \to \mathbb{K}, \qquad (u_0,\mu,\nu) \to \overset{\bullet}{y}(1,u_0,\mu,\nu).$$

As mentioned before, for the principal eigenvalues of elliptic operators of the zero Dirichlet boundary condition, the corresponding continuous dependence on potentials and weights has been established in [5]. For the elliptic problems of PDE with some positive measures and the zero Dirichlet boundary condition, the dependence of solutions and all eigenvalues orders on measures have been studied in some literature like [1, 2]. The results there are based on the corresponding Green's functions and are more restrictive than the ODE case we are considering this paper.

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