Letters in Mathematical Physics **60:** 9–17, 2002.

Twist Property of Periodic Motion of an Atom Near a Charged Wire

JINZHI LEI and MEIRONG ZHANG*
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China. e-mail: {jzlei, mzhang}@math.tsinghua.edu.cn.

(Received: 27 November 2001)

Abstract. We study the Lyapunov stability of periodic motion of an atom in the vicinity of an infinite straight wire with an oscillating uniform charge, which serves as a mechanism for trapping cold neutral atoms. It is proved by King and Leśniewski that the system has classical periodic motion for a certain range of parameters. In this Letter, we will prove, using the Birkhoff Normal Forms and Morse Twist Theorem, that such a periodic state is of twist type. As a result, besides the stability of the periodic state in the sense of Lyapunov, the system has infinitely many interesting bound states such as subharmonics and quasi-periodic states.

Mathematics Subject Classifications (2000). 70K43, 37J40, 34C15.

Key words. invariant closed curve, oscillating charge, periodic motion, stability, subharmonics, twist coefficient.

1. Introduction

The systems that can be used to trap cold atoms have aroused interest in recent years. The Paul trap [10] can be integrated explicitly and is well studied. Another interesting electromagnetic trapping mechanism involves the interaction of a neutral atom with a charged wire [1]. This model is, in general, not integrable. Using numerical simulations, Hau *et al.* [1] predicted the possibility of bound states for the both classical and quantum problems for a range of parameters. Later, King *et al.* [2] studied this model and proved the existence of classical periodic bound states for certain range of parameters. They also proved that the periodic states are *marginally stable*, while the stability in the sense of Lyapunov of these periodic states remains as an interesting problem [2, p. 368].

In this Letter, we will study the stability of such periodic states. Since the model is a nonlinear, singular Hamiltonian system, the stability of these periodic states cannot be obtained directly from their linear stability. In fact, the stability is related also with the nonlinear terms in the Taylor expansions of the system along the periodic states. We found that the approximation method up to the third order is well studied in some cases [8, 9]. The idea for this method is to find the Birkhoff

*Supported in part by the National 973 Project of China, the National NSF of China, and the Excellent Personnel Supporting Plan of the Ministry of Education of China.

Normal Form of the Poincaré map associated with the system. If some coefficients of the normal form, which are called *twist coefficients*, are nonzero, the Moser Twist Theorem [11] does imply the stability in the sense of Lyapunov. Moreover, the theorem asserts also the existence of more complicated bound states such as subharmonics and quasi-periodic states. As for the model here, we will use some perturbation method to prove that the first twist coefficient is nonzero when the parameters are in certain range (see Theorem 1). So the stability of the periodic states found in [2] are actually stable and some further classical bound states can be found. These are stated in Theorem 2.

The Letter is organized as follows. In Section 2, we introduce the trap to be considered. In Section 3, we prove firstly that the periodic states are linearly stable and the results are then proved by estimating the first twist coefficients.

2. The Trapping Mechanism

We follow [1, 2] to introduce the system we are considering. This system is a neutral atom moving in a vicinity of a rigid, straight wire carrying a uniformly distributed time-dependent charge $q(t) = Q\cos(\omega t/2)$. Note that the atom moves freely in the direction parallel to the wire and only the perpendicular motion is of interest to us. Now the radial motion of the atom is described by the following Hamiltonian:

$$H = \frac{p_r^2}{2M} + \frac{1}{r^2}\left[\frac{L^2}{2M} - \alpha Q^2\right] - \frac{\alpha Q}{r^2}\cos\omega t,$$

where p_r is the radial conjugate momentum. The parameters involved are as follows: L is the fixed angular momentum, α and M are the polarizability and the mass of the atom, respectively. Thus the equation governing the motion of the atom is the following Hamiltonian system of degree of freedom 1:

$$\frac{d^2 r}{dt^2} + \frac{A + B\cos\omega t}{r^3} = 0, \tag{2.1}$$

where

$$A = \frac{2\alpha Q^2}{M} - \frac{L^2}{M^2}, \quad B = \frac{2\alpha Q^2}{M}.$$

More generally, let $p(t)$ be T-periodic and consider the following differential equation with singularity:

$$r'' + \frac{p(t)}{r^3} = 0. \tag{2.2}$$

By a bound state $r(t)$ of (2.2), we mean that $r(t)$ is a solution of (2.2) such that $r(t)$ is bounded away from both the origin and the infinity, i.e.,

$$0 < \inf_t r(t) \leqslant \sup_t r(t) < \infty.$$

As observed in [2], there are some restrictions on $p(t)$ if (2.2) admits bound states. Let us write $p(t) = p_+(t) - p_-(t)$, where $p_\pm(t) = \max\{0, \pm p(t)\}$. Then (2.2) admits a bound state only if

$$\int_{[0,T]} p_+(t)\,\mathrm{d}t > \int_{[0,T]} p_-(t)\,\mathrm{d}t > 0. \tag{2.3}$$

Applying (2.3) to (2.1), we know that (2.1) can admit bound states only if

$$B > A > 0. \tag{2.4}$$

As for the system (2.1), it is proved that a bound periodic state $r(t)$ does occur if the parameter $A > 0$ is sufficiently small [2, Theorem 3.1]. The proof is based on the Implicit Function Theorem. The obtained periodic motion $r(t)$ is far away from the origin and $r(t) - r(0)$, $\dot{r}(t)$ are small if $A > 0$ is small.

In the following, we always assume that (2.4) holds and A is small. Let us introduce, as in [2], a small parameter

$$\beta = (2A)/(3B) > 0.$$

Rescale (2.1) as

$$\sigma(t) = \beta^{1/2}\omega B^{-1/2} r^2(t/\omega). \tag{2.5}$$

Conversely,

$$r(t) = \beta^{-1/4}\omega^{-1/2} B^{1/4}\sigma^{1/2}(\omega t). \tag{2.6}$$

Now the equation for $\sigma(t)$ reads as

$$\sigma\ddot{\sigma} - \tfrac{1}{2}\dot{\sigma}^2 + 3\beta^2 + 2\beta\cos t = 0. \tag{2.7}$$

Mathematically, even when $\beta = 0$, Equation (2.7) is well defined on the phase plane

$$(r, \dot{r}) \in \{(x, y) : x > 0,\ y \in R\} =: R^+ \times R.$$

Let $Q_\beta\colon R^+ \times R \to R^+ \times R$ be the Poincaré map associated with the system (2.7), where $\beta \geqslant 0$. Note that when $\beta = 0$, Q_0 has an invariant manifold $M_0 := R^+ \times \{0\}$ consisting completely of fixed points of Q_0. It is easy to obtain that the tangent map of Q_0 along M_0

$$DQ_0|_{M_0} = \begin{pmatrix} 1 & 2\pi\omega^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus M_0 is not normally hyperbolic and we cannot assert from the invariant manifolds theory the existence of periodic states of (2.7) when $\beta > 0$ is small. Considering this, the existence of periodic solutions in [2, Theorem 3.1], proved by applying the Implicit Function Theorem to some fixed point problem related with (2.7), is a remarkable result. The positive 2π-periodic state of (2.7), denoted by $\sigma_\beta(t)$, satisfies that $\sigma_\beta(0) - 1$ and $\dot{\sigma}_\beta(0)$ are small provided that $\beta > 0$ is small. Using the rescaling (2.6), we know that the corresponding periodic state $r_\beta(t)$ of (2.1) is of order $O(\beta^{-1/4})$ which is large. We can obtain from (2.7) the expansion of $\sigma_\beta(t)$:

$$\sigma_\beta(t) = 1 + 2\beta\cos t + \beta^2\left[-\frac{\cos 2t}{4} + \frac{29}{32}\right] + O(\beta^3). \tag{2.8}$$

Thus we have, via (2.5) and (2.6),

$$r = r_\beta(t) = \beta^{-1/4}B^{1/4}\omega^{-1/2}[1 + \beta\cos\omega t + O(\beta^2)]. \tag{2.9}$$

Note that the coefficient of β^2 in expansion (3.6) of $\sigma_\beta(t)$ in [2] contains an error.

3. Twist Property of Periodic States

In this section, we will use the third order approximation of (2.1) near $r_\beta(t)$ to prove the twist property of $r_\beta(t)$. Note that such an approximation technique is studied in Ortega's papers [8, 9]. See also [6, 7, 12] for further developments and applications. Since the first twist coefficient is given explicitly only when the linearization equation is R-elliptic, this adds the difficulty in computation of the twist coefficient. We will use the perturbation method [5] to obtain necessary expansions as series of small parameter β and then find the order of the twist coefficient.

As for our problem, the parameter ω is irrelevant. So we assume that $\omega = 1$. In the following we keep the same terminology and notation as in [9, 12]. Denote

$$p_\beta(t) = A + B\cos\omega t = B(\cos\omega t + 3\beta/2),$$

and let $r = r_\beta(t) + s$ in (2.1). Then the third-order approximation of (2.1) near $r_\beta(t)$ is

$$\ddot{s} + a(t)s + b(t)s^2 + c(t)s^3 + \cdots = 0, \tag{3.1}$$

where the dots denote terms of orders higher than $o(s^3)$ and $a(t)$, $b(t)$, $c(t)$ are given by

$$a(t) = -3p_\beta(t)r_\beta^{-4}(t), \quad b(t) = 6p_\beta(t)r_\beta^{-5}(t), \quad c(t) = -10p_\beta(t)r_\beta^{-6}(t).$$

Using the expansion (2.9), we have

$$a(t) = -3\beta\cos t + \beta^2(3/2 + 6\cos 2t) + O(\beta^3),$$
$$b(t) = 6\beta^{5/4}B^{-1/4}\cos t - 3\beta^{9/4}B^{-1/4}(2 + 5\cos 2t) + O(\beta^{13/4}),$$
$$c(t) = -10\beta^{3/2}B^{-1/2}\cos t + 15\beta^{5/2}B^{-1/2}(1 + 2\cos 2t) + O(\beta^{7/2}).$$

The linearization equation of (2.1) near $r_\beta(t)$ is

$$\ddot{s} + a(t)s = 0. \tag{3.2}$$

PROPOSITION 1. *The linearization equation* (3.2) *is stable when* $\beta > 0$ *is sufficiently small. Actually,* $a(t)$ *is in the first stability zone.*

Proof. We will apply the Lyapunov stability criterion [3, 13] to prove this. The criterion asserts that (3.2) is in the first stability zone, which means that the zeroth periodic eigenvalue with potential $a(t)$ is negative and the first antiperiodic eigenvalue is positive, if the following two conditions are satisfied:

$$\int_{[0,T]} a(t)\,\mathrm{d}t > 0 \tag{3.3}$$

and

$$\int_{[0,T]} a_+(t)\,\mathrm{d}t \leqslant 4/T, \tag{3.4}$$

where $T = 2\pi$ is the period and $a_+(t)$ is, as before, the positive part of $a(t)$.

Since $a(t)$ and $a_+(t)$ are of order $\mathrm{O}(\beta)$, one sees that (3.4) is satisfied if $\beta > 0$ is small. As for (3.3), we have

$$\begin{aligned}\int_{[0,2\pi]} a(t)\,\mathrm{d}t &= \int_{[0,2\pi]} (-3\beta\cos t + \beta^2(3/2 + 6\cos 2t))\,\mathrm{d}t + \mathrm{O}(\beta^3)\\ &= 3\pi\beta^2 + \mathrm{O}(\beta^3),\end{aligned}$$

which is positive if $\beta > 0$ is small. Note that the expansion of $a(t)$ up to order $\mathrm{O}(\beta)$ yields only

$$\int_{[0,2\pi]} a(t)\,\mathrm{d}t = -3\beta \int_{[0,2\pi]} \cos t\,\mathrm{d}t + \mathrm{O}(\beta^2) = \mathrm{O}(\beta^2)$$

and we have no information for the sign of this integral. This is why we have expanded $a(t)$ up to order $\mathrm{O}(\beta^2)$. □

Since $a(t)$ is of order $\mathrm{O}(\beta)$, we know from Proposition 1 that the Floquet multipliers of (3.2) are $\lambda_{1,2} = \exp(\pm i\theta_\beta)$, $i = \sqrt{-1}$, where $\theta_\beta > 0$ and θ_β is of order $\mathrm{O}(\beta)$. For the exact expansion of θ_β up to order $\mathrm{O}(\beta)$, see (3.10) below. Therefore, equation (3.2) is 4-elementary, i.e., $\lambda_{1,2}^m \neq 1$ for all $1 \leqslant m \leqslant 4$.

Let P_β be the Poincaré map associated with (3.1). Then P_β has 0 as an elliptic fixed point which is also 4-elementary. Using the Birkhoff Normal Form of P_β, the (first) twist coefficient $\mathcal{T}_\beta = \mathcal{T}(P_\beta, 0)$ is well-defined. See [8, 9]. We say that the solution $r_\beta(t)$ of (2.1), or the solution $s(t) \equiv 0$ of (3.1), is of *twist type* if $\mathcal{T}(P_\beta, 0) \neq 0$.

As explained before, the explicit formula for $\mathcal{T}(P_\beta, 0)$ is not available. However, it is known from [8, Proposition 7] that there exist $t_0 \in R$ and $\alpha > 0$ such that the change of variables

$$\xi = s, \qquad \tau = \alpha^{-2}(t - t_0), \tag{3.5}$$

transforms (3.2) into an R-elliptic equation

$$\ddot{\xi} + a^*(\tau)\xi = 0, \tag{3.6}$$

where $a^*(\tau) = \alpha^4 a(t_0 + \alpha^2\tau)$ is periodic of period $T^* = 2\pi\alpha^{-2}$. The R-ellipticity here means that the Poincaré matrix associated with (3.6) is a rigid rotation. Note that Equation (3.1) is transformed, under the change of variables (3.5), into

$$\ddot{\xi} + a^*(\tau)\xi + b^*(\tau)\xi^2 + c^*(\tau)\xi^3 + \cdots = 0, \tag{3.7}$$

where $a^*(\tau)$ is as above and

$$b^*(\tau) = \alpha^4 b(t_0 + \alpha^2\tau), \quad c^*(\tau) = \alpha^4 c(t_0 + \alpha^2\tau),$$

and the new period is $T^* = 2\pi\alpha^{-2}$. So the Poincaré map P^*_β and the corresponding twist coefficient $\mathcal{T}^*_\beta = \mathcal{T}(P^*_\beta, 0)$ for (3.7) are defined. An important relation is

$$\operatorname{sign}\mathcal{T}(P_\beta, 0) = \operatorname{sign}\mathcal{T}(P^*_\beta, 0). \tag{3.8}$$

THEOREM 1. *The periodic state $r_\beta(t)$ is of twist type if $\beta > 0$ is sufficiently small.*

Proof. Using (3.8), it suffices to prove that $\mathcal{T}^*_\beta \neq 0$ when $\beta > 0$ is sufficiently small.

Let $\psi(t)$ be the Floquet solution of (3.2), i.e., $\psi(t)$ satisfies $\psi(t + 2\pi) = e^{i\theta_\beta}\psi(t)$. Using the perturbation method, we have

$$\psi(t) = \phi(t)\exp(i\theta_\beta t/2\pi),$$

where

$$\phi(t) = 1 - 3\beta\cos t + \beta^2\left(\frac{21}{8}\cos 2t + i6\sqrt{6}\sin t\right) + O(\beta^3) \tag{3.9}$$

and

$$\theta_\beta = 2\sqrt{6}\pi\beta + O(\beta^2). \tag{3.10}$$

From the proof of [8, Proposition 7], one can choose t_0 as a zero of $d/dt|\phi(t)|^2$ and α as $\alpha = |\dot{\phi}(t_0)/\phi(t_0)|^{-1/2}$.

By (3.9) it is easy to see that

$$t_0 = O(\beta^2), \qquad \alpha^2 = \frac{1}{6\sqrt{6}}\beta^{-2} + O(\beta^{-1}).$$

Denote $\Psi^*(\tau) = R^*(\tau)\exp(i\varphi^*(\tau))$ the solution of (3.6) with initial conditions $\Psi^*(0) = 1$, $\dot{\Psi}^*(0) = i$. Let $\Psi_0(t) = \Psi^*(\alpha^{-2}(t - t_0))$. Then $\Psi_0(t)$ is the solution of (3.2) with initial conditions

$$\Psi_0(t_0) = 1, \qquad \dot{\Psi}_0(t_0) = \alpha^{-2}i = 6\sqrt{6}\beta^2 i + O(\beta^3).$$

It is not difficult to find that $\Psi_0(t)$ can be expressed as

$$\Psi_0(t) = 1 + 3\beta(1 - \cos(t - t_0)) + O(\beta^2), \quad t \in [t_0 - 2\pi, t_0 + 2\pi].$$

Write $\Psi_0(t) = R_0(t)\exp(i\varphi_0(t))$. Then, for $t \in [t_0 - 2\pi, t_0 + 2\pi]$,

$$R_0(t) = 1 + 3\beta(1 - \cos(t - t_0)) + O(\beta^2),$$

$$\cos\varphi_0(t) = 1 + O(\beta^2), \quad \sin\varphi_0(t) = O(\beta^2).$$

Note that (3.2) and (3.6) have the same Floquet multipliers $\exp(\pm i\theta_\beta)$. Since (3.6) is R-elliptic, the twist coefficient of (3.7) is (cf. [9, 12])

$$\begin{aligned}\mathcal{T}^*_\beta = &-\frac{3}{8}\int_{[0,T^*]} c^*(\tau)R^{*4}(\tau)\mathrm{d}\tau+\\ &+\frac{1}{8}\iint_{[0,T^*]^2} b^*(\tau)b^*(\zeta)R^{*3}(\tau)R^{*3}(\zeta)[2+\cos 2(\varphi^*(\tau)-\varphi^*(\zeta))]\times\\ &\times\sin(|\varphi^*(\tau)-\varphi^*(\zeta)|)\mathrm{d}\tau\mathrm{d}\zeta+\\ &+\frac{1}{16}\frac{3\sin\theta_\beta}{1-\cos\theta_\beta}\left|\int_{[0,T^*]} b^*(\tau)R^{*3}(\tau)\exp(-i\varphi^*(\tau))\mathrm{d}\tau\right|^2+\\ &+\frac{1}{16}\frac{\sin 3\theta_\beta}{1-\cos 3\theta_\beta}\left|\int_{[0,T^*]} b^*(\tau)R^{*3}(\tau)\exp(3i\varphi^*(\tau))\mathrm{d}\tau\right|^2.\end{aligned}$$

Using the change of variables (3.5) again, we express $\mathcal{T}^*_\beta$, using the original coefficients together with t_0 and α, as

$$\begin{aligned}\mathcal{T}^*_\beta = &-\frac{3}{8}\alpha^2\int_{[t_0,t_0+2\pi]} c(t)R_0^4(t)\mathrm{d}t+\\ &+\frac{1}{8}\alpha^4\iint_{[t_0,t_0+2\pi]^2} b(t)b(s)R_0^3(t)R_0^3(s)[2+\cos 2(\varphi_0(t)-\varphi_0(s))]\times\\ &\times\sin(|\varphi_0(t)-\varphi_0(s)|)\,\mathrm{d}t\,\mathrm{d}s+\\ &+\frac{1}{16}\alpha^4\frac{3\sin\theta_\beta}{1-\cos\theta_\beta}\left|\int_{[t_0,t_0+2\pi]} b(t)R_0^3(t)\exp(-i\varphi_0(t))\,\mathrm{d}t\right|^2+\\ &+\frac{1}{16}\alpha^4\frac{\sin 3\theta_\beta}{1-\cos 3\theta_\beta}\left|\int_{[t_0,t_0+2\pi]} b(t)R_0^3(t)\exp(3i\varphi_0(t))\,\mathrm{d}t\right|^2.\end{aligned}$$

Recall that we have the necessary expansions for t_0, α, θ_β, $a(t)$, $b(t)$, $c(t)$, $R_0(t)$, $\cos\varphi_0(t)$, and $\sin\varphi_0(t)$. Now we can calculate the terms in $\mathcal{T}^*_\beta$ as follows.

Using the expansions above, we have

$$c(t)R_0^4(t) = -10\beta^{3/2}B^{-1/2}\cos t - 120\beta^{5/2}B^{-1/2}(1-\cos(t-t_0))\cos t + \mathrm{O}(\beta^{7/2}).$$

Thus the first term in $\mathcal{T}^*_\beta$ is

$$\begin{aligned}\mathcal{T}^*_1 &= \frac{3}{8}\alpha^2\int_{[t_0,t_0+2\pi]} 120\beta^{5/2}B^{-1/2}(1-\cos(t-t_0))\cos t\,\mathrm{d}t + \alpha^2\times\mathrm{O}(\beta^{7/2})\\ &= \mathrm{O}(\beta^{1/2})\end{aligned}$$

because $\alpha^2 = \mathrm{O}(\beta^{-2})$. In the second term $\mathcal{T}^*_2$, noticing simply that

$$\alpha^4 = \mathrm{O}(\beta^{-4}),\qquad b(t),\ b(s) = \mathrm{O}(\beta^{5/4}),\quad \text{and}\quad \sin(|\varphi_0(t)-\varphi_0(s)|) = \mathrm{O}(\beta^2),$$

we have $\mathcal{T}^*_2 = \mathrm{O}(\beta^{1/2})$. Note that these first two terms are small when β is small.

In the last two terms, we have

$$\frac{1}{16}\alpha^4 = \frac{1}{16\times 6^3}\beta^{-4} + \mathrm{O}(\beta^{-3}),$$
$$\frac{3\sin\theta_\beta}{1-\cos\theta_\beta} = \frac{3}{\sqrt{6}\pi}\beta^{-1} + \mathrm{O}(1),$$
$$\frac{\sin 3\theta_\beta}{1-\cos 3\theta_\beta} = \frac{1}{3\sqrt{6}\pi}\beta^{-1} + \mathrm{O}(1).$$

Moreover, $\exp(ik\varphi_0(t)) = 1 + \mathrm{O}(\beta^2)$, where $k = -1$ or 3. Thus, we have

$$\begin{aligned} & b(t)R_0^3(t)\exp(ik\varphi_0(t)) \\ &\quad = 6\beta^{5/4}B^{-1/4}\cos t + 3\beta^{9/4}B^{-1/4}[18(1-\cos(t-t_0))\cos t - 2 - 5\cos 2t] + \mathrm{O}(\beta^{13/4}). \end{aligned}$$

Consequently,

$$\begin{aligned} & \left| \int_{[t_0,t_0+2\pi]} b(t)R_0^3(t)\exp(ik\varphi_0(t))\,\mathrm{d}t \right|^2 \\ &\quad = \left| \int_{[t_0,t_0+2\pi]} 3\beta^{9/4}B^{-1/4}[18(1-\cos(t-t_0))\cos t - 2 - 5\cos 2t]\,\mathrm{d}t + \mathrm{O}(\beta^{13/4}) \right|^2 \\ &\quad = 4356\pi^2\beta^{9/2}B^{-1/2} + \mathrm{O}(\beta^{11/2}). \end{aligned}$$

From these computations, we have

$$\mathcal{T}_3^* = \frac{121\pi}{32\sqrt{6B}}\beta^{-1/2} + \mathrm{O}(\beta^{1/2}),$$
$$\mathcal{T}_4^* = \frac{121\pi}{288\sqrt{6B}}\beta^{-1/2} + \mathrm{O}(\beta^{1/2}).$$

Note that these last two terms are large when β is small.

Finally, we know that the twist coefficient of (3.7) is

$$\mathcal{T}_\beta^* = \frac{605\pi}{144\sqrt{6B}}\beta^{-1/2} + \mathrm{O}(\beta^{1/2}).$$

So $\mathcal{T}_\beta^* \gg 1$ if $\beta > 0$ is small. By (3.8), the theorem is proved. □

Since $r_\beta(t)$ is of twist type, we have the following results on (2.1) from the Moser Twist Theorem [4, 11].

THEOREM 2. *There exists a constant* $\beta_0 > 0$ *such that for any* $0 < \beta < \beta_0$, *the Poincaré map* P_β *of system* (2.1) *has infinitely many invariant closed curves in the neighborhood of* $(r_\beta(0), \dot r_\beta(0))$, *on which* P_β *is conjugate to the rigid irrational rotations. So system* (2.1) *admits infinitely many quasi-periodic bound states near* $r_\beta(t)$. *Furthermore, system* (2.1) *has infinitely subharmonics* $r_m(t)$ *between those invariant closed curves with minimal periods of* $r_m(t)$ *tending to infinity as* $m \to \infty$.

Note that those classical bound states from Theorem 2 are complicated than the periodic one found in [2] and may be of more interest.

Acknowledgement

The authors would like to thank P. J. Torres for bringing their attention to the work [2].

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