An Abstract Result on Asymptotically Positively Homogeneous Differential Equations*

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Following from the proof of spectrum theory of completely continuous linear operators, we give an abstract result concerning the estimates of norms of positively homogeneous operators in this paper. Using this result and some duality theorems for coincidence degree due to A. Capitello, J. Mawhin, and F. Zanolin (Trans. Amer. Math. Soc. 329, 1992, 41–72), the periodic boundary value problem of asymptotically positively homogeneous differential equations will be studied and the present results sharply improve those obtained earlier. © 1997 Academic Press

1. INTRODUCTION

From the property of invariance under homotopies for Leray–Schauder degree, it is known that if

\[ \sup_{x \in \partial \Omega} \| G(x) - F(x) \| < \inf_{x \in \partial \Omega} \| x - F(x) \|, \]

then

\[ d_{LS}(I - G, \Omega, 0) = d_{LS}(I - F, \Omega, 0), \]

where \( d_{LS} \) means the Leray–Schauder degree.

As a result, a quantitative estimate for lower bounds of \( \inf_{x \in \partial \Omega} \| x - F(x) \| \) will benefit the computation of degrees. In this paper, it is proved that such a quantitative estimate for \( \inf_{x \in \partial \Omega} \| x - F(x) \| \) can be realized

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when \( F \) is positively 1-homogeneous. See Theorem 1 in the next section. This result is a natural generalization of one property of the Riesz–Schauder theory of completely continuous linear operators to positively homogeneous operators.

Positively homogeneous systems originate naturally from many boundary value problems of nonlinear differential equations. The first example is the so-called “jumping nonlinearity,” which is the very beginning step from linearity to nonlinearity. By jumping nonlinearity is meant the following differential equation

\[
\ddot{x} + \lambda x^+ + \mu x^- = 0,
\]

and its generalizations. Here, \( \lambda \) and \( \mu \) are constants and \( x^+ = \max(x, 0), x^- = \min(x, 0) \). See [1, 2, 5, 11]. Another source for positively homogeneous systems is the nonresonance problem in many boundary value problems of different types of nonlinear differential equations. See, e.g., [3, 6, 12] for the Dirichlet problem of \( p \)-Laplacian.

In Section 3, we will use Theorem 1 and some duality theorems for coincidence degree to consider the periodic boundary value problem of asymptotically positively homogeneous systems in details. The existence results obtained in this section improve the corresponding ones in Krasnosel’skii and Zabreiko [4] and Capietto, Mawhin, and Zanolin [1].

2. AN ABSTRACT RESULT

Let \( X \) be a real normed space with the norm \( \| \cdot \| \). By a completely continuous mapping \( M \) from \( X \) to \( X \) is meant that \( M \) is continuous and maps bounded sets to relatively compact sets.

We say that \( M: X \to X \) is positively \( \alpha \)-homogeneous, \( \alpha \in \mathbb{R} \), if

\[
M(kx) = k^\alpha M(x), \quad \forall k > 0, \forall x \in X.
\]

**Theorem 1.** Let \( P, M: X \to X \) be completely continuous. Assume that \( P \) is linear and \( M \) is positively 1-homogeneous and the equation

\[
x - Px - Mx = 0
\]

(1)

has a unique solution \( x = 0 \) in \( X \). Then there exists a constant \( c > 0 \) such that

\[
\|x - Px - Mx\| \geq c \|x\|, \quad \forall x \in X.
\]

(2)

**Proof.** By the assumption that Eq. (1) has only a trivial solution and a standard property of completely continuous perturbations of identity (see, e.g., the remark at the end of Section 19.4 of [4]), there exists \( c > 0 \) such that

\[
\|y - Py - My\| \geq c
\]

(3)
for all \( y \in X \) with \( \| y \| = 1 \). Then, for each nonzero \( x \in X \), it follows from (3) applied to \( y = x/\| x \| \) and from the positive homogeneity of \( P \) and \( M \) that (2) holds.  

Remark. When \( M = 0 \), Theorem 1 follows from the fact that nonzero spectra of completely continuous linear operators must be eigenvalues in the Riesz–Schauder theory.

3. PERIODIC BOUNDARY VALUE PROBLEMS

Consider the following periodic boundary value problem

\[
\begin{aligned}
\dot{x} &= f(t, x), \\
x(0) &= x(\omega),
\end{aligned}
\tag{4}
\]

where \( \omega > 0 \) is a constant and \( f = f(t, x) \) is a Carathéodory function from \([0, \omega] \times \mathbb{R}^n \) to \( \mathbb{R}^n \), i.e.,

1. \( t \mapsto f(t, x) \) is measurable for each \( x \in \mathbb{R}^n \);
2. \( x \mapsto f(t, x) \) is continuous for a.e. \( t \in [0, \omega] \); and
3. for any \( r > 0 \) there is \( h_r \in L^1([0, \omega]; \mathbb{R}) \) such that

\[ |f(t, x)| \leq h_r(t), \quad \text{a.e. } t \in [0, \omega], \forall |x| \leq r. \]

As is known in Mawhin [7], Eq. (4) can be rewritten as a coincidence equation in normed spaces. To this end, let \( X = \{ x : [0, \omega] \to \mathbb{R}^n : x \text{ is continuous and } x(0) = x(\omega) \} \) with the maximum norm \( \| \cdot \| \) and \( Z = L^1([0, \omega]; \mathbb{R}^n) \) with the \( L^1 \)-norm \( \| \cdot \|_1 \). Let \( D(L) = \{ x \in X : x(\cdot) \text{ is absolutely continuous on } [0, \omega] \} \) and define \( L : D(L) \subset X \to Z \) by

\[ (Lx)(t) = \dot{x}(t). \]

Define \( N : X \to Z \) by

\[ (Nx)(t) = f(t, x(t)). \]

Then Eq. (4) is equivalent to the coincidence equation

\[ Lx = Nx, \quad x \in X. \tag{5} \]

As \( L \) is a Fredholm operator of index zero, we can take projectors \( P : X \to X \) and \( Q : Z \to Z \) as

\[
(Px)(t) = x(0), \quad (Qz)(t) = \frac{1}{\omega} \int_0^\omega z(s) \, ds.
\]
Then \( \text{Im}(P) = \text{Ker}(L) \) and \( \text{Ker}(Q) = \text{Im}(L) \). Let \( Jc = \omega c \) be the isomorphism from \( \text{Im}(Q) (\cong \mathbb{R}^n) \) onto \( \text{Ker}(P) (\cong \mathbb{R}^n) \). It follows from [7] that Eq. (5) can be reduced to a fixed point equation in \( X \):

\[
x(t) = x(0) + \frac{\omega - t}{\omega} \int_0^\omega f(s, x(s)) \, ds + \int_0^t f(s, x(s)) \, ds =: (Fx)(t)
\]

for all \( t \in [0, \omega] \). Note that \( F \) is positively 1-homogeneous if \( f(t, \cdot) \) is positively 1-homogeneous for all \( t \in [0, \omega] \).

In this Section we consider a periodic problem of asymptotically positively homogeneous differential equations of the form

\[
\begin{cases}
\dot{x} = g(t, x) + h(t, x), & t \in [0, \omega], \\
x(0) = x(\omega),
\end{cases}
\]

(7)

where \( g = g(t, x) \) is a continuous function from \([0, \omega] \times \mathbb{R}^n \) to \( \mathbb{R}^n \) and \( g(t, \cdot) \) is positively \( \alpha \)-homogeneous for all \( t \in [0, \omega] \), and \( h = h(t, x) \) is a Carathéodory function from \([0, \omega] \times \mathbb{R}^n \) to \( \mathbb{R}^n \) such that

\[
\limsup_{|x| \to \infty} \frac{|h(t, x)|}{|x|^{\alpha}} = 0
\]

uniformly in a.e. \( t \in [0, \omega] \).

The existence problem for (7) has been studied by many authors [4, 5, 8, 10]. Especially, the cases \( \alpha > 1 \) and \( 0 < \alpha < 1 \) have been well solved in [4], see Theorems 41.7, 41.8, and Theorem 41.10, respectively. A much more difficult problem raised in [4, p. 253] is to give the corresponding results for the case \( \alpha = 1 \). After giving a general duality theorem on coincidence degree, a partial answer to this problem is given in [1, Corollary 8] when \( g = g(x) \) is autonomous. See also [9]. Using Theorem 1, the result in [1] will be sharply improved, i.e., under the same conditions on \( g \) as in [1, Corollary 8], our theorems allow such perturbations \( h(t, x) \) that

\[
\|\varphi\|_1 < c_0
\]

for some positive constant \( c_0 \), where

\[
\varphi(t) = \limsup_{|x| \to \infty} \frac{|h(t, x)|}{|x|}
\]

\(^{1}\) The formula \( \Phi_3(x(t)) = x(0) + (\omega - t) \int_0^\omega f_0(x(s)) \, ds + \int_0^t f_0(x(s)) \, ds \) in [1, p. 49] seems to be incorrect because \( \Phi_3 \) does not map \( X \) into \( X \).
exists in the sense that for any given \( \varepsilon > 0 \), there is \( \psi_{\varepsilon}(t) \in L^1([0, \omega], \mathbb{R}) \) such that

\[
|h(t, x)| \leq (\varphi(t) + \varepsilon)|x| + \psi_{\varepsilon}(t) \quad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } t \in [0, \omega],
\]

(8)

and \( \varphi \in L^1([0, \omega], \mathbb{R}) \).

At first we consider the autonomous case \( g = g(x) \) in (7). From (6), it is known that problem (7) is now equivalent to

\[
x = Px + Gx + Hx, \quad x \in X,
\]

(9)

where \( P, G, \) and \( H \) are operators from \( X \) to \( X \) given by

\[
(Px)(t) = x(0),
\]

\[
(Gx)(t) = \frac{\omega - t}{\omega} \int_0^\omega g(x(s)) \, ds + \int_0^t g(x(s)) \, ds,
\]

\[
(Hx)(t) = \frac{\omega - t}{\omega} \int_0^\omega h(s, x(s)) \, ds + \int_0^t h(s, x(s)) \, ds,
\]

respectively.

**Theorem 2.** Assume that \( g = g(x) \) in (7) is positively 1-homogeneous such that

\[
(g_1) \quad \text{The problem }
\]

\[
\begin{align*}
\dot{x} &= g(x) \\
x(0) &= x(\omega)
\end{align*}
\]

has no \( \omega \)-periodic solution other than \( x = 0 \); and

\[
(g_2) \quad \text{deg} g, B(0, r), 0 \neq 0 \text{ for some } r > 0, \text{ where } \text{deg} \text{ means the Brouwer degree and } B(0, r) = \{x \in \mathbb{R}^n: |x| < r\}.
\]

Then there is a constant \( c_0 > 0 \) such that if \( \varphi(t) \) in (8) satisfies

\[
\|\varphi\|_1 < c_0,
\]

the problem (7) has at least one \( \omega \)-periodic solution.

**Proof.** We need only consider the problem (9). Condition \((g_1)\) means that

\[
x - Px - Gx = 0
\]
has a unique solution $x = 0$ in $X$. Thus, by Theorem 1, there exists a constant $c > 0$ such that

$$\|x - Px - Gx\| \geq c \|x\|, \quad \forall x \in X. \tag{10}$$

On the other hand, for any fixed $\varepsilon > 0$ and any $x \in X$,

$$|(Hx)(t)| = \frac{\omega - t}{\omega} \int_0^\omega h(s, x(s)) \, ds + \int_0^t h(s, x(s)) \, ds \leq 2 \int_0^\omega ((\varphi(s) + \varepsilon) |x(s)| + \psi_\varepsilon(s)) \, ds \leq 2(\|\varphi\|_1 + \varepsilon) \|x\| + 2 \|\psi_\varepsilon\|_1$$

for all $t \in [0, \omega]$. Namely,

$$\|Hx\| \leq 2(\|\varphi\|_1 + \varepsilon) \|x\| + 2 \|\psi_\varepsilon\|_1. \tag{11}$$

From (10) and (11), if

$$\|\varphi\|_1 < c_0 := c/2$$

and letting $\varepsilon > 0$ be sufficiently small and

$$r > \frac{2 \|\psi_\varepsilon\|_1}{c - 2(c_0 + \varepsilon)},$$

then

$$\sup_{\|x\| = r} \|H(x)\| < \inf_{\|x\| = r} \|x - P(x) - G(x)\|.$$

Thus we can use the homotopy property of Leray–Schauder degree to obtain

$$d_{LS}(I - P - G - H, \tilde{B}(0, r), 0) = d_{LS}(I - P - G, \tilde{B}(0, r), 0),$$

where $\tilde{B}(0, r) = \{x \in X : \|x\| < r\}$. From the duality theorem of Capietto, Mawhin, and Zanolin [1, Corollary 1]

$$d_{LS}(I - P - G, \tilde{B}(0, r), 0) = (-1)^n \deg(g, \tilde{B}(0, r) \cap \mathbb{R}^n, 0).$$

Thus it is nonzero by $(g_2)$ and Eq. (9) has at least one solution in $X$.

For nonautonomous $g(t, x)$, we can use a homotopy from $g(t, x)$ to the averaged $\bar{g}(x) = (1/\omega)\int_0^\omega g(t, x) \, dt$ and obtain the following result.
**Theorem 3.** Assume that $g = g(t, x)$ in (7) is positively 1-homogeneous such that

- $(g'_1)$ The problem
  \[
  \begin{cases}
  \dot{x} = \lambda g(t, x) \\
  x(0) = x(\omega)
  \end{cases}
  \]
  has no $\omega$-periodic solution other than $x = 0$ for any $0 < \lambda \leq 1$; and
  \[
  (g'_2) \quad \deg(g, B(0, r), 0) \neq 0 \text{ for some } r > 0.
  \]

Then there is a constant $c_0 > 0$ such that if

\[
\|\varphi\|_1 < c_0,
\]

the problem (7) has at least one $\omega$-periodic solution.

**Proof.** See [4, 1].

We can state the analogue of Theorem 2 for $m$th order systems of the form

\[
\begin{cases}
  x^{(m)} = g(x, \dot{x}, \ldots, x^{(m-1)}) + h(t, x, \dot{x}, \ldots, x^{(m-1)}), & t \in [0, \omega], \\
  x^{(i)}(0) = x^{(i)}(\omega), & i = 0, 1, \ldots, m - 1,
  \end{cases}
\]

(12)

where

\[
g(kx, k\dot{x}, \ldots, kx^{(m-1)}) = kg(x, \dot{x}, \ldots, x^{(m-1)})
\]

for all $k > 0$, $(x, \dot{x}, \ldots, x^{(m-1)}) \in \mathbb{R}^m$, and

\[
\varphi(t) = \lim_{|x| + |\dot{x}| + \cdots + |x^{(m-1)}| \to \infty} \frac{|h(t, x, \dot{x}, \ldots, x^{(m-1)})|}{|x| + |\dot{x}| + \cdots + |x^{(m-1)}|}
\]

exists and $\varphi \in L^1([0, \omega]; \mathbb{R})$.

**Corollary.** Assume that

- $(g''_1)$ The problem
  \[
  \begin{cases}
  x^{(m)} = g(x, \dot{x}, \ldots, x^{(m-1)}), & t \in [0, \omega], \\
  x^{(i)}(\omega) = x^{(i)}(0), & i = 0, 1, \ldots, m - 1,
  \end{cases}
  \]
  has no $\omega$-periodic solution other than $x = 0$; and
  \[
  (g''_2) \quad \deg(\tilde{g}, B(0, r), 0) \neq 0 \text{ for some } r > 0, \text{ where } \tilde{g}(x) = g(x, 0, \ldots, 0).
  \]

Then there is a constant $c_0 > 0$ such that if

\[
\|\varphi\|_1 < c_0,
\]

the problem (12) has at least one $\omega$-periodic solution.
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