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THE BEST BOUND ON THE ROTATIONS IN THE STABILITY OF PERIODIC SOLUTIONS OF A NEWTONIAN EQUATION

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Abstract

In most cases, the third order approximation of a scalar Newtonian equation can lead to the Lyapunov stability of a periodic solution through the obtaining of a nonzero twist coefficient. Recently, Ortega obtained the twist property of a periodic solution when the second order coefficient does not change sign and the third one is negative under a crucial limitation to the rotation of the linearization equation. The paper finds that the best bound on the limitation of the rotations is $\theta_0^* = \arccos(-1/4)$.

1. *Introduction*

In this paper we study the stability (in the sense of Lyapunov) of T-periodic solutions of the scalar Newtonian equation

$$x'' + f(t,x) = 0, \tag{1.1}$$

where $f(t,x)$ is T-periodic in t and is sufficiently smooth, for example

$$f \in C^{0,4}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R}).$$

Suppose that $x_0(t)$ is a T-periodic solution of (1.1). An 'almost' necessary condition for $x_0(t)$ to be stable is that it is linearly stable (or elliptic), which means that the linearization equation of (1.1) along $x_0(t)$,

$$x'' + a(t)x = 0, \tag{1.2}$$

has the Floquet multipliers lying on $S^1 \backslash \{\pm 1\}$, where

$$a(t) = \frac{\partial f}{\partial x}(t, x_0(t)).$$

However, the stability of $x_0(t)$ cannot be determined by the linear stability of (1.2) because the first Lyapunov method cannot be applied to this case. In fact, the Lyapunov stability is related to the nonlinear terms in the Taylor expansion of f evaluated at $x_0(t)$.

There has been a long history of studying the Lyapunov stability of nonlinear equations. Theoretically, if one can construct the Birkhoff normal form for the Poincaré map of (1.1), then the stability can be obtained using the Moser twist theorem [**5**, **12**] by proving that the twist coefficients are nonzero. Moreover, equation (1.1) has subharmonic solutions with minimal periods tending to ∞ and infinitely many quasi-periodic solutions in the neighborhood of $x_0(t)$.

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In most cases it is possible to use the third order approximation of f to study the stability. We are hence led to the stability problem of $x(t) = 0$ (as a T-periodic solution) of

$$x'' + a(t)x + b(t)x^2 + c(t)x^3 + \dots = 0, \tag{1.3}$$

where $a, b, c \in C(\mathbb{R}/T\mathbb{Z})$ and the dots denote the remainder of order $o(x^3)$ uniformly with respect to t. (We have translated the solution $x_0(t)$ to the origin.) Recently, Ortega [**9**–**11**] derived the formula for the first twist coefficient $\beta = \beta(a, b, c)$ of (1.3) and then obtained some stability criteria.

An important result of Ortega is [**11**, Theorem 3.2], which asserts that the solution $x = 0$ of (1.3) is stable for all those coefficients b and c in the following set:

$$BC = \left\{ (b, c) : b, c \in C(\mathbb{R}/T\mathbb{Z}),\ c \leqslant 0,\ b \geqslant 0 \text{ or } b \leqslant 0, \text{ and } \int_{[0,T]} |b| + |c| > 0 \right\},$$

provided that the linearization equation (1.2) satisfies assumptions (1.4) and (1.5). Suppose that (1.2) has the Floquet multipliers $e^{\pm i\theta}$, $\theta \in \mathbb{R}$. The value θ is related to the rotation number of (1.2) (see (2.4)). The first assumption on (1.2) is that there exists a number θ^* in the interval $(\pi/2, 2\pi/3)$ such that θ satisfies

$$0 < |\theta| \leqslant \theta^* \quad \text{or} \quad 2\pi/3 < |\theta| < \pi. \tag{1.4}$$

See [**11**, condition (3.6)]. The second assumption is as follows:

The distance between two consecutive zeros of a nontrivial solution of (1.2) is at least T. (1.5)

See [**11**, condition (3.4)].

[**11**, Example 3.5] shows that a certain limitation on the rotation θ is necessary in obtaining a nonzero twist coefficient. In [**11**], Ortega gave an admissible value for θ^* which is the zero of the following function in the interval $(\pi/2, 2\pi/3)$:

$$\Phi(\theta) = \frac{3\sin\theta}{1-\cos\theta} + \frac{\sin 3\theta}{1-\cos 3\theta}\frac{1}{\cos^2(\theta/2)} \equiv \frac{6\cos^2\theta + 13\cos\theta + 1}{(2\cos\theta + 1)\sin\theta}.$$

See [**11**, p. 510]. Thus the bound θ^* in [**11**] is

$$\theta^* = \arccos((\sqrt{145} - 13)/12) \doteq 1.6507$$

However, the best upper bound on θ^* is unknown, as mentioned by Ortega [**11**].

This stability criterion of Ortega is of interest for the following two reasons:

(1) Equation (1.3) is a perturbation of *nonintegrable* equation (1.2).

(2) The stability condition on (b, c) does not involve *any small parameter*.

Núñez [**7**] then partially generalized [**11**, Theorem 3.2] to the cases where b, c may change sign under a more conservative limitation on the rotation of (1.2). Analogously, the corresponding best bound on the rotations is not known. Notice that such a best bound is important in applications [**8**, **14**].

In this paper, we study the following two problems.

Problem 1. What is the best bound θ_0^* on θ^* such that $\beta(a, b, c) \neq 0$ for all $(b, c) \in BC$ under assumptions (1.4) and (1.5)?

Problem 2. For any θ satisfying $\theta_0^* < |\theta| < 2\pi/3$, is there a collection of a, b, c such that the rotation of (1.2) is θ and $(b, c) \in BC$, while $\beta(a, b, c) = 0$?

Our solutions are as follows:

(1) The best bound θ_0^* in Problem 1 is

$$\theta_0^* = \arccos(-1/4) \doteq 1.8234$$

This is obtained by proving the positivity of some compact linear operator.

(2) The solution to Problem 2 is positive for any θ satisfying $\theta_0^* < |\theta| < 2\pi/3$.

As a result, we know that the set of rotations θ of (1.2) for which $\beta(a, b, c) \neq 0$ for all $(b, c) \in BC$ is exactly the same as

$$\Theta^* = \{\theta : 0 < |\theta| \leqslant \theta_0^* \text{ or } 2\pi/3 < |\theta| < \pi\}.$$

The paper is organized as follows. In Section 2, we give the solutions of Problems 1 and 2. In Section 3, we give some explicit conditions on the rotations of Hill equations (1.2).

2. *The best bound*

Let us introduce some notations as in **[11]**. First we consider the linear part of (1.3), that is, the Hill equation (1.2). Denote by $\Psi(t) = \phi_1(t) + i\phi_2(t)$ the (complex) solution of (1.2) with the initial data $\Psi(0) = 1$ and $\Psi'(0) = i$, where ϕ_1 and ϕ_2 are, respectively, the real and imaginary parts of Ψ. Sometimes we need the form of Ψ in the polar coordinates: $\Psi(t) = r(t)e^{i\varphi(t)}$, where $r, \varphi \in C^2(\mathbb{R})$ and $r(0) = 1$, $\varphi(0) = 0$. In this case $r(t) > 0$ and $\varphi'(t) \equiv 1/r^2(t)$ for all t. Thus $\varphi(t)$ is strictly increasing.

The *Floquet multipliers* of (1.2) are the eigenvalues λ, μ of the monodromy matrix

$$M = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \phi_1'(T) & \phi_2'(T) \end{pmatrix}.$$

Since M is symplectic, $\lambda \cdot \mu = 1$.

We are only interested in the case where (1.2) is *elliptic*, that is, $\lambda = \bar{\mu} \in S^1 \backslash \{\pm 1\}$. We say that (1.2) is *R-elliptic* if there is some $\lambda \in S^1 \backslash \{\pm 1\}$ such that

$$\Psi(t + T) = \bar{\lambda}\Psi(t) \quad \forall\ t \in \mathbb{R},$$

in which case the monodromy matrix M is a rigid rotation (different from $\pm I$) and the multipliers of (1.2) are λ and $\bar{\lambda}$. According to [**9**, Proposition 7], every elliptic equation (1.2) can be transformed to an R-elliptic one by rescaling and translating time t. Such a change has no influence on the twist character of the nonlinear equation (1.3) (see [**9**]).

In the following, we fix an $a \in C(\mathbb{R}/T\mathbb{Z})$ such that (1.2) is elliptic and (1.5) is satisfied. Without loss of generality, we assume also that (1.2) is R-elliptic with respect to some Floquet multiplier $\lambda \in S^1 \backslash \{\pm 1\}$. Let us assume that $\lambda = e^{-i\theta}$ definitely, where $\theta \in (0, \pi)$, because the case $\lambda = e^{i\theta}$, $\theta \in (0, \pi)$, can be treated similarly.

When (1.2) is R-elliptic with $\lambda = e^{-i\theta}$, $\theta \in (0, \pi)$, $r(t)$ is T-periodic, and $\varphi(t)$ satisfies

$$\varphi(t + T) = \varphi(t) + \theta + 2n\pi \quad \forall\ t \in \mathbb{R},$$

where n is a nonnegative integer. Under assumption (1.5), we know that 0 is in the first stability zone of the parameterized Hill equation

$$x'' + (\lambda + a(t))x = 0. \tag{2.1}$$

Thus n above is 0. Therefore we have

$$\varphi(t+T) = \varphi(t) + \theta \quad \forall\ t \in \mathbb{R}.$$

We will return to the discussion of assumption (1.5) in Section 3.

Let us now use the rotation number of (1.2) to explain the value θ in the Floquet multipliers. Let $x = \rho\cos\psi$ and $x' = -\rho\sin\psi$ in (1.2). Then the equation for $\psi(t)$ is

$$\psi' = \sin^2\psi + a(t)\cos^2\psi. \tag{2.2}$$

As in [**6**, **11**, **16**], define the *rotation number* of (1.2) by

$$\alpha = \lim_{t\to\infty} \psi(t)/t, \tag{2.3}$$

where $\psi(t)$ is any solution of (2.2). In the case where (1.2) is elliptic, it is well known that the Floquet multipliers of (1.2) are $e^{\pm iT\alpha}$.

Since assumptions (1.4) and (1.5) imply that 0 is in the first stability zone of (2.1), it is known that $T\alpha \in (0, \pi)$. See, for example, [**2**, Proposition 2.1] or [**16**, Theorem 3.3]. In our case, θ is related to the rotation number α in the following way:

$$\theta = T\alpha. \tag{2.4}$$

From now on we consider the nonlinear equation (1.3). Let

$$\hat{F}(x_0, y_0) = (\hat{F}_1(x_0, y_0), \hat{F}_2(x_0, y_0))$$

be the Poincaré map of (1.3). Write $\hat{F}$ in the complex form

$$\begin{aligned}(z, \bar{z}) &= \hat{F}_1((z+\bar{z})/2, (z-\bar{z})/(2i)) + i\hat{F}_2((z+\bar{z})/2, (z-\bar{z})/(2i))\\ &= \lambda z + F_2(z,\bar{z}) + F_3(z,\bar{z}) + \dots,\end{aligned}$$

where

$$F_2(z,\bar{z}) = Az^2 + Bz\bar{z} + C\bar{z}^2,$$

$$F_3(z,\bar{z}) = Mz^3 + Nz^2\bar{z} + Pz\bar{z}^2 + Q\bar{z}^3.$$

Some of the coefficients are given in [**9**]:

$$A = -\frac{i\lambda}{4}\int_{[0,T]} b(t)\bar{\Psi}^2(t)\Psi(t)\,dt,$$

$$C = -\frac{i\lambda}{4}\int_{[0,T]} b(t)\Psi^3(t)\,dt,$$

$$\begin{aligned}N = &-\frac{3i\lambda}{8}\int_{[0,T]} c(t)|\Psi(t)|^4\,dt\\ &-\frac{i\lambda}{4}\iint_{\Delta_T} G(t,s)b(t)b(s)[2|\Psi(t)|^2|\Psi(s)|^2 + \Psi^2(t)\bar{\Psi}^2(s)]\,dt\,ds,\end{aligned}$$

where

$$\Delta_T = \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\},$$

and

$$G(t,s) = \phi_1(t)\phi_2(s) - \phi_1(s)\phi_2(t)$$

is the Green function of the problem

$$x'' + a(t)x + f(t) = 0$$

with the T-periodic boundary condition.

By the known results from [**11**, Theorem 3.2], we are left to consider the case $\theta \in (\pi/2, 2\pi/3)$. Therefore (1.2) is 4-*elementary*, which means that $\mu = e^{\pm i\theta}$ satisfy $\mu^p \neq 1$ for all $1 \leqslant p \leqslant 4$. Now the first twist coefficient is [**9**]:

$$\beta = \operatorname{Im}(\bar{\lambda}N) + \frac{3\sin\theta}{1-\cos\theta}|A|^2 + \frac{\sin 3\theta}{1-\cos 3\theta}|C|^2. \tag{2.5}$$

We say that the solution $x(t) = 0$ (as a T-periodic solution) of (1.3) is of *twist type* if β is nonzero. Therefore $x(t) = 0$ is necessarily stable in the sense of Lyapunov if it is of twist type.

The main result of this paper is the following theorem.

THEOREM 1. (i) *The best upper bound in Problem* 1 *is* $\theta_0^* = \arccos(-1/4)$.
(ii) *Given a* θ *satisfying* $\theta_0^* < |\theta| < 2\pi/3$, *there must be some* $a \in C(\mathbb{R}/T\mathbb{Z})$ *with the rotation* θ *and some* $(b, c) \in BC$ *such that* $\beta = 0$, *that is, the solution* $x = 0$ *of* (1.3) *is not of twist type.*

Proof. As before, we need only to consider the case where (1.2) is R-elliptic and $\theta \in (\pi/2, 2\pi/3)$. Using the notations above, we have

$$G(t, s) = -r(t)r(s)\sin(\varphi(t) - \varphi(s)),$$

$$2|\Psi(t)|^2|\Psi(s)|^2 + \Psi^2(t)\bar{\Psi}^2(s) = r^2(t)r^2(s)(2 + e^{i2(\varphi(t)-\varphi(s))}).$$

Now we get from (2.5) that

$$\begin{aligned}
\beta &= \beta(b, c) \\
&= -\frac{3}{8}\int_{[0,T]} c(t)r^4(t)\,dt \\
&= +\frac{1}{4}\iint_{\Delta_T} b(t)b(s)r^3(t)r^3(s)(2 + \cos 2(\varphi(t) - \varphi(s)))\sin(\varphi(t) - \varphi(s))\,dt\,ds \\
&\quad + \frac{1}{16}\frac{3\sin\theta}{1-\cos\theta}\left|\int_{[0,T]} b(t)r^3(t)e^{-i\varphi(t)}\,dt\right|^2 \\
&\quad + \frac{1}{16}\frac{\sin 3\theta}{1-\cos 3\theta}\left|\int_{[0,T]} b(t)r^3(t)e^{3i\varphi(t)}\,dt\right|^2.
\end{aligned} \tag{2.6}$$

Observe that

$$\beta(b, c) = \beta(b, 0) + \beta(0, c),$$

where

$$\beta(0, c) = -\frac{3}{8}\int_{[0,T]} c(t)r^4(t)\,dt \geqslant 0$$

because $c \leqslant 0$. If $b = 0$, then $\beta(b, c) = \beta(0, c) > 0$, because $c \ll 0$. Here and henceforth, $c \ll 0$ means that $c \leqslant 0$ and $c(t) < 0$ on a subset of positive measure.

In the following we always assume that $(b, c) \in BC$ with $b \neq 0$. In this case

$$\beta(b, c) \geqslant \beta(b, 0) =: \beta_*(b), \tag{2.7}$$

where $\beta_*(b)$ is the sum of the latter three terms in (2.6). We are led to finding a condition on θ such that $\beta_*(b) > 0$ holds for all b in the following set:

$$B = B^+ \cup B^-, \quad B^\pm = \{b \in C(\mathbb{R}/T\mathbb{Z}) : \pm b \gg 0\}.$$

Recall that $\varphi(t)$ is strictly increasing. Using the change of variables $u = \varphi(t)$, $v = \varphi(s)$, we have

$$\iint_{\Delta_T} b(t)b(s)r^3(t)r^3(s)(2 + \cos 2(\varphi(t) - \varphi(s)))\sin(\varphi(t) - \varphi(s))\,dt\,ds$$

$$= \iint_{\Delta_\theta} h(u)h(v)(2 + \cos 2(u - v))\sin(u - v)\,du\,dv,$$

where $\Delta_\theta = \{(u, v) : 0 \leqslant v \leqslant u \leqslant \theta\}$ and

$$h(u) = b(\varphi^{-1}(u))r^3(\varphi^{-1}(u))/\varphi'(\varphi^{-1}(u)).$$

Note that h satisfies either $h \gg 0$ or $h \ll 0$. By the equalities

$$\left|\int_{[0,T]} b(t)r^3(t)e^{in\varphi(t)}\,dt\right|^2 = \left|\int_{[0,\theta]} h(u)e^{inu}\,du\right|^2$$

$$= 2\iint_{\Delta_\theta} h(u)h(v)\cos n(u - v)\,du\,dv \quad n = -1, 3,$$

we have

$$\beta_*(b) = \beta_\theta(h) := \iint_{\Delta_\theta} h(u)h(v)\chi_\theta(u - v)\,du\,dv$$

$$= \frac{1}{2}\iint_{[0,\theta]^2} h(u)h(v)\chi_\theta(|u - v|)\,du\,dv, \tag{2.8}$$

where the kernel $\chi_\theta(x)$ is given by

$$\chi_\theta(x) = \tfrac{1}{4}(2 + \cos 2x)\sin x + \frac{1}{8}\frac{3\sin\theta}{1 - \cos\theta}\cos x + \frac{1}{8}\frac{\sin 3\theta}{1 - \cos 3\theta}\cos 3x$$

$$= \frac{3}{8}\frac{\cos(x - \theta/2)}{\sin(\theta/2)} + \frac{1}{8}\frac{\cos 3(x - \theta/2)}{\sin(3\theta/2)} \quad x \in [0, \theta].$$

We will make extensive use of the first integral expression (2.8) over Δ_θ.

Our best bound is obtained by seeking those $\theta \in (\pi/2, 2\pi/3)$ such that

$$\min_{x\in[0,\theta]} \chi_\theta(x) \geqslant 0.$$

When $\theta \in (\pi/2, 2\pi/3)$, it is not difficult to verify that the function $\chi_\theta(x)$ possesses the following properties:

(1) $\chi_\theta(x)$ is symmetric with respect to the line $x = \theta/2$, that is, $\chi_\theta(\theta - x) = \chi_\theta(x)$.

(2) $\chi_\theta(x)$ is strictly increasing on $[0, \theta/2]$ and strictly decreasing on $[\theta/2, \theta]$.

As a result,

$$\min_{x\in[0,\theta]} \chi_\theta(x) = \chi_\theta(0) = \chi_\theta(\theta) = \frac{1}{8}\frac{3\sin\theta}{1 - \cos\theta} + \frac{1}{8}\frac{\sin 3\theta}{1 - \cos 3\theta} =: \Phi^*(\theta).$$

The function $\Phi^*(\theta)$ can be rewritten, by a computation, as

$$\Phi^*(\theta) = \frac{\sin\theta}{4(1-\cos\theta)(2\cos\theta+1)}(4\cos\theta+1).$$

Therefore, the (unique) zero of $\Phi^*(\theta)$ in the interval $(\pi/2,\, 2\pi/3)$ is

$$\theta_0^* = \arccos(-1/4).$$

Note that $\Phi^*(\theta) > 0$ on $(\pi/2,\, \theta_0^*)$ and $\Phi^*(\theta) < 0$ on $(\theta_0^*,\, 2\pi/3)$.

If $\theta \in (\pi/2,\, \theta_0^*]$, we know that $\chi_\theta(u-v) > 0$ for all $(u,v) \in \operatorname{int}\Delta_\theta$, the interior of Δ_θ. It follows from (2.8) that $\beta_*(b) = \beta_\theta(h)$ must be positive because h satisfies either $h \gg 0$ or $h \ll 0$. Thus we get from (2.7) that $\beta(b,c) > 0$ for all $(b,c) \in BC$.

Next we prove that θ_0^* is just the solution to Problem 1 in Section 1. Fix a $\theta \in (\theta_0^*,\, 2\pi/3)$. Let us consider the following quadratic equation:

$$x'' + x + b(t)x^2 = 0, \tag{2.9}$$

where $b \in C(\mathbb{R}/T\mathbb{Z})$ and $T = \theta$. In this case, $\Psi(t) = e^{it}$ and $h(u) \equiv b(u)$. We intend to find some $b \in B$ such that $\beta(b,0) = \beta_*(b) = 0$. This will be realized in three steps.

Step 1: Let $b_0(t) \equiv 1$. Then

$$\beta_*(b_0) = \iint_{\Delta_\theta} \chi_\theta(u-v)\,du\,dv = \frac{1}{2}\iint_{\Delta'} \chi_\theta(\xi)\,d\xi\,d\eta,$$

where

$$\Delta' = \{(\xi,\eta) : 0 \leqslant \xi \leqslant \theta,\ \xi \leqslant \eta \leqslant 2\theta - \xi\}.$$

Thus

$$\beta_*(b_0) = \frac{1}{2}\int_{[0,\theta]} d\xi \int_{[\xi,2\theta-\xi]} \chi_\theta(\xi)\,d\eta = \frac{5\theta}{12} > 0.$$

Step 2: Let $\varepsilon = \varepsilon(\theta)$ be the first positive zero of $\chi_\theta(t)$. Then $\chi_\theta(t) > 0$ on $(\varepsilon, \theta-\varepsilon)$ and $\chi_\theta(t) < 0$ on $[0,\varepsilon) \cup (\theta-\varepsilon, \theta]$. Let $b_1 \in C(\mathbb{R}/T\mathbb{Z})$ be such that $b_1(t) > 0$ on $(0,\varepsilon)$, and $b_1(t) \equiv 0$ on $\{0\} \cup [\varepsilon, \theta]$. Then $b_1 \in B^+$. Due to the special form of $b_1(t)$, we find that the twist coefficient is

$$\beta_*(b_1) = \iint_{\Delta_\varepsilon} \chi_\theta(u-v)\,du\,dv < 0,$$

because $\chi_\theta(u-v) < 0$ for all $(u,v) \in \operatorname{int}\Delta_\varepsilon = \{(u,v) : 0 < v < u < \varepsilon\}$.

Step 3: The twist coefficient $\beta_*((1-\tau)b_0 + \tau b_1)$ is continuous in $\tau \in [0,1]$. We obtain from steps 1 and 2 that $\beta_*((1-\tau_0)b_0 + \tau_0 b_1) = 0$ for some $\tau_0 \in (0,1)$. Let $b_*(t) = (1-\tau_0)b_0(t) + \tau_0 b_1(t) \in C(\mathbb{R}/T\mathbb{Z})$. Then b_* is strictly positive, while $\beta(b_*,0) = \beta_*(b_*) = 0$.

These examples show that the best bound in Problem 1 is θ_0^*. On the other hand, for any $\theta \in (\theta_0^*, 2\pi/3)$, we have found $b \in B^+$ such that the rotation of (2.9) is θ and equation (2.9) is not of twist type. □

Remark 1. (i) When a is fixed and $c = 0$, the twist coefficient $\beta_*(b)$ is a quadratic form of b (cf. (2.6) or (2.8)). If we introduce a compact symmetric linear operator in $C[0,\theta]$ by

$$\mathscr{L}_\theta h(u) = h^*(u) := \int_{[0,\theta]} \tfrac{1}{2}\chi_\theta(|u-v|)h(v)\,dv,$$

then the twist coefficient $\beta_*(b)$ of (1.3) is given by

$$\beta_*(b) = \beta_\theta(h) = \langle \mathscr{L}_\theta h,\, h\rangle_{L^2(0,\theta)},$$

where $h = (b\circ\varphi^{-1})(r\circ\varphi^{-1})^3/(\varphi'\circ\varphi^{-1}) = (b\circ\varphi^{-1})(r\circ\varphi^{-1})^5$. Now the best bound is obtained by requiring that $\mathscr{L}_\theta$ be a strictly positive operator, that is, $h \gg 0$ implies that $\min h^* > 0$.

(ii) Such a solution to Problems 1 and 2 may be viewed as the first attempt towards the solution of some delicate problem on the limitation of rotations for wider classes of coefficients b and c. See the remarks following [**7**, Theorem 2.2].

3. *Conditions on the linearization equations*

In order to apply Theorem 1 to the stability problem of (1.3), a central problem is to analyze the Hill equation (1.2).

Let us introduce some notation for eigenvalues from [**2**]. Consider eigenvalue problems of

$$x'' + (\lambda + a(t))x = 0 \tag{3.1}$$

with respect to the T-periodic boundary condition:

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{3.2}$$

or with respect to the T-anti-periodic boundary condition

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{3.3}$$

It is well known [**4**, Theorem 2.1] that there exist two sequences $\{\underline{\lambda}_n(a) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(a) : n \in \mathbb{Z}^+\}$ of the reals with the following properties:

(i) They have the following order

$$\overline{\lambda}_0(a) < \underline{\lambda}_1(a) \leqslant \overline{\lambda}_1(a) < \underline{\lambda}_2(a) \leqslant \overline{\lambda}_2(a) < \ldots < \underline{\lambda}_n(a) \leqslant \overline{\lambda}_n(a) < \ldots.$$

(ii) λ is an eigenvalue of (3.1) and (3.2) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ with n even; λ is an eigenvalue of (3.1) and (3.3) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ with n odd.

(iii) Equation (3.1) is stable if λ is in the following intervals:

$$S_n(a) = \{\lambda \in \mathbb{R} : \overline{\lambda}_{n-1}(a) < \lambda < \underline{\lambda}_n(a)\} \quad n = 1, 2, \ldots .$$

In this work, we see that 0 is in the first stability interval $S_1(a)$ of equation (1.2). In fact, if $\theta \in \Theta^*$, then (1.2) is elliptic and hence is stable. Moreover, assumption (1.5) is equivalent to the non-negativeness of the first anti-periodic eigenvalue $\underline{\lambda}_1(a)$:

$$\underline{\lambda}_1(a) \geqslant 0$$

by the comparison theorem of solutions. For a proof, see [**15**, Lemma 2.1]. Combining these, we see that we have assumed in Theorem 1 that $0 \in S_1(a)$, that is,

$$\overline{\lambda}_0(a) < 0 \quad \text{and} \quad \underline{\lambda}_1(a) > 0. \tag{3.4}$$

Therefore, we are led to a classical problem for Hill equations, namely the problem of the first stability zone [**3**, **13**].

Applying the well known stability criteria of Lyapunov and Zukovskii [**3**, **13**], Ortega gave some efficient conditions on $a(t)$ in [**11**, Corollary 3.3] so that Theorem 1 is applicable. We remark that the bound θ^* there can now be replaced by the best bound θ_0^*.

A very recent result in a joint work of Li with the present author [**17**] has interpolated the Lyapunov criterion (the L^1 case) and the Zukovskii criterion (the L^∞ case) using the L^p $(1 < p < \infty)$ norms of $a^+(t) = \max\{a(t), 0\}$.

Let us now use this idea to give some conditions on (1.2) which sharpen [**11**, Corollary 3.3]. Introduce constants $K_*(p)$, $1 \leqslant p \leqslant \infty$, by

$$K_*(p) = \frac{\pi(p-1)}{p}\left(\frac{p-1}{2p-1}\right)^{1/p}\left(\frac{\Gamma((p-1)/(2p))}{\Gamma((2p-1)/(2p))}\right)^2,$$

where $\Gamma(\cdot)$ is the Gamma function. When $p = 1$ or $p = \infty$, the constants are

$$K_*(1) = 4, \quad K_*(\infty) = \pi^2.$$

Now we give a preliminary result on the distribution of zeros of solutions of (1.2), along the lines of [**11**, Lemma 3.4].

LEMMA 1. *Let $t_0 < t_1 < \ldots < t_m$, $m \in \mathbb{N}$, be $(m+1)$ consecutive zeros of any nontrivial solution $x(t)$ of* (1.2). *Then*

$$(t_m - t_0)^{(2p-1)/p}\|a^+\|_{L^p(t_0,t_m)}/K_*(p) \geqslant m^2 \tag{3.5}$$

for any $1 \leqslant p \leqslant \infty$.

Proof. We will use the method of estimating the lower bounds of $\underline{\lambda}_1(a)$ in [**17**] to give a simpler proof.

For any $i = 1, \ldots, m$, $x(t)$ is a solution of

$$x'' + a(t)x = 0, \quad t \in I_i = [t_{i-1}, t_i],$$

satisfying $x(t_{i-1}) = x(t_i) = 0$ and $x(t) \neq 0$ for all $t \in I_i$. This means that 0 is the first eigenvalue of

$$x'' + (\lambda + a(t))x = 0 \quad (t \in I_i), \quad x(t_{i-1}) = x(t_i) = 0.$$

It follows from the proof of [**17**, Theorem 4] that we have, for any $1 \leqslant p \leqslant \infty$,

$$0 \geqslant \frac{\|x'\|^2_{L^2(I_i)}}{\|x\|^2_{L^2(I_i)}}\left(1 - \frac{\|a^+\|_{L^p(I_i)}}{K_*(p)/|I_i|^{(2p-1)/p}}\right),$$

that is,

$$|I_i|^{(2p-1)/p}\|a^+\|_{L^p(I_i)} \geqslant K_*(p) \quad 1 \leqslant i \leqslant m.$$

Note that

$$\sum_{i=1}^m |I_i| = |I|, \tag{3.6}$$

where $I = [t_0, t_m]$. Now we have

$$\|a^+\|_{L^p(I)} = \left(\sum_{i=1}^{m} \|a^+\|^p_{L^p(I_i)}\right)^{1/p}$$

$$\geqslant \left(\sum_{i=1}^{m} \frac{(K_*(p))^p}{|I_i|^{2p-1}}\right)^{1/p} =: g(|I_1|, \dots, |I_m|).$$

Under constraint (3.6), it is easy to check that

$$g(|I_1|, \dots, |I_m|) \geqslant g(|I|/m, \dots, |I|/m) = m^2 K_*(p)/|I|^{(2p-1)/p}.$$

Thus we have (3.5). □

Remark 2. When $p = 1$, (3.5) recovers [**11**, Lemma 3.4], in which case the inequality above is strict. Moreover, as in [**17**, Theorem 1], the estimates in (3.5) are the best possible for all $1 < p \leqslant \infty$.

Now we give an improvement of [**11**, Corollary 3.3].

Theorem 2. *Assume that $a \in C(\mathbb{R}/T\mathbb{Z})$ satisfies one of the following conditions*:

(i)

$$\int_{[0,T]} a > 0 \tag{3.7}$$

and

$$T^{(2p-1)/p} \|a^+\|_{L^p(0,T)}/K_*(p) \leqslant (\theta_0^*/\pi)^2 \tag{3.8}$$

for some $1 \leqslant p \leqslant \infty$.

(ii)

$$(2\pi/(3T))^2 \tag{3.9}$$

and

$$T^{(2p-1)/p} \|a^+\|_{L^p(0,T)}/K_*(p) < 1 \tag{3.10}$$

for some $1 \leqslant p \leqslant \infty$.

Then (3.4) *is satisfied and the rotation θ of* (1.2) *satisfies $0 < \theta \leqslant \theta_0^*$ in case* (i) *and $2\pi/3 < \theta < \pi$ in case* (ii). *Moreover, the solution $x = 0$ of* (1.3) *is of twist type for all $(b, c) \in BC$.*

Proof. It is well known [**4**] that the zeroth periodic eigenvalue has the following upper bound:

$$\overline{\lambda}_0(a) \leqslant -\frac{1}{T} \int_{[0,T]} a.$$

Thus $\overline{\lambda}_0(a) < 0$ by assumption (3.7) or assumption (3.9). On the other hand, if (3.8) or (3.10) is satisfied, then we know from [**17**, formula (13)] that

$$\underline{\lambda}_1(a) \geqslant \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{T^{(2p-1)/p} \|a^+\|_{L^p(0,T)}}{K_*(p)}\right) > 0.$$

Thus (3.4) is fulfilled.

Let us prove the result on the rotation θ using the rotation numbers (2.3), as in [**11**, proof of Corollary 3.3]. We assert that if

$$T^{(2p-1)/p}\|a^+\|_{L^p(0,T)}/K_*(p) \leqslant \alpha_0^2 \tag{3.11}$$

for some $\alpha_0 > 0$, where $p \in [1,\infty]$, then the rotation number α of (1.2) satisfies

$$\alpha \leqslant \pi\alpha_0/T. \tag{3.12}$$

In fact, let $x(t)$ be any nontrivial solution of (1.2). Then the rotation number (2.3) is also equal to

$$\alpha = \lim_{\tau\to+\infty} \pi N(\tau)/\tau,$$

where

$$N(\tau) = \#\{t \in [0,\tau] : x(t) = 0\}.$$

For any $n \in \mathbb{N}$, let $0 \leqslant t_0 < t_1 < \ldots < t_m \leqslant nT$ be all zeros of $x(t)$ within $[0, nT]$. By Lemma 1 and condition (3.11), we have

$$\begin{aligned} m^2 &\leqslant (t_m - t_0)^{(2p-1)/p}\|a^+\|_{L^p(t_0,t_m)}/K_*(p) \\ &\leqslant (nT)^{(2p-1)/p}\|a^+\|_{L^p(0,nT)}/K_*(p) \\ &= n^2 T^{(2p-1)/p}\|a^+\|_{L^p(0,T)}/K_*(p) \\ &\leqslant n^2\alpha_0^2. \end{aligned}$$

Thus

$$N(nT) = m + 1 \leqslant n\alpha_0 + 1,$$

from which we have

$$\alpha = \lim_{n\to+\infty} \pi N(nT)/(nT) \leqslant \pi\alpha_0/T.$$

When Theorem 2(i) or (ii) is satisfied, the estimates on θ can be obtained by using relation (2.4) and inequality (3.12) and noticing that (3.9) implies that $\theta > 2\pi/3$.

Finally, the twist character of (1.3) follows from Theorem 1, because all conditions on (1.2) are now fulfilled. □

REMARK 3. The results in [**11**, Corollary 3.3] correspond to the cases $p = 1$ and $p = \infty$ of Theorem 2.

In applications of Theorem 2, the existence of a periodic solution to (1.1) can be found using the principle of lower and upper solutions, and especially the principle for the reversely ordered lower and upper solutions [**1**, **15**]. This also yields the bounds for the periodic solution so that the coefficients a, b, c can be estimated and then the stability of the periodic solution can be proved when the conditions of Theorem 1 or that of Theorem 2 are fulfilled. See [**7**, **8**, **11**, **14**] for some possible applications.

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