THE ROTATION NUMBER APPROACH TO EIGENVALUES OF THE ONE-DIMENSIONAL p-LAPLACIAN WITH PERIODIC POTENTIALS

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Abstract

The paper studies the periodic and anti-periodic eigenvalues of the one-dimensional p-Laplacian with a periodic potential. After a rotation number function $\rho(\lambda)$ has been introduced, it is proved that for any non-negative integer n, the endpoints of the interval $\rho^{-1}(n/2)$ in $\mathbb{R}$ yield the corresponding periodic or anti-periodic eigenvalues. However, as in the Dirichlet problem of the higher dimensional p-Laplacian, it remains open if these eigenvalues represent all periodic and anti-periodic eigenvalues. The result obtained is a partial generalization of the spectrum theory of the one-dimensional Schrödinger operators with periodic potentials.

1. *Introduction*

Let $1 < p < \infty$ be fixed. The p-Laplacian $-\Delta_p$, acting on a function $u(x)$, $x \in \Omega \subset \mathbb{R}^N$, is

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u),$$

where ∇u is the gradient of u. The eigenvalues of the p-Laplacian Δ_p with the zero Dirichlet boundary condition are, as in the usual Laplacian $-\Delta_2$, those λ such that the following problem

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has non-zero solutions.

Such an eigenvalue problem was first proposed by Ôtani [**21**] and has received much attention recently. Many generalizations to various classes of eigenvalue problems with certain homogeneity have been developed, see [**1**, **3**, **14**] and the references therein. However the structure of eigenvalues of (1.1) with a general domain $\Omega \subset \mathbb{R}^N$ has not been understood completely even when the dimension N is 2. For general domains, it is known that (1.1) has a smallest eigenvalue λ_1, which is positive, simple and has an eigenfunction strictly positive in the domain [**14**]. Recently, the second eigenvalue λ_2 ($> \lambda_1$) has been characterized using the variational method [**1**]. It is also well known that (1.1) has a sequence of so-called *variational eigenvalues* λ_n, which are characterized using the minimax principle. However, it is not known if this sequence presents a complete list of eigenvalues of (1.1). Some properties of the p-Laplacian have been revealed. These results show that the p-Laplacian has some different properties from its linear counterpart $-\Delta_2$. For example, the Fredholm alternative may not hold for $-\Delta_p$, see [**2**, **5**, **8**, **10**]. On

Received 4 July 2000; revised 8 November 2000.

2000 *Mathematics Subject Classification* 34L30, 37E45.

Supported by the National Natural Science Foundation of China, the National 973 Project of China, and the Excellent Personnel Supporting Plan of the Ministry of Education of China.

J. London Math. Soc. (2) 64 (2001) 125–143.

the other hand, many applications of eigenvalues of the p-Laplacian, for example, to the non-resonance problem, have been developed in the literature; see [**4**, **6**, **7**, **18**, **19**, **23**, **24**].

When the dimension N is 1, eigenvalues of (1.1) can be given explicitly because (1.1) is now integrable. In fact, if $\Omega = (0, T) \subset \mathbb{R}$, (1.1) has a sequence of eigenvalues given by

$$\lambda_n = (n\pi_p/T)^p, \quad n = 1, 2, \dots$$

where

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{(1 - s^p/(p-1))^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)},$$

cf. [**19**].

Let $\phi_p(s)$ be the mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $s \longmapsto |s|^{p-2}s$. For the one-dimensional case, the structure of eigenvalues of

$$(\phi_p(x'))' + (\lambda + q(t))\phi_p(x) = 0 \quad \left(' = \frac{d}{dt}\right) \tag{1.2}$$

with more general two-point boundary conditions can be characterized completely, although equation (1.2) may be no longer integrable. See Theorems 4.1 and 4.2 below.

For the one-dimensional case, it is more interesting to consider eigenvalues of (1.2) with respect to the periodic boundary condition:

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{1.3}$$

or to the anti-periodic boundary condition:

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{1.4}$$

Again, when no potentials are present in (1.2): $q(t) \equiv 0$, the periodic or the anti-periodic eigenvalues of (1.2) are known because (1.2) is integrable. However, when some potentials are present in (1.2), the (anti-) periodic eigenvalues have not been studied adequately. In fact, I think this problem, which may be referred as the $1\frac{1}{2}$-*dimensional problem*, is the first non-trivial one in studying eigenvalues of p-Laplacian. Let us introduce some notation. Denote by $\mathscr{P}$ and $\mathscr{A}$ the eigenvalues of the problem (1.2) + (1.3) and the problem (1.2) + (1.4), respectively. Let $\mathscr{P}_* = \mathscr{P} \cup \mathscr{A}$. Before explaining the difficulty in understanding the structure of $\mathscr{P}_*$, let us recall some classical results for the linear counterpart of (1.2), that is, the periodic and anti-periodic eigenvalues of the following linear Schrödinger operator

$$(Lx)(t) := -x''(t) - q(t)x(t) = \lambda x(t), \tag{1.5}$$

where $q(t)$ is 2π-periodic and $q \in L^1(0, 2\pi)$. (We assume here that the period is $T = 2\pi$.) Then one has the following classical result, which is part of [**16**, Theorem 2.1].

THEOREM A [**16**]. *There exist two sequences* $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$ *and* $\{\overline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ *of the reals with the following properties:*

(i) *They have the following order:*

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leqslant \overline{\lambda}_1(q) < \underline{\lambda}_2(q) \leqslant \overline{\lambda}_2(q) < \dots < \underline{\lambda}_n(q) \leqslant \overline{\lambda}_n(q) < \dots. \tag{1.6}$$

(ii) λ *is an eigenvalue of* (1.5) + (1.3) *if and only if* $\lambda = \underline{\lambda}_n(q)$ *or* $\overline{\lambda}_n(q)$ *for some* *even integer n; and* λ *is an eigenvalue of* (1.5) + (1.4) *if and only if* $\lambda = \underline{\lambda}_n(q)$ *or* $\overline{\lambda}_n(q)$ *for some odd integer n.*

The proof of Theorem A is essentially based on the Floquet theory for *linear* periodic equations [**11**] and on the classification of symplectic 2×2 matrices. Returning to the $1\frac{1}{2}$-dimensional problem, we know that there is no hope of giving the complete structure of $\mathscr{P}_*$, because a Floquet-like theory for periodic equations (1.2) of the *p-Laplacian-type* is not available, although the Floquet theory has been generalized to various linear systems such as functional differential equations [**17**] and differential algebraic equations [**13**]. On the other hand, unlike the Dirichlet problem, there would be a coexistence problem for eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$:

$$\underline{\lambda}_n(q) = \overline{\lambda}_n(q).$$

Such a coexistence problem is extraordinarily difficult for general potentials $q(t)$. This also adds to the difficulty in understanding the structure of $\mathscr{P}_*$.

In this paper, we try to give a partial generalization of Theorem A to eigenvalues $\mathscr{P}_*$ of (1.2) for p-Laplacian using a more geometric approach, that is, the rotation number approach, which has been well developed for linear periodic systems; see Johnson and Moser [**12**] and Moser [**20**]. This approach applies also to (1.5) with quasi-periodic or almost periodic potentials and is very useful in understanding the spectrum and the dynamics aspect of (1.5).

For the linear case (1.5) with periodic potential $q(t)$, the rotation number function in [**12**] is an analytical function on the upper half-plane. When real parameters are considered, the rotation number function $\rho(\lambda)$ is as follows. Let $x = r\cos\theta$ and $x' = -r\sin\theta$ in (1.5). Then θ satisfies the following equation:

$$\theta' = (\lambda + q(t))\cos^2\theta + \sin^2\theta =: \Phi(t, \theta; \lambda). \tag{1.7}$$

As $\Phi(t, \theta; \lambda)$ is 2π-periodic in both t and θ, it is well known from [**11**, Theorem 2.1, Chapter 2] that the rotation number of (1.7)

$$\rho(\lambda) = \rho(\lambda; q) = \lim_{t\to\infty} \frac{\theta(t; \theta_0, \lambda) - \theta_0}{t}$$

exists and is independent of θ_0, where $\theta(t; \theta_0, \lambda)$ is the solution of (1.7) satisfying the initial condition: $\theta(0; \theta_0, \lambda) = \theta_0$. Using the function $\rho(\lambda)$, all eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ can be characterized in the following way [**12**, **20**]. A simple proof based on [**16**, Theorem 2.1] is given in [**9**].

THEOREM B [**20**, Theorem 4.3]. *Let q and* $\rho(\lambda; q)$ *be as above. Then*

$$\begin{aligned} \underline{\lambda}_n(q) &= \min\{\lambda : \rho(\lambda; q) = n/2\}, \quad n \in \mathbb{N}, \\ \overline{\lambda}_n(q) &= \max\{\lambda : \rho(\lambda; q) = n/2\}, \quad n \in \mathbb{Z}^+. \end{aligned} \tag{1.8}$$

In fact, the graph of $\rho(\lambda)$ looks like a staircase which is continuous and is strictly increasing except on the possible platforms $\rho^{-1}(n/2)$, $n \in \mathbb{Z}^+$, while the endpoints of the platforms $\rho^{-1}(n/2)$ are exactly the periodic and anti-periodic eigenvalues of (1.5).

In this paper, when $q(t)$ is a periodic potential, we follow the idea in [**9**, **20**, **25**] of introducing a rotation number function $\rho(\lambda)$ for (1.2). Then we use (1.8) and (1.9) to define two sequences $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$, $\{\overline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$. It will be proved that $\{\underline{\lambda}_n(q), \overline{\lambda}_n(q) : n \text{ is even}\}$ are the periodic eigenvalues of (1.2)+(1.3) and $\{\underline{\lambda}_n(q), \overline{\lambda}_n(q) :$

n is odd$\}$ are the anti-periodic eigenvalues of (1.2) + (1.4). Namely, the 'if' part of Theorem A(ii) holds for such a $1\frac{1}{2}$-dimensional problem (Theorem 3.3). Differently from the variational structure of (1.1) in defining the variational eigenvalues of higher dimensional p-Laplacian, we extensively make use of the Hamiltonian structure of (1.2) for the $1\frac{1}{2}$-dimensional problems (1.2) + (1.3) and (1.2) + (1.4). However, as in the higher dimensional Dirichlet problem for p-Laplacian, it remains open as to whether the 'only if' part of Theorem A(ii) also holds for the $1\frac{1}{2}$-dimensional p-Laplacian, namely, whether those eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ represent a complete list of eigenvalues of (1.2) + (1.3) and (1.2) + (1.4). For this reason, we will call these periodic and anti-periodic eigenvalues $\underline{\lambda}_n(q), \overline{\lambda}_n(q)$, constructed using the rotation number function, the *rotational periodic eigenvalues* of (1.2).

This paper is organized as follows. In Section 2, we will introduce the p-polar coordinates and then derive the counterpart of (1.7) for (1.2). The rotation number function $\rho(\lambda)$ of (1.2) will be introduced. Some properties of $\rho(\lambda)$ will be proved. In Section 3, the (anti-) periodic eigenvalues of (1.2) will be studied. The 'if' part of Theorem A(ii) for p-Laplacian with periodic potentials will be proved in this section, see Theorem 3.3. In Section 4, we study the eigenvalues of the Sturm–Liouville problems of (1.2) and will give another characterization of rotational periodic eigenvalues using eigenvalues of Sturm–Liouville problems. In Section 5, we present some examples which show that the 'only if' part of Theorem A(ii) also holds.

2. *p-polar coordinates and rotation number functions*

Let $1 < p < \infty$ be fixed. Define $\phi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R}$. Then ϕ_p is an odd homeomorphism of the real line $\mathbb{R}$ and is $(p-1)$-homogeneous in the following sense:

$$\phi_p(ks) = k^{p-1}\phi_p(s), \quad k \geqslant 0, \ s \in \mathbb{R}.$$

The inverse of ϕ_p is ϕ_{p^*}, where $p^* = p/(p-1)$ is the conjugate exponent of p.

Let $q(t)$ in (1.2) be a periodic function such that $q \in L^1_{\text{loc}}(\mathbb{R})$. We always assume that the period of $q(t)$ is $T = 2\pi_p$. Before studying eigenvalues of (1.2), we introduce some coordinates in the plane $\mathbb{R}^2$, which may be called the p-polar coordinates. This will be realized using the solutions of the following differential equation:

$$(\phi_p(x'))' + \phi_p(x) = 0. \tag{2.1}$$

Set $\phi_p(x') = -y$ in (2.1). Then (2.1) is equivalent to the following planar system

$$x' = -\phi_{p^*}(y), \quad y' = \phi_p(x), \tag{2.2}$$

which is an integrable Hamiltonian system with the Hamiltonian

$$H(x, y) = |x|^p/p + |y|^{p^*}/p^*.$$

Let $(C_p(t), S_p(t))$ be the unique solution of (2.2) with the initial value $(x(0), y(0)) = (1, 0)$. Then $C_p(t)$ and $S_p(t)$ are well defined on the whole line $\mathbb{R}$. These functions $C_p(t)$ and $S_p(t)$ are called the *p-cosine* and the *p-sine*, respectively, because they share many similar properties of the cosine and sine functions [**15**]. Some properties of $C_p(t)$ and $S_p(t)$ are summarized in the following lemma.

LEMMA 2.1 [**15**]. *The functions $C_p(t)$ and $S_p(t)$ have the following properties:*

(i) *Both* $C_p(t)$ *and* $S_p(t)$ *are* $2\pi_p$*-periodic.*

(ii) $C_p(t)$ *is even in* t *and* $S_p(t)$ *is odd in* t.

(iii) $C_p(t+\pi_p) = -C_p(t)$, $S_p(t+\pi_p) = -S_p(t)$.

(iv) $C_p(t) = 0$ *if and only if* $t = \pi_p/2 + m\pi_p$, $m \in \mathbb{Z}$; *and* $S_p(t) = 0$ *if and only if* $t = m\pi_p$, $m \in \mathbb{Z}$.

(v) $C_p'(t) = -\phi_{p^*}(S_p(t))$ *and* $S_p'(t) = \phi_p(C_p(t))$.

(vi) $|C_p(t)|^p/p + |S_p(t)|^{p^*}/p^* \equiv 1/p$.

Let us define the p-polar coordinates in $\mathbb{R}^2$ by

$$x = r^{2/p} C_p(\theta), \quad y = r^{2/p^*} S_p(\theta). \tag{2.3}$$

Using Lemma 2.1, it is easy to check that

$$dx \wedge dy = (2/p) r\, dr \wedge d\theta. \tag{2.4}$$

Now we consider differential equation (1.2). As before, let

$$\phi_p(x') = -y \tag{2.5}$$

in (1.2). Then (1.2) is equivalent to the following Hamiltonian system:

$$x' = -\phi_{p^*}(y) = -\frac{\partial H(t,x,y)}{\partial y}, \quad y' = (\lambda + q(t))\phi_p(x) = \frac{\partial H(t,x,y)}{\partial x}, \tag{2.6}$$

where the Hamiltonian is

$$H(t,x,y;\lambda) = (\lambda + q(t))|x|^p/p + |y|^{p^*}/p^*.$$

In the p-polar coordinates (2.3), it is not difficult to check that r and θ satisfy the following equations:

$$r' = (p/2)(\lambda + q(t) - 1)\phi_p(C_p(\theta))\phi_{p^*}(S_p(\theta)) r =: \Psi(t,\theta,r;\lambda), \tag{2.7}$$

$$\theta' = p((\lambda + q(t))|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*) =: \Xi(t,\theta;\lambda). \tag{2.8}$$

Note that system (2.6) preserves the area $dx \wedge dy$. Hence system (2.7)+(2.8) preserves the area $r\, dr \wedge d\theta$ because of the equality (2.4).

For any $(x_0, y_0) \in \mathbb{R}^2$, let $(x(t; x_0, y_0, \lambda), y(t; x_0, y_0, \lambda))$ be the unique solution of (2.6) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Similarly, for $\theta_0 \in \mathbb{R}$, let $(r(t;\theta_0,\lambda), \theta(t;\theta_0,\lambda))$ be the unique solution of (2.7) + (2.8) satisfying the initial conditions $r(0) = 1$, $\theta(0) = \theta_0$. As $\Xi(t,\theta;\lambda)$ is bounded in θ, we know that $r(t;\theta_0,\lambda)$ and $\theta(t;\theta_0,\lambda)$ are defined on the whole line $\mathbb{R}$. Due to the oddness of (2.6) in (x,y) and the homogeneity of (2.7) in r, we have the following relation between the solutions $(x(t;x_0,y_0,\lambda), y(t;x_0,y_0,\lambda))$ and $(r(t;\theta_0,\lambda), \theta(t;\theta_0,\lambda))$:

$$\begin{aligned}(x(t;\varphi_p(r_0)C_p(\theta_0), \varphi_{p^*}(r_0)S_p(\theta_0),\lambda), y(t;\varphi_p(r_0)C_p(\theta_0), \varphi_{p^*}(r_0)S_p(\theta_0),\lambda))\\ \equiv (\varphi_p(r_0) r^{2/p}(t;\theta_0,\lambda) C_p(\theta(t;\theta_0,\lambda)), \varphi_{p^*}(r_0) r^{2/p^*}(t;\theta_0,\lambda) S_p(\theta(t;\theta_0,\lambda)))\end{aligned} \tag{2.9}$$

for all r_0, $\theta_0 \in \mathbb{R}$, where

$$\varphi_p(r_0) = |r_0|^{2/p} \operatorname{sign} r_0, \quad r_0 \in \mathbb{R}.$$

In particular, the Poincaré map P_λ of (2.6) is given by

$$\begin{aligned} &P_\lambda(\varphi_p(r_0)C_p(\theta_0), \varphi_{p^*}(r_0)S_p(\theta_0)) \\ &\quad = (x(2\pi_p; \varphi_p(r_0)C_p(\theta_0), \varphi_{p^*}(r_0)S_p(\theta_0), \lambda), y(2\pi_p; \varphi_p(r_0)C_p(\theta_0), \varphi_{p^*}(r_0)S_p(\theta_0), \lambda)) \\ &\quad = (\varphi_p(r_0)R_\lambda^{2/p}(\theta_0)C_p(\Theta_\lambda(\theta_0)), \varphi_{p^*}(r_0)R_\lambda^{2/p^*}(\theta_0)S_p(\Theta_\lambda(\theta_0))) \end{aligned} \tag{2.10}$$

for r_0, $\theta_0 \in \mathbb{R}$, where $R_\lambda(\theta_0) := r(2\pi_p; \theta_0, \lambda)$, and

$$\Theta_\lambda(\theta_0) := \theta(2\pi_p; \theta_0, \lambda). \tag{2.11}$$

Thus $\Theta_\lambda(\theta_0)$ is the corresponding Poincaré map of (2.8).

Note that $\Xi(t, \theta; \lambda)$ is $2\pi_p$-periodic in both θ and t. In fact, $\Xi(t, \theta; \lambda)$ is actually π_p-periodic in θ, cf. Lemma 2.1(iii). Equation (2.8) is a family of $2\pi_p$-periodic equations on the circle $\mathbb{S} := \mathbb{R}/\pi_p\mathbb{Z}$. By the periodicity of $\Xi(t, \theta; \lambda)$, we have the following equalities on the solutions $\theta(t; \theta_0, \lambda)$ of (2.8):

$$\theta(t + 2m\pi_p; \theta_0, \lambda) \equiv \theta(t; \theta(2m\pi_p; \theta_0, \lambda), \lambda) \tag{2.12}$$

$$\theta(t; \theta_0 + m\pi_p, \lambda) \equiv \theta(t; \theta_0, \lambda) + m\pi_p \tag{2.13}$$

for all $\theta_0 \in \mathbb{R}$ and all $m \in \mathbb{Z}$. In particular, the rotation number of (2.8)

$$\begin{aligned} \rho(\lambda) = \rho(\lambda; q) &:= \lim_{|t|\to\infty} \frac{\theta(t; \theta_0, \lambda) - \theta_0}{t} \\ &= \lim_{|m|\to\infty} \frac{\Theta_\lambda^m(\theta_0) - \theta_0}{2m\pi_p} \end{aligned} \tag{2.14}$$

exists and is independent of θ_0; see [**11**, Theorem 2.1, Chapter 2].

Notice that the vector field $\Xi(t, \theta; \lambda)$ is increasing when λ increases. We have the following monotonicity result on the solutions $\theta(t; \theta_0, \lambda)$ of (2.8), which is a consequence of the comparison theorem for differential equations.

LEMMA 2.2. *If $\lambda_1 > \lambda_2$, then $\theta(t; \theta_0, \lambda_1) \geqslant \theta(t; \theta_0, \lambda_2)$ for all $t \geqslant 0$ and all θ_0. Moreover, $\theta(t; \theta_0, \lambda_1) > \theta(t; \theta_0, \lambda_2)$ for all $t \geqslant 2\pi_p$ and all θ_0.*

Proof. This is essentially similar to [**24**, Lemma 2.2]. □

For later purposes, we give here another preliminary result. Let $w(t)$ be a $2\pi_p$-periodic function such that $w \in L^1(0, 2\pi_p)$. Consider the equation

$$\theta' = p(w(t)|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*) =: \tilde{\Xi}(t, \theta) \tag{2.15}$$

and its solutions $\theta(t; \theta_0)$ satisfying $\theta(0; \theta_0) = \theta_0$.

LEMMA 2.3. (i) *if $\theta_0 \geqslant \pi_p/2 + m\pi_p$ for some $m \in \mathbb{Z}$, then $\theta(t; \theta_0) > \pi_p/2 + m\pi_p$ for all $t > 0$.* (ii) *Assume that $w(t) < 0$. If $\theta_0 \leqslant m\pi_p$ for some $m \in \mathbb{Z}$, then $\theta(t; \theta_0) < m\pi_p$ for all $t > 0$.*

Proof. For the exponent $1 < p < \infty$, let us introduce the p-tangent and the p-cotangent functions

$$T_p(\theta) = \pm\frac{|S_p(\theta)|}{|C_p(\theta)|^{p/p^*}}, \quad G_p(\theta) = \pm\frac{|C_p(\theta)|}{|S_p(\theta)|^{p^*/p}}, \tag{2.16}$$

where the signs $\pm$ are determined on which quadrant $(C_p(\theta), S_p(\theta))$ lies; see [**15**].

Using the properties in Lemma 2.1, one has

$$\frac{dT_p(\theta)}{d\theta} \equiv \frac{1}{|C_p(\theta)|^p}, \quad \frac{dG_p(\theta)}{d\theta} \equiv -\frac{1}{(p/p^*)|S_p(\theta)|^{p^*}}. \tag{2.17}$$

The result (i) follows essentially from the fact that the vector field $\tilde{\Xi}(t,\theta) > 0$ at $\theta = \pi_p + m\pi_p$, $m \in \mathbb{Z}$. By (2.13) we need only to prove (i) for the case $\theta_0 \geqslant \pi_p/2$. We first prove that there exists some $\delta_0 > 0$ such that

$$\theta(t;\theta_0) > \pi_p/2 \quad \text{for all } 0 < t < \delta_0. \tag{2.18}$$

It is obvious that (2.18) holds if $\theta_0 > \pi_p/2$, so let us assume that $\theta_0 = \pi_p/2$ at the moment. Choose $\delta_0 > 0$ such that

$$0 < \theta(t;\theta_0) < \pi_p \quad \text{for all } 0 \leqslant t < \delta_0.$$

Then the function

$$\xi(t) = -G_p(\theta(t;\theta_0))$$

is well defined on $[0,\delta_0)$. Moreover, by (2.15), $\xi(t)$ satisfies, for almost everywhere $t \in [0,\delta_0)$,

$$\xi'(t) = (p^*/p)w(t)|\xi(t)|^p + 1 = a(t)\xi(t) + 1, \tag{2.19}$$

where

$$a(t) = (p^*/p)w(t)\phi_p(\xi(t)).$$

Since $\xi(0) = 0$, we get from (2.19)

$$\xi(t) = \int_0^{\delta_0} \exp\left(\int_s^t a(\tau)d\tau\right) ds > 0$$

for all $t \in (0,\delta_0)$. Thus (2.18) is proved.

Suppose that $\theta(t;\theta_0) > \pi_p/2$ fails for some $t_0 > 0$, that is, $\theta(t_0;\theta_0) \leqslant \pi_p/2$. Using the fact (2.18) we can define

$$t_* = \min\{t > 0 : \theta(t;\theta_0) = \pi_p/2\}.$$

Then $t_* > 0$ and $\theta(t_*;\theta_0) = \pi_p/2$. Let $\delta_* \in (0,t_*)$ be such that $\pi_p/2 < \theta(t;\theta_0) < 3\pi_p/2$ for all $t \in [\delta_*,t_*)$. Then the function $\zeta(t) = -T_p(\theta(t;\theta_0))$ is well defined for $t \in [\delta_*,t_*)$. By (2.15), we have, for almost everywhere $t \in [\delta_*,t_*)$,

$$\zeta'(t) = -w(t) - (p/p^*)|\zeta(t)|^p \leqslant -w(t).$$

Thus

$$\zeta(t) \leqslant \zeta(\delta_*) + \int_{\delta_*}^{t_*} |w(s)|\, ds < +\infty$$

for all $t \in [\delta_*,t_*)$. This contradicts the fact that $\lim_{t\to t_*-} \zeta(t) = +\infty$.

The conclusion (ii) can be proved similarly. □

Now we give some properties of the rotation number function $\rho(\lambda)$.

THEOREM 2.1. *The rotation number has the following properties:*

(i) *$\rho(\lambda)$ is continuous in $\lambda \in \mathbb{R}$.*

(ii) *$\rho(\lambda)$ is non-decreasing in $\lambda \in \mathbb{R}$.*

(iii) *$\rho(\lambda) = 0$ if $\lambda \ll -1$, and $\lim_{\lambda\to+\infty} \rho(\lambda) = +\infty$.*

Proof. (i) Note that the vector field $\Xi(t, \theta; \lambda)$ depends on λ continuously. Then so does the Poincaré map $\Theta_\lambda(\theta_0)$. Now the continuity of $\rho(\lambda)$ follows from **[11**, Corollary 2.1, Chapter 2].

(ii) The monotonicity of $\rho(\lambda)$ follows simply from Lemma 2.2.

(iii) Let us first prove that $\lim_{\lambda\to+\infty}\rho(\lambda) = +\infty$. To this end, assume that $\lambda > 0$. Let us consider differential equation (1.2) again. Let $\hat{x} = x$, $\hat{y} = -\lambda^{-1/p^*}\phi_p(x')$. Then (1.2) is equivalent to the following system:

$$\hat{x}' = -\lambda^{1/p}\phi_{p^*}(\hat{y}), \quad \hat{y}' = (\lambda^{1/p} + \lambda^{-1/p^*}q(t))\phi_p(\hat{x}). \tag{2.20}$$

Let $\hat{x} = \hat{r}^{2/p}C_p(\hat{\theta})$, $\hat{y} = \hat{r}^{2/p^*}S_p(\hat{\theta})$ in (2.20). Then $\hat{r}$, $\hat{\theta}$ satisfy the following equations:

$$\hat{r}' = (p/2)\lambda^{-1/p^*}q(t)\phi_p(C_p(\hat{\theta}))\phi_{p^*}(S_p(\hat{\theta}))\hat{r}, \tag{2.21}$$
$$\hat{\theta}' = \lambda^{1/p} + \lambda^{-1/p^*}q(t)|C_p(\hat{\theta})|^p/p. \tag{2.22}$$

As before, let $\hat{\theta}(t; \hat{\theta}_0, \lambda)$ denote the solution of (2.22) with the initial condition: $\hat{\theta}(0) = \hat{\theta}_0$. By (2.22), one has

$$\lambda^{1/p}t - \lambda^{-1/p^*}\int_0^t q_-(s)\,ds \leqslant \hat{\theta}(t; \hat{\theta}_0, \lambda) - \hat{\theta}_0 \leqslant \lambda^{1/p}t + \lambda^{-1/p^*}\int_0^t q_+(s)\,ds \tag{2.23}$$

for all $t \geqslant 0$ and all $\hat{\theta}_0 \in \mathbb{R}$, where

$$q_+(t) = \max\{q(t), 0\}, \quad q_-(t) = \max\{-q(t), 0\}.$$

Comparing (2.20) with (2.6), we have the following relation:

$$(x, y) = (\hat{x}, \lambda^{1/p^*}\hat{y}).$$

In the corresponding p-polar coordinates, we have

$$r^{2/p}C_p(\theta) = \hat{r}^{2/p}C_p(\hat{\theta}), \quad r^{2/p^*}S_p(\theta) = \lambda^{1/p^*}\hat{r}^{2/p^*}S_p(\hat{\theta}).$$

As a result,

$$C_p(\theta) = \frac{C_p(\hat{\theta})}{(|C_p(\hat{\theta})|^p + (p/p^*)\lambda|S_p(\hat{\theta})|p^*)^{1/p}}, \tag{2.24}$$

$$S_p(\theta) = \frac{\lambda^{1/p^*}C_p(\hat{\theta})}{(|C_p(\hat{\theta})|^p + (p/p^*)\lambda|S_p(\hat{\theta})|p^*)^{1/p}}. \tag{2.25}$$

Let us define a homeomorphism $\mathscr{H}_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$, where $\theta = \mathscr{H}_\lambda(\hat{\theta})$ is determined by (2.24) and (2.25). Note that $\mathscr{H}_\lambda$ fixes the points $\{m\pi_p, \pi_p/2 + m\pi_p : m \in \mathbb{Z}\} \subset \mathbb{R}$ and

$$\lim_{|\hat{\theta}|\to\infty} \frac{\mathscr{H}_\lambda(\hat{\theta})}{\hat{\theta}} = 1. \tag{2.26}$$

Comparing (2.22) with (2.8), $\mathscr{H}_\lambda$ preserves solutions

$$\theta(t; \mathscr{H}_\lambda(\hat{\theta}_0), \lambda) \equiv \mathscr{H}_\lambda(\hat{\theta}(t; \hat{\theta}_0, \lambda)). \tag{2.27}$$

By (2.26) and (2.27) we have

$$\begin{aligned}\rho(\lambda) &= \lim_{t\to+\infty}\frac{\theta(t;\,\mathscr{H}_\lambda(\hat\theta_0),\lambda)}{t}\\ &= \lim_{t\to+\infty}\frac{\mathscr{H}_\lambda(\hat\theta(t;\hat\theta_0,\lambda))}{t}\\ &= \lim_{t\to+\infty}\frac{\mathscr{H}_\lambda(\hat\theta(t;\hat\theta_0,\lambda))}{\hat\theta(t;\hat\theta_0,\lambda)}\frac{\hat\theta(t;\hat\theta_0,\lambda)}{t}\\ &= \lim_{t\to+\infty}\frac{\hat\theta(t;\hat\theta_0,\lambda)}{t},\end{aligned}\tag{2.28}$$

because, when $\lambda \gg 1$,

$$\hat\theta(t;\hat\theta_0,\lambda) \geqslant \hat\theta_0 + \lambda^{1/p}t - \lambda^{-1/p^*}\int_0^t q_-(s)\,ds \to +\infty \quad \text{as } t\to+\infty.$$

Next, by (2.23), we get from (2.28)

$$\rho(\lambda) = \lim_{t\to+\infty}\frac{\hat\theta(t;\hat\theta_0,\lambda)}{t} \geqslant \lambda^{1/p} - \hat q_-\lambda^{-1/p^*},$$

where

$$\bar q_- = \frac{1}{2\pi_p}\int_0^{2\pi_p} q_-(s)\,ds$$

is the mean value. Now it is obvious that $\rho(\lambda)\to+\infty$ when $\lambda\to+\infty$.

Next let us prove that $\rho(\lambda)\geqslant 0$ for all λ. Applying Lemma 2.3(i) to $w(t)=\lambda+q(t)$, we have $\theta(t;\pi_p/2,\lambda) > \pi_p/2$ for all $t>0$. Thus

$$\rho(\lambda) = \lim_{t\to+\infty}\frac{\theta(t;\pi_p/2,\lambda)}{t} \geqslant 0.$$

Finally we prove that $\rho(\lambda)=0$ if $\lambda \ll -1$. For simplicity, let us assume that the potential $q(t)$ is bounded on $\mathbb{R}$, that is, there is some $M_0>0$ such that $|q(t)|\leqslant M_0, t\in\mathbb{R}$. Suppose that $\lambda < -M_0$. Then $w(t)=\lambda+q(t)<0$ for all t. It follows from Lemma 2.3(ii) that $\theta(t;0,\lambda)<0$ for all $t>0$. Thus

$$\rho(\lambda) = \lim_{t\to+\infty}\frac{\theta(t;0,\lambda)}{t} \leqslant 0.$$

This, together with the conclusion $\rho(\lambda)\geqslant 0$, shows that $\rho(\lambda)=0$ for all $\lambda\ll-1$. For general periodic potentials $q(t)\in L^1_{\mathrm{loc}}(\mathbb{R})$, the result can be proved similarly by using some similar transformation as in the proof of the conclusion $\lim_{\lambda\to+\infty}\rho(\lambda)=+\infty$. □

REMARK 2.1. It follows from (2.23) that the rotation number function has the following estimates:

$$\lambda^{1/p} - \bar q_-\lambda^{-1/p^*} \leqslant \rho(\lambda) \leqslant \lambda^{1/p} + \bar q_+\lambda^{-1/p^*}, \quad \lambda>0,\tag{2.29}$$

where $\bar q_\pm$ are the mean values of $2\pi_p$-periodic functions $q_\pm(t)$.

3. *Periodic eigenvalues of the p-Laplacian*

In this section we will use the rotation number function $\rho(\lambda)$ to study the main problem in this paper, that is, the periodic eigenvalues $\mathscr{P}$ of (1.2) + (1.3) and the anti-periodic eigenvalues $\mathscr{A}$ of (1.2) + (1.4). We always assume that T in (1.3) and (1.4) is $2\pi_p$.

Note that $(x(t; x_0, y_0, \lambda), y(t; x_0, y_0, \lambda))$ is a solution of (1.2) + (1.3) or (1.2) + (1.4) if and only if (x_0, y_0) is a fixed point of the Poincaré map P_λ:

$$P_\lambda(x_0, y_0) = (x_0, y_0),$$

or an anti-fixed point of P_λ:

$$P_\lambda(x_0, y_0) = -(x_0, y_0).$$

Using expression (2.10) for P_λ, the following result is obvious.

THEOREM 3.1. *$\lambda \in \mathscr{P}_*$ if and only if there exist some $\theta_0 \in \mathbb{R}$ and some $n \in \mathbb{Z}^+$ such that*

$$\theta(2\pi_p; \theta_0, \lambda) = \theta_0 + n\pi_p \quad \textit{and} \quad r(2\pi_p; \theta_0, \lambda) = 1. \tag{3.1}$$

Moreover, the case n is even (n is odd, respectively) in (3.1) *corresponds to the periodic eigenvalues (the anti-periodic eigenvalues, respectively).*

Note that the vector field $\Xi(t, \theta; \lambda)$ is differentiable in θ. Thus the Poincaré map $\Theta_\lambda(\theta_0)$ of (2.8) (cf. (2.11)) is also differentiable in θ_0. The following proposition, which is a result of the area-preserving property of P_λ, is fundamental in obtaining some eigenvalues in $\mathscr{P}_*$.

PROPOSITION 3.1.

$$\frac{d\Theta_\lambda(\theta_0)}{d\theta_0} = \frac{1}{R_\lambda^2(\theta_0)}, \quad \theta_0 \in \mathbb{R}. \tag{3.2}$$

Proof. Consider the space $S = (0, \infty) \times (\mathbb{R}/2\pi_p\mathbb{Z})$ with the area $r\, dr \wedge d\theta$. Let θ_0 be any fixed real number. For any θ_1 ($> \theta_0$) which is near θ_0, consider the following domain in S:

$$\mathscr{D} = \{(r, \theta) : 0 \leqslant r \leqslant 1, \theta_0 \leqslant \theta \leqslant \theta_1\}.$$

With respect to $r\ dr \wedge d\theta$, $\mathscr{D}$ has area

$$|\mathscr{D}| = \int_0^1 \int_{\theta_0}^{\theta_1} r\, dr\, d\theta = \tfrac{1}{2}(\theta_1 - \theta_0).$$

Note that the Poincaré map P_λ, in the (r, θ)-coordinates, is

$$P_\lambda(r, \theta) = (rR_\lambda(\theta), \Theta_\lambda(\theta)).$$

The image $\tilde{\mathscr{D}}$ of $\mathscr{D}$ is the following domain in S:

$$\tilde{\mathscr{D}} = \{(\tilde{r}, \tilde{\theta}) : 0 \leqslant \tilde{r} \leqslant R_\lambda(\Theta_\lambda^{-1}(\tilde{\theta})), \Theta_\lambda(\theta_0) \leqslant \tilde{\theta} \leqslant \Theta_\lambda(\theta_1)\},$$

cf. (2.10). Here $\Theta_\lambda^{-1}(\cdot)$ is the inverse of $\Theta_\lambda(\cdot)$. Thus $\tilde{\mathscr{D}}$ has area

$$\begin{aligned} |\tilde{\mathscr{D}}| &= \int_{\Theta_\lambda(\theta_0)}^{\Theta_\lambda(\theta_1)} d\tilde{\theta} \int_0^{R_\lambda(\Theta_\lambda^{-1}(\tilde{\theta}))} \tilde{r}\, d\tilde{r} \\ &= \tfrac{1}{2}\int_{\Theta_\lambda(\theta_0)}^{\Theta_\lambda(\theta_\tau)} R_\lambda^2(\Theta_\lambda^{-1}(\tilde{\theta}))\, d\tilde{\theta} \\ &= \tfrac{1}{2}\int_{\theta_0}^{\theta_1} R_\lambda^2(\theta) \frac{d\Theta_\lambda(\theta)}{d\theta}\, d\theta. \end{aligned}$$

As P_λ preserves $r\,dr \wedge d\theta$, we have

$$\tfrac{1}{2}(\theta_1 - \theta_0) \equiv \tfrac{1}{2}\int_{\theta_0}^{\theta_1} R_\lambda^2(\theta) \frac{d\Theta_\lambda(\theta)}{d\theta}\, d\theta.$$

Differentiating this equality with respect to θ_1 at $\theta_1 = \theta_0$, we get the desired equality (3.2). □

REMARK 3.1. The equality (3.2) can be also proved using equations (2.7) and (2.8).

By Proposition 3.1, we can state an equivalent form of Theorem 3.1.

THEOREM 3.2. *$\lambda \in \mathscr{P}_*$ if and only if there exist some $\theta_0 \in \mathbb{R}$ and some $n \in \mathbb{Z}^+$ such that*

$$\Theta_\lambda(\theta_0) = \theta_0 + n\pi_p \quad \textit{and} \quad \left.\frac{d\Theta_\lambda(\theta)}{d\theta}\right|_{\theta=\theta_0} = 1. \tag{3.3}$$

Moreover, the case n is even (n is odd, respectively) in (3.3) *corresponds to the periodic eigenvalues (the anti-periodic eigenvalues, respectively).*

We will establish some relation between condition (3.3) with the rotation number function $\rho(\lambda)$. To this end, we need the following result.

PROPOSITION 3.2. *Let h be a homeomorphism of $\mathbb{R}$ satisfying*

$$h(\theta + m\pi_p) = h(\theta) + m\pi_p, \quad \theta \in \mathbb{R},\ m \in \mathbb{Z}. \tag{3.4}$$

Define the rotation number of h by

$$\rho(h) = \lim_{|m|\to\infty} \frac{h^m(\theta_0) - \theta_0}{2m\pi_p}.$$

Suppose that n is an integer. Then

(i) $\rho(h) \geqslant n/2$ *if and only if* $\max_{\theta_0\in\mathbb{R}}(h(\theta_0) - (\theta_0 + n\pi_p)) \geqslant 0$;

(ii) $\rho(h) \leqslant n/2$ *if and only if* $\min_{\theta_0\in\mathbb{R}}(h(\theta_0) - (\theta_0 + n\pi_p)) \leqslant 0$.

Proof. This is [**9**, Proposition 2.2]. □

Now we introduce for (1.2) two sequences $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ using the rotation number function $\rho(\lambda)$.

Definition 3.1. Let $q(t)$ be a $2\pi_p$-periodic potential with $q(t) \in L^1(0, 2\pi_p)$. Define

$$\underline{\lambda}_n(q) = \min\{\lambda \in \mathbb{R} : \rho(\lambda; q) = n/2\} \quad \text{for } n \in \mathbb{N}, \tag{3.5}$$

$$\overline{\lambda}_k(q) = \max\{\lambda \in \mathbb{R} : \rho(\lambda; q) = n/2\} \quad \text{for } n \in \mathbb{Z}^+. \tag{3.6}$$

Note that these sequences are well-defined by Theorem 2.1. Now we prove the main result in this paper.

Theorem 3.3. *If $n \in \mathbb{Z}^+$ is even, then $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are eigenvalues of* (1.2) + (1.3), *and if $n \in \mathbb{N}$ is odd, then $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are eigenvalues of* (1.2) + (1.4).

Proof. Let us consider the family of Poincaré maps Θ_λ of (2.8). By (2.13), Θ_λ satisfies (3.4) for each λ. By Lemma 2.2, for any fixed θ_0, the function $\Theta_\lambda(\theta_0) - \theta_0$ is strictly increasing in λ. Thus the functions

$$\max_{\theta_0 \in \mathbb{R}} (\Theta_\lambda(\theta_0) - \theta_0),$$

and

$$\min_{\theta_0 \in \mathbb{R}} (\Theta_\lambda(\theta_0) - \theta_0)$$

are strictly increasing in λ. Now it follows from (3.5) and (3.6) and from Proposition 3.2 that $\lambda = \underline{\lambda}_n(q)$ if and only if λ satisfies

$$\max_{\theta_0 \in \mathbb{R}} (\Theta_\lambda(\theta_0) - \theta_0) = n\pi_p, \tag{3.7}$$

and that $\lambda = \overline{\lambda}_n(q)$ if and only if λ satisfies

$$\min_{\theta_0 \in \mathbb{R}} (\Theta_\lambda(\theta_0) - \theta_0) = n\pi_p. \tag{3.8}$$

As a result, if $\lambda = \underline{\lambda}_n(q)$ or $\overline{\lambda}_n(q)$, we know from (3.7) or (3.8) that there must be some $\theta_0 \in \mathbb{R}$ such that (3.3) is satisfied. Now the result follows from Theorem 3.2. □

Remark 3.2. We do not know that if the converse of Theorem 3.3 also holds, that is, if all eigenvalues of (1.2)+(1.3) and (1.2)+(1.4) are necessarily given by those $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$. The gap from this problem is that all eigenvalues are characterized by (3.3), while rotational eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ are characterized by (3.7) and (3.8).

Remark 3.3. Since $\rho(\lambda)$ is non-decreasing, one sees that the order (1.6) holds also for the p-Laplacian. As $\rho(\underline{\lambda}_n(q)) = \rho(\overline{\lambda}_n(q)) = n/2$, it follows from the estimates (2.29) that one has the following asymptotic formula for rotational periodic eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$:

$$\underline{\lambda}_n(q), \overline{\lambda}_n(q) \approx (n/2)^p \quad \text{as } n \to +\infty. \tag{3.9}$$

4. *Sturm–Liouville problems*

In this section we will give another characterization of rotational periodic eigenvalues using the eigenvalues of (1.2) with respect to certain two-point boundary conditions. To this end, we consider the Sturm–Liouville problems of (1.2).

Before considering general two-point boundary conditions, we consider the eigenvalues of (1.2) with respect to the Dirichlet boundary condition:

$$x(0) = x(2\pi_p) = 0. \tag{4.1}$$

THEOREM 4.1. *Eigenvalue problem* (1.2) + (4.1) *has a sequence of eigenvalues:*

$$-\infty < \lambda_1^{\mathrm{D}}(q) < \lambda_2^{\mathrm{D}}(q) < \ldots < \lambda_n^{\mathrm{D}}(q) < \ldots .$$

Moreover, all eigenvalues $\lambda_n^{\mathrm{D}}(q)$ are simple, and associated with the first eigenvalue (*or the principal eigenvalue*) *$\lambda_1^{\mathrm{D}}(q)$, there is an eigenfunction $\varphi_1(t)$ such that $\varphi_1(t)$ is strictly positive on* $(0, 2\pi_p)$.

Proof. Note that if $x(t)$ is a solution of (1.2), then so is $mx(t)$ for any $m \in \mathbb{R}$. In order to consider eigenvalue problem (1.2)+(4.1), we need only consider the solution $(x(t), y(t))$ of (2.6) satisfying $(x(0), y(0)) = (0, 1)$. Now λ is an eigenvalue of problem (1.2) + (4.1) if and only if $x(2\pi_p; 0, 1, \lambda) = 0$, which, in the p-polar coordinates (2.3), is equivalent to

$$\theta(2\pi_p; \pi_p/2, \lambda) = \pi_p/2 + n\pi_p \quad \text{for some } n \in \mathbb{Z}, \tag{4.2}$$

cf. (2.9). By (2.13), it follows from (4.2) that

$$\theta(2m\pi_p; \pi_p/2, \lambda) = \pi_p/2 + nm\pi_p$$

for all $m \in \mathbb{Z}$. As a result,

$$\rho(\lambda) = \lim_{m\to\infty} \frac{\theta(2m\pi_p; \pi_p/2, \lambda) - \pi_p/2}{2m\pi_p} = n/2.$$

Now Theorem 2.1 shows that n in (4.2) is necessarily non-negative. Furthermore, it is also necessary that n in (4.2) is positive because $\theta(2\pi_p; \pi_p/2, \lambda) > \pi_p/2$ by applying Lemma 2.3(i) to $w(t) = \lambda + q(t)$.

Next let us prove that for any integer $n \in \mathbb{N}$, (4.2) has a unique solution $\lambda = \lambda_n^{\mathrm{D}}(q)$ which yields an eigenvalue of (1.2) + (4.1). Let us keep the notations in the proof of Theorem 2.1(iii). By (2.23), if $\lambda > 0$ then

$$\hat{\theta}(2\pi_p; \pi_p/2, \lambda) - \pi_p/2 \geqslant 2\pi_p\lambda^{1/p} - \lambda^{-1/p^*}\int_0^{2\pi_p} q_-(s)\, ds.$$

Thus

$$\lim_{\lambda\to+\infty} (\hat{\theta}(2\pi_p; \pi_p/2, \lambda) - \pi_p/2) = +\infty.$$

Now it follows from (2.26) and (2.27) that

$$\begin{aligned}\lim_{\lambda\to+\infty} &(\theta(2\pi_p; \pi_p/2, \lambda) - \pi_p/2) \\ &= \lim_{\lambda\to+\infty} (\mathscr{H}_\lambda(\hat{\theta}(2\pi_p; \pi_p/2, \lambda)) - \pi_p/2) = +\infty,\end{aligned} \tag{4.3}$$

because the convergence in (2.26) is in fact uniform in $\lambda \gg 1$.

On the other hand, as in the proof of Theorem 2.1(iii), if $\lambda \ll -1$, then

$$\theta(2\pi_p; \pi_p/2, \lambda) < \pi_p < \pi_p/2 + n\pi_p. \tag{4.4}$$

By Lemma 2.2, $\theta(2\pi_p; \pi_p/2, \lambda)$ is strictly increasing in λ. Now the existence and uniqueness of solution of (4.2) follows from the intermediate value theorem and from (4.3), (4.4).

The simplicity of all eigenvalues $\lambda_n^{\rm D}(q)$ can be obtained by the uniqueness result for solutions of (1.2).

As for the principal eigenvalue $\lambda = \lambda_1^{\rm D}(q)$, we will prove that

$$0 < \theta(t; \pi_p/2, \lambda) - \pi_p/2 < \pi_p \quad \text{for all } t \in (0, 2\pi_p). \tag{4.5}$$

Thus $\lambda_1^{\rm D}(q)$ has an eigenfunction $x(t) = -r^{2/p}(t; \pi_p/2, \lambda) C_p(\theta(t; \pi_p/2, \lambda)) > 0$ for all $t \in (0, 2\pi_p)$. The left inequality in (4.5) follows from Lemma 2.3(i). If the right inequality in (4.5) fails at some $t = t_0 \in (0, 2\pi_p)$, that is, $\theta(t_0; \pi_p/2, \lambda) - \pi_p/2 \geqslant \pi_p$, then we consider the equation

$$\breve\theta' = p((\lambda + q_{t_0}(t))|C_p(\breve\theta)|^p/p + |S_p(\breve\theta)|^{p^*}/p^*), \tag{4.6}$$

where $q_{t_0}(t) \equiv q(t+t_0)$. Let $\breve\theta(t; \breve\theta_0, \lambda)$ be the solution of (4.6) with the initial condition $\breve\theta(0) = \breve\theta_0$. By the uniqueness result of solutions, one has the following equality

$$\theta(t; \theta_0, \lambda) \equiv \breve\theta(t - t_0; \theta(t_0; \theta_0, \lambda), \lambda).$$

Applying Lemma 2.3(i) to $w(t) = \lambda + q_{t_0}(t)$, we have

$$\theta(2\pi_p; \pi_p/2, \lambda) = \breve\theta(2\pi_p - t_0; \theta(t_0; \pi_p/2, \lambda), \lambda) > \pi_p/2 + \pi_p,$$

because we have assumed that $\theta(t_0; \pi_p/2, \lambda) \geqslant \pi_p/2 + \pi_p$. As a result, $\lambda = \lambda_1^{\rm D}(q)$ does not satisfy (4.1) (with $n = 1$), which is a contradiction. □

REMARK 4.1. The Dirichlet eigenvalues of the p-Laplacian with weights are studied in [**24**]. Some application of weighted eigenvalues to non-resonance of semilinear equations of the p-Laplacian-type is given in [**24**].

The general two-point boundary conditions are of the form

$$\xi x(0) + \eta x'(0) = 0, \quad \sigma x(2\pi_p) + \tau x'(2\pi_p) = 0, \tag{4.7}$$

where ξ, η, α, τ are constants such that $\xi^2 + \eta^2 > 0$ and $\sigma^2 + \tau^2 > 0$. The conditions (4.7) can be rewritten as the following concise form:

$$\phi_{p^*}(S_p(\alpha))x(0) + C_p(\alpha)x'(0) = 0, \quad \phi_{p^*}(S_p(\beta))x(2\pi_p) + C_p(\beta)x'(2\pi_p) = 0, \tag{4.8}$$

where $\sigma, \beta \in [0, \pi_p)$. Under transformations (2.3) and (2.5), the first condition in (4.8) is equivalent to

$$\begin{aligned} &\phi_{p^*}(S_p(\alpha))x(0) - C_p(\alpha)\phi_{p^*}(y(0)) = 0 \\ &\quad \Longleftrightarrow x(0) = \varphi_p(r_0)C_p(\alpha), \quad y(0) = \varphi_{p^*}(r_0)S_p(\alpha) \quad \text{for some } r_0 \\ &\quad \Longleftrightarrow \theta(0) = \alpha + m\pi_p \quad \text{for some } m \in \mathbb{Z}. \end{aligned}$$

Similarly, the second condition in (4.8) is equivalent to

$$\theta(2\pi_p) = \beta + \ell\pi_p \quad \text{for some } \ell \in \mathbb{Z}.$$

By (2.13), one knows that λ is an eigenvalue of (1.2) + (4.8) if and only if

$$\theta(2\pi_p; \alpha, \lambda) = \beta + n\pi_p, \quad n \in \mathbb{Z}. \tag{4.9}$$

Note that n in (4.9) is non-negative.

In the following, we are only interested in the case $\beta = \alpha \in [0, \pi_p)$ in (4.8):

$$\phi_{p^*}(S_p(\alpha))x(0) + C_p(\alpha)x'(0) = 0, \quad \phi_{p^*}(S_p(\alpha))x(2\pi_p) + C_p(\alpha)x'(2\pi_p) = 0. \tag{4.10}$$

Hence λ is an eigenvalue of (1.2) + (4.10) if and only if

$$\theta(2\pi_p;\alpha,\lambda) = \alpha + n\pi_p, \quad n \in \mathbb{Z}^+. \tag{4.11}$$

Note that the case $\alpha = \pi_p/2$ in (4.10) corresponds to the Dirichlet condition (4.1), and the case $\alpha = 0$ in (4.10) corresponds to the Neumann condition. As in the Dirichlet case, one has the following result for eigenvalue problem (1.2)+(4.10). The only difference is that (1.2) + (4.10) has also a zeroth eigenvalue $\lambda = \lambda_0^\alpha(q)$.

THEOREM 4.2. *Let $\alpha \in [0,\pi_p/2) \cup (\pi_p/2,\pi_p)$ in* (4.10). *Then eigenvalue problem* (1.2) + (4.10) *has a sequence of eigenvalues*

$$-\infty < \lambda_0^\alpha(q) < \lambda_1^\alpha(q) < \ldots < \lambda_n^\alpha(q) < \ldots .$$

Moreover, all eigenvalues $\lambda_n^\alpha(q)$ are simple.

REMARK 4.2. It follows from (4.11) that $\rho(\lambda_n^\alpha(q)) = n/2$. Thus one has the following relation between $\lambda_n^\alpha(q)$ and the rotational periodic eigenvalues, which is well known for the linear case $p = 2$:

$$\underline{\lambda}_n(q) \leqslant \lambda_n^\alpha(q) \leqslant \overline{\lambda}_n(q), \quad \alpha \in [0,\pi_p), \ n \in \mathbb{N}. \tag{4.12}$$

By the asymptotic formula (3.9) for $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$, one has the following asymptotic formula for $\lambda_n^\alpha(q)$:

$$\lambda_n^\alpha(q) \approx (n/2)^p \quad \text{as } n \to +\infty. \tag{4.13}$$

Now we give the following characterization of rotational periodic eigenvalues using the eigenvalues of Sturm–Liouville problems.

THEOREM 4.3. *For any $n \in \mathbb{N}$,*

$$\underline{\lambda}_n(q) = \min_{s\in\mathbb{R}} \lambda_n^\alpha(q_s), \tag{4.14}$$

$$\overline{\lambda}_n(q) = \max_{s\in\mathbb{R}} \lambda_n^\alpha(q_s). \tag{4.15}$$

The eigenvalue $\overline{\lambda}_0(q)$ can be characterized using the Neumann eigenvalues

$$\overline{\lambda}_0(q) = \max_{s\in\mathbb{R}} \lambda_0^0(q_s). \tag{4.16}$$

Here $q_s(t)$ are translations of $q(t)$, that is, $q_s(t) = q(t+s)$.

Proof. For any fixed $s \in \mathbb{R}$, let $\theta(t;\theta_0,\lambda,q_s)$ be the solution of the following equation

$$\theta' = p((\lambda + q_s(t))|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*)$$

satisfying the initial condition $\theta(0) = \theta_0$. Then one has the following equality:

$$\theta(t;\theta(-s;\theta_0,\lambda,q_s),\lambda,q) \equiv \theta(t-s;\theta_0,\lambda,q_s)$$

for all t, s, θ_0 and λ. Hence $\rho(\lambda;q) = \rho(\lambda;q_s)$ for all λ and all s. Consequently, $\underline{\lambda}_n(q_s) = \underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q_s) = \overline{\lambda}_n(q)$ for all s. By (4.12), one has

$$\underline{\lambda}_n(q) \leqslant \lambda_n^\alpha(q_s) \leqslant \overline{\lambda}_n(q), \quad \text{for all } s \in \mathbb{R}.$$

Moreover, the eigenvalues $\lambda_n^\alpha(q_s)$, as functions of s, are continuous in s, cf. (4.11).

Now we are going to prove (4.15). Assume first that $n \in \mathbb{N}$ in even. Then $\lambda = \overline{\lambda}_n(q)$ is an eigenvalue of the periodic problem (1.2)+(1.3), that is, equation (1.2) has a non-zero $2\pi_p$-periodic function $x(t)$. We claim that there exists $s_0 \in \mathbb{R}$ such that

$$\phi_{p^*}(S_p(\alpha))x(s_0) + C_p(\alpha)x'(s_0) = 0. \tag{4.17}$$

Case 1: $\alpha = 0$. As $x(t)$ is $2\pi_p$-periodic, there exists s_0 such that $x'(s_0) = 0$. Thus (4.17) is satisfied.

Case 2: $\alpha = \pi_p/2$. If (4.17) does not hold, then either $x(t) > 0$ for all t or $x(t) < 0$ for all t. Let r_0 ($\neq 0$) and θ_0 be such that

$$x(0) = \varphi_p(r_0)C_p(\theta_0), \quad y(0) = -\phi_p(x'(0)) = \varphi_{p^*}(r_0)S_p(\theta_0).$$

Then $\theta(t;\theta_0,\lambda,q)$ is bounded because $x(t) \neq 0$ for all t. As a result, we have $\rho(\lambda) = 0$. This is a contradiction because $\rho(\lambda) = \rho(\overline{\lambda}_n(q)) = n/2 > 0$.

Case 3: $\alpha \in (0,\pi_p/2) \cup (\pi_p/2,\pi_p)$. If (4.17) does not hold, then the function

$$\xi(t) = x'(t) + \gamma x(t) \tag{4.18}$$

is a $2\pi_p$-periodic function and satisfies $\xi(t) \neq 0$ for all t, where $\gamma = \phi_p(S_p(\alpha))/C_p(\alpha) \neq 0$. Without loss of generality, we assume that $\xi(t) > 0$ for all t. It is easy to prove that for any given $2\pi_p$-periodic function $\xi(t)$, the linear equation (4.18) has a unique $2\pi_p$-periodic solution $x(t)$. In fact, such a $2\pi_p$-periodic solution is given by

$$x(t) = \int_{\varepsilon\infty}^{0} \xi(s+t)\exp(\gamma s)\,ds,$$

where $\varepsilon = -$ if $\alpha < \pi_p/2$, or $\varepsilon = +$ if $\alpha > \pi_p/2$. As $\xi(t) > 0$ for all t, we have $x(t) > 0$ for all t if $\alpha < \pi_p/2$, or $x(t) < 0$ for all t if $\alpha > \pi_p/2$. Now we can use a similar argument as in case 2 to prove that (4.17) holds for some $s_0 \in \mathbb{R}$.

Let now $z(t) = x(t+s_0)$, where s_0 satisfies (4.17). Then $z(t)$ satisfies differential equation

$$(\phi_p(z'(t)))' + (\lambda + q_{s_0}(t))\phi_p(z(t)) = 0. \tag{4.19}$$

Moreover, by condition (4.17) and the $2\pi_p$-periodicity of $x(t)$, $z(t)$ satisfies the boundary condition (4.10). This means that $\lambda = \overline{\lambda}_n(q)$ is an eigenvalue of (4.15) + (4.10). As we have shown $n/2 = \rho(\overline{\lambda}_n(q);q) = \rho(\overline{\lambda}_n(q);q_{s_0})$, therefore $\overline{\lambda}_n(q) = \lambda_n^\alpha(q_{s_0})$. This proves that the maximum in (4.15) can be attained when $n > 0$ is even.

The characterization (4.16) also holds because we have (4.17) in this case.

Now we consider (4.15) when $n \in \mathbb{N}$ is odd. In this case, for $\lambda = \overline{\lambda}_n(q)$, (1.2) has a non-zero solution $x(t)$ satisfying the anti-periodic boundary condition (1.4). As (1.2) is odd in x, it is easy to check that $x(t)$ satisfies $x(t+2\pi_p) \equiv -x(t)$ and $x(t)$ is a $4\pi_p$-periodic solution of (1.2). Now a similar argument shows that (4.15) holds also in this case.

Equality (4.14) can be proved similarly. □

Remark 4.3. Let us write $q_1 \prec q_2$ if $q_1(t) \leqslant q_2(t)$ for almost everywhere $t \in [0,2\pi_p]$ and $q_1(t) < q_2(t)$ for t in some subset of $[0,2\pi_p]$ of positive measure. One has the following comparison results for eigenvalues $\lambda_n^\alpha(q)$, $\underline{\lambda}_n(q)$, $\overline{\lambda}_n(q)$ with respect to the potentials q. If $q_1 \prec q_2$, then

$$\lambda_n^\alpha(q_1) > \lambda_n^\alpha(q_2),$$
$$\underline{\lambda}_n(q_1) > \underline{\lambda}_n(q_2),$$
$$\overline{\lambda}_n(q_1) > \overline{\lambda}_n(q_2),$$

for all $n \in \mathbb{N}$. The reason is that the following monotonicity of solutions $\theta(t;\theta_0,\lambda,q)$ of (2.8) on the potentials q, that is, if $q_1 \prec q_2$, then $\theta(t;\theta_0,\lambda,q_1) < \theta(t;\theta_0,\lambda,q_2)$ for all $t \geqslant 2\pi_p$ and all θ_0, cf. Lemma 2.2. Now the comparison results follow from (4.11) and (3.7), (3.8). These comparison results for eigenvalues are useful in the study of non-resonance of *non-autonomous* equations of the p-Laplacian-type, see [**18**, **24**].

5. *Examples and concluding remarks*

In this section, we present some examples such that the converse of Theorem 3.3 holds.

EXAMPLE 5.1. Let $q(t)$ in (1.2) be a constant, say $q(t) \equiv 0$. Then (2.8) is

$$\theta' = p(\lambda|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*).$$

In this case, $\rho(\lambda) = 0$ if $\lambda \leqslant 0$, and $\rho(\lambda) = \lambda^{1/p}$ if $\lambda > 0$, cf. (2.22). Thus we can recover the classical eigenvalues of the p-Laplacian without potential: $\overline{\lambda}_0(0) = 0$ and

$$\underline{\lambda}_n(0) = \lambda_n^\alpha(0) = \overline{\lambda}_n(0) = (n/2)^p, \quad n \in \mathbb{N},\ \alpha \in [0,\pi_p).$$

In this case, one sees that the converse of Theorem 3.3 holds.

EXAMPLE 5.2. Let us consider the 2-step potential $q(t)$:

$$q(t) = \begin{cases} c_1 & t \in [0,t_1), \\ c_2 & t \in [t_1, 2\pi_p), \end{cases}$$

where $c_1 \neq c_2$ and $0 < t_1 < 2\pi_p$. When $p = 2$, the eigenvalues $\underline{\lambda}_n(q)$ and $\overline{\lambda}_n(q)$ have been studied in detail so that the global structure of resonance pockets of Hill's equations can be given [**9**, **25**]. For general p, equation (2.8) reads as

$$\theta' = ((\lambda + c_1)|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*), \qquad t \in [0,t_1), \tag{5.1}$$

$$\theta' = p((\lambda + c_2)|C_p(\theta)|^p/p + |S_p(\theta)|^{p^*}/p^*), \quad t \in [t_1, 2\pi_p). \tag{5.2}$$

Using the p-tangent function $T_p(\theta)$ in (2.16) and the formula in (2.17), equations (5.1) and (5.2) can be written as

$$\frac{d}{dt}T_p(\theta) = (\lambda + c_1) + (p/p^*)|T_p(\theta)|^{p^*}, \quad t \in [0,t_1), \tag{5.3}$$

$$\frac{d}{dt}T_p(\theta) = (\lambda + c_2) + (p/p^*)|T_p(\theta)|^{p^*}, \quad t \in [t_1, 2\pi_p), \tag{5.4}$$

respectively. Equations (5.3) and (5.4) can be integrated as for the case $p = 2$. Using a similar technique as in [**9**], one can prove that the converse of Theorem 3.3 also holds in this case. We will not give the details here and refer the interested reader to [**9**].

EXAMPLE 5.3. Let $q(t) = \cos(\pi t/\pi_p)$ in (1.2). Then $q(t)$ is $2\pi_p$-periodic. In this case, a numerical simulation shows that the converse of Theorem 3.3 also holds.

We conjecture that the converse of Theorem 3.3 holds for general potentials, that is, the sequences $\{\underline{\lambda}_n(q) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(q) : n \in \mathbb{Z}^+\}$ represent all periodic and anti-periodic eigenvalues of (1.2). As explained in the introduction, this is an interesting problem. As (1.2) is an ordinary differential equation, it would be a relatively easier problem than the famous problem of the higher dimensional Dirichlet eigenvalues of the p-Laplacian.

We end the paper with two remarks.

REMARK 5.1. The approach developed here also applies to the eigenvalues of the following type:

$$(r(t)\phi_p(x'))' + (\lambda w(t) + q(t))\phi_p(x) = 0, \tag{5.5}$$

where $r(t)$, $w(t)$, $q(t)$ are $2\pi_p$-periodic and $r(t) > 0$, $w(t) > 0$. Note that the radial eigenvalues of higher dimensional p-Laplacian (1.1) will lead to (5.5). Therefore, by Theorem 4.2, the radial Dirichlet eigenvalues of (1.1) is clear when Ω is an annulus.

REMARK 5.2. The idea in this paper is also useful in another famous spectrum problem, that is, the Fučik spectrum of

$$x'' + (\lambda_+ + q_+(t))x_+ - (\lambda_- + q_-(t))x_- = 0, \tag{5.6}$$

where $q_\pm(t)$ are $2\pi_p$-periodic potentials. The spectrum of (5.6) + (1.3) has not been well understood. We will develop the rotation number approach and give a partial understanding of (5.6)+(1.3) in [**22**]. Because of the asymmetry in (5.6), the spectrum of (5.6) + (1.4) is more complicated.

Acknowledgements. This work was completed when the author was visiting the Center for Dynamical Systems and Nonlinear Studies, Georgia Institute of Technology. He would like to thank Professors Shi Jin and K. Mischaikow for their kind hospitality. The author thanks the referee for pointing out the important reference [**21**].

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