Extremal values of smallest eigenvalues of Hill’s operators with potentials in $L^1$ balls

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ABSTRACT

Given a $1$-periodic real potential $q \in L^1(\mathbb{R}/\mathbb{Z})$. We use $\lambda_0(q)$ to denote the smallest $1$-periodic eigenvalue of the Hill’s equation $x'' + (\lambda + q(t))x = 0$. Let $B_1[r]$ be the ball centered at $0$ of radius $r$ in the $L^1$ space $L^1(\mathbb{R}/\mathbb{Z})$. It is trivial that $\sup\{\lambda_0(q): q \in B_1[r]\} = r$ for all $r \geq 0$. Based on continuity of $\lambda_0(q)$ in $q$ with the weak topology and continuous differentiability of $\lambda_0(q)$ in $q$ with the $L^1$ norm $\|\cdot\|_1$, we will apply scaling technique, variational approach to extremal values in $L^p$ balls, singular integrals and the limiting approach as $p \downarrow 1$ to obtain (i) $\lambda_0(q)$ is bounded for $q$ in any bounded set of $(L^1(\mathbb{R}/\mathbb{Z}), \|\cdot\|_1)$, and (ii) the minimal value

$$L_1(r) := \inf\{\lambda_0(q): q \in L^1(\mathbb{R}/\mathbb{Z}), \|q\|_1 \leq r\}$$

is simply $Z_0^{-1}(r)$, where $Z_0(x) := 2\sqrt{-x}\tanh(\sqrt{-x}/2)$, $x \leq 0$. The extremal values of the smallest Neumann eigenvalues for potentials in $L^1$ balls are also found explicitly.

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1. Introduction

Eigenvalues and their estimates are important in many problems in mathematics and applied sciences. In this paper, we will use continuity of eigenvalues in weak topologies, some topological facts on $L^p$ spaces, variational method, dynamical systems and singular integrals to give some deep results on the smallest periodic eigenvalues of Hill’s operators.

Let $q$ be a 1-periodic (real) potential from the Lebesgue space $L^p := L^p(S_1)$, where $S_1 = \mathbb{R}/\mathbb{Z}$ and $1 \leq p \leq \infty$. The eigenvalue problem

$$x'' + (\lambda + q(t))x = 0$$

has a double-sequence of eigenvalues

$$\bar{\lambda}_0(q) < \lambda_1(q) \leq \bar{\lambda}_1(q) < \cdots \leq \lambda_m(q) < \bar{\lambda}_m(q) < \cdots,$$

where $\lambda_m(q)$, $\bar{\lambda}_m(q)$ are 1-periodic eigenvalues of (1.1) for $m$ even, and $\lambda_{2m}(q)$, $\bar{\lambda}_{2m}(q)$ are 1-anti-periodic eigenvalues of (1.1) for $m$ odd. See [15,24], or [25] for a rotation number approach.

As a functional of potentials $q \in L^p$, each of these eigenvalues is continuous in the usual $L^p$ topology $\|q\|_p := \|q\|_{L^p(S_1)}$. Moreover, since $\overline{\lambda}_0(q)$ is simple, $q \in (L^p, \| \cdot \|_p) \rightarrow \overline{\lambda}_0(q)$ is actually continuously differentiable [13]. A recent result by the author shows that eigenvalues have very strong continuous dependence on potentials.

**Theorem 1.1.** (See Zhang [27].) Let $1 \leq p \leq \infty$ and $m \geq 0$. The functionals

$$(L^p, w_p) \rightarrow \mathbb{R}, \quad q \rightarrow \lambda_m(q), \quad q \rightarrow \bar{\lambda}_m(q)$$

are continuous, where $w_p$ indicates the topology of weak convergence in the space $(L^p, \| \cdot \|_p)$ for $1 \leq p < \infty$ and $w_\infty$ is the topology of weak* convergence in the space $(L^\infty, \| \cdot \|_\infty)$. Here $\lambda_0(q)$ is void.

For case $p = \infty$, see also [17]. For case $2 \leq p \leq \infty$, see also [18]. For some continuity results of solutions in weak topologies, see [11,20].

Such a strong continuity of eigenvalues has some important implications. In case $1 < p \leq \infty$, it is well known [23] that any bounded subset of $(L^p, \| \cdot \|_p)$ is sequentially relatively compact in $(L^p, w_p)$. By Theorem 1.1, both $\lambda_m(q)$ and $\bar{\lambda}_m(q)$ are bounded for $q$ in any bounded subset of $(L^p, \| \cdot \|_p)$. Since bounded subsets of $(L^1, \| \cdot \|_1)$ may lack compactness even in $w_1$, the boundedness of $\lambda_m(q)$ and $\bar{\lambda}_m(q)$ for $q$ in bounded subsets of $(L^1, \| \cdot \|_1)$ cannot be deduced from Theorem 1.1 in a direct way. In this paper, we will completely solve this for the smallest periodic eigenvalues $\overline{\lambda}_0(q)$.

For $1 \leq p \leq \infty$ and $r \geq 0$, let

$$B_p[r] := \{ q \in L^p : \|q\|_p \leq r \}$$

be the ball of the $L^p$ space. Since we are mainly concerned with the smallest periodic eigenvalues in this paper, for simplicity, we write $\overline{\lambda}_0(q)$ as $\lambda_0(q)$ for $q \in L^p$. Let us define the following extremal values

$$L_p(r) := \inf_{q \in B_p[r]} \lambda_0(q), \quad M_p(r) := \sup_{q \in B_p[r]} \lambda_0(q).$$

For the maximal values $M_p(r)$, let us recall a trivial upper bound of $\lambda_0(q)$

$$\lambda_0(q) \leq -\int_0^1 q(t) \, dt, \quad q \in L^p.$$
Moreover, (1.3) becomes an equality if and only if \( q(t) \) is constant for a.e. \( t \). See, for example, [26, Lemma 3.6]. By the Hölder inequality, we have \( \lambda_0(q) \leq \|q\|_1 \leq \|q\|_p \). Hence one can obtain

\[
M_p(r) = \lambda_0(-r) = r, \quad \forall 1 \leq p \leq \infty, \quad \forall r \geq 0.
\]  

(1.4)

Recall that \( L_p(r) \) is finite when \( 1 < p \leq \infty \). For the most interesting case \( p = 1 \), in this paper we will obtain the following main theorem.

**Theorem 1.2.**

(i) The minimal value \( L_1(r) \) is finite for each \( r \geq 0 \).

(ii) Let us introduce the following elementary function

\[
Z_0(x) = 2\sqrt{-x}\tanh(\sqrt{-x}/2), \quad x \leq 0,
\]

which is a decreasing diffeomorphism from \((-\infty, 0]\) onto \([0, \infty)\). Then one has

\[
L_1(r) = Z_0^{-1}(r), \quad \forall r \geq 0.
\]  

(1.6)

Let us give the outline of the proof of Theorem 1.2.

**Step 1.** By the Hölder inequality, we know that \( B_2[r] \subset B_{p'}[r] \subset B_1[r] \) for all \( 1 \leq p' \leq p \leq \infty \). Hence, for any \( r \) given, \( L_p(r) \) is a well-defined, non-decreasing function of \( p \in (1, \infty) \). One topological fact is that the union

\[
\bigcup_{p \in (1, \infty)} B_p[r]
\]

is dense in \( B_1[r] \) in the \( L^1 \) topology \( \| \cdot \|_1 \). See Lemma 2.1. By continuity of eigenvalues in potentials, we can obtain the following limiting equality

\[
L_1(r) = \lim_{p \downarrow 1} L_p(r) = \inf_{p \in (1, \infty)} L_p(r) \quad \text{for all} \quad r \geq 0.
\]  

(1.7)

See Lemma 2.2.

**Step 2.** Given \( p \in (1, \infty) \) and \( r \in (0, \infty) \). We will study \( L_p(r) \) in Sections 3 and 4. In this case, \( L_p(r) \) has minimal potentials \( q_{p,r} \in B_p[r] \). From basic properties of eigenvalues, \( q_{p,r} \) is actually on the \( L^p \) sphere, i.e., \( \|q_{p,r}\|_p = r \). Recall that \( \lambda_0 : (C^p, \| \cdot \|_p) \to \mathbb{R} \) is continuously differentiable. Hence we can use the variational method to obtain the critical equation for \( q_{p,r} \). It is interesting to notice that the critical equation is a nonlinear Schrödinger equation in the form of the associated eigen-functions of \( \lambda_0(q_{p,r}) = L_p(r) \). See Eq. (3.6) and Proposition 3.1. Using some temporal and spatial changes of potentials, a crucial observation is that minimal potentials are either constant or have minimal period 1. See Lemmas 2.11 and 2.12 and Remark 3.2.

**Step 3.** In Section 4, two of the main critical values of \( L_p(r) \), \( p \in (1, \infty) \), will be converted into singular integrals and many fundamental estimates will be given by exploiting typical techniques in qualitative theory of ODEs. The main results of this section are Lemmas 4.12 and 4.13 where some asymptotical comparisons are given for these two critical values. Our concern is also on the uniformity of these estimates in \( p \) which is close to 1. It is also observed that \( L_p(r) \) has constant minimal potentials when the radius \( r \) is small, while the minimal potentials have minimal period 1 when the radius \( r \) is large enough. See Propositions 3.5 and 4.14. Since we are not able to give a complete comparison between these two critical values, the estimates there look complicated.

**Step 4.** In Section 5, we will study \( \lim_{p \downarrow 1} L_p(r) \) and give a complete proof of Theorem 1.2. The explicit expressions (1.5) and (1.6) for \( L_1(r) \) are obtained by finding the limits of the two critical values as \( p \downarrow 1 \).
In Section 6, we will use Theorem 1.2 to obtain some estimates on the smallest periodic eigenvalues. See Corollary 6.3. Examples show that the choice of the $L^1$ norm for potentials will significantly improve the usual results on estimates of eigenvalues. Based on the relation between periodic and Neumann eigenvalues, we will use the scaling technique to give the extremal values of the smallest Neumann eigenvalues of (1.1) for $q$ in $L^1$ balls. See Theorem 6.5.

By exploiting the results and the approaches of this paper, the corresponding extremal values of higher-order periodic, Dirichlet and Neumann eigenvalues in $L^1$ balls will be constructed explicitly in another paper [22]. Again, these extremal values are also elementary functions of $r$.

We remark that Theorem 1.2 has given not only the minimal value of $\lambda_0(q)$ for $q$ in the $L^1$ ball $B_1[r]$, but also the uniform minimal value of $\lambda_0(q)$ for $q$ in all $L^p$ balls $B_p[r]$, $p \in [1, \infty]$, of the same radius $r$. That is,

$$\inf \{\lambda_0(q) : q \in B_p[r], \ 1 \leq p \leq \infty \} = Z_0^{-1}(r).$$

2. General results on eigenvalues and extremal values

2.1. A topological fact on $L^p$ spaces

Denote

$$S_p[r] := \{ q \in L^p : \|q\|_p = r \}$$

the $L^p$ sphere of radius $r$. We have the following topological fact.

**Lemma 2.1.** Given $r > 0$. Let $q \in S_1[r]$. Then there exists $q_p \in S_p[r]$ such that $\lim_{p \downarrow 1} \|q_p - q\|_1 = 0$.

**Proof.** Let $q \in L^1$ with $\|q\|_1 = r > 0$. Define a family of measurable functions

$$q_p(t) = r^{1/p^*} |q(t)|^{1/p} \cdot \text{sign}(q(t)), \quad p \in (1, \infty).$$

Then $q_p \in L^p \subset L^1$. Note that $\|q_p\|_p = r^{1/p^*} \|q\|_1^{1/p} = r$. One has

$$|q_p(t) - q(t)| \leq |q(t)| + (1 + r)(1 + |q(t)|) \in L^1, \quad p \in (1, \infty).$$

Moreover, for any $t$ fixed, one has $\lim_{p \downarrow 1} |q_p(t) - q(t)| = 0$. Now the Lebesgue dominated convergence theorem shows that

$$\lim_{p \downarrow 1} \|q_p - q\|_1 = \lim_{p \downarrow 1} \int_0^1 |q_p(t) - q(t)| \, dt = 0.$$

That is, as $p \downarrow 1$, $q_p$ converges to $q$ in $(L^1, \| \cdot \|_1)$. \qed

The Hölder inequality $\|q\|_{p_1} \leq \|q\|_{p_2}$, $1 \leq p_1 \leq p_2 \leq \infty$, implies that $B_{p_2}[r] \subset B_{p_1}[r]$, which, in turn, implies the monotonicity of $L_p(r)$ in exponent $p$

$$1 \leq p_1 \leq p_2 \leq \infty \quad \Rightarrow \quad L_{p_1}(r) \subset L_{p_2}(r). \tag{2.1}$$

Note that, at this moment, we do not know whether $L_1(r) > -\infty$ for all $r > 0$. 

Lemma 2.2. For any \( r \geq 0 \), one has the limiting equality (1.7).

Proof. By (2.1), let us introduce

\[
L(r) := \lim_{p \uparrow 1} L_p(r) = \inf_{p \in (1, \infty)} L_p(r) \in [\infty, 0].
\]

We have \( L_1(r) \leq L(r) \), following from (2.1). Conversely, let \( q \in S_1[r] \). We have \( \lim_{p \downarrow 1} q_p = q \) in \( (L^1, \| \cdot \|_1) \). By the continuity of eigenvalues in the usual \( L^1 \) topology, we have

\[
\lambda_0(q) = \lim_{p \downarrow 1} \lambda_0(q_p) \geq \lim_{p \downarrow 1} L_p(r) = L(r) = L(\|q\|_1).
\]

Thus

\[
L_1(r) = \inf_{q \in B_1[r]} \lambda_0(q) \geq \inf_{q \in B_1[r]} L(\|q\|_1) \geq L(r),
\]

because \( \|q\|_1 \leq r \) and \( L(\cdot) \) is non-increasing. Hence (1.7) is always true. \( \square \)

2.2. Scaling and differentials of eigenvalues

Let \( q \in L^p(S_T) \), where \( S_T = \mathbb{R}/T \mathbb{Z} \) and \( 1 \leq p \leq \infty \), with the \( L^p \) norm denoted by \( \|q\|_{p,T} = \|q\|_{L^p(S_T)} \). We use \( \lambda_{0,T}(q) \) to denote the smallest \( T \)-periodic eigenvalue of (1.1). One has the following simple observation on \( \lambda_{0,T}(q) \).

Lemma 2.3. Let \( \lambda \in \mathbb{R} \) be a \( T \)-periodic eigenvalue of (1.1) and \( E(t) \) be a corresponding eigen-function. Then \( \lambda = \lambda_{0,T}(q) \) if and only if \( E(t) \) has no any zero. Moreover, \( \lambda_{0,T}(q) \) is simple.

Let \( E_T(t) \) be an eigen-function associated with \( \lambda_{0,T}(q) \)

\[
E_T'' + (\lambda_{0,T}(q) + q(t))E_T = 0.
\]

Suppose that \( S > 0 \). Define the \( S \)-periodic function \( E_S(t) = E_T(Tt/S) \). Then \( E_S(t) \) satisfies

\[
E_S'' + ((T/S)^2 \lambda_{0,T}(q) + q_S(t))E_S = 0, \quad q_S(t) := (T/S)^2 q(Tt/S) \in L^p(S_S).
\]

Since \( E_S(t) \) does not change sign, we obtain from Lemma 2.3 the following scaling equalities

\[
\lambda_{0,S}(q_S) = (T/S)^2 \lambda_{0,T}(q), \quad \|q_S\|_{p,S} = (T/S)^{2-1/p} \|q\|_{p,T}.
\]  \hspace{1cm} (2.2)

Let us introduce the minimal values

\[
L_p(r, T) := \inf\{\lambda_{0,T}(q) : q \in L^p(S_T), \|q\|_{p,T} \leq r\}.
\]  \hspace{1cm} (2.3)

where \( T > 0, r \geq 0 \) and \( p \in [1, \infty] \). Inherited from (2.2), we have the following scaling results for \( L_p(r, T) \).

Lemma 2.4. There hold the following equalities

\[
L_p(r, T) = (S/T)^2 L_p((T/S)^{2-1/p}T, S), \quad \forall r, T, S > 0,
\]  \hspace{1cm} (2.4)

\[
L_p(r, T) = T^{-2} L_p(\hat{r}, 1) = T^{-2} L_p(\hat{r}), \quad \hat{r} := T^{2-1/p} r.
\]  \hspace{1cm} (2.5)
Note that the smallest $T$-periodic eigenvalue $\lambda_{0,T}(q)$ is always simple. As a functional of potentials in the space $(L^p(S_T), \| \cdot \|_{p,T})$, it is well known that $\lambda_{0,T}(q)$ is continuously differentiable in $q$. See, for example, [13,16,24]. The differentials of eigenvalues can be computed using the corresponding eigen-functions.

**Lemma 2.5.** (See [13,24].) The differential $\partial_q\lambda_{0,T}(q)$ at $q \in (L^p(S_T), \| \cdot \|_{p,T})$ is given by

$$
\partial_q\lambda_{0,T}(q) = -\frac{d}{d q} \left( E(\cdot; q) \right) = \left( E(\cdot; q) \right) = \left( E(\cdot; q) \right),
$$

(2.6)

where $E(t; q)$ is a normalized $T$-periodic eigen-function associated with $\lambda_{0,T}(q)$

$$
E'' + \left( \lambda_{0,T}(q) + q(t) \right) E = 0, \quad \| E(\cdot; q) \|_{2,T} = 1.
$$

(2.7)

Precisely, formula (2.6) means that $\partial_q\lambda_{0,T}(q)$ is the following bounded linear functional of $(L^p(S_T), \| \cdot \|_{p,T})$

$$
\partial_q\lambda_{0,T}(q) \cdot h = -\int_0^T E^2(t; q) h(t) \, dt, \quad h \in L^p(S_T).
$$

(2.8)

Since $E(t; q)$ does not change sign, we can choose $E(t; q)$ so that $E(t; q) > 0$ for all $t$.

**Remark 2.6.** By (2.6) and (2.8), $\partial_q\lambda_{0,T}(q) \in (L^p(S_T), \| \cdot \|_{p,T})^*$ has the norm

$$
\| \partial_q\lambda_{0,T}(q) \| = \| E^2(\cdot; q) \|_{p^*,T} = \| E(\cdot; q) \|_{2p^*,T}^2.
$$

(2.9)

Here $p^* := p/(p-1) \in [1, \infty]$ is the conjugate exponent. When $p = \infty$, one has $\| \partial_q\lambda_{0,T}(q) \| = 1$ for all $q \in (L^\infty(S_T), \| \cdot \|_{\infty,T})$. See the normalization condition in (2.7). Thus the boundedness of $\lambda_{0,T}(q)$ in bounded subsets of $(L^\infty(S_T), \| \cdot \|_{\infty,T})$ can be obtained. However, when $p \in [1, \infty)$, we have $2p^* > 2$. It seems that the norms (2.9) are not bounded. Hence the boundedness of $L_1(r)$ cannot be obtained by finding upper bounds of differentials (2.6). See Lemma 2.8(i) and Remark 6.2.

As (2.8) defines negative functionals, Lemma 2.5 can yield the following comparison results for eigenvalues $\lambda_{0,T}(q)$.

**Lemma 2.7.** If $q_1(t) \geq q_2(t)$ a.e. $t$, we have $\lambda_{0,T}(q_1) \leq \lambda_{0,T}(q_2)$. In case $q_1(t) \geq q_2(t)$ a.e. $t$ and $q_1(t) > q_2(t)$ for $t$ in a subset of positive measure, we have $\lambda_{0,T}(q_1) < \lambda_{0,T}(q_2)$.

These results are quite standard in eigenvalue theory. They can be proved using many different approaches. See, for example, [25].

Due to the so-called coexistence [3,9], higher-order periodic and anti-periodic eigenvalues $\lambda_{2m,T}(q)$ and $\lambda_{2m,T}(q)$ are not continuously differentiable at some potentials $q$. This will add the difficulty in finding the corresponding extremal values of these eigenvalues [22].

**2.3. Basic properties for minimal values**

As we have the scaling equalities (2.4) and (2.5), from here we always assume that the period $T$ is 1. By considering constant potentials in $B_p(r)$, we can obtain the following trivial upper bounds of $L_p(r)$.

**Lemma 2.8.**

(i) In case $p = \infty$, $L_\infty(r) = -r$ for all $r \geq 0$.

(ii) In case $1 < p < \infty$, $L_p(r) \leq -r$ for all $r \geq 0$.

(iii) In case $p = 1$, $L_1(r) < -r$ for all $r > 0$. 
Proof. Both (i) and (ii) are trivial. For (iii), let
\[ q_r(t) = \frac{r}{2} \quad \text{for } 0 \leq t < 1/2, \quad q_r(t) = 3r/2 \quad \text{for } 1/2 \leq t < 1. \]
Then \( \|q_r\|_L^1 = r \). It follows from (1.3) that \( L_1(r) \leq -\lambda_0(q_r) < -\int_0^1 q_r = -r. \)

In the following, we always assume that \( p \in (1, \infty) \). From Theorem 1.1, for \( r > 0 \) fixed, there always exists some \( q \in B_p[r] \) such that \( \lambda_0(q) = \lambda_p(r) \). Such a \( q \) is called a minimizer of problem (1.2).

By Lemma 2.7, we have the following properties of minimizers.

**Lemma 2.9.** Let \( p \in (1, \infty) \) and \( r > 0 \). Then any minimizer of (1.2) must be non-negative and be on the sphere \( S_p[r] \). Consequently,
\[ L_p(r) = \min \{ \lambda_0(q); \ q \in B_p[r] \} = \min \{ \lambda_0(q); \ q \in S_p[r] \}. \]  

**Proof.** Let \( q \in B_p[r] \) be a minimizer. Define \( q_+(t) = \max(q(t), 0) \). Then \( q_+ \in B_p[r] \) because \( \|q_+\|_p \leq \|q\|_p \leq r \). We assert that \( q(t) = q_+(t) \geq 0 \) a.e. \( t \). Otherwise, we have \( q(t) < q_+(t) \) a.e. \( t \) and \( q(t) < q_+(t) \) on a subset of positive measure. Thus, by Lemma 2.7, \( \lambda_0(q_+) < \lambda_0(q) = L_p(r) \), a contradiction with the definition of \( L_p(r) \).

We assert that \( \|q\|_p = r. \) Otherwise, we have \( \|q\|_p < r \). Let then \( \hat{q} := (r/\|q\|_p)q \). We have \( \hat{q} \in B_p[r] \) and \( \hat{q}(t) \geq q(t) \) and \( \hat{q}(t) > q(t) \) at all \( t \) for which \( q(t) > 0 \). By Lemma 2.7 again, \( \lambda_0(\hat{q}) < \lambda_0(q) = L_p(r) \), a contradiction with the definition of \( L_p(r) \). □

The following property on the minimal values \( L_p(r) \) can be obtained using Theorem 1.1, the sequential compactness of balls \( B_p[r] \) in \( (L^p, \| \cdot \|_p) \) and Lemma 2.7.

**Lemma 2.10.** Let \( p \in (1, \infty) \). Then the function
\[ [0, \infty) \ni r \rightarrow L_p(r) \in (-\infty, 0] \]
is continuous and is non-increasing in \( r \). Moreover, \( L_p(\cdot) \) has \( (-\infty, 0] \) as its range.

The continuity of \( L_p(r) \) in \( r \) follows simply from the continuity of \( \lambda_0(q) \) in \( L^p \) topology. The monotonicity of \( L_p(r) \) with respect to \( r \) will be improved in the next lemma.

**Lemma 2.11.** Let \( q \in L^p, p \in (1, \infty), \) and \( \Lambda(\tau) := \lambda_0(\tau q), \tau > 0. \) Then the function \( \Lambda(\tau)/\tau \) is non-increasing in \( \tau \in (0, \infty). \) In fact, we have the following results.

(i) In case \( q \) is constant, \( \Lambda(\tau) \equiv c\tau \) for some constant \( c. \)
(ii) In case \( q \) is non-constant, \( \Lambda(\tau)/\tau \) is strictly decreasing in \( \tau \in (0, \infty). \)
(iii) Consequently, one has the following stronger monotonicity result on \( L_p(r) \):
\[ \frac{L_p(r_1)}{r_1} \geq \frac{L_p(r_2)}{r_2}, \quad 0 < r_1 < r_2. \]  

In particular, \( r \in [0, \infty) \rightarrow L_p(r) \in (-\infty, 0] \) is a decreasing homeomorphism.

**Proof.** Let \( E(t; \tau q) \) be the corresponding normalized eigen-functions associated with \( \lambda_0(\tau q) = \Lambda(\tau). \) That is,
\[ E''(t; \tau q) + (\Lambda(\tau) + \tau q(t))E(t; \tau q) = 0, \quad \|E(\cdot; \tau q)\|_2 = 1. \]
The function $\Lambda(\tau)$ is differentiable in $\tau$ with the derivative
\[
\frac{d\Lambda(\tau)}{d\tau} = -\int_0^1 q(t)E^2(t; \tau q)\,dt.
\]

See (2.8). Multiplying (2.12) by $E(t; \tau q)$ and then integrating $t$ over $S_1$, we get
\[
e(\tau) := \int_0^1 \left(E'(t; \tau q)\right)^2\,dt = \Lambda(\tau) \int_0^1 \left(E(t; \tau q)\right)^2\,dt + \tau \int_0^1 q(t)(E(t; \tau q))^2\,dt
\]
\[
= \Lambda(\tau) - \tau \frac{d\Lambda(\tau)}{d\tau}.
\]
That is
\[
\frac{d}{d\tau} \frac{\Lambda(\tau)}{\tau} = -\frac{e(\tau)}{\tau^2} \leq 0, \quad \tau \in (0, \infty).
\]
Hence $\Lambda(\tau)/\tau$ is always non-increasing in $\tau \in (0, \infty)$. When $q$ is non-constant, the eigen-functions $E(t; \tau q)$ are also non-constant for $\tau > 0$. Hence $e(\tau) > 0$ for $\tau > 0$, and $\Lambda(\tau)/\tau$ is strictly decreasing in $\tau \in (0, \infty)$. These prove (i) and (ii).

For (iii), let $q \in S_p[1]$. We have now
\[
\lambda_0(r_1 q)/r_1 \geq \lambda_0(r_2 q)/r_2, \quad 0 < r_1 < r_2.
\]
Notice that $r_i q \in S_p[r_i]$. By (2.10), we have
\[
\lambda_0(r_1 q)/r_1 \geq L_p(r_2)/r_2, \quad \forall q \in S_p[1].
\]
Taking the infimum over $q \in S_p[1]$, we obtain inequality (2.11). As $L_p(r) < 0$ for $r \in (0, \infty)$, (2.11) implies that $L_p(r)$ is strictly decreasing in $r$. As $L_p(r)$ is continuous in $r$, $L_p : [0, \infty) \to (-\infty, 0]$ is necessarily a homeomorphism. \qed

Now we can give an important result on minimal problem (2.10).

**Lemma 2.12.** Let $1 < p < \infty$ and $r > 0$ be given. Let $q(\in S_p[r])$ be a minimizer of problem (2.10) and $E(t)$ $(> 0)$ be an eigen-function associated with $\lambda_0(q) = L_p(r)$. Then $E(t)$ is either constant or has 1 as its minimal period.

**Proof.** Note that $E(t)$ is a positive 1-periodic solution of
\[
E'' + (L_p(r) + q(t))E = 0.
\]
Suppose that $E(t)$ is non-constant. As $E(t)$ is 1-periodic, $E(t)$ has the minimal period $1/n$ for some $n \in \mathbb{N}$. By (2.13),
\[
q(t) = -E''(t)/E(t) - L_p(r)
\]
is also $1/n$-periodic. We need to prove that $n = 1$. Otherwise, assume $n > 1$. As did in the scaling result for $L_p(r)$, let
\[ \hat{q}(t) := n^{-2}q(t/n), \quad \hat{E}(t) := E(t/n). \]
Then $\hat{q}(t)$ and $\hat{E}(t)$ are 1-periodic. By (2.13), we can obtain
\[ \hat{E''} + \left( L_p(r)/n^2 + \hat{q}(t) \right) \hat{E} = 0. \]
As $\hat{E}(t) > 0$, one has $\lambda_0(\hat{q}) = L_p(r)/n^2$. Applying Lemma 2.11(ii) to non-constant potential $\hat{q}(t)$, $\lambda_0(\tau \hat{q})/\tau$ is strictly decreasing in $\tau \in (0, \infty)$. In particular, we have
\[ \lambda_0(n^2 \hat{q})/n^2 < \lambda_0(\hat{q}) = L_p(r)/n^2, \quad \text{i.e.,} \quad \lambda_0(n^2 \hat{q}) < L_p(r). \quad (2.14) \]
Note that $n^2 \hat{q} \in \mathcal{L}^p$ and
\[ \left\| n^2 \hat{q} \right\|_p = \left( \int_0^1 |q(t/n)|^p \, dt \right)^{1/p} = \left( n \int_0^{1/n} |q(t)|^p \, dt \right)^{1/p} = \left\| q \right\|_p = r, \]
where the $1/n$-periodicity of $q(t)$ is used. By the definition of $L_p(r)$, it is impossible for the last inequality of (2.14) to be true. \hfill \Box

Note that Lemma 2.11 concerns with some spatial transformations for potentials, while the proof of Lemma 2.12 is to use some temporal transformations of potentials.

Most of the results here for $\lambda_0(q)$ are also true for higher-order eigenvalues of Hill’s operators and Sturm–Liouville operators, after some minor changes for the statements.

3. Variational approach to minimal problems in $L^p$ balls

In the following, we always assume that $T = 1$, $p \in (1, \infty)$ and $r \in (0, \infty)$.

3.1. Critical equations

From characterization (2.10), in order to study $L_p(r)$, we need only to consider the following minimal problem
\[ \text{Minimize } \lambda_0(q) \quad \text{subject to } \quad q \in S_p[r]. \quad (3.1) \]
We write the constraint $q \in S_p[r]$ as the following equation
\[ \left\| q \right\|_p^p = \int_0^1 |q(t)|^p \, dt = r^p. \quad (3.2) \]
Since $p \in (1, \infty)$, the functional $q \to \left\| q \right\|_p^p$ is continuously differentiable in $q \in (\mathcal{L}^p, \left\| \cdot \right\|_p)$ and has the differential
\[ \partial_q \left\| q \right\|_p^p = p |q(t)|^{p-2} q(t) = p \phi_p(q(t)). \quad (3.3) \]
Here $\phi_p(x) := |x|^{p-2}x$ is the function in defining the so-called $p$-Laplacian [26].
Lemma 2.12. For the sufficiency, let us notice that

$$E^2(t; q) = c_q \phi_p(q(t)),$$

where $c_q \neq 0$ is the multiplier. Here $E(t) = E(t; q) > 0$ is a normalized eigen-function associated with $\lambda_0(q) = L_p(r)$. As $q(t) \geq 0$, cf. Lemma 2.9, we have necessarily $c_q > 0$ and (3.4) shows that $q(t) = \phi_p(E^2(t)/c_q) > 0$ for all $t$. Let us introduce

$$\mu := -L_p(r) > 0, \quad y(t) := E(t; q)/\sqrt{c_q}.$$

By (3.4), we have

$$q(t) = (y(t))^{2/(p-1)} = (y(t))^{2\rho - 2}, \quad y(t) = (q(t))^{(p-1)/2}.$$  \hfill (3.5)

Note that $y(t)$ is also an eigen-function of $\lambda_0(q) = L_p(r) = -\mu$, i.e., $y(t)$ satisfies

$$y'' - \mu y + q(t) y = 0.$$  \hfill (3.6)

Using the expression of $q(t)$ in (3.5), we know that $y(t)$ satisfies the following nonlinear Schrödinger equation

$$y'' - \mu y + y^{2\rho - 1} = 0.$$  \hfill (3.7)

As $q = y^{2\rho - 2}$, the constraint (3.2) for $q$ is the same as

$$\|y\|^{2\rho/p} = r.$$  \hfill (3.7)

In the following we consider $\mu$ in Eq. (3.6) as a parameter, taking value in $(0, \infty)$.

**Proposition 3.1.** In order that $\mu$ is equal to $-L_p(r)$, it is necessary that Eq. (3.6) has a strictly positive periodic solution $y(t)$ such that (i) $y(t)$ is either constant or has 1 as its minimal period, and (ii) $y(t)$ satisfies (3.7).

Conversely, $-L_p(r)$ is the maximum of those parameters $\mu > 0$ for which Eq. (3.6) has solutions which possess properties (i) and (ii) above.

**Proof.** For the necessity, we need only to notice that property (i) follows immediately from Lemma 2.12. For the sufficiency, let us notice that $L_p(r)$ can be attained by some $q \in S_p[r]$. Now the statement for $-L_p(r)$ is an easy consequence of the necessity. \hfill $\square$

**Remark 3.2.** From the proof of Lemma 2.12, when the minimizer $q \in S_p[r]$ is non-constant, we cannot infer that $q(t)$ has minimal period 1. However, from relation (3.5) between $q(t)$ and $y(t)$, we know from Lemma 2.12 that any non-constant minimizer $q(t)$ of (3.1) must have 1 as its minimal period.

We call (3.6) the critical equation of problem (3.1), expressed in eigen-functions $y(t)$ of the critical potentials $q(t)$. For case $p = 2$, the critical equation for (3.1) is written out in [18] using potentials $q(t)$ themselves. As noted there, the minimizers are related with the Weierstrass functions for $p = 2$.

**Remark 3.3.** When extremal problems for higher-order eigenvalues in $L^p$ balls are considered, we need to consider sign-changing eigen-functions $y(t)$. In this case, Eq. (3.6) shall be replaced by

$$y'' + vy \pm \phi_{2p^*}(y) = 0.$$
See [22]. The PDE counterpart of this equation is the following nonlinear Schrödinger equation

\[ -\Delta u + \nu u \pm \phi_{2p^*}(u) = 0, \]

which has a lot of applications like the flame propagation [10] and the Yamabe problem [19]. Different from the usual bifurcation analysis, we need to study solutions of (3.6) in a quantitative way.

3.2. Critical values

By Proposition 3.1, we need only to consider positive constant solutions and positive periodic solutions of (3.6) of the minimal period 1, where \( \mu > 0 \) is a parameter.

Let \( \mu > 0 \) be given. Obviously, Eq. (3.6) has the unique positive constant solution

\[ y_0^\mu(t) := E_0 + (\mu) = \mu(p - 1)/2. \]

It is an equilibrium of (3.6). Note that \( E_0 = 0 \) can be considered as another equilibrium. Evidently, \( E_0 \) is hyperbolic and \( E_+ (\mu) \) is elliptic, because the linearization equations of (3.6) at \( E_0 \) and \( E_+ (\mu) \) are, respectively,

\[ y'' - \mu y = 0, \quad y'' + 2(p - 1)^{-1} \mu y = 0. \]

Moreover, \( E_+ (\mu) \) is surrounded by a family of non-constant positive periodic solutions, which can be parameterized as \( y_{\mu}(t; a) \), the solution of (3.6) satisfying the initial condition \( (y(0), y'(0)) = (a, 0) \), where \( a \in (0, E_+ (\mu)) \). Let the minimal period of \( y_{\mu}(t; a) \) be denoted by \( T_{p, \mu}(a) \), the period function of (3.6), can be expressed using an integral with singularities. It will be studied in detail in Section 4. As a function of \( a \in (0, E_+ (\mu)) \), it is well known that \( T_{p, \mu}(a) \) is strictly decreasing in \( a \in (0, E_+ (\mu)) \).

Moreover, \( E_+ (\mu) \) is surrounded by a family of non-constant positive periodic solutions, which can be parameterized as \( y_{\mu}(t; a) \), the solution of (3.6) satisfying the initial condition \( (y(0), y'(0)) = (a, 0) \), where \( a \in (0, E_+ (\mu)) \). See Fig. 1 in the next section.

Let the minimal period of \( y_{\mu}(t; a) \) be denoted by \( T_{p, \mu}(a) \), the period function of (3.6), can be expressed using an integral with singularities. It will be studied in detail in Section 4. As a function of \( a \in (0, E_+ (\mu)) \), it is well known that \( T_{p, \mu}(a) \) is strictly decreasing in \( a \in (0, E_+ (\mu)) \).

Moreover, \( E_+ (\mu) \) is surrounded by a family of non-constant positive periodic solutions, which can be parameterized as \( y_{\mu}(t; a) \), the solution of (3.6) satisfying the initial condition \( (y(0), y'(0)) = (a, 0) \), where \( a \in (0, E_+ (\mu)) \). Moreover,

\[ \lim_{a \to 0} T_{p, \mu}(a) = +\infty, \quad \lim_{a \to E_+ (\mu)} T_{p, \mu}(a) = \pi \sqrt{2(p - 1)/\mu}. \]

Here the second limiting period can be derived from the second equation of (3.8).

Now we consider positive periodic solutions of (3.6) of the minimal period 1. In order that (3.6) has such solutions, by (3.9), it is necessary and sufficient that \( \mu > 2(p - 1)^2 =: R_p \).

Under (3.10), there exists a unique \( a \in (0, E_+ (\mu)) \) such that \( T_{p, \mu}(a) = 1 \). That is, \( a = T_{p, \mu}^{-1}(1) \). Now, up to a translation of times, (3.6) has the unique positive periodic solution of the minimal period 1

\[ y_{\mu}^1(t) := y_{\mu}(t; T_{p, \mu}^{-1}(1)). \]

Correspondingly, the potentials are

\[ q_{\mu}^1(t) := \left( y_{\mu}(t) \right)^{2p^* - 2} \quad \text{for } \mu > R_p. \]

Denote

\[ q_{\mu}^0(t) := \left( y_{\mu}(t) \right)^{2p^* - 2} \equiv \mu \quad \text{for } \mu > 0. \]

Now we can introduce the following functions
\[
V_{p}^{0}(\mu) := \|y_{\mu}^{0}\|_{2p^{*}/p}^{2p^{*}/p} = \|q_{\mu}^{0}\|_{p} = \mu \quad \text{for } \mu > 0, \\
V_{p}^{1}(\mu) := \|y_{\mu}^{1}\|_{2p^{*}/p}^{2p^{*}/p} = \|q_{\mu}^{1}\|_{p} \quad \text{for } \mu > R_{p}.
\]

(3.12)

By Proposition 3.1, we have the following characterization on the minimal values \(L_{p}(r)\).

**Proposition 3.4.** Let \(p \in (1, \infty)\) and \(r > 0\). Then \(\mu = -L_{p}(r) > 0\) is determined by one of the following equations

\[
\mu > 0 \quad \text{and} \quad V_{p}^{0}(\mu) = r, \\
\mu > R_{p} \quad \text{and} \quad V_{p}^{1}(\mu) = r.
\]

(3.13) \quad (3.14)

Eq. (3.13) is always solvable and the solution is \(\mu = r\). The first condition in (3.14) comes from (3.10). The solvability of Eq. (3.14) will impose some restriction on \(r\). A numerical simulation to \(V_{p}^{1}(\mu)\) shows that (3.14) is solvable for all \(r > R_{p}\). See Remark 4.15 and Fig. 2. However, a complete answer to solvability of (3.14) is unclear to the author. This will add the difficulty in the obtention of \(L_{p}(r)\) and \(L_{1}(r)\). At this stage, we give a preliminary result on \(L_{p}(r)\) for \(r\) small.

**Proposition 3.5.**

(i) If problem (3.1) has constant minimizers, then \(L_{p}(r) = -r\). Conversely, if \(L_{p}(r) < -r\), then problem (3.1) has non-constant minimizers.

(ii) Suppose that \(p\) and \(r\) satisfy

\[
0 < r \leq R_{p} = 2(p - 1)\pi^{2} \quad \text{and} \quad p \in (1, \infty).
\]

Then problem (3.1) has the constant potential \(q(t) \equiv r\) as its minimizer and therefore \(L_{p}(r) = -r\).

**Proof.** (i) It is obvious.

(ii) By Lemma 2.11, \(L_{p}(\cdot)\) is a decreasing homeomorphism. Note that \(L_{p}(0) = 0\). Thus there exists some \(r_{p} > 0\) such that

\[
0 < r \leq r_{p} \quad \implies \quad 0 < \mu := -L_{p}(r) \leq R_{p}.
\]

(3.15)

In this case, Eq. (3.6) has only the constant 1-periodic solution, we know from Lemma 2.12 and Remark 3.2 that the minimizer of (3.1) is constant. Hence one has the alternative (3.13) and \(L_{p}(r) = \lambda_{0}(r) = -r\) for all \(r \in [0, r_{p}]\). This shows that the largest \(r_{p}\) in (3.15) must satisfy \(r_{p} \geq R_{p}\). In particular, when \(r \leq R_{p}\), we have \(L_{p}(r) = \lambda_{0}(r) = -r\). \(\square\)

4. Singular integral approach to critical values

In order to apply Proposition 3.4 to determine the minimal values \(L_{p}(r)\), it is crucial to have a comparison between solutions of (3.13) and possible solutions of (3.14). This is equivalent to the comparison between critical values \(V_{p}^{0}(\mu)\) and \(V_{p}^{1}(\mu)\) themselves. In this section we will give a detailed analysis on \(V_{p}^{0}(\mu)\) for \(\mu > 0\) and \(V_{p}^{1}(\mu)\) for \(\mu > R_{p}\). Then we will give in Lemma 4.13 some asymptotical comparison results between them.

4.1. Normalization of critical equations

Let \(p \in (1, \infty)\) be given. For critical equation (3.6), we normalize the parameter \(\mu\) and the solution \(y(t)\) as

\[
\mu = \nu^{2}, \quad \nu > 0, \quad y(t) = \nu^{p-1}z(\nu t).
\]

(4.1)
Now the equation for \( z(t) \) is

\[
z'' - z + z^{2p^*-1} = 0, \quad z > 0,
\]

which depends on the exponent \( p \) only. We remark that when sign-changing solutions are considered, Eq. (4.2) reads as

\[
z'' - z + |z|^{2p^*-2}z = 0, \quad z \in \mathbb{R},
\]

whose phase portrait is as in Fig. 1.

Note that (4.1) has transformed the solutions \( y_{\mu}^1(t) \) of (3.6) to the following solutions of (4.2)

\[
z_0^*(t) \equiv 1 \quad \text{for } \nu > 0, \quad z_{\nu}(t) := \nu^{1/p} y_1^{1/\nu}(t/\nu) \quad \text{for } \nu > \sqrt{R_p}.
\]

Now \( z_{\nu}(t) \) is a positive periodic solution of (4.2) of the minimal period \( \nu \). Since \( z_{\nu}(0) \in (0, 1) \), \( z_{\nu}'(0) = 0 \) and \( z_{\nu}(t) \) satisfies the autonomous equation (4.2), \( z_{\nu}(t) \) is even in \( t \) and is symmetric with respect to \( t = \nu/2 \), i.e., \( z_{\nu}(\nu - t) \equiv z_{\nu}(t) \). Thus max, \( z_{\nu}(t) = z_{\nu}(1/2) \in (1, \infty) \).

Now we consider critical values \( V_{\nu}^* p(\mu) \). Under transformations (4.1), we have

\[
V_p^0(\nu^2) = \nu^2, \quad \nu > 0.
\]

By defining equality (3.12) and scaling equalities (4.1), we have

\[
V_p^1(\nu^2) = \left\| y_{\nu^2}^1 \right\|_{2p^*,1}^{2p^*/p} = \left\| y_{\nu^2}^{p-1} z_{\nu}(\nu t) \right\|_{2p^*,1}^{2p^*/p} = \nu^{2-1/p} \left\| z_{\nu} \right\|_{2p^*,\nu}^{2p^*/p}
\]

\[
= \nu^2 \left( \int_0^1 (z_{\nu}(\nu t))^{2p^*/p} \, dt \right)^{1/p} = \nu^{2-1/p} \left\| z_{\nu} \right\|_{2p^*,\nu}^{2p^*/p}
\]

\[
= \nu^2 \left( \left\| z_{\nu} \right\|_{2p^*,\nu}^{2p^*/p} \right)^{1/p} = \nu^2 (E_p(\nu)/\nu)^{1/p} \quad \text{for } \nu > \sqrt{R_p},
\]
where

\[ E_p(v) := \|z_v\|_{2p^*}^{2p^*} = \int_0^v (z_v(t))^{2p^*} \, dt \quad \text{for} \ v > \sqrt{R_p}. \]

That is, \( E_p(v) \) is related with the \( L^{2p^*} \) norm of the \( v \)-periodic solution \( z_v(t) \). It can be also understood as follows. Multiplying (4.2) by \( z_v(t) \) and then integrating over \( S_v \), we can use the \( v \)-periodicity of \( z_v(t) \) to obtain

\[ E_p(v) = \int_0^v (z_v(t))^{2p^*} \, dt = \int_0^v ((z'_v(t))^2 + (z_v(t))^2) \, dt. \quad (4.4) \]

That is, \( E_p(v) \) can also be computed using the \( H^1(S_v) \) norm of \( z_v(t) \).

4.2. Critical values of constant critical potentials

Eq. (4.2) has a hyperbolic equilibria \( z = 0 \) and an elliptic equilibrium \( z = 1 \). The first integral is

\[ z'^2 = F_p(z) - F_p(a), \quad a = z(0), \quad (4.5) \]

where

\[ F_p(x) := x^2 - (p^*)^{-1} x^{2p^*} \quad \text{for} \ x > 0. \]

As a function of \( x \), \( F_p(x) \) has different convexity on \((0, (p^*)^{-(p-1)/2})\) and \(( (p^*)^{-(p-1)/2}, \infty) \). This adds to the difficulty in the following estimates.

For any \( a \in (0, 1) \), the solution \( z(t) = z(t; a) \) of (4.2), satisfying \((z(0), z'(0)) = (a, 0)\), is a positive periodic solution with the minimal period denoted by \( T_p(a) \). From Eq. (4.5), let \( b = b_p(a) > 1 \) be determined by

\[ F_p(b) = F_p(a). \]

Then the minimal period \( T_p(a) \) is given by the following singular integral

\[ T_p(a) = 2 \int_a^b \frac{dx}{\sqrt{F_p(x) - F_p(a)}} \quad (> \sqrt{R_p}). \]

Note that \( b = b_p(a) \in (1, b_p) \) for all \( a \in (0, 1) \), where \( b_p > 1 \) satisfies \( F_p(b_p) = 0 \). Explicitly,

\[ b_p = (p^*)^{(p-1)/2} > 1. \]

Considering \( b_p \) as a function of \( p \in (1, \infty) \), one has

\[ \lim_{p \downarrow 1} b_p = 1, \quad \sup_{p \in (1, \infty)} b_p = e. \quad (4.6) \]

When \( a \) increases from 0 to 1, \( b_p(a) \) is strictly decreasing from \( b_p \) to 1. For the period function \( T_p(a) \), one has the following standard result.

**Lemma 4.1.** Given \( p \in (1, \infty) \). The period function \( T_p : (0, 1) \to (\sqrt{R_p}, \infty) \) is a decreasing homeomorphism.
In the following we choose $a \in (0, 1)$ as an independent variable and give necessary estimates on $T_p(a)$. At first we give a uniform (in $p \in (1, \infty)$) lower bound for $T_p(a)$.

**Lemma 4.2.** There holds

$$T_p(a) > 2 \log \frac{1 + \sqrt{1 - a^2}}{a} =: T_1(a), \quad \forall a \in (0, 1), \forall p \in (1, \infty). \quad (4.7)$$

**Proof.** Note that for $x \in (a, 1)$,

$$F_p(x) - F_p(a) = x^2 - a^2 - \left(x^{2p^*} - a^{2p^*}\right)/p^* < x^2 - a^2.$$

Thus (4.7) can be obtained as follows

$$\frac{1}{2} T_p(a) = \int_a^1 \frac{dx}{\sqrt{F_p(x) - F_p(a)}} + \int_1^b \frac{dx}{\sqrt{F_p(x) - F_p(a)}}$$

$$> \int_a^1 \frac{dx}{\sqrt{F_p(x) - F_p(a)}}$$

$$> \int_a^1 \frac{dx}{\sqrt{x^2 - a^2}} = \frac{1}{2} T_1(a). \quad \square$$

In order to give upper bounds for $T_p(a)$, let us take $\delta$ so that $0 < a < \delta < 1$. Write

$$\frac{T_p(a)}{2} = \int_a^\delta \frac{dx}{\sqrt{F_p(x) - F_p(a)}} + \int_\delta^1 \frac{dx}{\sqrt{F_p(x) - F_p(a)}} + \int_1^b \frac{dx}{\sqrt{F_p(x) - F_p(b)}}$$

$$=: I_1(a) + I_2(a) + I_3(a). \quad (4.8)$$

Note that

$$I_1(a) = \int_a^\delta \frac{dx}{\sqrt{x^2 - a^2} \cdot \sqrt{1 - G(x)}}, \quad G(x) = x^{2p^*} - a^{2p^*}/p^*(x^2 - a^2).$$

For any $x \in (a, \delta)$, there exists $\theta_x \in (a^2, x^2)$

$$G(x) = \theta_x^{p^* - 1} \leq x^{2p^* - 2} \leq \delta^{2p^* - 2} \leq \delta^{2/(p - 1)}.$$

Thus

$$I_1(a) < \frac{1}{\sqrt{1 - \delta^{2/(p - 1)}}} \int_a^\delta \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= \frac{1}{\sqrt{1 - \delta^{2/(p - 1)}}} \log \frac{\delta + \sqrt{\delta^2 - a^2}}{a} =: A_1(a, p, \delta). \quad (4.9)$$
As $F_p(x)$ is increasing in $x \in (\delta, 1)$,

$$I_2(a) \leq \frac{1 - \delta}{\sqrt{F_p(\delta) - F_p(a)}} =: A_2(a, p, \delta). \tag{4.10}$$

As $F_p(x)$ is concave in $x \in (1, b)$,

$$F_p(x) - F_p(b) > \frac{F_p(1) - F_p(b)}{b - 1}(b - x), \quad x \in (1, b),$$

and, consequently,

$$I_3(a) = \int_1^b \frac{dx}{\sqrt{F_p(x) - F_p(b)}}< \left( \frac{b - 1}{F_p(1) - F_p(b)} \right)^{1/2} \int_1^b \frac{dx}{\sqrt{b - x}} = \frac{2(b - 1)}{\sqrt{F_p(1) - F_p(a)}}< \frac{2(b_p - 1)}{\sqrt{F_p(1) - F_p(a)}} =: A_3(a, p). \tag{4.11}$$

We conclude from (4.8)–(4.11) the following upper bounds for $T_p(a)$.

**Lemma 4.3.** Suppose that $p \in (1, \infty)$ and $0 < a < \delta < 1$. One has the following upper bound of $T_p(a)$

$$T_p(a) < 2(A_1(a, p, \delta) + A_2(a, p, \delta) + A_3(a, p)). \tag{4.12}$$

From (4.7) and (4.12), we have the following asymptotical results.

**Lemma 4.4.**

(i) Given $p \in (1, \infty)$. There holds

$$\lim_{a \downarrow 0} \frac{T_p(a)}{2 \log(1/a)} = 1. \tag{4.13}$$

(ii) Let $T_1(a)$ be as (4.7). Then, for any closed interval $J \subset (0, 1)$, the following limit is uniform in $a \in J$

$$\lim_{p \downarrow 1} T_p(a) = T_1(a). \tag{4.14}$$

**Proof.** (i) Let $p \in (1, \infty)$ be fixed. Take any $\delta \in (0, 1)$. When $a \downarrow 0$, we obtain from (4.7) and (4.9)–(4.11) the following estimates

$$T_1(a) = 2 \log(1/a) + O(1),$$

$$2A_1(a, p, \delta) = \frac{1}{\sqrt{1 - \delta^{2/(p-1)}}} \cdot 2 \log(1/a) + O(1),$$

$$A_2(a, p, \delta) = \frac{1}{\sqrt{1 - \delta^{2/(p-1)}}} \cdot 2 \log(1/a) + O(1),$$

and

$$A_3(a, p) = \frac{2(b_p - 1)}{\sqrt{F_p(1) - F_p(a)}} = \frac{2(b_p - 1)}{\sqrt{F_p(1) - F_p(a)}} = \frac{2(b_p - 1)}{\sqrt{F_p(1) - F_p(a)}}.$$
\[ 2A_2(a, p, \delta) = O(1), \]
\[ 2A_3(a, p) = O(1). \]

Now (4.7) and (4.12) imply
\[ 1 \leq \liminf_{a \downarrow 0} \frac{T_p(a)}{2 \log(1/a)} \leq \limsup_{a \downarrow 0} \frac{T_p(a)}{2 \log(1/a)} \leq \frac{1}{\sqrt{1 - \delta^2/(p-1)}}. \]

By letting \( \delta \downarrow 0 \), we get (4.13).

(ii) Let us first prove that (4.14) is true when \( a \in (0, 1) \) is given. To this end, we can use the following limit
\[ \lim_{p \uparrow 1} F_p(x) = \lim_{p \uparrow 1} (x^2 - x^{2p}/p^a) = x^2 \text{ for all } x \in (0, 1). \]

(4.15)

Taking \( \delta \in (a, 1) \), we have inequality (4.12). By (4.15), we can obtain
\[ \lim_{p \uparrow 1} A_1(a, p, \delta) = \log \frac{\delta + \sqrt{\delta^2 - a^2}}{a}, \quad \lim_{p \uparrow 1} A_2(a, p, \delta) = \frac{1 - \delta}{\sqrt{\delta^2 - a^2}}. \]

(4.16)

By (4.6), (4.11) and (4.15), we have
\[ \lim_{p \uparrow 1} A_3(a, p) = 0. \]

(4.17)

Consequently, we get from (4.7) and (4.12) that
\[ T_1(a) \leq \liminf_{p \uparrow 1} T_p(a) \leq \limsup_{p \uparrow 1} T_p(a) \leq 2 \left( \log \frac{\delta + \sqrt{\delta^2 - a^2}}{a} + \frac{1 - \delta}{\sqrt{\delta^2 - a^2}} \right). \]

Letting \( \delta \uparrow 1 \), we obtain (4.14).

For \( a \in J \), in the proof above, one can take any \( \delta \in (\max\{a: a \in J\}, 1) \). The argument above shows that (4.14) is actually uniform in \( a \in J \).

In order to study the limit \( \lim_{p \uparrow 1} L_p(r) \), we will give a result on \( T_p(a) \) when \( a \uparrow 1 \). The following basic result in calculus can simplify some arguments below.

Lemma 4.5 (Dini theorem). Let \( \Omega \) be a compact metric space. Suppose that \( \{\Phi_n(x)\}_{n \in \mathbb{Z}^+} \) is a sequence of real-valued, continuous functions on \( \Omega \) such that

- for any \( x \in \Omega \), the real sequence \( \{\Phi_n(x)\}_{n \in \mathbb{N}} \) is decreasing in \( n \in \mathbb{N} \), i.e., \( \Phi_n(x) \leq \Phi_{n+1}(x) \) for all \( n \in \mathbb{N} \), and
- \( \lim_{n \to \infty} \Phi_n(x) = \Phi_0(x) \) for each \( x \in \Omega \).

Then, as \( n \to \infty \), \( \Phi_n(x) \) is uniformly convergent to \( \Phi_0(x) \) in \( x \in \Omega \).

Recall that \( T_p(a) \) is defined for \( (p, a) \in (1, \infty) \times (0, 1) \). In order to apply the Dini theorem, we will extend \( T_p(a) \) to \( (p, a) \in [1, \infty) \times (0, 1] \), still denoted by \( T_p(a) \). Note that, for any \( p \in (1, \infty) \) fixed, \( T_p(a) \) is decreasing in \( a \in (0, 1) \). By (3.9), define
\[ T_p(1) := \lim_{a \uparrow 1} T_p(a) = \pi \sqrt{2(p-1)}, \quad p \in (1, \infty). \]
Using the limit (4.14), $T_p(a)$ is also meaningful for $p = 1$ and $a \in (0, 1)$. Suggested by (4.18), one can define $T_1(1) = 0$. It is easy to see that for any $a \in (0, 1]$ fixed, $T_p(a)$ is a continuous function of $p \in [1, \infty)$.

**Lemma 4.6.** Let $p_0 > 1$ be given. As $a \uparrow 1$, $T_p(a)$ is uniformly convergent to $T_p(1)$ in $p \in \Omega = [1, p_0]$. That is, for any $\varepsilon > 0$, there exists $\Delta = \Delta_{p_0, \varepsilon} \in (0, 1)$ such that

$$a \in (\Delta, 1) \implies \max_{p \in [1, p_0]} |T_p(a) - T_p(1)| < \varepsilon. \quad (4.19)$$

**Proof.** It suffices to prove (4.19) for any sequence $a_n \in (0, 1)$ such that $a_n \uparrow 1$. To this end, let

$$\Phi_0(p) := T_p(1), \quad \Phi_n(p) := T_p(a_n), \quad n \in \mathbb{N}, \ p \in \Omega.$$

Then $\{\Phi_n\}_{n \in \mathbb{N}}$ is a sequence of continuous functions of $p \in \Omega$. By Lemma 4.1, $\Phi_n(p)$ is decreasing in $n \in \mathbb{N}$. Now (4.18) shows that $\lim_{n \to \infty} \Phi_n(p) = \Phi_0(p)$ for each $p \in (1, p_0]$. At $p = 1$, we have

$$\lim_{n \to \infty} \Phi_n(1) = \lim_{n \to \infty} T_1(a_n) = \lim_{n \to \infty} 2 \log \frac{1 + \sqrt{1 - a_n^2}}{a_n} = 0 = \Phi_0(1).$$

Now (4.19) follows from the Dini theorem. \(\Box\)

As a consequence of the preceding results, we can obtain the continuity of the extended period function $T_p(a)$.

**Lemma 4.7.** The function $T_p(a) : [1, \infty) \times (0, 1] \to \mathbb{R}$ is jointly continuous in $(p, a)$.

**Proof.** Let $(p_n, a_n) \to (p_\infty, a_\infty)$ in $[1, \infty) \times (0, 1]$. In case $p_\infty \in (1, \infty)$ and $a_\infty \in (0, 1)$, one has trivially

$$\lim_{n \to \infty} T_{p_n}(a_n) = T_{p_\infty}(a_\infty). \quad (4.20)$$

In case $p_\infty = 1$ and $a_\infty \in (0, 1)$, we have

$$|T_{p_n}(a_n) - T_{p_\infty}(a_\infty)| \leq |T_{p_n}(a_n) - T_1(a_n)| + |T_1(a_n) - T_1(a_\infty)| \to 0,$$

because the uniform convergence (4.14) shows that the first term tends to 0 and the continuity of $T_1(a)$ in $a$ shows that the second term also tends to 0. That is, (4.20) is also true for this case.

In case $p_\infty \in [1, \infty)$ and $a_\infty = 1$, we have

$$|T_{p_n}(a_n) - T_{p_\infty}(a_\infty)| \leq |T_{p_n}(a_n) - T_{p_\infty}(a_n)| + |T_{p_\infty}(a_n) - T_{p_\infty}(1)|.$$

As $a_n \to 1$, (4.19) shows that the first term tends to 0. As $T_{p_\infty}(a)$ is continuous in $a \in (0, 1]$, the second term tends to 0 as well. We have proved (4.20) for all cases. \(\Box\)

From Lemma 4.7, we have the following results.

**Lemma 4.8.** Given $p_0 > 1$. The following convergence results are uniform in $p \in (1, p_0]$

$$\lim_{v \uparrow \infty} T_p^{-1}(v) = 0, \quad (4.21)$$

$$\lim_{a \downarrow 0} T_p(a) = +\infty. \quad (4.22)$$
\textbf{Proof.} If (4.21) is false, there would exist $0 < \epsilon_0 < 1$, $p_n \in (1, p_0]$ and $v_n \rightarrow \infty$ such that $a_n := T_p^{-1}(v_n) \geq \epsilon_0$ for all $n$. That is, for all $n$,

$$v_n \leq T_{p_n}(\epsilon_0) \leq \max_{p \in [1, p_0]} T_p(\epsilon_0) < \infty,$$

following from Lemma 4.7. It is a contradiction with the assumption $v_n \rightarrow \infty$.

If (4.22) is false, there would exist $C > 1$, $p_n \in (1, p_0]$ and $a_n \downarrow 0$ such that $v_n := T_{p_n}(a_n) \leq C$ for all $n$. Without loss of generality, we may assume that $p_n \rightarrow p_\infty \in [1, p_0]$. For all $n > m$, we have $a_n \leq a_m$ and

$$T_{p_n}(a_m) \leq T_{p_n}(a_n) = v_n \leq C.$$

Letting $n \rightarrow \infty$, we know from Lemma 4.7 that

$$T_{p_\infty}(a_m) \leq C, \quad a_m \geq T_{p_\infty}^{-1}(C) > 0.$$

As $m \in \mathbb{N}$ is arbitrary, the latter is a contradiction with the assumption $a_n \downarrow 0$. \qed

\subsection*{4.3. Critical values of non-constant critical potentials}

In the following, we give the corresponding estimates for the function $E_p(\nu)$ defined in (4.4). We still take $a \in (0, 1)$ as an independent variable. The function $E_p(\nu)$ is transformed to

$$U_p(a) := E_p \circ T_p(a)$$

$$= \int_0^{T_p(a)} (z(t; a))^2 \, dt = \int_0^{T_p(a)/2} (z(t; a))^2 \, dt + \int_0^{T_p(a)/2} (z(t; a))^2 \, dt$$

$$= 2 \int_0^b (z(t; a))^2 \, dt = \int_0^{T_p(a)/2} (z(t; a))^2 \, dt + \int_0^{T_p(a)/2} (z(t; a))^2 \, dt$$

$$= 2 \int_a^b \frac{\chi^{2p^*}}{\sqrt{F_p(x) - F_p(a)}} \, dx = 2 \int_a^b \frac{G_p(x)}{\sqrt{F_p(x) - F_p(a)}} \, dx,$$

(4.23)

where $a \in (0, 1), b = b_p(a) \in (1, b_p)$, and

$$G_p(x) := F_p(x) - F_p(a) + x^2 = (2x^2 - a^2) - (\chi^{2p^*} - a^{2p^*})/p^*, \quad x > 0.$$

The last two equalities of (4.23) follow from the change of variables $x = z(t; a)$ for which one has

$$z'(t; a) = \sqrt{F_p(x) - F_p(a)}, \quad dx = z'(t; a) \, dt = \sqrt{F_p(x) - F_p(a)} \, dt.$$

See Eq. (4.5). In order to get better estimates of $U_p(a)$, we will use both integral expressions of (4.23). For simplicity, we always write

$$\nu = T_p(a) > \sqrt{R_p}.$$
Note that the solution $z(t; a)$ satisfies $z(0; a) = a \in (0, 1)$ and $z(v/2; a) = b > 1$. Hence there exists $\tau \in (0, v/2)$ such that $z(\tau; a) = 1$. By Eq. (4.5), we have
\[
z'(\tau; a) = \sqrt{F_p(1) - F_p(a)} = \sqrt{1/p - F_p(a)} =: A_4(a, p).
\end{equation}

Consider Eq. (4.2) on the intervals $[0, \tau]$ and $[\tau, v/2]$ separately. Multiplying Eq. (4.2) by $z(t) := z(t; a)$ and integrating on these intervals, we obtain
\[
\begin{align*}
\int_{0}^{\tau} z^2 p^* dt &= \int_{0}^{\tau} (z^2 + z^2) dt - z(t)z'(t)|_{t=0} = \int_{0}^{\tau} (z^2 + z^2) dt - A_4(a, p), \\
\int_{\tau}^{v/2} z^2 p^* dt &= \int_{\tau}^{v/2} (z^2 + z^2) dt - z(t)z'(t)|_{t=0} = \int_{\tau}^{v/2} (z^2 + z^2) dt + A_4(a, p),
\end{align*}
\]
where $z(\tau) = 1$ and $z'(\tau) = A_4(a, p)$ are used. By the change of variables $z(t; a) = x$, these equalities become
\[
\begin{align*}
\int_{a}^{\tau} \frac{x^2 p^*}{\sqrt{F_p(x) - F_p(a)}} dx &= \int_{a}^{\tau} \frac{G_p(x)}{\sqrt{F_p(x) - F_p(a)}} dx - A_4(a, p), \\
\int_{\tau}^{b} \frac{x^2 p^*}{\sqrt{F_p(x) - F_p(a)}} dx &= \int_{\tau}^{b} \frac{G_p(x)}{\sqrt{F_p(x) - F_p(a)}} dx + A_4(a, p). \tag{4.25}
\end{align*}
\]

From the first expression in (4.23) and equality (4.25), the function $U_p(a)$ can be also written as
\[
\frac{U_p(a)}{2} = \int_{a}^{\tau} \frac{x^2 p^*}{\sqrt{F_p(x) - F_p(a)}} dx + \int_{\tau}^{b} \frac{G_p(x)}{\sqrt{F_p(x) - F_p(a)}} dx + A_4(a, p), \tag{4.26}
\]
where $A_4(a, p)$ is as in (4.24). Here one shall notice that $F_p(a) = F_p(b)$.

Using expressions (4.23) and (4.26) for $U_p(a)$, we can give reasonable lower and upper bounds for $U_p(a)$.

**Lemma 4.9.** Let $0 < a < 1$ and $p \in (1, \infty)$. One has the following lower bound
\[
U_p(a) > \hat{U}_p(a) := 2\sqrt{1-a^2} - \frac{2}{p^*(1-a^2)}. \tag{4.27}
\]

Moreover, the lower bound $\hat{U}_p(a)$ is meaningful when $a$ is small, because $\lim_{a \downarrow 0} \hat{U}_p(a) = 2/p > 0$.

**Proof.** By (4.23), we have
\[
\frac{1}{2} U_p(a) = \int_{a}^{b} \frac{G_p(x)}{\sqrt{x^2 - a^2} - (x^2 p^* - a^2 p^*)/p^*} dx \geq \int_{a}^{b} \frac{G_p(x)}{\sqrt{x^2 - a^2} - (x^2 p^* - a^2 p^*)} dx
\]
\[
= \int_{a}^{b} \frac{2x^2 - a^2}{\sqrt{x^2 - a^2}} dx - \frac{1}{p^*} \int_{a}^{b} \frac{x^2 p^* - a^2 p^*}{\sqrt{x^2 - a^2}} dx.
\]
The first integral is $\sqrt{1-a^2}$. By the convexity of the function $y^{p^*}$,

$$x^{2p^*} - a^{2p^*} = (x^2)^{p^*} - (a^2)^{p^*} < \frac{1-a^{2p^*}}{1-a^2} (x^2 - a^2) < \frac{x^2 - a^2}{1-a^2}$$

for all $x \in (a, 1)$. Thus the second integral is

$$\int_a^1 \frac{x^{2p^*} - a^{2p^*}}{\sqrt{x^2-a^2}} \, dx < \frac{1}{1-a^2} \int_a^1 \sqrt{x^2-a^2} \, dx < \frac{1}{1-a^2}.$$ 

We have now the lower bound (4.27). □

Note that the function $\hat{U}_p(a)$ is well defined for all $a \in (0, 1)$. However, the lower bound (4.27) is useless for $a$ near 1 because $\lim_{a \to 1} \hat{U}_p(a) = -\infty$. We will use (4.27) mainly for $p$ to be near 1.

For the upper bounds of $U_p(a)$, as did for $T_p(a)$, we write, for $0 < a < \delta < 1,$

$$\frac{1}{2} U_p(a) = \int_a^\delta \frac{x^{2p^*}}{F_p(x) - F_p(a)} \, dx + \int_\delta^1 \frac{x^{2p^*}}{F_p(x) - F_p(a)} \, dx + \int_1^b \frac{G_p(x)}{F_p(x) - F_p(b)} \, dx + A_4(a, p)$$

$$=: J_1(a) + J_2(a) + J_3(a) + A_4(a, p).$$

It is trivial that

$$J_1(a) \leq \delta^{2p^*} I_1(a), \quad J_2(a) \leq I_2(a), \quad J_3(a) \leq A_5(a, p) I_3(a),$$

where

$$A_5(a, p) = \max_{x \in [1, b_p(a)]} G_p(x) = \max_{x \in [1, b_p(a)]} \left( \frac{(2x^2 - a^2) - (x^{2p^*} - a^{2p^*})/p^*}{p^*} \right)$$

$$< 2 \left( b_p(a) \right)^2 < 2b_p^2 < 2e < 6, \quad \forall p \in (1, \infty).$$

See (4.6). Using the upper bounds (4.9)–(4.11) for $I_1(a)$, we have the following result.

**Lemma 4.10.** Suppose that $p \in (1, \infty)$ and $0 < a < \delta < 1$. Then one has

$$U_p(a) < 2(\delta^{2p^*} A_1(a, p, \delta) + A_2(a, p, \delta) + 6A_3(a, p) + A_4(a, p)).$$  \hspace{1cm} (4.28)

From Lemmas 4.9 and 4.10, we can derive some asymptotical results for $U_p(a)$.

**Lemma 4.11.** Let $J \subset (0, 1)$ be any closed interval. Then the following limit is uniform in $a \in J$

$$\lim_{p \downarrow 1} U_p(a) = U_1(a) := 2\sqrt{1-a^2}.$$  \hspace{1cm} (4.29)

**Proof.** As in the proof of Lemma 4.4, we need only to prove (4.29) for any $a \in (0, 1)$ fixed.

From (4.27), we get $\liminf_{p \downarrow 1} U_p(a) \geq U_1(a)$. On the other hand, given $a \in (0, 1)$, we fix $\delta \in (a, 1)$. Then, for $p$ sufficiently near 1, one has $a\sqrt{p} < \delta < 1$. Recall that $\lim_{p \downarrow 1} A_i, \ 1 \leq i \leq 3,$ are given by (4.16) and (4.17). Note that $\lim_{p \downarrow 1} A_4(a, p) = \sqrt{1-a^2}$. We get from (4.28)
$$\limsup_{p \downarrow 1} U_p(a) \leq 2 \left( 0 + \frac{1 - \delta}{\sqrt{\delta^2 - a^2}} + 0 + \sqrt{1 - a^2} \right)$$

$$= 2\sqrt{1 - a^2} + \frac{2(1 - \delta)}{\sqrt{\delta^2 - a^2}}$$

for all \(a < \delta < 1\). By letting \(\delta \uparrow 1\), we obtain \(\limsup_{p \downarrow 1} U_p(a) \leq 2\sqrt{1 - a^2} = U_1(a)\). Hence we have (4.29). \(\Box\)

4.4. Comparison between critical values

The critical values \(V_p^n(\mu), n = 0, 1\), are related with \(T_p(a)\) and \(U_p(a)\) in the following way

$$V_p^0(\mu) = (T_p(a))^2, \quad V_p^1(\mu) = (T_p(a))^2 \left( \frac{U_p(a)}{T_p(a)} \right)^{1/p}. \quad (4.30)$$

Lemma 4.12. Given \(p_0 > 1\). Assume that

$$1 < p \leq p_0, \quad 0 < \delta < 1, \quad 0 < a < \delta/\sqrt{2p_0}. \quad (4.31)$$

One has the following estimate

$$\frac{U_p(a)}{T_p(a)} < \delta^{2p_0^*} + \frac{A_{p_0, \delta}}{T_1(a)} \quad A_{p_0, \delta} := 2((7 + 1/\delta)\sqrt{2p_0} + 1). \quad (4.32)$$

Proof. By the lower bound (4.7) for \(T_p(a)\) and the upper bound (4.28) for \(U_p(a)\), we have, for \(0 < a < \delta < 1\),

$$\frac{U_p(a)}{T_p(a)} = \frac{J_1(a)}{I_1(a) + I_2(a) + I_3(a)} + \frac{J_2(a) + J_3(a)}{T_p(a)}$$

$$< \frac{\delta^{2p_0^*}I_1(a)}{I_1(a) + I_2(a) + I_3(a)} + \frac{2(I_2(a) + 6I_3(a) + A_4(a, p))}{T_p(a)}$$

$$< \delta^{2p_0^*} + \frac{2(A_2(a, p, \delta) + 6A_3(a, p) + A_4(a, p))}{T_1(a)}.$$

It is trivial that \(\delta^{2p_0^*} \leq \delta^{2p_0^*}\). Let \(a, \delta\) be as in (4.31). As

$$F_p(\delta) - F_p(a) = \delta^2 - a^2 - \delta^{2p_0^*}/p^* + a^{2p_0^*}/p^*$$

$$> \delta^2 - a^2 - \delta^2/p_0^*$$

$$= \delta^2/p_0 - a^2 > \delta^2/(2p_0),$$

it follows from (4.10) that

$$A_2(a, p, \delta) < (1/\delta - 1)\sqrt{2p_0}.$$

Since \(F_p(1) - F_p(a) > 1/p_0 - a^2 > 1/(2p_0)\), it follows from (4.11) that

$$6A_3(a, p) < 12(b_p - 1)\sqrt{2p_0} < 8\sqrt{2p_0},$$
because sup_{p \in (1, \infty)} 12(b_p - 1) = 12(\sqrt{e} - 1) \approx 7.7847. See (4.6). Moreover, it is trivial that A_4(a, p) < 1. Combining these upper bounds with result (4.7), we obtain (4.32). □

Note that the upper bound (4.32) is independent of p \in (1, p_0].

**Lemma 4.13.** Given p_0 > 1. The following convergence results are uniform in p \in (1, p_0]

\[
\lim_{a \to 0} \frac{U_p(a)}{T_p(a)} = 0, \quad \lim_{\mu \to \infty} \frac{V^1_p(\mu)}{\mu} = 0.
\]

**Proof.** Since \lim_{a \to 0} T_1(a) = +\infty, (4.33) can be obtained simply from (4.32). By relations (4.30) and the uniform results (4.21) and (4.22), it is easy to see that (4.34) is the same as (4.33). □

Asymptotical result (4.34) shows that V^1_p(\mu) < \mu for \mu \gg 1. From this, we can obtain some result on minimal problems (3.1). Let p \in (1, \infty). By (4.34),

\[
r_p := \inf \{ r_0 \geq R_p : V^1_p(\mu) < V^0_p(\mu) = \mu \text{ for all } \mu > r_0 \}
\]

is well defined. By Proposition 3.5, one has r_p \geq R_p = 2(p - 1)\pi^2. The precise meaning of r_p is as follows.

**Proposition 4.14.** Let p \in (1, \infty). If r > r_p, problem (3.1) has non-constant minimizers.

**Proof.** It is not difficult to verify that \lim_{\mu \to \infty} V^1_p(\mu) = +\infty. Suppose that r > r_p. We have V^1_p(r_p +) \leq r_p < r and V^1_p(\mu) > r for \mu \gg 1. Hence there exists \mu_r > r_p such that V^1_p(\mu_r) = r. We have \mu_r > r, because \mu_r \leq r will imply r = V^1_p(\mu_r) < V^0_p(\mu_r) = \mu_r.

Consider then the potential q := q^1_{\mu_r} \in S_p[r]. See (3.11). Now Eq. (3.6) for y := y^1_{\mu_r} can be written as

\[
y'' - \mu_r y + q(t)y = 0.
\]

Since y(t) = y^1_{\mu_r}(t) > 0, we know from Lemma 2.3 that \lambda_0(q) = -\mu_r. Consequently, L_p(r) \leq -\mu_r < -r. Therefore problem (3.1) has non-constant minimizers. □

**Remark 4.15.** We conjecture that r_p = R_p. That is,

\[
V^1_p(\mu) < V^0_p(\mu) = \mu, \quad \forall \mu \in (R_p, \infty).
\]

By relations (4.30), this is completely equivalent to the following inequality for two singular integrals

\[
U_p(a) < T_p(a), \quad \forall a \in (0, 1).
\]

Lemma 4.13 shows that (4.36) is true for \mu \gg 1 and (4.37) is true for a near 0. Numerical simulations show that both (4.36) and (4.37) are true. See Fig. 2. Once these can be proved, some arguments in the proof of Theorem 1.2 can be simplified.
The critical values $V_n^p(\mu)$, $n = 0, 1$, where $p = 11/10$.

5. Limiting approach to minimal values $L_1(r)$

In this section, we use the estimates in the preceding section to give a complete proof of Theorem 1.2.

By (4.27), one has

$$U_p(a) > 2\sqrt{1 - a^2} - \frac{1}{1 - a^2} > \frac{1}{2}, \quad \forall p \in (1, 2], \forall a \in (0, 1/4).$$  \hfill (5.1)

Let $p_0 = 2$ in (4.32). We can get

$$\frac{U_p(a)}{T_p(a)} < \frac{A_2}{T_1(a)}, \quad 1 < p \leq 2, \quad 0 < \delta < 1, \quad 0 < a < \delta/2,$$  \hfill (5.2)

where $A_3 := A_{2,4} = 30 + 4/\delta$.

Given $r \in (0, \infty)$. By Lemma 2.8(iii), one has $-\infty < L_p(r) < -r$. From equality (1.7), there exists some $p_r \in (1, 2]$ such that $-\infty < L_p(r) < -r$ for all $p \in (1, p_r]$. In the following, we always assume that $p \in (1, p_r)$. Since $L_p(r) < -r$, problem (3.1) must have non-constant minimizers. Denote

$$v_p := (-L_p(r))^{1/2} \in (r^{1/2}, \infty), \quad a_p := T_p^{-1}(v_p) \in (0, 1).$$

By Proposition 3.4, $L_p(r)$ is necessarily determined by $V_p^1(L_p(r)) = r$. From (4.30), this is the same as $v_p^{2-1/p}(U_p(a_p))^{1/p} = r$. Thus $v_p$ and $a_p$ satisfy the following system of equations

$$T_p(a_p) = v_p,$$  \hfill (5.3)

$$U_p(a_p) = r^{p}v_p^{1-2p}.$$  \hfill (5.4)

Lemma 5.1. Given $r > 0$. Then $L_1(r) > -\infty$ iff $\inf_{p \in (1, p_r)} a_p > 0$. 

Fig. 2. The critical values $V_n^p(\mu)$, $n = 0, 1$, where $p = 11/10$. 

Proof. For the necessity, suppose that $\mathbf{L}_1(r) > -\infty$. As $\nu_p \leq \nu_1 := (-\mathbf{L}_1(r))^{1/2} < \infty$, it follows from (4.7) and (5.3) that

$$\nu_1 \geq \nu_p = T_p(a_p) > T_1(a_p).$$

Hence $a_p > T_1^{-1}(\nu_1) > 0$ for all $p \in (1, p_1]$.

For the sufficiency, suppose that $a_1 := \inf_{p \in (1, p_1]} a_p > 0$. By equality (5.3) again, we have

$$\sup_{p \in (1, p_1]} \nu_p \leq \max_{(p,a) \in [1, p_1] \times [a_1, 1]} T_p(a) < \infty.$$

See Lemma 4.7. Now (1.7) shows that $\mathbf{L}_1(r) > -\infty$. □

Lemma 5.2. Given $r > 0$. One has $a_1 := \inf_{p \in (1, p_1]} a_p > 0$.

Proof. By eliminating $\nu_p$ in (5.3) and (5.4), we have the following equality

$$\frac{U_p(a_p)}{T_p(a_p)} = \frac{(U_p(a_p))^{(2p)^*}}{r_p/(2p-1)}.$$  \hspace{1cm} (5.5)

Note that $(2p)^* \in (1, 2)$. It is easy to see that

$$D_r := \inf_{u \in (1/2, \infty), p \in (1, 2]} \frac{u^{(2p)^*}}{r_p/(2p-1)} > 0.$$

Define

$$\delta_r := \min(1/2, \sqrt[4]{D_r/2}) \in (0, 1).$$

One sees that

$$D_r - \delta_r^4 \geq D_r/2 > 0.$$

Now we return to the estimate of $a_p$. Let $\delta = \delta_r$ in (5.1) and (5.2). If

$$a_p < a_0 := \min(1/4, \delta_r/2),$$  \hspace{1cm} (5.6)

we know from (5.1) and (5.2) that, for all $p \in (1, p_1] \subset (1, 2],$

$$\frac{U_p(a_p)}{T_p(a_p)} < \delta_r^4 + \frac{A_{\delta_r}}{T_1(a_p)}, \quad \frac{(U_p(a_p))^{(2p)^*}}{r_p/(2p-1)} \geq D_r.$$

By equality (5.5), we obtain

$$\frac{A_{\delta_r}}{T_1(a_p)} > D_r - \delta_r^4 \geq D_r - \frac{D_r}{2}.$$  \hspace{1cm} (5.7)

That is, if $a_p$ satisfies (5.6), we will have $T_1(a_p) < b_0 := 2A_{\delta_r}/D_r$ and $a_p > T_1^{-1}(b_0)$. These arguments show that $\inf_{p \in (1, p_1]} a_p \geq \min(a_0, T_1^{-1}(b_0)) > 0$. □
It follows from Lemmas 5.1 and 5.2 that $L_1(r) > -\infty$ for $r > 0$. We have proved Theorem 1.2(i).

Next we are going to derive formulas (1.5) and (1.6). They are obtained from the limiting analysis of system (5.3)–(5.4) as $p \downarrow 1$.

By Theorem 1.2(i) and (1.7), at this moment, we have

$$
\lim_{p \downarrow 1} \nu_p = \nu_1 := \left( -L_1(r) \right)^{1/2} \in (0, \infty). \tag{5.7}
$$

By Lemma 5.1, one has

$$a_p \in [a_1, 1], \quad \forall p \in (1, p_r].
$$

Here $a_1 \in (0, 1)$. Let $p_n \in (1, p_r]$ be such that $p_n \downarrow 1$. Without loss of generality, we may assume that $\lim_{n \to \infty} a_{p_n} = a_0 \in [a_1, 1]$. As $a_0 > 0$, we can apply the joint continuity of $T_p(a)$ in Lemma 4.7 to obtain

$$
\lim_{n \to \infty} T_{p_n}(a_{p_n}) = T_1(a_0).
$$

By (5.7), we have

$$
\lim_{n \to \infty} T_{p_n}(a_{p_n}) = \lim_{n \to \infty} \nu_n = \nu_1.
$$

Thus

$$
T_1(a_0) = \nu_1. \tag{5.8}
$$

That is, $a_0 = T_1^{-1}(v_1)$ is actually in $(0, 1)$. Since $a_0$ is independent of sequences $\{p_n\}$, we know that

$$
\lim_{p \downarrow 1} a_p = a_0 = T_1^{-1}(v_1) \in (0, 1). \tag{5.9}
$$

Next, as $p \downarrow 1$, the right-hand side of (5.4) has limit $r/\nu_1$. See (5.7). Note that

$$
|U_p(a_p) - U_1(a_0)| \leq |U_p(a_p) - U_1(a_p)| + |U_1(a_p) - U_1(a_0)|.
$$

By (5.9), we know that $a_p \to a_0 \in (0, 1)$. Now the uniform result of Lemma 4.11 shows that $\lim_{p \downarrow 1} |U_p(a_p) - U_1(a_p)| = 0$. As $a_0 \in (0, 1)$ and $U_1(a)$ is continuous in $a \in (0, 1)$, we know from (5.9) that $\lim_{p \downarrow 1} |U_1(a_p) - U_1(a_0)| = 0$. Consequently, as $p \downarrow 1$, the left-hand side of (5.4) tends to $U_1(a_0)$. Thus we have

$$
U_1(a_0) = r/\nu_1. \tag{5.10}
$$

Eliminating $a_0$ from (5.8) and (5.10), we obtain

$$
r = \nu_1 \cdot U_1(T_1^{-1}(v_1)) = \nu_1 \cdot U_1(1/\cosh(\nu_1/2))
= \nu_1 \cdot 2\sqrt{1 - 1/\cosh^2(\nu_1/2)} = \nu_1 \cdot 2 \tanh(\nu_1/2).
$$

This is $r = Z_0(L_1(r))$ by recalling $v_1 = (-L_1(r))^{1/2}$. We have obtained the desired equality (1.6).

We have completed the proof of Theorem 1.2.
**Remark 5.3.** It is important to note that for any $r > 0$, $L_1(r) = \inf[q\in B_1[r]]$ cannot be attained by any $q \in B_1[r]$. In fact, as $p \downarrow 1$, the 'limiting potential' of the minimizers $q_{p,r} \in S_p[r]$ is the Dirac function $\delta_{1/2}$ which is not in the $L^1$ space.

Given $p \in (1, \infty)$. Let us see what is the minimal value $L_p(r)$. By Propositions 3.5 and 4.14, one has a maximal radius

$$\tilde{r}_p := \sup\{r_0 > 0: L_p(r) has constant minimizers for r \in (0, r_0]\}.$$ 

Note that $\tilde{r}_p$ is well defined and $\tilde{r}_p \geq R_p = 2(p - 1)\pi^2$. From $L_1(r) < -r$ and (1.7), it is easy to see that $\lim_{p \downarrow 1} \tilde{r}_p = 0$. Compared with the radius $r_p$ of (4.35), one has $\tilde{r}_p \leq r_p$. We conjecture that $\tilde{r}_p$ is also $R_p$. This is the same as the conjecture in Remark 4.15.

For any $r > \tilde{r}_p$, $L_p(r)$ has non-constant minimizers. Denote $x := L_p(r) < 0$. We have from (5.3) and (5.4)

$$r = (\sqrt{-x})^{2-1/p} \cdot (U_p \circ T_p^{-1}(\sqrt{-x}))^{1/p} := Z_{0,p}(x).$$

Thus

$$L_p(r) = Z_{0,p}^{-1}(r), \quad r > \tilde{r}_p, \quad p \in (1, \infty).$$

As $\lim_{p \downarrow 1} T_p(a) = T_1(a)$ and $\lim_{p \downarrow 1} U_p(a) = U_1(a)$, we obtain

$$\lim_{p \downarrow 1} Z_{0,p}(x) = \sqrt{-x} \cdot U_1 \circ T_1^{-1}(\sqrt{-x}) = Z_0(x), \quad x > 0.$$ 

This gives an explanation for the construction of the function $Z_0(x)$.

**6. Estimates of periodic eigenvalues and extremal Neumann eigenvalues**

**6.1. Estimates of smallest periodic eigenvalues**

Let us see what is the minimal value $L_1(r)$ when $r$ is large. Since

$$Z_0(x) = 2 \sqrt{-x} \tanh(\sqrt{-x}/2) < 2 \sqrt{-x} \quad for \ all \ x < 0,$$

we have always $L_1(r) = Z_0^{-1}(r) < -r^2/4$ for all $r > 0$. On the other hand, as $x \to -\infty$,

$$Z_0(x) = 2 \sqrt{-x} + O(\sqrt{-x}e^{-\sqrt{-x}}).$$

We have the following asymptotical formula for large radius

$$-r^2/4 - o(1) < L_1(r) < -r^2/4 \quad for \ all \ r \gg 1. \quad (6.1)$$

For example, one has

$$-r^2/4 - 0.02 < L_1(r) < -r^2/4 \quad for \ all \ r > 20.$$

The orders of the minimal functions $L_p(r), p \in (1, \infty)$, in large $r$ can be found explicitly.
Corollary 6.1. For any $p \in (1, \infty)$, one has the following order of $L_p(r)$ in $r \gg 1$

$$\lim_{r \uparrow \infty} \frac{-L_p(r)}{r^{(2p)\ast}} = C(p) := \left( \frac{(p - 1)^{p-1}}{p^p B(p, 1/2)} \right)^{2/(2p-1)}.$$  \hfill (6.2)

Here $B(\cdot, \cdot)$ is the Beta function of Euler.

**Proof.** Let $p \in (1, \infty)$ be fixed. When $a \downarrow 0$, we have $b = b_p(a) \uparrow b_p$ and

$$U_p(a) \to 2 \int_0^{b_p} \frac{x^{2p\ast}}{\sqrt{x^2 - x^{2p\ast}/p\ast}} \, dx = 2b_p^{2p\ast} \int_0^1 \frac{x^{2p\ast}}{\sqrt{x^2 - x^{2p\ast}}} \, dx = b_p^{2p\ast} (p - 1) B(p, 1/2) = \frac{p^p B(p, 1/2)}{(p - 1)^{p-1}}.$$  \hfill (6.3)

For $r$ large enough, we know that $\nu_{p, r} := \sqrt{-L_p(r)}$ is determined by

$$T_p(a_{p, r}) = \nu_{p, r} \quad \text{and} \quad (\nu_{p, r})^{2-1/p}(U_p(a_{p, r}))^{1/p} = r.$$  

Thus we have the equality

$$(-L_p(r))/r^{(2p)\ast} = 1/(U_p(a_{p, r}))^{2/(2p-1)}.$$  \hfill (6.4)

As $r \uparrow \infty$, we have $\lim_{r \uparrow \infty} a_{p, r} = 0$, and, by (6.3),

$$\lim_{r \uparrow \infty} U_p(a_{p, r}) = p^p B(p, 1/2)/(p - 1)^{p-1}.$$

Let $r \uparrow \infty$ in (6.4). We get the desired result (6.2). $\square$

The orders (6.2) of $L_p(r)$ in $r \gg 1$ are consistent with the known results for $p = 1$ and $p = \infty$. In fact, when $p = 1$, we know from (6.1) that

$$\lim_{r \uparrow \infty} \left(-L_1(r)/r^2 \right) = 1/4 = C(1).$$

When $p = \infty$, we have $L_\infty(r) \equiv -r$. Note that the coefficients $C(p)$ satisfy

$$\lim_{p \uparrow \infty} C(p) = 1 =: C(\infty).$$

Corollary 6.1 asserts that the minimal functions $L_p(r)$ have the orders $-r^{(2p)\ast}$ when $r \gg 1$, with the coefficients $C(p)$. Note that when $p$ increases from 1 to $\infty$, the power $(2p)\ast$ decreases from 2 to 1, while the coefficient $C(p)$ varies between 0.1974 and 1.

**Remark 6.2.** As noted in Remark 2.6, the boundedness result of $\lambda_0(q)$ in $B_1[r]$ cannot be derived in a direct way from the norms of differentials of $\lambda_0(q)$. This can be explained using (6.2), because, for $p \in [1, \infty)$, $L_p(r)$ grows in $r \gg 1$ in a superlinear way. Hence it is impossible to expect the boundedness of the norms $\|\partial q \lambda_0(q)\| = \|E(\cdot; q)\|^2_{2p^*}$ for $q \in L^p$ when $p \in [1, \infty)$. 


The estimates for eigenvalues of any specific problem and/or a family of problems with a specific class of potentials and/or weights are important in applied sciences. For example, the maximum principle has a close connection with positiveness of the first Dirichlet eigenvalues [2,5]. Some nice works on estimates of eigenvalues are [1,6–8,12,14,26,28]. Some connection between extremal values and optimal control method can be found in [4,21]. By the continuity of eigenvalues in weak topologies in [27] and the main theorem of this paper, the extremal problems do make sense for Sturm–Liouville operators for potentials and/or weights in any bounded subsets of the $L^p$ spaces, $1 \leq p \leq \infty$.

Results (1.3) and (1.6) can yield some nice estimates for the smallest periodic eigenvalues $\lambda_0(q)$ of (1.1). Let $q \in L^1$. Denote $\tilde{q}(t) = q(t) - \bar{q}$ and $\tilde{q}_+(t) = \max(\tilde{q}(t), 0)$. Then $\|\tilde{q}_+\|_1 = \|\tilde{q}\|_1/2$. By the comparison for eigenvalues, one has

$$\lambda_0(q) = -\bar{q} + \lambda_0(\tilde{q}) \geq -\bar{q} + \lambda_0(\tilde{q}_+).$$

From (1.3) and (1.6) we have the following results.

**Corollary 6.3.** There hold the following estimates for $\lambda_0(q)$

$$-\bar{q} + Z_0^{-1}(\|\tilde{q}\|_1/2) \leq \lambda_0(q) \leq -\bar{q}, \quad q \in L^1.$$  \hspace{1cm} (6.5)

**Example 6.4.** (i) Let $q_0(t) = \cos(2\pi t)$. By considering $q_0 \in L^\infty$, we have the following trivial lower bound

$$\lambda_0(q_0) \geq -\|q_0\|_\infty = -1.$$  \hspace{1cm} 

However, if we consider $q_0$ as in $L^1$, since $\bar{q}_0 = 0$, we can use (6.5) to obtain a new lower bound

$$\lambda_0(q_0) \geq Z_0^{-1}(\|q_0\|_1/2) = Z_0^{-1}(1/\pi) \approx -0.3269.$$  \hspace{1cm} 

It is much better.

(ii) Consider $q_n \in L^\infty$, $n \in \mathbb{N}$, which are defined by

$$q_n(t) = \begin{cases} n & \text{for } |t - 1/2| < 1/(2n), \\ 0 & \text{for } 1/(2n) \leq |t - 1/2| \leq 1/2. \end{cases}$$

Note that $\max_t q_n(t) = n$. If the $L^\infty$ norm is used, one can get

$$\lambda_0(q_n) \geq -n, \quad n \in \mathbb{N}.$$  \hspace{1cm} 

By considering $q_n \in L^p$ where $p \in (1, \infty)$, we have $\|q_n\|_p = n^{p-1}$. Thus

$$\lambda_0(q_n) \geq L_p(n^{p-1}) \approx -C(p)n^{(2p)^*(p-1)} \to -\infty \quad \text{as } n \to +\infty.$$  \hspace{1cm} 

See (6.2). Since $\|q_n\|_1 = 1$ for all $n \in \mathbb{N}$, we can get from Theorem 1.2

$$\lambda_0(q_n) \geq Z_0^{-1}(1) \approx -1.0892, \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} 

This is optimal in some sense, because one has

$$\lim_{n \to \infty} \lambda_0(q_n) = Z_0^{-1}(1).$$
These examples show that the choice of metrics for potentials will have an important effect on estimates of eigenvalues.

Let us give some physical explanation to the results of this paper. Consider $q(t)$ as the density of a string. Now eigenvalues of (1.1) represent frequencies of oscillation of the string. Mathematically, for any $p \in [1, \infty]$, the $L^p$ norm $\|q\|_p$ is a measurement for the string. This yields the extremal problems $L_p(r)$ and $M_p(r)$ for the smallest eigenvalues $\lambda_0(q)$. When $q(t) \geq 0$, the $L^1$ norm $\|q\|_1$ is the total mass of the string. Now Theorem 1.2 and result (1.4) have obtained the minimal and maximal values $L_1(r)$ and $M_1(r)$ of $\lambda_0(q)$ for all potentials $q(t)$ with mass $r$. Note that both $L_1(r)$ and $M_1(r)$ are elementary functions of $r$. However, using a mathematical measurement $\|\cdot\|_p$, $p \in (1, \infty)$, for potentials $q$, even though the minimal values $L_p(r)$ can be given using singular integrals, they are not elementary functions of $r$ even for the case $p = 2$.

In this paper we have only considered extremal values of the smallest periodic eigenvalues of (1.1). As mentioned in the introduction, we will study in [22] extremal values of higher-order eigenvalues $\lambda_{km}(q)$ and $\bar{\lambda}_{km}(q)$ for $q$ in $L^1$ balls.

6.2. Extremal values of smallest Neumann eigenvalues

Following from the relations between periodic eigenvalues and eigenvalues of Sturm–Liouville operators [9,25], extremal values $L_p(r)$ and $M_p(r)$ have some connections with other kinds of eigenvalues of (1.1). For example, given $q \in L^p([0,1]) = L^p(S_1) = L^p$, we use $\lambda_0^N(q)$ to denote the smallest Neumann eigenvalue of (1.1). Consider extremal values

$$L_p^N(r) := \inf\{\lambda_0^N(q): \ q \in B_p[r]\}, \quad M_p^N(r) := \sup\{\lambda_0^N(q): \ q \in B_p[r]\}.$$ 

By [25, Theorem 4.3], we have

$$\lambda_0(q) = \max\{\lambda_0^N(q_s): \ s \in \mathbb{R}\} \geq \lambda_0^N(q),$$

where $q_s(t) := q(t + s)$ are translations of $q(t)$. Note that $\|q_s\|_p = \|q\|_p$. Hence $L_p^N(r) \leq L_p(r)$ and $M_p^N(r) \leq M_p(r)$.

For the maximal values $M_p(r)$, we have $M_p^N(r) \leq M_p(r) = r$. Since $-r \in B_p[r]$ and $\lambda_0^N(-r) = r$, we can conclude

$$M_p^N(r) = M_p(r) = r, \quad p \in [1, \infty], \ r \geq 0. \quad (6.6)$$

That is, the maximal values of the smallest Neumann eigenvalues are the same as that of the smallest periodic eigenvalues.

However, the minimal values $L_p^N(r)$ are different from $L_p(r)$. Recall from [27] that $\lambda_0^N : (L^p, \omega_p) \to \mathbb{R}$ is continuous and from [13,24] that $\lambda_0^N : (L^p, \|\cdot\|_p) \to \mathbb{R}$ is continuously differentiable. Moreover, the differential is

$$\partial_t \lambda_0^N(q) = -(E^N(\cdot; q))^2,$$

where $E^N(t; q)$ is an eigen-function associated with $\lambda_0^N(q)$ and satisfies

$$\|E^N(\cdot; q)\|_2 = 1.$$ 

Now we can use the approaches above to obtain the following results.

**Theorem 6.5.** There holds

$$L_p^N(r) = L_p(4r)/4 \quad \text{for all } r \geq 0, \ p \in [1, \infty]. \quad (6.7)$$
In particular, for \( p = 1 \), we have
\[
L^N_p(r) = \hat{Z}_0^{-1}(r), \quad \hat{Z}_0(x) := \sqrt{-x} \tanh \sqrt{-x}, \quad x \leq 0.
\] (6.8)

**Proof.** We only give the sketch. Let \( r > 0 \) be given. For \( p \in (1, \infty) \), \( L^N_p(r) \) has a minimizer \( q_p \in B_p[r] \). Similarly, \( q_p \in S_p[r] \) and \( q_p(t) \geq 0 \). Let us take a normalized eigen-function \( E(t) = E^N(t; q_p), \) \( t \in [0, 1] \), of \( \lambda_N^p(q_p) = L^N_p(r) \) so that \( E(t) > 0 \). By taking some \( y(t) = c_p E(t), \) \( c_p > 0 \), we know that the critical equation for \( y(t) \) is still Eq. (3.6), where \( \mu \) is now \(-L^N_p(r)\). However, in the present case, \( y(t) \) satisfies the Neumann boundary condition
\[
y'(0) = y'(1) = 0.
\] (6.9)

Like Proposition 3.1, it can be proved that \( y(t) \) is either constant or \( t = 1 \) is the minimal positive time so that \( y'(t) = 0 \). This can also be seen from critical equation (3.6) which is autonomous. Using condition (6.9), \( y(t) \) can be extended to the interval \([1, 2] \) by
\[
y(t) := y(2 - t), \quad t \in [1, 2].
\]

In such a way, \( y(t) \) can be understood as a positive periodic solution of (3.6), which is either constant or has 2 as its minimal period. The corresponding potential \( \hat{q}_p(t) = (y(t))^{2p - 2} \) is in \( L^p(S_2) \) and
\[
\|\hat{q}_p\|_{p,2} = 2^{1/p} \|q_p\|_{p,1} = 2^{1/pr}.
\]

Next let us consider the following minimal problem of 2-periodic eigenvalues \( \lambda_{0,2}(q) \)
\[
L_{p,2}(2^{1/pr}) = \inf \{\lambda_{0,2}(q) \colon q \in L^p(S_2), \|q\|_{p,2} \leq 2^{1/pr} \}.
\]

See (2.3). The analysis above shows that the critical problem for \( L^N_p(r) \) is completely the same as that for \( L_p(2^{1/pr}, 2) \). Thus
\[
L^N_p(r) = L_p(2^{1/pr}, 2) = 2^{-2}L_p(\hat{r}), \quad \hat{r} := 2^{-1/p}(2^{1/pr}) = 4r.
\]

See the scaling equality (2.5). This proves (6.7) for \( p \in (1, \infty) \).

When \( p = \infty \), it is trivial that \( L^\infty_p(r) = \lambda_{0,2}^N(r) = -r \). Hence (6.7) is also true for \( p = \infty \).

For the most interesting case \( p = 1 \), one can obtain (6.7) by considering the limit \( \lim_{p \downarrow 1} L^N_p(r) \). By Theorem 1.2, one has
\[
L^N_1(r) = L_1(4r)/4 = Z_0^{-1}(4r)/4.
\]

Denote \( x = L^N_1(r) \). We have
\[
r = \frac{Z_0(4x)}{4} = \frac{2\sqrt{-4x} \tanh \sqrt{-4x}}{4} = \hat{Z}_0(x).
\]

This gives result (6.8). \( \square \)

In conclusion, (6.6) and (6.7) have given the extremal values of the smallest Neumann eigenvalues with potentials in \( L^p \) balls and on \( L^p \) spheres. The answers on extremal values of the Dirichlet and other Neumann eigenvalues in \( L^p \), \( p \in [1, \infty] \), balls will be given in [22].

The extremal values \( L_1(r) = Z_0^{-1}(r) \) and \( M_1(r) = r \) of periodic eigenvalues \( \lambda_0(q) \) and extremal values \( L^N_1(r) = Z_0^{-1}(r) \) and \( M^N_1(r) = r \) of Neumann eigenvalues \( \lambda_0^N(q) \) for \( q \) in \( L^1 \) balls \( B_1[r] \) are plotted in Fig. 3.
Fig. 3. The extremal values $L_{N1}(r) = Z_{N1}^{-1}(r)$, $L_1(r) = Z_1^{-1}(r)$ and $M_1(r) = M_1(r) = r$.

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