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Multiplicity of positive periodic solutions to superlinear repulsive singular equations

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Abstract

In this paper, we study positive periodic solutions to the repulsive singular perturbations of the Hill equations. It is proved that such a perturbation problem has at least two positive periodic solutions when the anti-maximum principle holds for the Hill operator and the perturbation is superlinear at infinity. The proof relies on a nonlinear alternative of Leray–Schauder type and on Krasnoselskii fixed point theorem on compression and expansion of cones.

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1. Introduction

In this paper, we consider the perturbation of the Hill equation

$$x'' + a(t)x = f(t, x). \quad (1.1)$$

The type of perturbations $f(t, x)$ we are mainly interested in is that $f(t, x)$ has a repulsive singularity near $x = 0$ and $f(t, x)$ is superlinear near $x = +\infty$, although the main results of this paper apply also to more general type of perturbations. The problem

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we are interested in is the existence and multiplicity of positive periodic solutions of (1.1), i.e., positive solutions of (1.1) satisfying the periodic boundary condition

$$x(0) = x(1), \quad x'(0) = x'(1). \tag{1.2}$$

(We have fixed the period as 1.)

From the physical explanation, (1.1) has a *repulsive singularity* at $x = 0$ if

$$\lim_{x \rightarrow 0^+} f(t, x) = +\infty \quad \text{uniformly in } t. \tag{1.3}$$

The *superlinearity* of $f(t, x)$ near $x = +\infty$ means that

$$\lim_{x \rightarrow +\infty} f(t, x)/x = +\infty \quad \text{uniformly in } t. \tag{1.4}$$

Such a type of singular equations appears in many problems of applications such as the Brillouin focusing system [1,6,23,24] and nonlinear elasticity [4,5]. It is found recently in [15] that positive periodic solutions of the Ermakov–Pinney equation

$$x'' + a(t)x = x^{-3},$$

which is of the type we are considering, play the fundamental role in studying the twist character and stability of periodic solutions of the scalar Newtonian equations.

Mathematically, Eq. (1.1) is a singular perturbation of a non-autonomous (and therefore, non-integrable) Hill equation. The existence and multiplicity of (strictly) positive periodic solutions of these singular equations have attracted many researches in the recent years. See, for example, [2,5,7,10,12,13,18,20]. See also [16] for an introduction to this topic.

For two classes of these equations, some interesting results have been proved in literature. The first case is the perturbation of superlinear autonomous (and hence, integrable) equations

$$x'' + g(x) = h(t), \tag{1.5}$$

where $g \in C((0, \infty), \mathbb{R})$ has a singularity at 0 satisfying the strong force condition (see (G_3) in Section 4) and is superlinear near $x = +\infty$. In this case, del Pino and Manásevich [4] and Fonda et al. [11] have obtained the existence of infinitely many periodic solutions to (1.5), and the existence of infinitely many subharmonics. These results are obtained by using the Poincaré–Birkhoff Theorem or Moser Twist Theorem [19]. In this sense, the dynamics of singular equations (1.5) are much similar to the regular case.

Another case is the singular forces have semilinear growth near $x = +\infty$. In this case, the singular forces may be non-autonomous (and hence, non-integrable):

$$x'' + g(t, x) = h(t), \tag{1.6}$$

where g satisfies some strong force condition near $x = 0$. del Pino et al. [5] and the third author of the present paper and his collaborators [22–24] have given a detailed study on the existence of at least one positive periodic solution to (1.6) when $g(t, x)$ grows semi-linearly near $x = +\infty$. It is proved in [22] that the periodic and anti-periodic eigenvalues play the same role in these existence results, which reveal some phenomenon different from the regular case. These results are obtained using the coincidence degree theory of Mawhin [16].

Besides the coincidence degree theory used in the existence problems, another tool—the method of upper and lower solutions [3]—is also used in some works mentioned above. These are also basic tools in dealing with regular equations [17].

On the other hand, some fixed point theorems in cones for completely continuous operators have been extensively employed in studying the existence of positive solutions, specially to the separated boundary value problems after the early works [8,9]. However, for the periodic problem, it is difficult to find much references, and only very recently, papers [18,20] are known to us. The reason for this contrast may be due to the fact that it is more difficult to perform a study of the sign of the Green functions for the corresponding linear periodic problems. In paper [20], Torres succeeded in overcoming this difficulty by using a new L^p -anti-maximum principle developed in [21] and obtained some new existence results to problem (1.1)–(1.2).

The aim of this paper is to show that these fixed point theorems in cones can be applied to the periodic problem. Based on the basic results in [20] on the unperturbed non-autonomous equation, i.e., the left-hand side of (1.1) is zero, we will study mainly the non-autonomous superlinear-repulsive perturbations (see (1.3) and (1.4)) and obtain the existence of two different positive periodic solutions to (1.1). See Theorems 3.3 and 4.4. The existence of the first solution is obtained using a nonlinear alternative of Leray–Schauder, and the second one is found using a fixed point theorem in cones.

The remaining part of the paper is organized as follows. In Section 2 some preliminary results will be given. In Section 3, we are devoted to the positone case, i.e., $f(t, x)$ is positive. In this case, we prove that the weak singularity of $f(t, x)$ at $x = 0$ is allowed, as revealed in [18,20]. In Section 4, the semi-positone case, i.e., $f(t, x) \geq -M$ for some $M > 0$, is studied. In this case, some strong force conditions are needed to obtain the existence and multiplicity of positive periodic solutions of (1.1).

Some illustrating examples will be given. In some sense, the results in this paper unify some previous works such as in [20,23,24].

2. Preliminaries and notation

Throughout this paper, we assume that the unperturbed part of (1.1), i.e.,

$$x'' + a(t)x = 0 \tag{2.1}$$

satisfies the following standing hypothesis (A):

- (A) The Green function, $G(t, s)$, associated with the following problem (2.2), is positive for all $(t, s) \in [0, 1] \times [0, 1]$,

where (2.2) is the following non-homogeneous problem:

$$x'' + a(t)x = h(t), \quad x(0) = x(1), \quad x'(0) = x'(1). \tag{2.2}$$

In other words, the (strict) anti-maximum principle holds for (2.2). In this case, the solution of (2.2) is given by

$$x(t) = (\mathcal{L}h)(t) := \int_0^1 G(t, s)h(s) ds. \tag{2.3}$$

In order to guarantee the positivity of $G(t, s)$, it is proved recently in [21] (see also [20]) that if $a(t)$ satisfies $a > 0$ then the positivity of $G(t, s)$ is equivalent to

$$\lambda_1(a) > 0, \tag{2.4}$$

where the notation $a > 0$ means that $a(t) \geq 0$ for all $t \in [0, 1]$ and $a(t) > 0$ for t in a subset of positive measure. The notation $\lambda_1(a)$ denotes the first anti-periodic eigenvalue of

$$x'' + (\lambda + a(t))x = 0 \tag{2.5}$$

subject to the anti-periodic boundary condition

$$x(0) = -x(1), \quad x'(0) = -x'(1). \tag{2.6}$$

Some classes of potentials $a(t)$ for (A) to hold have been found recently in [21]. To describe these, we use $\|\cdot\|_q$ to denote the usual L^q -norm over $(0,1)$ for any given exponent $q \in [1, \infty]$. The conjugate exponent of q is denoted by q^* : $\frac{1}{q} + \frac{1}{q^*} = 1$. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C \|u\|_q^2 \leq \|u'\|_2^2 \quad \text{for all } u \in H_0^1(0, 1).$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \frac{2\pi}{q} \left(\frac{2}{2+q}\right)^{1-2/q} \left(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2} + \frac{1}{q})}\right)^2 & \text{if } 1 \leq q < \infty, \\ 4 & \text{if } q = \infty, \end{cases} \tag{2.7}$$

where Γ is the Gamma function.

Lemma 2.1 (Torres [20]). Assume that $a(t) > 0$ and $a \in L^p[0, 1]$ for some $1 \leq p \leq \infty$. If

$$\|a\|_p < K(2p^*), \tag{2.8}$$

then (2.1) satisfies the standing hypothesis (A), i.e., $G(t, s) > 0$ for all $(t, s) \in [0, 1] \times [0, 1]$.

Remark 2.2. If $p = 1$, the corresponding condition in (2.8) can be weakened as $\|a\|_1 \leq K(\infty) = 4$ by the celebrated stability criterion of Lyapunov. In case $p = \infty$, condition (2.8) reads as $\|a\|_\infty < K(2) = \pi^2$, which is a well-known criterion for the anti-maximum principle yet used in the related literature. In this case, (2.8) can be weakened as $a(t) < \pi^2$.

Under hypothesis (A), we always denote

$$A = \min_{0 \leq s, t \leq 1} G(t, s), \quad B = \max_{0 \leq s, t \leq 1} G(t, s), \quad \sigma = A/B. \tag{2.9}$$

Thus $B > A > 0$ and $0 < \sigma < 1$. We also use $w(t)$ to denote the unique periodic solution of (2.2) with $h(t) = 1$, i.e., $w(t) = (\mathcal{L}1)(t)$. In particular, $A \leq \|w\|_\infty \leq B$.

In the obtention of the second periodic solution of (1.1), we need the following well-known fixed point theorem of compression and expansion of cones [14].

Theorem 2.3 (Krasnosel'skii [14, p. 148]). Let X be a Banach space and $K(\subset X)$ be a cone. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a continuous and compact operator such that either

- (i) $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In applications below, we take $X = C[0, 1]$ with the supremum norm $\|\cdot\|$ and define

$$K = \left\{ x \in X : x(t) \geq 0 \text{ for all } t \text{ and } \min_{0 \leq t \leq 1} x(t) \geq \sigma \|x\| \right\}, \tag{2.10}$$

where σ is as in (2.9).

One may readily verify that K is a cone in X . If $h \in L^1(0, 1)$ with $h(t) \geq 0$ a.e. t , it is easy to see that $\mathcal{L}h \in K$.

Suppose now that $F : [0, 1] \times \mathbb{R} \rightarrow [0, \infty)$ is a continuous function. Define an operator $T : X \rightarrow X$ by

$$(Tx)(t) = \int_0^1 G(t, s)F(s, x(s)) ds \tag{2.11}$$

for $x \in X$ and $t \in [0, 1]$.

Lemma 2.4. *T is well defined and maps X into K. Moreover, T is continuous and completely continuous.*

3. Positone case

In this section we establish the existence and multiplicity of positive solutions to (1.1) in the positone case.

Recall that (1.1) is positone if $f(t, x) > 0$ for all $t \in [0, 1]$ and all $x > 0$. Since we are mainly interested in the repulsive-superlinear nonlinearities (see (1.3) and (1.4)), without loss of generality, we may assume that $f(t, x)$ satisfies

(F₁) For each constant $L > 0$, there exists a function $\phi_L > 0$ such that $f(t, x) \geq \phi_L(t)$ for all $(t, x) \in [0, 1] \times (0, L]$.

Motivated by the nonlinearities like $f(t, x) = b(t)x^{-\alpha} + c(t)x^\beta + e(t)$ and $f(t, x) = b(t)x^{-\alpha} + c(t)e^x + e(t)$, $\alpha, \beta > 0$, we introduce the following assumption on $f(t, x)$:

(F₂) There exist continuous, non-negative functions $g(x)$ and $h(x)$ on $(0, \infty)$ such that

$$f(t, x) \leq g(x) + h(x) \quad \text{for all } (t, x) \in [0, 1] \times (0, \infty)$$

and $g(x) > 0$ is non-increasing and $h(x)/g(x)$ is non-decreasing in $x \in (0, \infty)$.

Theorem 3.1. *Suppose that a(t) satisfies (A) and f(t, x) satisfies (F₁) and (F₂). Suppose further that*

(F₃) *There exists a positive number r such that*

$$\frac{r}{g(\sigma r)(1 + h(r)/g(r))} > \|\omega\|,$$

where σ and $\omega(t)$ are as in Section 2.

Then Eq. (1.1) has at least one positive periodic solution with $0 < \|x\| < r$.

Proof. The existence is proved using the Leray–Schauder alternative principle, together with a truncation technique.

Let $N_0 = \{n_0, n_0 + 1, \dots\}$, where $n_0 \in \{1, 2, \dots\}$ is chosen such that

$$\|\omega\|g(\sigma r)(1 + h(r)/g(r)) + 1/n_0 < r.$$

See (F₃). Fix $n \in N_0$. Consider the family of equations

$$x'' + a(t)x = \lambda f_n(t, x(t)) + a(t)/n, \tag{3.1}$$

where $\lambda \in [0, 1]$ and $f_n(t, x) = f(t, \max\{x, 1/n\})$, $(t, x) \in [0, 1] \times \mathbb{R}$. Problem (3.1)–(1.2) is equivalent to the following fixed point problem in $C[0, 1]$:

$$x = \lambda T_n x + 1/n, \tag{3.2}$$

where T_n denotes the operator defined by (2.11), with $F(t, x)$ replaced by $f_n(t, x)$.

We claim that any fixed point x of (3.2) for any $\lambda \in [0, 1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that x is a solution of (3.2) for some $\lambda \in [0, 1]$ such that $\|x\| = r$. Note that $f_n(t, x) \geq 0$. By Lemma 2.4, for all t , $x(t) \geq 1/n$ and $r \geq x(t) \geq 1/n + \sigma\|x - 1/n\|$. By the choice of n_0 , $1/n \leq 1/n_0 < r$. Hence, for all t ,

$$x(t) \geq 1/n \quad \text{and} \quad r \geq x(t) \geq 1/n + \sigma\|x - 1/n\| \geq 1/n + \sigma(r - 1/n) > \sigma r. \tag{3.3}$$

Using (3.3), we have from condition (F₂), for all t ,

$$\begin{aligned} x(t) &= \lambda \int_0^1 G(t, s) f_n(s, x(s)) ds + 1/n \\ &\leq \int_0^1 G(t, s) f(s, x(s)) ds + 1/n \\ &\leq \int_0^1 G(t, s) g(x(s))(1 + h(x(s))/g(x(s))) ds + 1/n \\ &\leq g(\sigma r)(1 + h(r)/g(r)) \int_0^1 G(t, s) ds + 1/n_0 \\ &\leq \|\omega\|g(\sigma r)(1 + h(r)/g(r)) + 1/n_0. \end{aligned} \tag{3.4}$$

Therefore,

$$r = \|x\| \leq \|\omega\|g(\sigma r)(1 + h(r)/g(r)) + 1/n_0.$$

This is a contradiction to the choice of n_0 and the claim is proved.

From this claim, the nonlinear alternative of Leray–Schauder guarantees that (3.2) (with $\lambda = 1$) has a fixed point, denoted by x_n , in B_r , i.e., Eq. (3.1) (with $\lambda = 1$) has a periodic solution x_n with $\|x_n\| < r$. Since x_n satisfies (3.2), $x_n(t) \geq 1/n$ for all t and x_n is actually a positive periodic solution of (3.1) (with $\lambda = 1$).

Next we claim that these solutions x_n have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$\min_t x_n(t) \geq \delta \tag{3.5}$$

for all $n \in N_0$. To see this, we know from (F₁) that there exists a function $\phi_r > 0$ such that $f(t, x) \geq \phi_r(t)$ for $(t, x) \in [0, 1] \times (0, r]$. Now let $x_r(t)$ be the unique periodic solution to the problem (2.2) with $h = \phi_r(t)$. Then

$$x_r(t) = \int_0^1 G(t, s)\phi_r(s) ds \geq A\|\phi_r\|_1 > 0.$$

So we have

$$\begin{aligned} x_n(t) &= \int_0^1 G(t, s)f_n(s, x_n(s)) ds + 1/n = \int_0^1 G(t, s)f(s, x_n(s)) ds + 1/n \\ &\geq \int_0^1 G(t, s)\phi_r(s) ds + 1/n = x_r(t) + 1/n \geq A\|\phi_r\|_1 =: \delta. \end{aligned}$$

In order to pass the solutions x_n of the truncation equations (3.1) (with $\lambda = 1$) to that of the original equation (1.1), we need the following fact:

$$\|x'_n\| \leq H \tag{3.6}$$

for some constant $H > 0$ and for all $n \geq n_0$. To this end, by the boundary condition (1.2), $x'_n(t_0) = 0$ for some $t_0 \in [0, 1]$. Integrating (3.1) (with $\lambda = 1$) from 0 to 1, we obtain

$$\int_0^1 a(t)x_n(t) dt = \int_0^1 [f_n(t, x_n(t)) + a(t)/n] dt.$$

Then

$$\begin{aligned} \|x'_n\| &= \max_{0 \leq t \leq 1} |x'_n(t)| = \max_{0 \leq t \leq 1} \left| \int_{t_0}^t x_n''(s) ds \right| \\ &= \max_{0 \leq t \leq 1} \left| \int_{t_0}^t [f_n(s, x_n(s)) + a(s)/n - a(s)x_n(s)] ds \right| \\ &\leq \int_0^1 [f_n(s, x_n(s)) + a(s)/n] ds + \int_0^1 a(s)x_n(s) ds \\ &= 2 \int_0^1 a(s)x_n(s) ds < 2r\|a\|_1 =: H. \end{aligned}$$

The fact $\|x_n\| < r$ and (3.6) show that $\{x_n\}_{n \in N_0}$ is a bounded and equi-continuous family on $[0, 1]$. Now the Arzela–Ascoli Theorem guarantees that $\{x_n\}_{n \in N_0}$ has a subsequence, $\{x_{n_k}\}_{k \in \mathbb{N}}$, converging uniformly on $[0, 1]$ to a function $x \in C[0, 1]$. From the fact $\|x_n\| < r$ and (3.5), x satisfies $\delta \leq x(t) \leq r$ for all t . Moreover, x_{n_k} satisfies the integral equation

$$x_{n_k}(t) = \int_0^1 G(t, s)f(s, x_{n_k}(s)) ds + 1/n_k.$$

Letting $k \rightarrow \infty$, we arrive at

$$x(t) = \int_0^1 G(t, s)f(s, x(s)) ds,$$

where the uniform continuity of $f(t, x)$ on $[0, 1] \times [\delta, r]$ is used. Therefore, x is a positive periodic solution of (1.1).

Finally it is not difficult to show that $\|x\| < r$, by noting that if $\|x\| = r$, the argument similar to the proof of the first claim will yield a contradiction. \square

Corollary 3.2. *Let the nonlinearity in (1.1) be*

$$f(t, x) = b(t)x^{-\alpha} + \mu c(t)x^\beta + e(t), \quad 0 \leq t \leq 1, \tag{3.7}$$

where $\alpha > 0$, $\beta \geq 0$, $b(t)$, $c(t)$, $e(t) \in C[0, 1]$ are non-negative functions and $b(t) > 0$ for all t , and $\mu > 0$ is a positive parameter. Then

- (i) if $\beta < 1$, (1.1) has at least one positive periodic solution for each $\mu > 0$; and
- (ii) if $\beta \geq 1$, (1.1) has at least one positive periodic solution for each $0 < \mu < \mu_*$, where μ_* is some positive constant.

Proof. We will apply Theorem 3.1. To this end, assumption (F₁) is fulfilled by $\phi_L(t) = L^{-\alpha} \cdot \min_t b(t)$. To verify (F₂), one may simply take

$$g(x) = b_0x^{-\alpha}, \quad h(x) = \mu c_0x^\beta + e_0,$$

where

$$b_0 = \max_t b(t) > 0, \quad c_0 = \max_t c(t) \geq 0, \quad e_0 = \max_t e(t) \geq 0.$$

Now the existence condition (F₃) becomes

$$\mu < \frac{\sigma^\alpha r^{\alpha+1} / \|w\| - b_0 - e_0 r^\alpha}{c_0 r^{\alpha+\beta}}$$

for some $r > 0$. So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_* := \sup_{r>0} \frac{\sigma^\alpha r^{\alpha+1} / \|w\| - b_0 - e_0 r^\alpha}{c_0 r^{\alpha+\beta}}.$$

Note that $\mu_* = \infty$ if $\beta < 1$ and $\mu_* < \infty$ if $\beta \geq 1$. We have the desired results. \square

Theorem 3.1 applies to the example

$$f(t, x) = b(t)x^{-\alpha} + \mu c(t)e^x + e(t), \quad 0 \leq t \leq 1. \tag{3.8}$$

A result similar to Corollary 3.2(ii) holds for this example.

Theorem 3.1 also applies to regular examples like $f(t, x) = b(t)x^\beta + e(t)$, with $b \geq 0$ and $e > 0$. In particular, if $f(t, x) \equiv e(t)$, $e > 0$, the existence of positive periodic solutions of (1.1) coincides with our standing hypothesis, i.e., the anti-maximum principle holds for (2.2).

Another example is the Brillouin beam focusing equation

$$x'' + a(1 + \cos t)x = x^{-\alpha}.$$

This equation has been widely studied as a model for the motion of a magnetically focused axially symmetric electron beam with Brillouin flow (see [1] for a description of the model). The existence of positive periodic solutions has been considered by many authors. See for example [23,24], where the existence of one positive 2π -periodic solution is proved when $\alpha \geq 1$ and $a > 0$ is less than the first weighted anti-periodic eigenvalue of

$$x'' + \lambda(1 + \cos t)x = 0.$$

The latter condition is equivalent to the standing hypothesis (A) here. In [20], Torres studied the existence in the both strong and weak cases, $\alpha > 0$. Using Theorem 3.1, we can arrive at the same conclusion as in [20].

Next we will find another positive periodic solution to Eq. (1.1) by using Theorem 2.3 for certain nonlinearities.

Theorem 3.3. *Suppose that (A) and (F₁)–(F₃) are satisfied. Furthermore, assume that (F₄) There exist continuous, non-negative functions $g_1(x)$ and $h_1(x)$ on $(0, \infty)$ such that*

$$f(t, x) \geq g_1(x) + h_1(x) \quad \text{for all } (t, x) \in [0, 1] \times (0, \infty)$$

and $g_1(x) > 0$ is non-increasing and $h_1(x)/g_1(x)$ is non-decreasing in $x \in (0, \infty)$; and

(F₅) There exists a positive number $R > r$ such that

$$\frac{R}{\sigma g_1(R)(1 + h_1(\sigma R)/g_1(\sigma R))} \leq \|\omega\|,$$

where σ and $\omega(t)$ are as in Section 2.

Then, besides the periodic solution x constructed in Theorem 3.1, Eq. (1.1) has another positive periodic solution \tilde{x} with $r < \|\tilde{x}\| \leq R$.

Proof. Let $X = C[0, 1]$ and K be the cone in X in Section 2. Let $\Omega_1 = B_r$ and $\Omega_2 = B_R$ be balls in X . The operator $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is defined by (2.11), with $F(t, x)$ replaced by $f(t, x)$. The operator T is well defined on $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Note that any $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ satisfies $0 < \sigma r \leq x(t) \leq R$ by the definition of K .

First we have $\|Tx\| < \|x\|$ for $x \in K \cap \partial\Omega_1$. In fact, if $x \in K \cap \partial\Omega_1$, then $\|x\| = r$. Now the estimate $\|Tx\| < r$ can be obtained almost following the same ideas in proving (3.4). We omit the details.

Next we show that $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$. To see this, let $x \in K \cap \partial\Omega_2$. Then $\|x\| = R$ and $x(t) \geq \sigma R$. As a result, it follows from (F₄) and (F₅) that, for $0 \leq t \leq 1$,

$$\begin{aligned} (Tx)(t) &= \int_0^1 G(t, s)f(s, x(s)) ds \geq \int_0^1 G(t, s)g_1(x(s))(1 + h_1(x(s))/g_1(x(s))) ds \\ &\geq \int_0^1 G(t, s)g_1(R)(1 + h_1(\sigma R)/g_1(\sigma R)) ds \\ &= g_1(R)(1 + h_1(\sigma R)/g_1(\sigma R))\omega(t) \\ &\geq \sigma\|\omega\|g_1(R)(1 + h_1(\sigma R)/g_1(\sigma R)) \geq R = \|x\|. \end{aligned}$$

This implies $\|Tx\| \geq \|x\|$.

Now Theorem 2.3 guarantees that T has a fixed point $\tilde{x} \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Thus $r \leq \|\tilde{x}\| \leq R$. Clearly, \tilde{x} is a positive periodic solution of (1.1) and actually satisfies $\|\tilde{x}\| > r$. \square

Let consider again example (3.7) in Corollary 3.2 for the superlinear case, i.e., $\beta > 1$. We assume also that $c(t) > 0$ for all t . To verify (F₄), one may simply take

$$g_1(x) = b_1x^{-\alpha}, \quad h_1(x) = \mu c_1x^\beta + e_1,$$

where

$$b_1 = \min_t b(t) > 0, \quad c_1 = \min_t c(t) > 0, \quad e_1 = \min_t e(t) \geq 0.$$

Now the existence condition (F₅) becomes

$$\mu \geq \frac{R^{\alpha+1}/(\sigma\|\omega\|) - b_1 - e_1 R^\alpha}{c_1 R^{\alpha+\beta}}. \quad (3.9)$$

Since $\beta > 1$, the right-hand side goes to 0 as $R \rightarrow +\infty$. Thus, for any given $0 < \mu < \mu_*$, where μ_* is as in Corollary 3.2, it is always possible to find such $R \gg r$ that (3.9) is satisfied. Thus, (1.1) has an additional periodic solution \tilde{x} such that $\|\tilde{x}\| > r$.

Corollary 3.4. *Assume in (3.7) that $\beta > 1$ and $b(t) > 0$, $c(t) > 0$ for all t . Then, for each μ with $0 < \mu < \mu_*$, the corresponding equation (1.1) has at least two different positive periodic solutions.*

Such a multiplicity result holds also for the nonlinearity (3.8).

Remark 3.5. In the context of repulsive singularities, it is usual to assume some kind of strong force conditions (see, e.g., (G₃) in the next section), which means roughly that the potential is infinity at 0. Typically, this condition is employed to obtain a priori bounds of periodic solutions. In fact, the strong singularly condition cannot be dropped without further assumptions, and such a condition has become standard in the related literature. Recently, Rachunková et al. [18] and Torres [20] have obtained the existence results in the presence of weak singularities. In our case, we are able to deal also with weak singularities because the strong force conditions are not needed in Theorems 3.1 and 3.3, and their corollaries.

4. Semi-positone case

In this section, we establish the existence and multiplicity of positive periodic solutions of (1.1) in the semi-positone case. By the semi-positone case of (1.1), we mean that $f(t, x)$ may change sign and satisfies

(G₁) There exists a constant $M > 0$ such that $F(t, x) := f(t, x) + M \geq 0$ for all $(t, x) \in [0, 1] \times (0, \infty)$.

Again we assume that $f(t, x)$ is controlled as in Theorem 3.1.

(G₂) $F(t, x) \leq g(x) + h(x)$ for some continuous non-negative functions $g(x)$ and $h(x)$ with the properties that $g(x) > 0$ is non-increasing and $h(x)/g(x)$ is non-decreasing. In the semi-positone case, some strong force of $f(t, x)$ near $x = 0$ is needed. A typical one is:

(G₃) There exists a non-increasing positive continuous function $g_0(x)$ on $(0, \infty)$ and a constant $R_0 > 0$ such that $f(t, x) \geq g_0(x)$ for $(t, x) \in [0, 1] \times (0, R_0]$, where $g_0(x)$ satisfies $\lim_{x \rightarrow 0^+} g_0(x) = +\infty$ and $\lim_{x \rightarrow 0^+} \int_x^{R_0} g_0(u) du = +\infty$.

Theorem 4.1. *Suppose that $a(t)$ satisfies condition (A) and $f(t, x)$ satisfies (G₁)–(G₃). In addition, suppose that*

(G₄) *There exists $r > M\|\omega\|/\sigma$ such that $\frac{r}{g(\sigma r - M\|\omega\|)(1+h(r)/g(r))} > \|\omega\|$, where σ and $\omega(t)$ are the same as in Section 2.*

Then Eq. (1.1) has at least one positive periodic solution with $0 < \|x + M\omega\| < r$.

Proof. Since some parts of the proof are in the same line of that of Theorem 3.1, we will outline the proof with the emphasis on the difference.

Let $N_0 = \{n_0, n_0 + 1, \dots\}$, where $n_0 \in \{1, 2, \dots\}$ is chosen such that $1/n_0 < \sigma r - M\|\omega\|$ and

$$\|\omega\|g(\sigma r - M\|\omega\|)(1 + h(r)/g(r)) + 1/n_0 < r.$$

We will show first that

$$x'' + a(t)x = F(t, x(t) - M\omega(t)) \tag{4.1}$$

has a solution x satisfying (1.2) and $x(t) > M\omega(t)$ for $t \in [0, 1]$, $0 < \|x\| < r$. If this is true, it is easy to see that $u(t) = x(t) - M\omega(t)$ will be a positive solution of (1.1)–(1.2) with $0 < \|u + M\omega\| < r$.

For (4.1), we will consider, for each $n \in N_0$, the family of equations

$$x'' + a(t)x = \lambda F_n(t, x(t) - M\omega(t)) + a(t)/n, \tag{4.2}$$

where $F_n(t, x) := F(t, \max\{1/n, x\})$, $(t, x) \in [0, 1] \times \mathbb{R}$, and the parameter $\lambda \in [0, 1]$.

Problem (4.2)–(1.2) is equivalent to the following fixed point problem:

$$x(t) = \lambda \int_0^1 G(t, s)F_n(s, x(s) - M\omega(s)) ds + 1/n. \tag{4.3}$$

Using the constructions above, it is not difficult to prove that any solution x of (4.3) satisfies $\|x\| \neq r$ by the same argument as in the proof of Theorem 3.1. Using again the nonlinear alternative of Leray–Schauder, Eq. (4.2) (with $\lambda = 1$) has at least one periodic solution x_n for each $n \in N_0$ with the property $\|x_n\| < r$. One has also

$$\|x'_n\| \leq H$$

for some constant $H > 0$ and for all $n \geq n_0$ by the same argument as in the proof of (3.6).

In the next lemma, we will show that there exists a constant $\delta > 0$ such that

$$x_n(t) - M\omega(t) \geq \delta \quad \forall t \in [0, 1] \tag{4.4}$$

for n large enough.

By the standard argument as in the proof of Theorem 3.1, one can now extract a solution x of the fixed point equation

$$x(t) = \int_0^1 G(t, s)F(s, x(s) - M\omega(s)) ds,$$

i.e., a solution of (4.1)–(1.2) with the desired properties: $\|x\| < r$ and $x(t) - M\omega(t) \geq \delta$ for all $t \in [0, 1]$. \square

Lemma 4.2. *There exist a constant $\delta > 0$ and an integer $n_2 > n_0$ such that any solution x_n of (4.2) (with $\lambda = 1$) satisfies (4.4) for all $n \geq n_2$.*

Proof. The lower bound in (4.4) is established using the strong force condition of $f(t, x)$. By condition (G_3) , there exists $R_1 \in (0, R_0)$ and a continuous function \tilde{g}_0 such that

$$F(t, x) - a(t)x \geq \tilde{g}_0(x) > \max\{M, r\|a\|_1\} \tag{4.5}$$

for all $(t, x) \in [0, 1] \times (0, R_1]$, where \tilde{g}_0 satisfies also the strong force condition like in (G_3) .

Choose $n_1 \in N_0$ such that $1/n_1 \leq R_1$ and let $N_1 = \{n_1, n_1 + 1, \dots\}$. For $n \in N_1$, let

$$(0 <) \alpha_n = \min_{0 \leq t \leq 1} [x_n(t) - M\omega(t)] \quad \text{and} \quad \beta_n = \min_{0 \leq t \leq 1} [x_n(t) - M\omega(t)].$$

We claim first that $\beta_n > R_1$ for all $n \in N_1$. Otherwise, suppose that $\beta_n \leq R_1$ for some $n \in N_1$. Then it is easy to verify

$$F_n(t, x_n(t) - M\omega(t)) > r\|a\|_1. \tag{4.6}$$

In fact, if $1/n \leq x_n(t) - M\omega(t) \leq R_1$, we obtain from (4.5)

$$\begin{aligned} F_n(t, x_n(t) - M\omega(t)) &= F(t, x_n(t) - M\omega(t)) \\ &\geq a(t)(x_n(t) - M\omega(t)) + \tilde{g}_0(x_n(t) - M\omega(t)) \\ &\geq \tilde{g}_0(x_n(t) - M\omega(t)) > r\|a\|_1 \end{aligned}$$

and, if $x_n(t) - M\omega(t) \leq 1/n$, we have

$$F_n(t, x_n(t) - M\omega(t)) = F(t, 1/n) \geq a(t)/n + \tilde{g}_0(1/n) \geq \tilde{g}_0(1/n) > r\|a\|_1.$$

Integrating (4.2) (with $\lambda = 1$) from 0 to 1, we deduce that

$$\begin{aligned} 0 &= \int_0^1 [x_n''(t) + a(t)x_n(t) - F_n(t, x_n(t) - M\omega(t)) - a(t)/n] dt \\ &= \int_0^1 a(t)x_n(t) dt - (1/n) \int_0^1 a(t) dt - \int_0^1 F_n(t, x_n(t) - M\omega(t)) dt \\ &< \int_0^1 a(t)x_n(t) dt - r\|a\|_1 \leq 0, \end{aligned}$$

where estimate (4.6) and the fact $\|x_n\| < r$ are used. This is a contradiction. Thus the claim is proved and we have

$$\|x_n - M\omega\| > R_1 \quad \text{for all } n \in N_1. \tag{4.7}$$

Now we consider the minimum values α_n . Let $n \geq n_1$. We have two cases.

Case 1: $\alpha_n \geq R_1$. We have nothing to do when proving (4.4).

Case 2: $\alpha_n < R_1$, i.e.,

$$\alpha_n = \min_{0 \leq t \leq 1} [x_n(t) - M\omega(t)] = x_n(a_n) - M\omega(a_n) < R_1 \tag{4.8}$$

for some $a_n \in [0, 1]$. As $\alpha_n = x_n(a_n) - M\omega(a_n) < R_1$, by (4.7), there exists $c_n \in [0, 1]$ (without loss of generality, we assume $a_n < c_n$) such that $x_n(c_n) = M\omega(c_n) + R_1$ and $x_n(t) \leq M\omega(t) + R_1$ for $a_n \leq t \leq c_n$. It can be checked that

$$F_n(t, x_n(t) - M\omega(t)) > a(t)(x_n(t) - M\omega(t)) + M \quad \text{for } t \in [a_n, c_n]. \tag{4.9}$$

In fact, if $t \in [a_n, c_n]$ is such that $1/n \leq x_n(t) - M\omega(t) \leq R_1$, we have

$$\begin{aligned} F_n(t, x_n(t) - M\omega(t)) &= F(t, x_n(t) - M\omega(t)) \\ &\geq a(t)(x_n(t) - M\omega(t)) + \tilde{g}_0(x_n(t) - M\omega(t)) \\ &> a(t)(x_n(t) - M\omega(t)) + M, \end{aligned}$$

and, if $t \in [a_n, c_n]$ is such that $x_n(t) - M\omega(t) \leq 1/n$, we have

$$\begin{aligned} F_n(t, x_n(t) - M\omega(t)) &= F(t, 1/n) \geq a(t)/n + \tilde{g}_0(a/n) \\ &> a(t)(x_n(t) - M\omega(t)) + M. \end{aligned}$$

So (4.9) holds.

Using Eq. (4.2) (with $\lambda = 1$) for $x_n(t)$ and estimate (4.9), we have, for $t \in [a_n, c_n]$,

$$\begin{aligned} x_n''(t) - M\omega''(t) &= -a(t)x_n(t) + F_n(t, x_n(t) - M\omega(t)) + a(t)/n - M[1 - a(t)\omega(t)] \\ &> -a(t)x_n(t) + a(t)(x_n(t) - M\omega(t)) + M + a(t)/n - M[1 - a(t)\omega(t)] \\ &= a(t)/n \geq 0. \end{aligned}$$

As $x'_n(a_n) - M\omega'(a_n) = 0$, $x'_n(t) - M\omega'(t) > 0$ for all $t \in (a_n, c_n]$ and the function $y_n := x_n - M\omega$ is strictly increasing on $[a_n, c_n]$. We use ξ_n to denote the inverse function of y_n restricted to $[a_n, c_n]$.

In order to prove (4.4) in this case, we will first show that, for $n \in N_1$,

$$x_n(t) - M\omega(t) \geq 1/n. \tag{4.10}$$

Otherwise, suppose that $\alpha_n < 1/n$ for some $n \in N_1$. Then there would exist $b_n \in (a_n, c_n)$ such that $x_n(b_n) - M\omega(b_n) = 1/n$ and

$$x_n(t) - M\omega(t) \leq 1/n \quad \text{for } a_n \leq t \leq b_n, \quad 1/n \leq x_n(t) - M\omega(t) \leq R_1 \quad \text{for } b_n \leq t \leq c_n.$$

Multiplying (4.2) (with $\lambda = 1$) by $x'_n(t) - M\omega'(t)$ and integrating from b_n to c_n , we obtain

$$\begin{aligned} \int_{1/n}^{R_1} F(\xi_n(y), y) dy &= \int_{b_n}^{c_n} F(t, x_n(t) - M\omega(t))(x'_n(t) - M\omega'(t)) dt \\ &= \int_{b_n}^{c_n} F_n(t, x_n(t) - M\omega(t))(x'_n(t) - M\omega'(t)) dt \\ &= \int_{b_n}^{c_n} (x_n''(t) + a(t)x_n(t) - a(t)/n)(x'_n(t) - M\omega'(t)) dt \\ &= \int_{b_n}^{c_n} x_n''(t)(x'_n(t) - M\omega'(t)) dt \\ &\quad + \int_{b_n}^{c_n} (a(t)x_n(t) - a(t)/n)(x'_n(t) - M\omega'(t)) dt. \end{aligned}$$

By the facts $\|x_n\| < r$ and $\|x'_n\| \leq H$, one can easily obtain that the second term is bounded. The first term is

$$([x'_n(c_n)]^2 - [x'_n(a_n)]^2)/2 - M(x'_n(c_n)\omega'(c_n) - x'_n(a_n)\omega'(b_n)) + M \int_{a_n}^{c_n} x'_n(t)\omega''(t) dt,$$

which is also bounded. As a consequence, there exists $L > 0$ such that

$$\int_{1/n}^{R_1} F(\xi_n(y), y) dy \leq L. \tag{4.11}$$

On the other hand, by (G₃), we can choose $n_2 \in N_1$ large enough such that

$$\int_{1/n}^{R_1} F(\xi_n(y), y) dy \geq \int_{1/n}^{R_1} g_0(y) dy > L$$

for all $n \in N_2 = \{n_2, n_2 + 1, \dots\}$. So (4.10) holds for $n \in N_2$.

As a last step, we will show that (4.4) is right in Case 2. To this end, multiplying (4.2) (with $\lambda = 1$) by $x'_n(t) - M\omega'(t)$ and integrating from a_n to c_n , we obtain

$$\begin{aligned} \int_{\alpha_n}^{R_1} F(\xi_n(y), y) dy &= \int_{a_n}^{c_n} F(t, x_n(t) - M\omega(t))(x'_n(t) - M\omega'(t)) dt \\ &= \int_{a_n}^{c_n} F_n(t, x_n(t) - M\omega(t))(x'_n(t) - M\omega'(t)) dt \\ &= \int_{a_n}^{c_n} (x_n''(t) + a(t)x_n(t) - a(t)/n)(x'_n(t) - M\omega'(t)) dt. \end{aligned}$$

(We notice that estimate (4.10) is used in the second equality above.) In the same way as in the proof of (4.11), one may readily prove that the right-hand side of the above equality is bounded. On the other hand, if $n \in N_2$, by (G₃),

$$\int_{\alpha_n}^{R_1} F(\xi_n(y), y) dy \leq \int_{\alpha_n}^{R_1} g_0(y) dy + M(R_1 - \alpha_n) \rightarrow +\infty$$

if $\alpha_n \rightarrow 0^+$. Thus we know that $\alpha_n \geq \delta$ for some constant $\delta > 0$ in Case 2.

Combining these two cases, we have the desired statement in the lemma. \square

Corollary 4.3. *Let the nonlinearity in (1.1) be (3.7), where $\alpha \geq 1$, $\beta \geq 0$, $b(t)$, $c(t) \in C[0, 1]$ are non-negative functions and $b(t) > 0$ for all t , $e(t) \in C[0, 1]$ and $\mu > 0$*

is a positive parameter. Then

- (i) if $\beta < 1$, (1.1) has at least one positive periodic solution for each $\mu > 0$; and
- (ii) if $\beta \geq 1$, (1.1) has at least one positive periodic solution for each $0 < \mu < \mu_*$, where μ_* is some positive constant.

Proof. We will apply Theorem 4.1 with $M = e_0 = \max_t |e(t)|$ and

$$g(x) = b_0x^{-\alpha}, \quad h(x) = \mu c_0x^\beta + 2e_0.$$

Then conditions (G₁)–(G₃) are satisfied and the existence condition (G₄) becomes

$$\mu < \frac{r(\sigma r - M\|\omega\|)^\alpha/\|\omega\| - (b_0 + 2e_0r^\alpha)}{c_0r^{\alpha+\beta}}$$

for some $r > \|\omega\|/\sigma$. So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu_* := \sup_{r > M\|\omega\|/\sigma} \frac{r(\sigma r - M\|\omega\|)^\alpha/\|\omega\| - (b_0 + 2e_0r^\alpha)}{c_0r^{\alpha+\beta}}.$$

Note that $\mu_* = \infty$ if $\beta < 1$ and $\mu_* < \infty$ if $\beta \geq 1$. We have the desired results. \square

Theorem 4.1 applies to the example when $f(t, x)$ has representation (3.8). A result similar to Corollary 4.3 (ii) holds for this example.

As an application of Theorem 4.1 (or Corollary 4.3), let us consider the equation

$$x'' + a^2x = bx^{-\alpha} + e(t) \tag{4.12}$$

with $a \in (0, \pi)$ and $b, \alpha > 0, e(t) \in C[0, 1]$. Let us recall

$$e_1 = \min_t e(t), \quad e_0 = \max_t e(t).$$

In [18], it is proved that (4.12) has a positive periodic solution if the following inequality holds:

$$e_1 > -(\alpha + 1)b \left(\frac{\pi^2 - a^2}{\alpha b} \right)^{\alpha/(\alpha+1)}. \tag{4.13}$$

In particular, (4.13) is fulfilled when $e_1 \geq 0$. In paper [20], Torres gave another existence condition: $e_1 < 0$ and

$$e_0 \leq \frac{e_1}{\cos^\alpha(a/2)} + (a \sin a) \left(\frac{b}{|e_1|} \right)^{1/\alpha}. \tag{4.14}$$

Both conditions (4.13) and (4.14) describe the dependence of the range of $e(t)$ upon the parameter $\alpha > 0$. However, if $\alpha \geq 1$, i.e., the strong force condition is satisfied, we obtain from Corollary 4.3 that (4.12) always has at least one positive periodic solution for any forcing $e(t)$.

Next we will establish the existence of twin positive solutions to Eq. (1.1) by using Theorem 2.3.

Theorem 4.4. *Suppose that conditions (A) and (G₁)–(G₄) hold. In addition, it is assumed that the following two conditions are satisfied:*

(G₅) *There exist continuous, non-negative functions $g_1(x)$ and $h_1(x)$ on $(0, \infty)$ such that*

$$F(t, x) = f(t, x) + M \geq g_1(x) + h_1(x) \quad \text{for all } (t, x) \in [0, 1] \times (0, \infty),$$

and $g_1(x) > 0$ is non-increasing and $h_1(x)/g_1(x)$ is non-decreasing in $x \in (0, \infty)$; and

(G₆) *There exists a positive number $R > r$ such that*

$$\frac{R}{\sigma g_1(R)(1 + h_1(\sigma R - M\|\omega\|)/g_1(\sigma R - M\|\omega\|))} \leq \|\omega\|.$$

Then, besides the periodic solution x constructed in Theorem 4.1, Eq. (1.1) has another positive periodic solution $\tilde{x} \in C[0, 1]$ with $r < \|\tilde{x} + M\omega\| \leq R$.

Proof. As in the proof Theorem 4.1, we only need to show that Eq. (4.1) has a periodic solution $u \in C[0, 1]$ with $u(t) > M\omega(t)$ and $r < \|u\| \leq R$.

Let $X = C[0, 1]$ and K be as in Section 2. Set $\Omega_1 = \Omega_r$ and $\Omega_2 = \Omega_R$. The operator $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ is defined by (2.11), with $F(t, x)$ replaced by $F(t, x - M\omega(t))$. Note that any $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ satisfies $0 < \sigma r - M\|\omega\| \leq x(s) - M\omega(s) \leq R$. Thus T is well defined. Now the fixed points of T in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ are positive periodic solutions of (4.1). As in the proof of Theorem 3.3, Theorem 2.3 can be applied to this T in the domain $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ because our conditions (G₅)–(G₆) are imposed to ensure that the condition (ii) of Theorem 2.3 is satisfied. Thus T has a fixed point u , which lies actually in $\bar{\Omega}_2 \setminus \bar{\Omega}_1$. Finally, $\tilde{x} = u - M\omega$ will be the desired positive periodic solution of (1.1).

We omit the details because they are much similar to that in the proof of Theorem 3.3. \square

Let us consider again example (3.7) in Corollary 3.2 for the superlinear case, i.e., $\beta > 1$. We assume also that $c(t) > 0$ for all t . To verify (G₅), one may simply take

$$g_1(x) = b_1 x^{-\alpha}, \quad h_1(x) = \mu c_1 x^\beta.$$

Now the existence condition (G₆) becomes

$$\mu \geq \frac{R^{\alpha+1}/(\sigma\|w\|) - b_1}{c_1(\sigma R - M\|\omega\|)^{\alpha+\beta}}. \quad (4.15)$$

Since $\beta > 1$, the right-hand side of (4.15) goes to 0 as $R \rightarrow +\infty$. Thus, for any given $0 < \mu < \mu_*$, where μ_* is as in Corollary 4.3, it is always possible to find such $R \gg r$ that (4.15) is satisfied. Thus, (1.1) has an additional periodic solution \tilde{x} such that $\|\tilde{x} + M\omega\| > r$.

Corollary 4.5. *Assume in (3.7) that $\beta > 1$ and $b(t) > 0$, $c(t) > 0$ for all t . Then, for each μ with $0 < \mu < \mu_*$, the corresponding equation (1.1) has at least two different positive periodic solutions.*

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